POLYHEDRAL PRODUCTS, FLAG COMPLEXES AND MONODROMY REPRESENTATIONS

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ABSTRACT. This article presents a machinery based on polyhedral products that produces faithful representations of graph products of finite groups and direct products of finite groups into automorphisms of free groups $\mathrm{Aut}(F_n)$ and outer automorphisms of free groups $\mathrm{Out}(F_n)$, respectively, as well as faithful representations of products of finite groups into the linear groups $\mathrm{SL}(n, \mathbb{Z})$ and $\mathrm{GL}(n, \mathbb{Z})$. These faithful representations are realized as monodromy representations.

CONTENTS

1. INTRODUCTION

Studying the topology of a fibration sequence frequently involves the monodromy action, which is the action of the fundamental group π of the base space B on the fibre F. When using a spectral sequence one may need to consider the homology of the base with coefficients in the homology of the fibre regarded as an $R\pi$-module, where $R\pi$ is the group ring. Here the monodromy representation for a fibration $p : E \to B$ with fibre F will mean the representation $\rho : \pi_1(B) \to \mathrm{Out}(H_1(F))$. The goal of this paper is to study polyhedral products in connection with monodromy

representations for certain fibrations that arise naturally in the field of *toric topology*. The problem of explicitly describing such representations was studied by the author in [23].

Numerous results on graph products of groups and polyhedral products demonstrate that the underlying simplicial complex K plays an important role in their study. For example Droms [11] proved that two graph products of groups are isomorphic if and only if the graphs are isomorphic. Servatius, Droms, and Servatius [22] determine the simplicial complexes for which the commutator subgroup of a right-angled Artin groups is free. Moreover, if K is chosen carefully, one obtains classifying spaces for various important families of discrete groups, including right-angled Artin and Coxeter groups from geometric group theory [23, 9]. In another application Grbić, Panov, Theriault and Wu [13] give conditions on the 1-skeleton of a flag complex K that determine when the face ring of K is a Golod ring, or equivalently the corresponding moment-angle complex has the homotopy type of a wedge of spheres. Recently Panov and Veryovkin [20] studied polyhedral products that have the homotopy type of classifying spaces of right-angled Artin groups and right-angled Coxeter groups.

In the present article we study further properties of the monodromy representations associated to the (homotopy) fibration sequences

$$(1) \qquad (\underline{EG}, \underline{G})^K \to (\underline{BG}, \underline{1})^K \to \prod_{i=1}^n BG_i.$$

Each space in (1) is a polyhedral product, depending on a simplicial complex K, together with a sequence of finite groups $\underline{G} := \{G_1, \ldots, G_n\}$, their classifying spaces $\underline{BG} := \{BG_1, \ldots, BG_n\}$, and corresponding universal covers $\underline{EG} := \{EG_1, \ldots, EG_n\}$, see Definition 2.1.

We give explicit descriptions of monodromy representations for simplicial complexes K with more than two vertices, which were described geometrically in [23]. To do this we generalize and use some results of Panov and Veryovkin [20]. We give applications, in particular to spaces of commuting elements in *commutative transitive (CT) finite groups*, where commutativity is a transitive relation, studied in a celebrated paper of M. Suzuki [25], and an application to a problem related to the Feit-Thompson theorem, which states that all groups of odd order are solvable. Finally, we give a couple of examples, which can be generalized using Magma [3].

Main results. For given finite discrete groups $G_1, \ldots, G_n$, we use polyhedral products to construct monodromy representations $\Phi : G_1 \times \cdots \times G_n \to \mathrm{Out}(F_N)$ into outer automorphism groups of free groups. In particular, we obtain explicit faithful representations of graph products of finite groups into automorphism groups of free groups, and faithful representations of their direct products into linear groups $\mathrm{SL}(k, \mathbb{Z})$ or $\mathrm{GL}(k, \mathbb{Z})$. This article presents a machinery based on polyhedral products to achieve this. The first result is the following theorem.

Theorem 1.1. *Let $G_1, \ldots, G_n$ be finite groups and K a simplicial complex with n vertices with 1-skeleton K^1 a chordal graph. Then there are faithful representations*

$$\Theta_K : \prod_{K^1} G_i \to \mathrm{Aut}(F_{\rho_K})$$

and faithful monodromy representations

$$\Phi_K : G_1 \times \cdots \times G_n \to \mathrm{Out}(F_{\rho_K}),$$

where ρ_K is the rank of the fundamental group of the fibre in equation (1).

The case when the groups $G_1, \ldots, G_n$ are abelian the representations can be described explicitly and convenient models of polyhedral products can then be used to show that the corresponding monodromy representations obtained for non-abelian finite groups are also faithful.

Theorem 1.2. *Let $G_1, \ldots, G_n$ be finite abelian groups. Then the faithful monodromy representation Φ_K induces a faithful representation*

$$\Phi_K : G_1 \times \cdots \times G_n \to \mathrm{SL}(\rho_K, \mathbb{Z}).$$

If $G_1, \ldots, G_n$ are non-abelian then Φ_K maps into $\mathrm{GL}(\rho_K, \mathbb{Z})$.

Let $E(2, G) \subseteq EG$ and $B(2, G) \subseteq BG$ be the spaces defined in §7 that classify commuting elements in a group G. In particular, we use polyhedral products to study the class of finite transitively commutative (CT) groups, a class of groups where commtutativity is transitive. The following theorem is then an application of polyhedral products to group theory.

Theorem 1.3. *Finite CT groups with trivial center are solvable if and only if the induced map $H_1(E(2, G); \mathbb{Z}) \to H_1(B(2, G); \mathbb{Z})$ is not surjective.*

This theorem is motivated from a result of Adem, Cohen and Torres-Giese [1], which states an equivalent topological condition to the the Feit-Thompson theorem, namely that the theorem is true if and only if the map $H_1(E(2, G); \mathbb{Z}) \to H_1(B(2, G); \mathbb{Z})$ is not surjective.

Structure of paper. In Section 2 we define polyhedral products and the fibration sequences we work with. We study the commutator subgroup of graph products of groups in Section 3, and find bases for the free groups, wich will be used in Section 4 to provide examples. We prove Theorems 1.1 and 1.2 in Section 5. Finally applications are given in Sections 6 and 7, where we also prove Theorem 1.3.

2. POLYHEDRAL PRODUCTS AND RELATED FIBRATIONS

Moment-angle complexes, originally invented by Davis and Januszckiewicz [8], appeared also in the work of Buchstaber and Panov [5] in the context of *toric topology*. Polyhedral products are a generalization of moment-angle complexes and were introduced and popularized by work of Bahri, Bendersky, Cohen and Gitler [2], and are the main objects of study in toric topology; see the more recent monograph by Buchstaber and Panov [6].

Definition 2.1. Let $(\underline{X}, \underline{A})$ denote a sequence of pointed CW-pairs $\{(X_i, A_i)\}_{i=1}^n$ and $[n]$ denote the sequence of integers $\{1, 2, \ldots, n\}$.

- A **simplex** σ is given by an increasing sequence of integers $\sigma = \{1 \leq i_1 < \cdots < i_q \leq n\} \subseteq [n]$. A **simplicial complex** K is a collection of simplices such that if $\tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$. In particular $\emptyset \in K$. Geometrically K is a subcomplex of the the $(n-1)$-simplex $\Delta[n-1]$. If K has only 0- and 1-simplices we call it a simplicial graph.
- The **polyhedral product** $(\underline{X}, \underline{A})^K$ is the subspace of the product $X_1 \times \cdots \times X_n$ given by the colimit

$$(\underline{X}, \underline{A})^K := \operatorname*{colim}_{\sigma \in K} \mathcal{D}(\sigma) = \bigcup_{\sigma \in K} \mathcal{D}(\sigma) \subseteq \prod_{i=1}^{n} X_i,$$

  where $\mathcal{D}(\sigma) = \{(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i | x_i \in A_i \text{ if } i \notin \sigma\}$, the maps are the inclusions, and the topology is the subspace topology of the product. Another standard notation for polyhedral products is $Z_K(\underline{X}, \underline{A})$. Sometimes polyhedral products are called K**-powers**. Since $\emptyset$ is in any K, we have $\prod_{i=1}^{n} A_i \subset (\underline{X}, \underline{A})^K \subset \prod_{i=1}^{n} X_i$.
- If the pairs (X, A) are (D^2, S^1) or (D^1, S^0), then the polyhedral products are called **moment-angle complexes** and **real moment-angle complexes**, respectively.
- If all the pairs in the sequence $(X_1, A_1), \ldots, (X_n, A_n)$ are equal to (X, A), then we omit the underline in the notation of the polyhedral product, and write simply $(X, A)^K$.

Definition 2.2. Next we give some relevant definitions and notation:
- For a simplicial complex K, the complex K^i denotes the i**-skeleton** of K.
- A simplicial complex K is called a **flag complex** if for any complete subgraph $\Gamma \subset K^1$, it also contains the simplex spanned by these vertices. The structure of moment angle complexes (or polyhedral products in general) is better understood when K is a flag complex [6, §8.5].
- For a simplicial complex K, let $\mathrm{Flag}(K)$ denote the **clique complex** of K^1, i.e. the simplicial complex whose simplices are complete subgraphs of K^1. For example a flag complex is the clique complex of its 1-skeleton.
- A graph is called **chordal** if every cycle of length greater than three has an edge (called a *chord*) connecting two nonconsecutive vertices. Chordal graphs are also called **triangulated** graphs.
- For any group G denote its abelianization by $\mathscr{A}(G) := G/[G, G]$, and the abelianization map by $\mathrm{ab}_G : G \twoheadrightarrow \mathscr{A}(G)$.
- Let $G_1, \ldots, G_n$ be a sequence of groups and Γ a simplicial graph on $[n]$. The **graph product** of $G_1, \ldots, G_n$ over Γ is the quotient of their free product by the normal closure of the relations $R_\Gamma := \{[g_i, g_i] : \{i, j\} \text{ is an edge in } \Gamma\}$. The group obtained this way will be called a **graph group**, even though in the literature this name is sometimes used for right-angled Artin groups. We denote it by

$$\prod_\Gamma G_i := (G_1 * \cdots * G_n)/\langle R_\Gamma \rangle.$$

  In this notation right-angled Artin groups are graph products of the group $\mathbb{Z}$.

Example 2.3.
1. Let X be the unit interval $[0, 1]$ and $A \subset [0, 1]$ be the subset $\{0, 1/2, 1\}$. Let K be the simplicial complex consisting of only two vertices $\{\{v_1\}, \{v_2\}\}$. Then

$\mathcal{D}(\{v_1\}) = X \times A \subset [0,1]^2$ and $\mathcal{D}(\{v_2\}) = A \times X \subset [0,1]^2$. Therefore, $(X, A)^K = \mathcal{D}(\{v_1\}) \cup \mathcal{D}(\{v_2\}) = X \times A \cup A \times X$ is a graph inside the square $[0,1]^2$, homotopy equivalent to a wedge of 4 circles $\bigvee_4 S^1$. Similarly, we can choose A to be any finite subset of the unit interval and we obtain similar graphs.

2. Let $(X, A) = (D^2, S^1)$. If K is a the boundary of the n-simplex then the moment-angle complex $(D^2, S^1)^K = \bigcup_{\sigma_i} \mathcal{D}(\sigma_i)$ is homeomorphic to the sphere $S^{2n+1} = \partial D^{2(n+1)}$. It is also known [13, Theorem 4.6] that if K is a flag complex, then $(D^2, S^1)^K$ has the homotopy of a wedge of spheres if and only if K^1 is chordal.

3. If K is any simplicial complex on n vertices, and $\underline{*} = \{*_1, \ldots, *_n\}$ is the sequence of basepoints then $\underline{X}$ then $(\underline{X}, \underline{*})^{K^0} = \bigvee_{i=1}^{n} X_i$. Therefore, in general $\bigvee_{i=1}^{n} X_i \subseteq (\underline{X}, \underline{*})^K \subseteq \prod_{i=1}^{n} X_i$.

Let G be a topological group with basepoint its identity element $* = 1$, BG be the classifying space of G, with $B1 \simeq 1 = *$, and EG be a (weakly) contractible space with a free action of G such that the quotient map $EG \to BG$ is a principal G-bundle. G. Denham and A. Suciu [10, Lemma 2.3.2] gave a natural fibration relating the polyhedral product for the pair $(BG, 1)$ to the polyhedral product for the pair (EG, G). That is, for a simplicial complex K with n vertices, the polyhedral product $(BG, 1)^K$ fibres over the product $(BG)^n$ as follows

$$(2) \qquad (EG, G)^K \to (EG)^n \times_{G^n} (EG, G)^K \to (BG)^n,$$

where the total space is homotopy equivalent to $(BG, 1)^K$. Note that the group G acts coordinate–wise on the fibre $(EG, G)^K \subset (EG)^n$.

This fibration is a generalization of the Davis-Januszkiewicz space [8], with topological group the circle S^1, given by the Borel construction

$$\mathcal{DJ}(K) = (ES^1)^n \times_{(S^1)^n} (ES^1, S^1)^K,$$

which describes a cellular realization of the Stanley-Reisner ring of K, in the sense that the cohomology ring of $\mathcal{DJ}(K)$ is precisely the Stanley-Reisner ring of K defined as the quotient of the polynomial ring $R[K] = R[x_1, \ldots, x_n]/I_K$ by the Stanley-Reisner ideal $I_K = \langle x_{i_1} \cdots x_{i_t} | \{i_1, \ldots, i_t\} \neq K \rangle$. A later result of V. Buchstaber and T. Panov [4] showed that $\mathcal{DJ}(K) \simeq (BS^1, 1)^K$ and the homotopy fibre of the natural inclusion $\mathcal{DJ}(K) \to (\mathbb{C}P^\infty)^n$ is equivalent to the polyhedral product $(ES^1, S^1)^K$.

If $G_1, \ldots, G_n$ is a sequence of topological groups, then for any simplicial complex K with n vertices, the fibration sequence (2) can be generalized to obtain

$$(3) \qquad (\underline{EG}, \underline{G})^K \to (\underline{BG}, \underline{1})^K \to \prod_{i=1}^{n} BG_i.$$

Similarly, the fundamental group of the base space acts naturally coordinate-wise on the homotopy fibre. The monodromy representation of this fibration is the main object of study in this article.

Note that the homotopy type of a polyhedral product depends only on the relative homotopy type of the pairs (X, A), as observed in [10]. We are mainly interested in the cases when $G_1, \ldots, G_n$ are finite discrete groups. If the pairs $(\underline{EG}, \underline{G})$

are replaced by $(\underline{I}, \underline{F})$, where I is the unit interval and $F_i \subset I$ has the cardinality of G_i, then there is a homotopy equivalence $(\underline{EG}, \underline{G})^K \simeq (\underline{I}, \underline{F})^K$. Moreover, if $K = K^0$ is the 0-skeleton of K, then it follows from Example 2.3 that $(\underline{BG}, \underline{1})^{K^0} = BG_1 \vee \cdots \vee BG_n$. The homotopy fibre $(\underline{I}, \underline{F})^{K^0}$ has the homotopy type of a finite wedge of circles $(\underline{I}, \underline{F})^{K^0} \simeq \bigvee_{\rho_{K^0}} S^1$, as shown in [24], where

$$(4) \qquad \rho_{K^0} = (n-1)\prod_{i=1}^{n} |G_i| - \sum_{i=1}^{n}(\prod_{j \neq i} |G_j|) + 1.$$

This gives a topological proof of a classical theorem of J. Nielsen [17] concerning the rank of the free group in the following short exact sequence of groups

$$1 \to F_{\rho_{K^0}} \to G_1 * \cdots * G_n \to \prod_{1 \leq i \leq n} G_i \to 1.$$

Therefore, the rank ρ_{K^0} depends only on the order of the groups $G_1, \ldots, G_n$, and not on their group structure. For simplicity we simply write ρ_{K^0} for the rank, when the orders of G_i are clear from the context.

More generally, it was shown in [24] that if $G_1, \ldots, G_n$ are countable discrete groups then the spaces in the fibration sequence (3) are Eilenberg-MacLane spaces of the type $K(\pi, 1)$ if and only if K is a flag complex. Moreover, for any K, the fundamental group of the polyhedral product $(\underline{BG}, \underline{1})^K$ is determined by the 1-skeleton K^1 and is isomorphic to $\pi_1((\underline{BG}, \underline{1})^K) \cong \prod_{K^1} G_i$, the graph product of the groups $G_1, \ldots, G_n$. Hence we obtain a short exact sequence of groups, which is true also for K not necessarily a flag complex:

$$(5) \qquad 1 \to \pi_1((\underline{EG}, \underline{G})^K) \to \prod_{K^1} G_i \to \prod_{1 \leq i \leq n} G_i \to 1.$$

We want to study simplicial complexes K for which the kernel of the short exact sequence above is a free group. The following theorem shows exactly which simplicial complexes K have this property.

Theorem 2.4. *Let $G_1, \ldots, G_n$ be (countable) discrete groups and K be a flag complex on n vertices. Then $(\underline{EG}, \underline{G})^K$ has the homotopy type of a graph if and only if K^1 is a chordal graph.*

Proof. It was shown in [24, Theorem 1.1] that the space $(\underline{EG}, \underline{G})^K$ is a $K(\pi, 1)$ if and only if K is a flag complex. Therefore, we get the short exact sequence of groups in (5). Panov and Veryovkin [20, Theorem 4.3] showed that $\pi_1((\underline{EG}, \underline{G})^K)$ is free if and only if the graph K^1 is a chordal graph, which completes the proof. See also [22, Theorem 4.2] for a relevant result. $\square$

3. Commutator subgroups of graph groups

The *commutator subgroup* $[G, G]$ of a group G is generated by commutators $[g, h] := ghg^{-1}h^{-1}$ with $g, h \in G$. Let $G_1, \ldots, G_n$ be finite groups and K be a flag complex with K^1 a chordal graph. From the previous section we know that under these assumptions the kernel of the projection map

$$(6) \qquad p : \prod_{K^1} G_i \to \prod_{1 \leq i \leq n} G_i$$

is a free group. Denote the rank of $\ker(p)$ by ρ_K. This kernel is generated by iterated commutators of the form

$$[g_{j_1}, [g_{j_2}, [\ldots, [g_{j_k}, g_{j_{k+1}}] \cdots]]],$$

where g_{j_i} belong to distinct G_{j_i}. The kernel $\ker(p)$ is not necessarily the commutator subgroup of $\prod_{K^1} G_i$, if at least one of the G_i is not abelian. However, $\ker(p)$ coincides with the commutator subgroup of the graph group if the groups $G_1, \ldots, G_n$ are all abelian.

In this section we describe a basis for the free group $\ker(p) = F_{\rho_K}$ in terms of iterated commutators. A basis was given in [20, Lemma 4.7], where the groups under consideration had order 2, that is the graph groups were right-angled Coxeter groups. Another version of this basis of commutators was studied by Grbić, Panov, Theriault, and Wu in the context of exterior algebras in [13, Theorem 4.3].

Before we proceed it is important to note that the commutator subgroup of a free group can also be described by a generating set not consisting of commutators. One can obtain new presentations not involving commutators using Tietze transformations [16, 27].

Recall that the fibre in (3) depends only on the order of the finite groups $G_1, \ldots, G_n$, since its homotopy type depends on the relative homotopy type of the pairs (EG_i, G_i) (this is true for any polyhedral product – see [10, p.31]). Therefore, it suffices to describe the basis elements (i.e. iterated commutators) of $\ker(p) = F_{\rho_K}$ only when $G_1, \ldots, G_n$ are cyclic groups. The basis for the general case of any finite groups $G_1, \ldots, G_n$ can be obtained by considering the basis when G_i are all cyclic and then replacing the entries in the commutators with the nontrivial elements of G_i. This observation will be used in Section 5.

Proposition 3.1. *Let $G_1, \ldots, G_n$ be finite groups and $K = K^0$. Then the fundamental group of the fibre $(\underline{EG}, \underline{G})^K$ in (3) is the free group with basis consisting of the following iterated commutators*

(7) $$[g_j, g_i], \ [g_{k_1}, [g_j, g_i]], \ \ldots, [g_{k_1}, \ldots, [g_{k_l}, [g_j, g_i]] \ldots],$$

where $g_t \in G_t$, with $k_1 < \cdots < k_l < j$ and $j > i \neq k_r$, for all r.

Proof. We need to show that (1) this set of elements generates the fundamental group, and that (2) the number of elements in the set equals the rank of the free group in equation (4). Since the first part of the proof is essentially the proof of [20, Lemma 4.7], we only give an outline here. First recall the *Hall identities* for group elements a, b, c

(8) $$[a, bc] = [a, c][a, b][[a, b], c], \text{ and } [ab, c] = [a, c][[a, c], b][b, c],$$

and if x is a commutator we can write

(9) $$[g_j, [g_i, x]] = [g_j, x][x, [g_i, g_j]][g_j, g_i][x, g_i][g_i, [g_j, x]][x, g_j][g_i, g_j][g_i, x].$$

Therefore, given the equations (8) and (9) we proceed as follows:

– we can use the identities above to switch between the commutators $[g_j, [g_i, x]]$ and $[g_i, [g_j, x]]$ by using other commutators of lower degrees, we can change the order of $g_{k_1}, \ldots, g_{k_l}, g_j$ in the commutator $[g_{k_1}, \ldots, [g_{k_l}, [g_j, g_i]] \ldots]$ to have them in increasing order, so we can thus obtain the inequalities in the proposition;

– we can use the identities to eliminate commutators with two entries from the same group, since we can reorder the terms to have these two entries next to

each other, then their product is in the same group, hence having a lower degree commutator. For example, from (8) we can write $[a, [b, c]]$ in terms of $[ca, b]$ and other lower degree comutators, and if a, c are from the same group, then we have reduced the degree of the original commutator (also verifying the statement following (6) about the kernel for K^0);

– we can thus assume that the commutators have the prescribed order, and that g_k, g_l are from different groups if $k \neq l$;

– finally we obtain a generating set for the the free group $F_{\rho(n)}$ in terms of commutators $[g_{k_1}, \ldots, [g_{k_l}, [g_j, g_i]] \ldots]$, with $k_1 < \cdots < k_l < j$ and $j > i \neq k_r$, for all r. Call this set $\mathcal{S}$.

Note that $\mathcal{S}$ does not generate the commutator subgroup, unless all G_i are abelian. Now we need to show that this generating set is minimal, that is $|\mathcal{S}| = \rho_K$ defined in (4). For this we use induction on the number n of vertices of K. Let us denote $\rho_K = \rho(n)$ when $K = K^0$ consists of only n vertices.

Assume $G_1, \ldots, G_n$ have orders $m_1, \ldots, m_n$, respectively. For $n = 2$, clearly $|\mathcal{S}| = |\{(g_j, g_i)|g_j, g_i \neq 1\}| = (m_1 - 1)(m_2 - 1) = \rho_K = \rho(2)$. Suppose this is true for $n = k$. For $n = k + 1$ we claim that

$$\rho(k + 1) = m_{k+1}\rho(k) + (m_{k+1} - 1)(\prod_{1 \leq i \leq k} m_i - 1).$$

When we introduce a new group G_{k+1}, since it has the highest index, according to our assumption, its elements come only second from the last in the iterated commutator. This yields $m_{k+1}\rho(k)$ generators, by taking a commutator and placing the elements of G_{k+1} second from last in the iterated commutators, giving a higher degree commutator. The insertion of a new non-trivial element, gives more freedom to the last element in the iterated commutator. For each non-trivial element $1 \neq g_i \in G_{k+1}$, the last entry can take $\prod_{1 \leq i \leq k} m_i - 1$ values. Now counting for each element of the new group, gives the second term in our claim, hence proving the claim. Combining equation (4) and the claim, and rearranging the terms, the minimality of $\mathcal{S}$ follows. □

Example 3.2. Let $G_1 = \mathbb{Z}_2 = \{1, x\}$, $G_2 = \mathbb{Z}_2 = \{1, y\}$, $G_3 = \mathbb{Z}_3 = \{1, z, z^2\}$, and $K = \{\{1\}, \{2\}, \{3\}\}$. Then the fibre of the fibration has fundamental group the free group F_9 with a minimal generating set $\mathcal{S}$ given by

$$\mathcal{S} = \{[z, x], [z^2, x], [z, y], [z^2, y], [y, x], [x, [z, y]], [x, [z^2, y]], [y, [z, x]], [y, [z^2, x]]\}.$$

For simplicial complexes K strictly larger than their 0-skeleton the following proposition holds.

Proposition 3.3. *Let $G_1, \ldots, G_n$ be finite groups and K be a flag complex with n vertices such that K^1 is a chordal graph. Then the fundamental group of the fibre in (3) is a free group with a basis the iterated commutators*

$$[g_j, g_i], \ [g_{k_1}, [g_j, g_i]], \ \ldots, [g_{k_1}, \ldots, [g_{k_l}, [g_j, g_i]] \ldots],$$

where $g_t \in G_t$, with $k_1 < \cdots < k_l < j$ and $j > i \neq k_r$, for all r, and i is the smallest vertex in a component not containing j in the subcomplex of K restricted to $\{k_1, \ldots, k_l, j, i\}$.

When we start introducing edges in K^0, then we start introducing commutator relations $[g_i, g_j]$ whenever $\{i, j\}$ is an edge. In the iterated commutator

$[g_{k_1}, \ldots, [g_{k_l}, [g_j, g_i]] \ldots]$ if i, j are in the same connected component of K restricted to $\{k_1, \ldots, k_l, j, i\}$, then there is a path from i to j with coordinates from $\{k_1, \ldots, k_l\}$, hence we can consider the iterated commutator induced by these vertices. Using relations from the edges we can reduce this commutator to another commutator of shorter length etc. Thus we can choose i, j to be in different path components. If we have two commutators where the last coordinate is in the same component, one can show that we can write one in terms of the other. Hence, we choose the smallest between them. We leave it to the reader to check that the detailed arguments in the proof of [20, Theorem 4.5] work also for any selection of finite groups.

Example 3.4. Let us consider an example with the symmetric group on 3 letters. Let $G_1 = \Sigma_3 := \langle s, t | s^2 = t^2, (st)^3 \rangle = \{1, x_1, x_2, x_3, x_4, x_5\}$, $G_2 = \mathbb{Z}_2 = \{1, y\}$, $G_3 = \mathbb{Z}_3 = \{1, z, z^2\}$, and $K'' = \{\{1, 2\}, \{3\}\}$ in Figure 1. Then the fibre of the fibration has fundamental group the free group F_{22} with $\mathcal{S}$ given by

$$\mathcal{S} = \{[z, x_1], [z, x_2], [z, x_3], [z, x_4], [z, x_5], [z^2, x_1], [z^2, x_2], [z^2, x_3], [z^2, x_4], [z^2, x_5],$$
$$[z, y], [z^2, y], [x_1, [z, y]], [x_2, [z, y]], [x_3, [z, y]], [x_4, [z, y]], [x_5, [z, y]]$$
$$[x_1, [z^2, y]], [x_2, [z^2, y]], [x_3, [z^2, y]], [x_4, [z^2, y]], [x_5, [z^2, y]]\}.$$

Note that the structure of the symmetric group Σ_3 was not needed to write the generating set $\mathcal{S}$. Therefore, if we replace Σ_3 with the cyclic group of order six C_6, then the corresponding generating set $\mathcal{S}$ has the same number and types of generators where in the commutators in S we replace the elements of the symmetric group with those of the cyclic group.

4. Examples of monodromy representations

Let $G_1, \ldots, G_n$ be finite discrete groups and K be a flag complex with K^1 a chordal graph. Consider the following commutative diagram

(10)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_{\rho_K} & \longrightarrow & \prod_{K^1} G_i & \longrightarrow & \prod_i G_i & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \Psi=\text{iso}} & & \downarrow{\scriptstyle \Theta_K} & & \downarrow{\scriptstyle \Phi_K} & & \\ 1 & \longrightarrow & \text{Inn}(F_{\rho_K}) & \longrightarrow & \text{Aut}(F_{\rho_K}) & \longrightarrow & \text{Out}(F_{\rho_K}) & \longrightarrow & 1, \end{array}$$

where $\Theta(g)(h) = ghg^{-1}$ and $\Psi(g)(h) = ghg^{-1}$. We are interested in describing the maps Θ_K and Φ_K.

For examples concerning only two finite groups, i.e. $n = 2$, see [23], where explicit answers are given. We can explicitly describe faithful representations (eg. by using `Magma`)

$$\Phi_K : G_1 \times \cdots \times G_n \to \text{SL}(\rho_K, \mathbb{Z}),$$

where $G_1, \ldots, G_n$ are finite abelian groups and $n \geq 3$. In general, if G_i are any finite groups (not necessarily abelian), we obtain faithful representations of graph products of finite groups

$$\Theta_K : \prod_{K^1} G_i \to \text{Aut}(F_{\rho_K})$$

as well as faithful monodromy representations of direct products of finite groups

$$\Phi_K : G_1 \times \cdots \times G_n \to \text{GL}(\rho_K, \mathbb{Z})$$

as will be shown below. These include many interesting classes of discrete groups, such as right-angled Coxeter groups. If one of the groups is infinite discrete, then additional examples include hyperbolic groups as described in [14, Theorem 5.1], braid groups, right-angled Artin groups and more. Thus such representations can be realized as monodromy representations.

The rank ρ_K increases very fast (4) if we increase the order and the number of the groups in consideration. We concentrate on a couple of examples including right-angled Coxeter groups. In addition we select the simplicial complexes K and K' in Figure 1, to keep the rank of the free group small. It is certainly possible to obtain many more explicit examples, which we leave to the interested reader. However, note that the basis generated in Magma is different (yet equivalent) from the basis we describe in Section 3. To do the following examples it suffices to have Theorem 2.4 and the basis generated by Magma, but we need Propositions 3.1 and 3.3 to have an explicit basis in general.
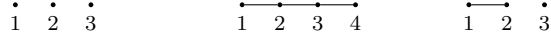
$$
\begin{array}{ccc}
\bullet \quad \bullet \quad \bullet & \bullet\!\!-\!\!\bullet \quad \bullet\!\!-\!\!\bullet & \bullet\!\!-\!\!\bullet \quad \bullet \\
1 \quad 2 \quad 3 & 1 \quad 2 \quad 3 \quad 4 & 1 \quad 2 \quad 3
\end{array}
$$

FIGURE 1. K, K', and K''

Example 4.1. Consider three groups of order 2 and the following short exact sequence obtained from the fibration sequence (3)

$$1 \to F_5 \to \mathbb{Z}_2 * \mathbb{Z}_2, *\mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2, \times\mathbb{Z}_2 \to 1,$$

corresponding to the simplicial complex K in Figure 1, where each of the cyclic groups is generated by a, b and c, respectively. Recall that the rank of the fibre is given by equation (4) and in this case is 5. Then F_5 has a generating set (thus a presentation) given by

$$F_5 := \langle (ba)^2, (ca)^2, (cb)^2, acbcba, bcacba \rangle = \langle x_1, x_2, x_3, x_4, x_5 \rangle.$$

The action of $\mathbb{Z}_2 \times \mathbb{Z}_2, \times\mathbb{Z}_2$ on F_5 is determined by the following:

$$
a \cdot x_i = \begin{cases} x_1^{-1} \\ x_2^{-1} \\ x_4 \\ x_3 \\ x_5^{-1} \end{cases}
\quad
b \cdot x_i = \begin{cases} x_1^{-1} \\ x_5 x_1^{-1} \\ x_3^{-1} \\ x_1 x_4^{-1} x_1^{-1} \\ x_2 x_1^{-1} \end{cases}
\quad
c \cdot x_i = \begin{cases} x_3 x_5 x_4^{-1} x_2^{-1} & \text{if } i = 1 \\ x_2^{-1} & \text{if } i = 2 \\ x_3^{-1} & \text{if } i = 3 \\ x_2 x_4^{-1} x_2^{-1} & \text{if } i = 4 \\ x_3 x_1 x_4^{-1} x_2^{-1} & \text{if } i = 5, \end{cases}
$$

where a, b, c act by conjugation. This gives also a faithful representation of the right-angled Coxeter group to the automorphism group $\mathrm{Aut}(F_5)$. It is straightforward to check that the induced action on the abelianization $\mathbb{Z}^5$ gives a faithful representation of the right-angled Coxeter group to the special linear group over the integers:

$$
\Phi_K : \mathbb{Z}_2 \times \mathbb{Z}_2, \times\mathbb{Z}_2 \to \mathrm{SL}(5, \mathbb{Z})
$$
$$
a, b, c \mapsto X, Y, Z,
$$

where X, Y, Z are the following matrices, respectively:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

Note that the generators of the fundamental group of the polyhedral product $(E\mathbb{Z}_2, \mathbb{Z}_2)^K$ can be described using the loops in Figure 2 sitting in the space $(I, F)^K$.
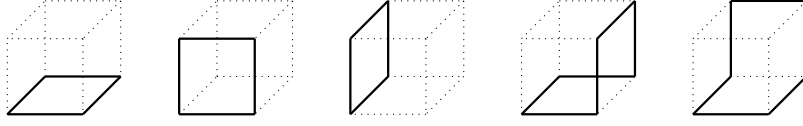

FIGURE 2. The generators x_1, x_2, x_3, x_4, and x_5, respectively

Example 4.2. Now we consider four cyclic groups. Construct the right-angled Coxeter group over the simplicial complex K' given in Figure 1. Then equation (3) gives the following short exact sequence of groups

$$1 \to F_5 \to \prod_{K'} \mathbb{Z}_2 \to \mathbb{Z}_2^4 \to 1,$$

where $F_5 = \langle x_1, \ldots, x_5 \rangle := \langle (ca)^2, (da)^2, (db)^2, adbdba, cdadca, \rangle$. The conjugation action is then described as follows:

$$a \cdot x_i = \begin{cases} x_1^{-1} \\ x_2^{-1} \\ x_4 \\ x_3 \\ x_5^{-1} \end{cases} \quad b \cdot x_i = \begin{cases} x_1 \\ x_3^{-1} x_2 x_4 \\ x_3^{-1} \\ x_4^{-1} \\ x_3^{-1} x_5 x_4 \end{cases} \quad c \cdot x_i = \begin{cases} x_1^{-1} \\ x_5 x_1^{-1} \\ x_3 \\ x_1 x_4 x_1^{-1} \\ x_2 x_1^{-1} \end{cases} \quad d \cdot x_i = \begin{cases} x_5 x_2^{-1} \\ x_2^{-1} \\ x_3^{-1} \\ x_2 x_4^{-1} x_2^{-1} \\ x_1 x_2^{-1} \end{cases}$$

for all values of $i = 1, 2, 3, 4, 5$, respectively. Then there is a representation of the right-angled Coxeter group $\prod_{K'} \mathbb{Z}_2$ into the automorphism group $\mathrm{Aut}(F_5)$.

$$\prod_{K'} \mathbb{Z}_2 \hookrightarrow \mathrm{Aut}(F_5).$$

The induced action on the abelianization of the free group gives a faithful representation

$$\Phi_{K'} : \mathbb{Z}_2^4 \to \mathrm{SL}(5, \mathbb{Z})$$
$$a, b, c, d \mapsto A, B, C, D$$

where A, B, C, D are the following matrices, respectively:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

One can start with any finite groups $G_1, \ldots, G_n$ with given presentations and any simplicial complex K with on n vertices such that K^1 is a chordal graph. If either of the groups is not abelian, then the representations obtained in the abelianization may not have images in $\mathrm{SL}(\rho_K, \mathbb{Z})$, but rather in $\mathrm{GL}(\rho_K, \mathbb{Z})$ as shown in [24, Example 2].

5. GRAPH PRODUCTS OF ABELIAN GROUPS

Every finite abelian group can be written as a finite direct sum of finite cyclic subgroups with order a power of a prime. Here we will describe how to think of a graph product of finite abelian groups over K as a graph product of cyclic groups over a new simplicial complex $K_{\underline{G}}$, at the expense of having more vertices in the simplicial complex.

In [23, Theorem 2.2] it was shown that two cyclic subgroups yield a faithful monodromy representation

$$C_n \times C_m \to \mathrm{Out}(F_{\rho_K})$$

and a faithful representation

$$\Phi_K : C_n \times C_m \to \mathrm{SL}(\rho_K, \mathbb{Z}),$$

where $K = \{\{1\}, \{2\}\}$.

Consider two finite abelian groups G, H. Then we can write them as direct sums if cyclic groups

$$G \cong \bigoplus_{i \in I} C_{n_i}, \quad H \cong \bigoplus_{j \in J} C_{m_j}.$$

We can then replace the simplicial complex $K = \{\{1\}, \{2\}\}$ by the union of two simplices

$$K' = \Delta[|I| - 1] \sqcup \Delta[|J| - 1].$$

This does not change the monodromy representation $G \times H \to \mathrm{Out}(F_{\rho_K})$, because the short exact sequences of groups

$$1 \to F_{\rho_K} \to G * H \to G \times H \to 1,$$

and

$$1 \to F_{\rho_{K'}} \to \prod_{(K')^1} C_{n_i, m_j} \to \prod_{i,j} C_{n_i, m_j} \to 1,$$

are equivalent; note that $\prod_{i,j} C_{n_i, m_j} \cong G \times H$, and

$$\prod_{(K')^1} C_{n_i, m_j} \cong \prod_{(\Delta[|I|-1])^1} C_{n_i} * \prod_{(\Delta[|J|-1])^1} C_{m_j} \cong G * H.$$

More generally, if K is a flag complex with K^1 a chordal graph, and $G_1, \ldots, G_n$ are finite abelian, write the direct sum decomposition of each group G_i

$$G_i \cong \bigoplus_{j \in J_i} C_{n_j^i}.$$

Replace each vertex i on the chordal graph K^1 by the simplex $\Delta[|J_i| - 1]$. Give unique names to all vertices in all these different simplices. Note that, if $\{i, j\}$ is an edge in K, and $\Delta[|J_i| - 1] = \{\{v_0^i, \cdots, v_{|J_i|-1}^i\}\}$, then we need to add $\{v_k^i, v_t^j\}$ to the new simplicial complex for all k, t since the G_i, G_j commute if and only if all their subgroups commute. We can now define the following simplicial complex.

Definition 5.1. Let $\underline{G} := \{G_1, \ldots, G_n\}$ be finite abelian groups. Let K be a simplicial complex on n vertices with 1-skeleton K^1 a chordal graph. Define the **simplicial complex** $K_{\underline{G}}$ to be the flag complex obtained from K by the following procedure: replace each vertex i of K with the full simplex $\Delta[|J_i| - 1] = \{\{v_0^i, \cdots, v_{|J_i|-1}^i\}\}$, add an edge between the vertices in $\Delta[|J_k| - 1]$ and the vertices in $\Delta[|J_l| - 1]$ if G_k and G_l commute in the graph product $\prod_{K^1} G_i$, and take the corresponding clique complex of the 1-skeleton of this new simplicial complex.

We then have the following lemma.

Lemma 5.2. *With the same assumptions, the graph $(K_{\underline{G}})^1$ is chordal.*

Proof. This follows from the definition: The 1-skeleton of $\Delta[|J_i| - 1]$ and K^1 are chordal graphs. If c is a cycle of length greater than 3, then its edges are either all in $(\Delta[|J_i|-1])^1$ for some i or it moves between various 1-skeleta $(\Delta[|J_k|-1])^1$. Suppose c has length 4. If vertices of c are all in a single $(\Delta[|J_i|-1])^1$ we are done. If vertices of c lie in two distinct $(\Delta[|J_i|-1])^1$'s, then there is one edge between $(\Delta[|J_k|-1])^1$ and $(\Delta[|J_l|-1])^1$, thus there is an edge between all the vertices between these two simplices, in particular between nonconsecutive vertices. If vertices of c lie in three distinct $(\Delta[|J_i|-1])^1$'s, the same argument holds. If vertices of c lie in four distinct $(\Delta[|J_i|-1])^1$'s, then c is a replica of a cycle in K. The same arguments show the triangulation of longer cycles c . $\qquad\square$

Theorem 5.3. *Let $G_1, \ldots, G_n$ be finite abelian groups and K^1 a chordal graph. Then the faithful monodromy representation $G_1 \times \cdots \times G_n \to \mathrm{Out}(F_{\rho_K})$ induces a faithful representation*

$$\Phi_K : G_1 \times \cdots \times G_n \to \mathrm{SL}(\rho_K, \mathbb{Z}).$$

Proof. By Lemma 5.2, the graph $(K_{\underline{G}})^1$ is chordal and by definition $K_{\underline{G}}$ is a flag complex. Therefore the spaces in fibration (3) are Eilenberg-MacLane spaces. Furthermore, the monodromy representation

$$\Phi_K : G_1 \times \cdots \times G_n \to \mathrm{Out}(F_{\rho_K})$$

is equivalent to the monodromy representation

$$\Phi_{K_{\underline{G}}} : G_1 \times \cdots \times G_n \to \mathrm{Out}(F_{\rho_{K_{\underline{G}}}}),$$

where $F_{\rho_{K_{\underline{G}}}} = F_{\rho_K}$ and we rewrite

$$G_i \cong \bigoplus_{j \in J_i} C_{n_j^i}.$$

Each element in G_i lies in a cyclic group, which by [23, Theorem 2.2] maps faithfully into $\mathrm{SL}(\rho_K, \mathbb{Z})$. Since $G_1 \times \cdots \times G_n$ is abelian the theorem follows. $\qquad\square$

Corollary 5.4. *If K^1 is a chordal graph, then there is a faithful representation of the graph product $\prod_{K^1} G_i$ into the automorphism group $\mathrm{Aut}(F_{\rho_K})$ of the free groups of rank ρ_K. In particular, this is true for any right-angled Coxeter group.*

Proof. This follows by considering the commutative diagram (10) since the left vertical map is an isomorphism and the right vertical map is an injection. $\qquad\square$

Recall that there is a short exact sequence of groups

$$1 \to \mathrm{IA}_N \to \mathrm{Aut}(F_N) \to \mathrm{GL}(N, \mathbb{Z}) \to 1$$

induced by the abelianization of the automorphisms of free groups, that is the induced map on the first homology $H_1(F_N)$. The group IA_N is the analogue of the Torelli group in mapping class groups of surfaces. Then we have the following immediate corollary.

Corollary 5.5. *If K^1 is a chordal graph and G_i are finite discrete groups, then the images of $\prod_{K^1} G_i$ under the faithful representations above are not in IA_{ρ_K}.*

6. Induced maps in homology

In this section we prove the following proposition.

Proposition 6.1. *Let K be a flag complex and $G_1, \ldots, G_n$ be finite groups. Then the induced map on first homology groups*

$$H_1((\underline{EG}, \underline{G})^K; \mathbb{Z}) \to H_1((\underline{BG}, \underline{1})^K; \mathbb{Z})$$

is the zero map.

Proof. The main ingredient in this proof is the fact that the abelianizations of both the fundamental group $\pi_1((\underline{BG}, \underline{1})^K) \cong \prod_{K^1} G_i$ and the product $\prod_{1 \leq i \leq n} G_i$ are the same. For a group G denote $\mathscr{A}(G) := G/[G, G]$ and the abelianization map $G \to \mathscr{A}(G)$ by ab_G. Note that the abelianization of $\prod_{K^1} G_i$ factors through the group $\prod_{1 \leq i \leq n} G_i$:

$$\prod_{K^1} G_i \to \prod_{1 \leq i \leq n} G_i \xrightarrow{\mathrm{ab}_G} \mathscr{A}(\prod_{K^1} G_i).$$

Let $G := \prod_{K^1} G_i$, $H := \prod_{1 \leq i \leq n} G_i$, and $N = \pi_1((\underline{EG}, \underline{G})^K)$. Note that in general, for any abelian group A and a surjection $h : G \twoheadrightarrow A$, there is a unique map $\phi : \mathscr{A}(G) \twoheadrightarrow A$ such that $\phi \circ \mathrm{ab}_G = h$.

Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & N & \xrightarrow{\ i\ } & G & \xrightarrow{\ p\ } & H & \longrightarrow & 1, \\
& & \downarrow{\scriptstyle \mathrm{ab}_N} & & \downarrow{\scriptstyle \mathrm{ab}_G} & & \downarrow{\scriptstyle \mathrm{ab}_H} & & \\
& & \mathscr{A}(N) & \xrightarrow{\ f\ } & \mathscr{A}(G) & \xrightarrow{\ =\ } & \mathscr{A}(H) & &
\end{array}
$$

where $p \circ i = 1$ (or 0 if H is abelian), and $\mathrm{ab}_G \circ i = \mathrm{ab}_H \circ p \circ i = 0$.

Now, since $\mathscr{A}(G)$ is an abelian group, there is a unique map $f : \mathscr{A}(N) \to \mathscr{A}(G)$ such that $\mathrm{ab}_G \circ i = f \circ \mathrm{ab}_N$. Since $\mathrm{ab}_G \circ i = 0 = f \circ \mathrm{ab}_N$, and ab_N is clearly not trivial, then f cannot be onto. Actually f is the zero map since the composition in the bottom row $\mathscr{A}(N) \to \mathscr{A}(G) \to \mathscr{A}(H)$ is the zero map, and the second map is an isomorphism. Therefore, $H_1((\underline{EG}, \underline{G})^K; \mathbb{Z}) \to H_1((\underline{BG}, \underline{G})^K; \mathbb{Z})$ is the zero map. $\qquad\square$

This proposition is in the spirit of the induced maps in homology introduced in the next section, concerning the spaces $B(2, G)$ and $E(2, G)$ defined below. We seek a similar result in that case, too.

7. CT GROUPS AND FEIT-THOMPSON THEOREM

In this section we study commutative transitive groups defined below, and use some methods from polyhedral products to understand the interplay between topology and group theory, and characterize some group properties using topology. For any group π the descending central series is given by a sequence of normal subgroups

$$\pi = \Gamma^1 \rhd \Gamma^2 \rhd \cdots \rhd \Gamma^{n+1} \rhd \cdots$$

where inductively $\Gamma^{n+1} = [\pi, \Gamma^n]$ for $n \geq 2$. If $\pi = F_n$ is the free group of rank n, then for any topological group G there is a filtration

$$\mathrm{Hom}(F_n/\Gamma^2, G) \subset \mathrm{Hom}(F_n/\Gamma^3, G) \subset \cdots \subset G^n.$$

The sequences of spaces given by

$$B_k(q, G) := \mathrm{Hom}(F_k/\Gamma^q, G) \subset G^k$$

and

$$E_k(q, G) := G \times \mathrm{Hom}(F_k/\Gamma^q, G) \subset G^{k+1}$$

have the structure of simplicial complexes ([1]), respectively, with respective geometric realizations defined as follows

$$B(q, G) := |B_*(q, G)|, \text{ and } E(q, G) := |B_*(q, G)|.$$

The projections $E_k(q, G) \twoheadrightarrow B_k(q, G)$ induce a fibration

$$(11) \qquad\qquad E(q, G) \to B(q, G) \to BG$$

and in particular, for $q = 2$ we have

$$(12) \qquad\qquad E(2, G) \to B(2, G) \to BG.$$

The total space $B(2, G)$ and the homotopy fibre $E(2, G)$ were studied by A. Adem, F. Cohen and E. Torres Giese [1]. They posed the question whether for finite G the space $B(2, G)$ is always a $K(\pi, 1)$, having showed that these spaces are occasionally $K(\pi, 1)$ for the case of *commutative transitive groups*. C. Okay [18, 19] gave classes of groups for which $B(2, G)$ is not a $K(\pi, 1)$, such as extraspecial 2-groups of order 2^{2n+1}, for $n \geq 2$, hence answering their question. A brief survey is given in [7, §9].

Definition 7.1. A group G is **commutative transitive** or CT if commutativity is a transitive relation in G. That is, if $[a, b] = [b, c] = 1$, then $[a, c] = 1$ for all non-central elements a, b, $c \in G$.

The class of CT groups played an important role in the classification of finite simple groups and were studied by M. Suzuki [25, 26], among many others, who showed that every non-abelian simple CT-group is of even order and isomorphic to $\mathrm{PSL}(2, 2^f)$ for some $f \geq 2$. Finite CT groups have been classified, see for example [21, p. 519, Theorem 9.3.12].

In particular, if G is a finite CT group with trivial center then the following is true.

Proposition 7.2 ([1, Cor. 8.5]). *If there are maximal abelian subgroups $G_1, \ldots, G_n$ of G that cover G, then there is a homotopy equivalence $B(2, G) \simeq \bigvee_i BG_i$.*

With the assumptions of this proposition we have the following corollary.

Corollary 7.3. $B(2, G)$ *has the homotopy type of the polyhedral product* $(\underline{BG}, \underline{1})^{K^0}$.

In what follows G is assumed to be finite.

Using the five term short exact sequence from the Lyndon-Hochschild-Serre spectral sequence Adem, Cohen and Torres Giese [1, Proposition 7.2] showed that the non-surjectivity of the induced map on first homology of the fibration (12)

$$H_1(E(2, G)) \to H_1(B(2, G))$$

is equivalent to the Feit-Thompson theorem that groups of odd order are solvable. Hence the study of the fibration encodes fundamental information about the group G. We would like to use polyhedral products, i.e. topology, to extract more information about this equivalent form of the Feit-Thompson theorem [12], which is algebraic in nature.

Let G be a finite CT group with trivial center and let $G_1, \ldots, G_n$ be its cover by maximal abelian subgroups as above (n is called the *covering number* of G). Since all spaces are $K(\pi, 1)$'s, we will move frequently between fundamental groups and their classifying spaces. Note that there are two commutative diagrams of short exact sequences of groups:

$$
(13) \quad
\begin{array}{ccccc}
\pi_1(E(2,G)) & \longrightarrow & \mathcal{H} & \longrightarrow & [G,G] \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(E(2,G)) & \longrightarrow & \pi_1(B(2,G)) & \longrightarrow & \pi_1(BG) \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & H_1(BG) & \longrightarrow & H_1(BG),
\end{array}
$$

and

$$
(14) \quad
\begin{array}{ccccc}
N & \longrightarrow & \pi_1(\underline{EG}, \underline{G})^{K^0} & \longrightarrow & [G,G] \\
\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow \\
\pi_1(E(2,G)) & \xrightarrow{i_1} & \pi_1(B(2,G)) & \xrightarrow{p_2} & \pi_1(BG) \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_3} & & \downarrow \\
Q & \longrightarrow & \pi_1(\prod_i BG_i) & \longrightarrow & H_1(BG),
\end{array}
$$

where Q is a finite abelian group and N is a free group (we omit the trivial groups on each side of the short exact sequences). The existence of the first diagram is clear, whereas for the second diagram, even though for CT groups the map $\pi_1(B(2,G)) \to G$ does not factor through the product $\prod_i G_i$, the composition $\pi_1(B(2,G)) \to G \to H_1(G)$, being an epimorphism onto an abelian group, factors uniquely through the abelianization of $\pi_1(B(2,G))$, which is the direct product $\prod_i G_i$.

The map p_1 factors uniquely through the abelianization $H_1(E(2,G))$, hence there is a map q_1 such that $p_1 = q_1 \circ$ ab. Hence there is a diagram

(15)
$$
\begin{array}{ccc}
\pi_1(E(2,G)) & \xrightarrow{\ i_1\ } & \pi_1(B(2,G)) \\
\downarrow{\scriptstyle \mathrm{ab}} & & \downarrow{\scriptstyle p_3} \\
H_1(E(2,G)) & \xdashrightarrow{\ (i_1)_*\ } & \prod_i G_i \\
\downarrow{\scriptstyle q_1} & & \| \\
Q & \longrightarrow & \prod_i G_i,
\end{array}
$$

where the dotted map is the one we are interested in (we are not claiming that the lower square commutes). By [1, Proposition 8.8] the group $\pi_1(E(2,G))$ is free, with rank

$$
\mathcal{N}_G = 1 - |G : Z(G)| + \sum_{1 \le i \le n} \left(|G : Z(G)| - |G : G_i| \right).
$$

Since $Z(G) = \{1_G\}$, by rearranging the terms of $\mathcal{N}_G$ we obtain the following:

$$
\begin{aligned}
\mathcal{N}_G &= 1 - |G : Z(G)| + \sum_{1 \le i \le n} \left(|G : Z(G)| - |G : G_i| \right) \\
&= 1 - |G| + \sum_{1 \le i \le n} \left(|G| - |G|/|G_i| \right) \\
&= 1 - |G| + n|G| - \sum_{1 \le i \le n} |G|/|G_i| \\
&= 1 + (n-1)|G| - \sum_{1 \le i \le n} |G|/|G_i|.
\end{aligned}
$$

Note that this is a more general version of the formula for ρ_K in equation (4), with the special case of $|G| = \prod_i |G_i|$ giving the rank ρ_K when K is only a set of n points; let us use the notation $\rho_K = \rho(n)$ since K becomes irrelevant. In general $l.c.m.(|G_1|, \ldots, |G_n|) \le |G| < \prod_i |G_i|$, since the groups G_i cover G. Actually they divide each other from left to right. Since $|G|$ divides $\prod_i |G_i|$, then $\prod_i |G_i| = C|G|$. Therefore, we have

$$
\rho(n) - \mathcal{N}_G = (n-1)|G|(C-1) - \sum_{1 \le j \le n} (|G|(C-1))/|G_j|
$$

$$
= |G|(C-1) \left(n - 1 - \sum_{1 \le j \le n} 1/|G_j| \right).
$$

Since all $|G_i| \ge 2$, then we have $\rho(n) - \mathcal{N}_G \ge |G|(C-1)(n/2 - 1)$. If $n = 2$ then G is abelian, so assume that $n \ge 3$. Then we get $\rho(n) - \mathcal{N}_G > 0$.

Lemma 7.4. *Let G be a finite CT group with trivial center. Then $\rho(n) > \mathcal{N}_G$.*

Proof. In addition to the above argument, this is also a direct consequence of the fact that the index of N in each of the free groups is given by the following formula [15, p.16]:

$$[\pi_1(E(2,G)) : N] = \frac{\mathrm{rank}(N) - 1}{\mathrm{rank}(\pi_1(E(2,G))) - 1} = |Q| = \frac{\prod_i |G_i|}{|H_1(G)|}, \text{ and,}$$

$$[\pi_1((\underline{EG},\underline{G})^{K_0}) : N] = \frac{\mathrm{rank}(N) - 1}{\mathrm{rank}(\pi_1((\underline{EG},\underline{G})^{K_0}) - 1} = |[G,G]| = \frac{|G|}{|H_1(G)|}.$$

Since $|G| < \prod_i |G_i|$ the lemma follows. $\square$

Indeed the proof of this lemma tells us that $\rho(n)/\mathcal{N}_G \sim |Q|/|[G,G]|$.

Before we proceed, it is clear from the diagrams (13,14) that if $G/[G,G] = 1$, then the induced map on homology $H_1(E(2,G)) \to H_1(B(2,G))$ is onto (without using the 5-term sequence in homology).

Proposition 7.5. *If G is simple, then the following map is a surjection*

$$H_1(E(2,G);\mathbb{Z}) \to H_1(B(2,G);\mathbb{Z}).$$

Instead, using only topology we want to prove the following equivalent statements: *if the map $H_1(E(2,G)) \to H_1(B(2,G))$ is onto, then $|G|$ is even,* or equivalently, *if $|G|$ is odd, then the map $H_1(E(2,G)) \to H_1(B(2,G))$ is not onto.*

Now, if $|G|$ is odd, then all Q, $\prod_i G_i$ and $[G,G]$ are odd. Also N is a free group of odd index in both free groups $\pi_1(E(2,G))$ and $\pi_1(EG_i,G_i)^{K_0}$. The following results are immediate:

Lemma 7.6. *Either all $\rho(n)$, $\mathcal{N}_G$, $\mathrm{rank}(N)$ are even, or, all $\rho(n)$, $\mathcal{N}_G$, $\mathrm{rank}(N)$ are odd, such that the ratios*

$$\frac{\mathrm{rank}(N) - 1}{\mathcal{N}_G - 1}, \frac{\mathrm{rank}(N) - 1}{\rho(n) - 1}, \frac{\rho(n) - 1}{\mathcal{N}_G - 1}$$

are odd.

Proof. Use the formulas in the proof of the previous Lemma. $\square$

Next note that the map $(i_1)_*$ in (15) can be a surjection only if $Q \lneq \mathrm{Im}((i_1)_*)$ (if not then their intersection is at most Q and the image cannot be everything). Consider the following diagram

$$
\begin{array}{ccc}
\mathrm{Ker}_1 & & \\
\downarrow & & \\
\mathrm{Ker}_2 \longrightarrow H_1(E(2,G)) & \xrightarrow{(i_1)_*} & \mathrm{Im}((i_1)_*) \\
\downarrow{\scriptstyle q_1} & & \downarrow \\
Q \xrightarrow{\quad i \quad} & & \mathrm{Im}((i_1)_*).
\end{array}
$$

The image $\mathrm{Im}((i_1)_*)$ has odd order. Since both kernels have full rank (the quotients are both finite groups) we have that $\mathrm{Ker}_2 \leq \mathrm{Ker}_1$. The kernels have bases as follows

$$\mathrm{Ker}_1 = span\{\alpha_i e_i : i \in [\mathcal{N}_G]\}, \text{ and } \mathrm{Ker}_2 = span\{\beta_i e_i : \beta_i | \alpha_i, \forall i \in [\mathcal{N}_G]\}.$$

Here all α_i, β_i have to be odd numbers such that $\alpha_i | \beta_i$ for all i. Indeed this can be done for any (finite) sequence of subgroups

$$Q < Q_1 < \cdots < \operatorname{Im}((i_1)_*) < \cdots < \prod_i G_i$$

as there are kernels $K_0, K_1, \ldots,$ of full ranks corresponding to projections. The following theorem shows that $\operatorname{Im}((i_1)_*) \lneqq \prod_i G_i$ for the case of CT groups with trivial center. We conclude this section with the following corollary.

Corollary 7.7. *Finite CT groups with trivial center are solvable if and only if the induced map $H_1(E(2, G); \mathbb{Z}) \to H_1(B(2, G); \mathbb{Z})$ is not a surjection.*

Of course this is a special case of the condition in [1, Proposition 7.2], but for this corollary we use only the diagrams (14,15).

Proof of Corollary 7.7. Consider the commutative diagram (14) and (15). If the map $(i_1)_*$ is a surjection, then the composition

$$H_1(E(2, G); \mathbb{Z}) \twoheadrightarrow H_1(B(2, G); \mathbb{Z}) \twoheadrightarrow G/[G, G]$$

is a surjection. On the other hand, from diagram (14), the group $H_1(E(2, G); \mathbb{Z})$ maps trivially, hence $G/[G, G]$ is trivial. On the other hand, if $G/[G, G]$ is trivial, then $Q = \prod_i G_i$ and $(i_1)_*$ is a surjection. $\qquad\qquad\square$

The following question is still open for CT groups: *Use Corollary 7.7 to show that if G is a simple finite TC group with trivial center, then G has even order.*

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