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Displacement of biased random walk in a one-dimensional percolation model

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Zusammenfassung. In Anlehnung an das Standardbeispiel für die einfache Irrfahrt betrachten wir das folgende Modell: Nach einer durchzechten Nacht findet sich eine betrunkene Person auf ihrem Nach-Hause-Weg aus unerfindlichen Gründen in einem unendlichen Irrgarten wieder. Aufgrund ihres Alkoholspiegels weiß die Person weder, an welchem Ort sie sich befindet noch wo sie sich vorher aufhielt, und so torkelt sie auf der Suche nach ihrer Wohnung durch das Labyrinth. Wir fassen den Weg des Betrunkenen als Irrfahrt in einem zufälligen Graphen auf und nehmen ferner an, dass die Irrfahrt einen Drift in eine bestimmte, fest gewählte Richtung aufweist. Ein Grund für diesen Drift könnte beispielsweise sein, dass das Labyrinth ein leichtes Gefälle in diese Richtung besitzt, wodurch der Betrunkene unwissentlich mit höherer Wahrscheinlichkeit bergab anstelle von bergauf torkelt.
Wir betrachten dieses Modell für den Spezialfall, dass die Umgebung der Irrfahrt durch ein ein-dimensionales Perkolations-Cluster gegeben ist. Die lineare Geschwindigkeit der Irrfahrt konvergiert fast sicher gegen eine Konstante $\overline{\mathrm{v}}$, welche deterministisch vom Drift-Parameter λ der Irrfahrt abhängt. Dieser Grenzwert ist für kleine Werte von λ strikt positiv, und es existiert ein kritischer Wert λ_{c}, sodass die Geschwindigkeit $\overline{\mathrm{v}}$ für alle $\lambda \geq \lambda_{\mathrm{c}}$ den Wert null annimmt. Im ballistischen Fall bestimmen wir die typische Größenordnung der Abweichung der Irrfahrt von ihrer linearen Geschwindigkeit $\overline{\mathrm{v}}$. Des Weiteren bestimmen wir im kritischen und im subballistischen Fall die Größenordnung der Entfernung der Irrfahrt vom Ursprung. Außerdem zeigen wir im subdiffusiven Fall ein Gesetz des iterierten Logarithmus.

Abstract. Suppose an ant is placed in a randomly generated, infinite maze. Having no orientation whatsoever, it starts to move along according to a nearest neighbour random walk. Now furthermore, suppose the maze is slightly tilted, such that the ant makes a step along the slope with higher probability than in the opposite direction. Tracking the ant's position, we are interested in the long-term behaviour of the corresponding random walk.
We study this model in the context that the maze is given by a one-dimensional percolation cluster. Depending on the bias parameter λ of the walk, its linear speed converges almost surely towards a deterministic value $\overline{\mathrm{v}}$. This limit exhibits a phase transition from positive value to zero at a critical value of λ. We investigate the typical order of fluctuations of the walk around $\overline{\mathrm{v}}$ in the ballistic speed regime, and the order of displacement from the origin in the critical and subballistic speed regimes. Additionally, we show a law of iterated logarithm in the subdiffusive speed regime.

Preface

This thesis was written from April 2014 to October 2018 under the supervision of Matthias Meiners and Volker Betz at Technische Universität Darmstadt. Parts of this work have been presented in the preprint

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which is listed as reference [**35**] in the bibliography.

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CHAPTER 1

Introduction

In this work, we will examine a variant of the 'ant in the labyrinth' model. The 'ant in the labyrinth' was introduced in a popular science article in 1976 by the physicist De Gennes [**17**] and can be described as follows: Suppose we place a blindfolded ant in an infinite, randomly generated maze. At each time step, the ant randomly chooses a direction along which it tries to make a step in the labyrinth. If the step is permitted by the labyrinth, the ant steps into a new site, time advances and it continues with choosing the direction of its next step. If the path is blocked, the ant hits a wall, cannot make a step and thus stays put. Then again, time advances and the ant continues with choosing the direction of its next step.
To describe this procedure in terms of mathematical objects, suppose we are given a graph (maze) $G = (V, E)$ and a random subset $E' \subseteq E$ of its edges. A particle (the ant) is then placed on one of the graph's nodes and performs a lazy nearest-neighbour random walk. This random walk is such that at each time, it chooses the direction of its next step (the direction of the ant) among those vertices that are neighboured to its current position according to the complete edge set E, but only changes its position if the selected edge is among those edges that are contained in E'. At times when a neighbouring vertex was selected due to an edge not contained in E', the walk stays put. That is, the ant hits a wall.
We are interested in those instances where the underlying random walk is not the simple random walk but tends to slightly prefer steps in a pre-specified direction over those in opposite direction. In terms of the blindfolded ant, we might imagine that the labyrinth is slightly tilted such that due to gravity, the ant unwittingly takes a step along the slope with increased probability.
In terms of real-world applications, the described process can for example be used to model the diffusion of a particle in large chromatographic columns, as indicated by Barma and Dhar in [**6**] or more generally to study dissipation of a gas in a porous medium under the influence of an external field inducing a bias direction.
In particular, we are interested in a precise description of the long-term behaviour of this process when the environment is given by the infinite open cluster of a conditional percolation model on the ladder graph.

1.1. Percolation

The processes that we want to analyse involve two sources of randomness. One being the random walk itself, while the other one is given by the environment. In case of the latter, we look for the most straightforward way to randomize a given graph. That is, for each edge of the graph, we flip an independent coin. Based on the outcome of the coin flip, we retain the edge if the coin shows heads, and delete it otherwise. The corresponding mathematical subject is known as *percolation*. While some aspects of percolation are covered in general textbooks about probability theory, e. g. [**32**], the standard reference for the subject is [**25**].
The subject started with the paper [**15**] by Broadbent and Hammersley in the 1950s and was motivated by the following scenario: Suppose we submerge a large porous stone in water. We can imagine the stone as a mixture of actual matter and a variety of tunnels of different diameter.

For each tunnel, there is a probability $p \in [0,1]$ that it is wide enough to allow the intruding water to flow along it, independent of all other tunnels. With probability $1-p$, it is too narrow to make this possible. A natural question to ask then is what is the probability of the interior of the stone to become wet. In other words, whether water which enters the stone at its outer boundary can percolate.
To describe this in less vague terms, we can think of the tunnels as edges of a graph, and the intersections of tunnels in the stone can be seen as its vertices. We say the edges are *open* if they allow flow of water, and call them *closed* otherwise. The aforementioned probability of the interior of the stone to become wet is then connected to the event that there exists a path from a node on the boundary of the graph to a node in the interior of the graph such that every edge of the path is open.
In mathematical terms, suppose we are given an infinite graph $G=(V,E)$. The most common examples are the lattices $\mathbb{Z}^d$, that is, the graphs with $V=\mathbb{Z}^d$ whose vertices share an edge if and only if their euclidean distance equals 1.
In *(Bernoulli) bond percolation*, for each edge e we flip an independent coin. Depending on the outcome of the coin flip, we assign to the edge a value of either 0 or 1, where 0 is to be interpreted as the edge being closed and 1 as the edge being open.
To be precise, we look at $\Omega=\{0,1\}^E$, endowed with the product σ-algebra $\mathcal{F}$. The elements $\omega=(\omega(e))_{e\in E}\in\Omega$ are called *configurations*. For $p\in[0,1]$, we define a probability measure P_p on $(\Omega,\mathcal{F})$ by

$$\mathrm{P}_p(\omega) := \prod_{e\in E} \mu_{p,e}(\omega(e)),$$

where $\mu_{p,e}$ is a probability measure on $(\{0,1\},\mathcal{P}(\{0,1\}))$ where $\mathcal{P}(A)$ is the power set of a set A, with $\mu_{p,e}(\{1\})=p$ and $\mu_{p,e}(\{0\})=1-p$.

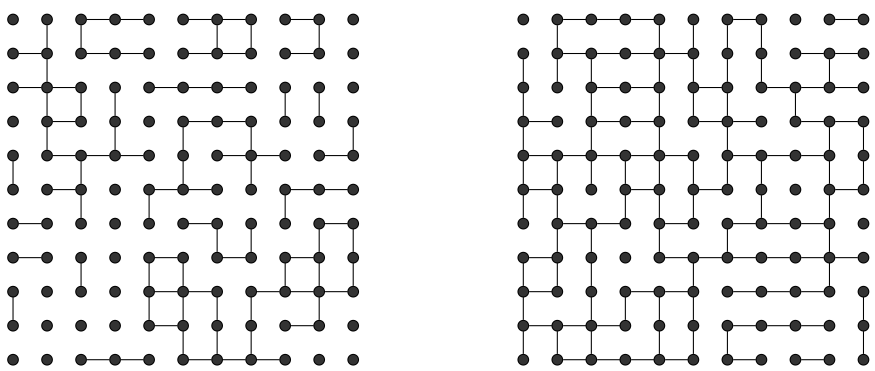


FIGURE 1. A subgraph of the lattice in $\mathbb{Z}^2$ where only the open edges for Bernoulli bond percolation with $p=0.3$ (left) and $p=0.6$ (right) are drawn.

For $v\in V$, denote by $\mathcal{C}(v)$ the connected component of v in the subgraph of G whose edge set only consists of the open edges. Speaking in terms of the porous stone which is submerged in water, if water enters the stone at vertex v, then it can percolate to some arbitrarily distant part of the stone if and only if $\mathcal{C}(v)$ is infinite. The *critical percolation threshold* p_{c} denotes - in terms of the edge retention parameter p - the critical value above which this event occurs with positive probability. More precisely, it is defined as

$$p_{\mathrm{c}} := \sup\{p\in[0,1] : \mathrm{P}_p(|\mathcal{C}|=\infty)=0\},$$

where $\mathcal{C}$ denotes the cluster $\mathcal{C}(\mathbf{0})$ at the origin.
For bond percolation on the lattice $\mathbb{Z}^d$, it is known that $p_c \in (0,1)$ when $d \geq 2$. Furthermore, exact values of p_c are only known for special cases, e. g. $p_c = 1/2$ for bond percolation on $\mathbb{Z}^2$ or $p_c = 2\sin(\pi/18)$ for bond percolation on the triangular lattice, see e. g. [**25**, Section 3].
In the *supercritical* case of $p > p_c$, there P_p-almost surely exists an infinite open cluster, which further is unique. For this thesis, we are interested in the properties of biased random walk whose environment is given by a sample of this infinite open cluster.

1.2. Random walk on the supercritical percolation cluster

When percolation in general can be thought of as water spreading in a random media, random walk on a percolation cluster can be thought of as the investigation of *how fast* this dispersion takes place by tracking a single water particle. This particle will perform a nearest-neighbour random walk on the infinite cluster of the graph. To spice things up, we will assume that there exists an external field that affects the transition probabilities of the particle such that one direction becomes more likely. A useful picture for this is that of a dried sponge whose lower end is submerged in water and that subsequently begins to suck up water, creating a small bias towards its dry component for water particles that enter at its bottom. Alternatively and closer to the original description of the topic in [**17**], we might imagine an ant that is placed in a random, slightly tilted labyrinth.
Suppose we are given an infinite graph $G = (V, E)$ and a probability measure P_p on $(\{0,1\}^E, \mathcal{F})$ where $\mathcal{F}$ is the product σ-algebra on $\Omega := \{0,1\}^E$. As indicated, for a *configuration* $\omega \in \Omega$, we say that an edge $e = \langle u, v\rangle \in E$ between vertices $u, v \in V$ is *open* in ω if $\omega(e) = 1$, and *closed* otherwise. We assume that with P_p-probability 1, there exists an infinite open cluster $\mathcal{C}$. Given a configuration $\omega \in \Omega$, we define a random walk $(Y_n)_{n\in\mathbb{N}_0}$ on $\mathcal{C}$ by putting $Y_0 = u$ for some vertex $u \in \mathcal{C}$ and then performing a nearest-neighbour random walk on the cluster according to some law P_ω on $(V^{\mathbb{N}_0}, \mathcal{G})$, where $\mathcal{G}$ is the product σ-algebra on $V^{\mathbb{N}_0}$. We choose the distribution P_ω to depend on ω such that the walk $(Y_n)_{n\in\mathbb{N}_0}$ is only allowed to take steps along edges that are open in ω. We call P_ω the *quenched* law of $(Y_n)_{n\in\mathbb{N}_0}$, that is the law of $(Y_n)_{n\in\mathbb{N}_0}$ given some fixed ω. The corresponding so-called *annealed* law $\mathbb{P}$ is then obtained by averaging the quenched laws P_ω over $\omega \in \Omega$ using P_p. That is, $\mathbb{P}$ is a probability measure on $\{0,1\}^E \times V^{\mathbb{N}_0}$ defined by setting, for $A \in \mathcal{F}, B \in \mathcal{G}$,

$$\mathbb{P}(A \times B) := \int_A P_\omega(B)\, \mathrm{P}_p(\mathrm{d}\omega). \tag{1.2.1}$$

We start with an excerpt of the existing literature on this topic. While the most interesting case is clearly given by biased random walk on the infinite cluster of supercritical percolation in $\mathbb{Z}^d$, this is also the most technically challenging. Due to interest in the topic from physics, there exists a large number of physics papers on this topic, some of whom provide a very useful intuition to explain the phenomena that occur. We sum up some of these findings in Section 1.2.1.1, and the known mathematical features in Section 1.2.1.2.
A more accessible instance of random walk on a random infinite graph is given when the environment is provided by an infinite Galton-Watson tree. In this case, biased random walk exhibits a very similar phenomenology, but technical properties of the Galton-Watson tree facilitate the analysis. In addition, in some cases properties can be described in a more transparent fashion. We summarize the most important results for biased random walk on Galton-Watson trees in Section 1.2.2.
In their papers [**5**] and [**4**], Axelson-Fisk and Häggström introduced a model for biased random walk on a conditional percolation model on the ladder graph. In their model, a similar phenomenology as in the two aforementioned models occurs, while simultaneously, the environment

takes a very simple form. We return to this toy model - whose analysis amounts to the main part of this thesis - in Section 1.2.3.

1.2.1. Supercritical bond percolation in $\mathbb{Z}^d$. The most general percolation setting which has been studied in the aforementioned context is supercritical bond percolation on the lattice $\mathbb{Z}^d$. In this case, for P_p we take the i.i.d. bond percolation measure on the d-dimensional lattice with $p > p_{\mathrm{c}}$, conditioned on the event that the infinite open cluster contains the origin $\mathbf{0}$. Given a configuration ω, a bias direction $l \in \mathbb{S}^{d-1}$ where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and a bias parameter $\lambda \in \mathbb{R}$, biased random walk $(Y_n)_{n\in\mathbb{N}_0}$ on the infinite cluster starts at $\mathbf{0}$ and its quenched transition probabilities are defined as

$$P_{\omega,\lambda,l}(Y_{n+1} = v \mid Y_n = u) = \frac{e^{\lambda l\cdot v}\omega(\langle u, v\rangle)}{Z_{u,\omega}}, \tag{1.2.2}$$

where $u{\cdot}v$ denotes the scalar product of $u, v \in \mathbb{Z}^d$, $Z_{u,\omega} := \sum_{w:w\sim u} e^{\lambda l\cdot w}\omega(\langle u, w\rangle)$ is a normalizing constant and $u \sim v$ denotes that u and v are adjacent vertices. Note that for $\lambda = 0$, this reduces to simple random walk on the infinite open cluster, where the transitions to all neighbouring vertices of the cluster are equally likely.
The annealed law of the random walk is obtained as in (1.2.1) via averaging $P_{\omega,\lambda,l}$ over all possible configurations using P_p.

1.2.1.1. Random walk on the percolation cluster from the physics perspective. After being introduced in [**17**], simple and biased random walk on the supercritical bond percolation cluster in $\mathbb{Z}^d$ were studied in the phyics literature in the 1980s, cf. [**44, 41, 40, 43, 46**]. Most relevant in the context of this thesis are the papers [**6**], [**18**] and [**19**].
In [**6**], it was first argued that the linear velocity

$$\overline{\mathrm{v}} := \lim_{n\to\infty} \frac{|Y_n \cdot l|}{n}$$

of the walk in the direction of the bias vanishes for large values of λ. To justify this, a heuristic argument for the computation of the expected time which the walk spends in dead-end regions of the graph in the direction of the bias was given. In this computation, a critical value λ_{c} for the bias parameter λ appeared such that the expected time spent in dead-end regions of the cluster becomes infinite for $\lambda > \lambda_{\mathrm{c}}$, subsequently leading to a value of $\overline{\mathrm{v}} = 0$ for $\lambda > \lambda_{\mathrm{c}}$.
This was further investigated in [**18**], where it was argued that, indeed, a phase transition of $\overline{\mathrm{v}}$ as a function of the bias parameter occurs. That is, the critical value λ_{c} is such that for $\lambda \in (0, \lambda_{\mathrm{c}})$ the linear speed $\overline{\mathrm{v}}$ is positive, whereas for $\lambda > \lambda_{\mathrm{c}}$ its value is 0. The somewhat accurate heuristics given in the paper is that up to time n, the random walk on the one hand moves with linear speed in those parts of the graph that permit travel in the direction of the bias without the need to backtrack. On the other hand, the walk spends a large amount of time being trapped in dead-end regions of the graph. Those dead ends are the parts of the graph where from each vertex of the dead-end, only finitely many other vertices of the infinite cluster can be reached without having to take backtracking steps against the bias direction. From the distribution of the length of dead-end regions in the direction of the bias, first the typical length of such traps encountered up to time n, and then the critical bias parameter λ_{c} are derived. The parameter λ_{c} marks the critical point at which the time spent in a 'typical' trap seen up to time n reaches an order higher than n. Based on this argument, it was also conjectured that for $\lambda > \lambda_{\mathrm{c}}$ the displacement $|Y_n \cdot l|$ from the origin is of order n^α for some $\alpha \in (0, 1)$.
In a later paper [**19**] by Dhar and Stauffer, a refinement of this argument led to the conjecture that at the critical bias $\lambda = \lambda_{\mathrm{c}}$ the displacement from the origin at time n is of order $n/\log n$. This notion was reinforced by simulation results in the same paper.

1.2.1.2. Mathematical analysis. The mathematical analysis of this particular instance of biased random walk in random environment started later. For simple random walk on the lattice in $\mathbb{Z}^d$, it is known that the random walk is recurrent for $d = 1, 2$, and transient otherwise. Therefore, due to Rayleigh's monotonicity law, simple random walk on the infinite open cluster of i.i.d. supercritical bond percolation is recurrent for $d = 1, 2$, too. In [**24**], using electrical analysis and a tree-like subgraph of the infinite cluster, it was shown that simple random walk on the supercritical percolation cluster in $\mathbb{Z}^d$ is transient for $d \geq 3$.
Asking how random sparsing of the graph affects the transition of the behaviour of the simple random walk from recurrence to transience, the (fractal) dimension at which the change from recurrence to transience occurs on the supercritical percolation cluster was further investigated e. g. in [**27, 10, 9, 3**].
The analysis of biased random walk on the superciritical percolation cluster in $\mathbb{Z}^d$ was first done in the parallel papers [**11**] and [**45**]. In [**11**], biased random walk on the infinite open cluster of supercritical percolation in $\mathbb{Z}^2$ which has transition probability proportional to e^λ along open edges in positive x-direction and proportional to 1 along open edges in any other direction was investigated. Using a regeneration argument and information about the shape of the cluster, it was shown that this biased random walk is $\mathbb{P}$-almost surely transient for $\lambda \neq 0$, and that there exist different speed regimes. More precisely, the limit

$$\overline{\mathrm{v}} := \lim_{n\to\infty} \frac{X_n}{n}$$

where $X_n := \mathsf{x}(Y_n)$ denotes the x-coordinate of the walk at time n, is a $\mathbb{P}$-almost surely deterministic constant with $\overline{\mathrm{v}} > 0$ for small, and $\overline{\mathrm{v}} = 0$ for large values of λ.
With a more analytic approach, the same was shown in [**45**] for biased random walk on the infinite cluster in $\mathbb{Z}^d$ with arbitrary bias direction and transition probabilities as in (1.2.2). More precisely, it was shown that there exist $\lambda_1 \leq 1 \leq \lambda_2$ such that

$$\overline{\mathrm{v}} := \lim_{n\to\infty} \frac{Y_n}{n}$$

is a $\mathbb{P}$-almost surely deterministic vector with $\overline{\mathrm{v}} \cdot l > 0$ for $\lambda \in (0, \lambda_1)$ and $\overline{\mathrm{v}} = 0$ for $\lambda > \lambda_2$. Additionally, it was shown that for small values of λ a central limit theorem for a suitable renormalisation of the walk holds.
In both papers, however, it was left open whether a sharp phase transition for $\overline{\mathrm{v}}$ as a function of λ holds. That is, whether there exists a critical bias parameter λ_c such that $\overline{\mathrm{v}} > 0$ for $\lambda \in (0, \lambda_\mathrm{c})$ and $\overline{\mathrm{v}} = 0$ for $\lambda \geq \lambda_\mathrm{c}$. The existence of such a λ_c was later confirmed by Fribergh and Hammond in [**22**].

1.2.2. Galton-Watson trees. Switching to a model that is more accessible than random walk on the supercritical percolation cluster in $\mathbb{Z}^d$ but which remains closely related leads to (biased) random walk on trees, in particular on Galton-Watson trees. In this case, most properties that are known for biased random walk on the supercritical percolation cluster in $\mathbb{Z}^d$ are known, too, but can be described in a similar or more transparent fashion.
Let $\xi, (\xi_{k,l})_{k,l\in\mathbb{N}}$ be a family of i.i.d. $\mathbb{N}_0$-valued random variables on a joint probability space $(\Omega, \mathcal{F}, \mathrm{P}')$ with generating function $f(z) = \sum_{m=0}^{\infty} p_m z^m$, where $p_m := \mathrm{P}'(\xi = m)$. Consider a population that evolves as follows. The first generation consists of a single individual which gives rise to $\xi_{1,1}$ children in generation 2, and then dies. Subsequently, in generation k, each living individual of the population independently gives birth to a random number of children in the following generation before dying. The number of descendants of each individual is distributed as an independent copy of ξ. More precisely, for the the l-th individual (given it exists) of the k-th generation of the population, we sample the number of its children from $\xi_{k,l}$. The size X_{k+1}

of the population in generation $k+1$ can be written as $X_{k+1} = \sum_{l=1}^{X_k} \xi_{k,l}$, starting at $X_1 = 1$. The population dies out if $X_k = 0$ for some (and subsequently all following) k.
We use this process to construct a (random) genealogical tree T known as *Galton-Watson tree.* Therefor, we number the individuals of generation k by $1, \dots, X_k$, and for each individual of the population in generation k, we introduce a vertex (k,l) where l corresponds to its number in its generation. Then, the vertex set V of the genealogical tree consists of the union of all sets of vertices $\{(k,1), \dots, (k, X_k)\}$ over all generations $k = 1, 2, \dots$. On the other hand, the edge set E of the tree is such that each vertex (k,l) is connected to the vertex that corresponds to its parent in the preceding generation, and to all vertices that correspond to its $\xi_{k,l}$ children in the following generation. As root of the tree, we take the vertex $(1,1)$ corresponding to the very first member of the population and denote it by $\mathbf{0}$.

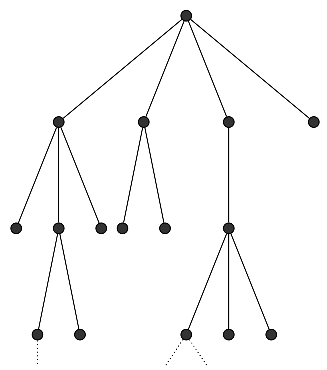


FIGURE 2. A Galton-Watson tree at the root with the first three offspring generations.

A Galton-Watson tree is called *supercritical* if the expected number of children $\mathrm{E}(\xi) = f'(1)$ of each individual is larger than 1. In this case, with positive probability the population survives, leading to an infinite genealogical tree. We let this infinite tree then serve as the environment of a biased random walk.
Denote by P the law of a supercritical Galton-Watson tree conditioned on nonextinction of the underlying population. Let $T(\omega)$, $\omega \in \Omega$ be a sample of an infinite tree according to P, and $\beta > 0$. The β*-biased* random walk $(Y_n)_{n\in\mathbb{N}_0}$ on $T(\omega)$ is defined as follows. The walk starts at the root $\mathbf{0}$ of the tree, and the (quenched) transition probabilities of $(Y_n)_{n\in\mathbb{N}_0}$ at $u \in V \setminus \{\mathbf{0}\}$ are given by

$$P_{\omega,\beta}(Y_{n+1} = \hat{u} \mid Y_n = u) = \frac{1}{1+\beta k_u},$$
$$P_{\omega,\beta}(Y_{n+1} = v_i \mid Y_n = u) = \frac{\beta}{1+\beta k_u}, \quad i = 1, \dots, k_u,$$

where $\hat{u}$ denotes the parent node of u, k_u denotes the number of children of u and $v_1, \dots, v_{k_u}$ denote the children of u, respectively. At the root, the walk transitions to each of the children of the root with equal probability.
The annealed law $\mathbb{P}$ of biased random walk on the Galton-Watson tree is again defined as in (1.2.1) as a probability measure on $\Omega \times V^{\mathbb{N}_0}$, with P and $P_{\omega,\beta}$ replacing P_p and P_ω, respectively. In [**36**], upon investigating the branching number of general trees - which is roughly the typical number of children of a vertex of the tree - a criterion for transience of the β-biased random walk on a tree was derived. In the case of a Galton-Watson tree conditioned on nonextinction, this criterion takes the form that the biased random walk is recurrent if $\beta\mathrm{E}(\xi) < 1$ and transient if $\beta\mathrm{E}(\xi) > 1$.

For $v \in V$, denote by $|v|$ the graph distance of v from the root of the tree, that is the length of the (unique) shortest path connecting v and $\mathbf{0}$. In [**38**] it was shown that analogously to the situation on the supercritical percolation cluster in $\mathbb{Z}^d$, for values of β such that $(Y_n)_{n\in\mathbb{N}_0}$ is transient, the linear speed limit

$$\overline{\mathrm{v}} := \lim_{n\to\infty} \frac{|Y_n|}{n}$$

of the walk is a $\mathbb{P}$-almost surely deterministic value. Further, with a suitably defined family $(\tau_k)_{k\in\mathbb{N}}$ of regeneration times, for $\beta < 1$, the linear speed limit can be written as $\overline{\mathrm{v}} = \mathbb{E}(|Y_{\tau_2}| - |Y_{\tau_1}|)/\mathbb{E}(\tau_2 - \tau_1)$. In addition, given the case that an individual of the population may produce zero offspring, a phase transition of $\overline{\mathrm{v}}$ occurs at $\beta_{\mathrm{c}} := 1/f'(q)$, where q is the extinction probability of the Galton-Watson tree. Namely, if $\beta < \beta_{\mathrm{c}}$, the walk satisfies $\overline{\mathrm{v}} > 0$, whereas for $\beta \geq \beta_{\mathrm{c}}$, we have $\overline{\mathrm{v}} = 0$. This is due to trapping of the walk in finite subtrees, where the walk has to take steps against the direction of the bias in order to proceed arbitrarily far away from the root. To show this, it was utilised that a Galton-Watson tree can be constructed by first drawing its *backbone*, that is an infinite Galton-Watson tree where each node has at least one child, and then attaching to each node of the backbone a random number of independent *leaves*, that is of almost surely finite Galton-Watson trees, cf. [**37**]. The generating functions of the offspring distributions of the backbone and the leaves directly depend on the generating function f of the offspring distribution of the original tree. Using this procedure, the time spent between regenerations of the walk can be decomposed into the time spent on the backbone and the time spent on independent excursions into leaves of the tree. For large values of β, the (annealed) expected time spent in a single leaf of the tree increases with β, and becomes infinite as soon as β reaches β_{c} which leads to the phase transition of $\overline{\mathrm{v}}$.

For Galton-Watson trees without leaves, that is for Galton-Watson trees whose offspring distribution satisfies $p_0 = 0$, a quenched central limit theorem was derived in [**42**]. On the one hand, for $\beta = 1/\mathrm{E}(\xi)$ and P-almost every Galton-Watson tree, the displacement of Y_n from the root converges in the quenched law under suitable renormalisation towards the absolute value of a Brownian motion. On the other hand, for $\beta > 1/\mathrm{E}(\xi)$, the increments of the regeneration times of the walk have arbitrary power moments. From this it follows that - given the offspring distribution of the Galton-Watson tree has exponential moments - for P-almost every tree, the walk converges under the usual scaling in the quenched law towards a Brownian motion. More precisely, there exists $\sigma^2 > 0$ such that for P-almost every ω

$$\left(\frac{|Y_{\lfloor nt\rfloor}| - nt\overline{\mathrm{v}}}{\sqrt{\sigma^2 n}}\right)_{t\geq 0} \xrightarrow{\mathrm{d}} (B(t))_{t\geq 0}$$

under $P_{\omega,\beta}$ as $n \to \infty$, where $(B(t))_{t\geq 0}$ is a standard Brownian motion and $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution.

In [**8**], using a coupling with the β-biased random walk on $\mathbb{Z}$ in order to derive regenerations of the walk that are independent of the environment, the order of displacement of the walk from the origin in the subballistic speed regime, that is for values of β such that $\overline{\mathrm{v}} = 0$, was derived. In particular, for $\beta > \beta_{\mathrm{c}}$ the laws of

$$\left(\frac{|Y_n|}{n^\gamma}\right)_{n\in\mathbb{N}}$$

are tight under $\mathbb{P}$, where $\gamma := \ln\beta_{\mathrm{c}}/\ln\beta$. Additionally, converging subsequences of the walk were identified. The results of [**8**] were later extended to a larger class of offspring distributions in [**14**].

1.2.3. Conditional percolation on the ladder graph. For the remainder of this thesis, we resort to a further simplification of the environment. For biased random walk on the supercritical percolation cluster on $\mathbb{Z}^d$ as well as for biased random walk on Galton-Watson trees, a crucial part in their analysis is played by the amount of time the walk is trapped. More precisely, with increasing strength of the bias, an increasingly large proportion of the time is spent in dead-end regions of the environment that stretch in the direction of the bias such that the walk is required to take multiple backtracking steps against the bias to leave. In order to analyse this behaviour in a more accessible graph, Axelson-Fisk and Häggström in [**4**] and [**5**] introduced biased random walk on a one-dimensional percolation model in which dead-ends in the direction of the bias take the simplest possible form. Apart from the simple geometry of traps, Axelson-Fisk and Häggström's model also features a description of the critical bias parameter as an elementary function of the percolation parameter while still mirroring large parts of the behaviour of biased random walk on the 'full' cluster in $\mathbb{Z}^d$.

We introduce the model in Chapter 2. There, in the model of Axelson-Fisk and Häggström, we deduce the order of the speed of biased random walk in the critical bias case, which has not yet been established in similar models apart from simulations, e. g. in the aforementioned paper [**19**] by Dhar and Stauffer, or for general random walk in random environment, cf. [**30**]. Furthermore, we describe the typical order of fluctuations of the random walk around its linear speed in the ballistic, nondiffusive speed regime, and the order of displacement from the origin in the subballistic speed regime. Together with existing results in [**23**], our results therefore suffice to describe the asymptotic behaviour of biased random walk on the one-dimensional percolation model at hand for all values of the bias parameter except at zero bias, which amounts to the case of simple random walk and requires a different approach due to recurrence. We also prove a law of iterated logarithm for the displacement of the walk from the origin in the nondiffusive parameter range.

Parts of Chapter 2 have been presented in the preprint [**35**] by the author and supervisor Matthias Meiners which can be found on the arxiv.

The remainder of this thesis is structured as follows. Due to the fact that we mainly operate with processes which can be well described in terms of electrical networks, and as we make use of this relation on several occasions, we dedicate the remainder of the introduction to first gathering some frequently used notation and then giving a short overview of the stochastic subject of electrical analysis.

In Chapter 2, we introduce the model of Axelson-Fisk and Häggström and derive the aforementioned results. First, we give a definition of the laws of the environment of the biased random walk and the biased random walk itself in Sections 2.1 and 2.2. This is followed by a section about regeneration times for the walk (Section 2.3), and a section about properties of the traps of the model (Section 2.4). After that, a section is devoted to the proof of tail estimates of the regeneration times (Section 2.5). Then, we apply these estimates to prove our main results in Section 2.6.

1.2.4. Preliminaries and notation. For random variables X and Y with distribution functions F and G, respectively, we say that X is *stochastically dominated* by Y, and write $X \preccurlyeq Y$, if $F(t) \geq G(t)$ for all $t \in \mathbb{R}$.

For a random variable Z and $\hat{p} \in (0,1)$, we write $Z \sim \text{geom}(\hat{p})$ if Z is geometric with success parameter $\hat{p}$, i.e., $\mathbb{P}(Z = k) = \hat{p}(1-\hat{p})^k$, $k \in \mathbb{N}_0$.

Convergence in distribution of a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables towards a random variable X is denoted by $X_n \stackrel{d}{\to} X$. Analogously, convergence in probability of $(X_n)_{n\in\mathbb{N}}$ to X under $\mathbb{P}$ is denoted by $X_n \stackrel{\mathbb{P}}{\to} X$.

As usual, for sequences $a, b : \mathbb{N} \to [0,\infty)$, we write $a = o_n(b)$ or $a_n = o(b_n)$ as $n \to \infty$ if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ with $a_n \leq \epsilon b_n$ for all $n \geq n_0$. We say that a and b are *asymptotically equivalent* and write $a \sim b$ or $a_n \sim b_n$ as $n \to \infty$ if $a_n, b_n > 0$ for all sufficiently large n and

$\lim_{n\to\infty} a_n/b_n = 1$. Furthermore, we write $a = \mathcal{O}_n(b)$ or $a_n = \mathcal{O}(b_n)$ as $n \to \infty$ if there exists some $C > 0$ such that $a_n \le Cb_n$ for all sufficiently large n.
For a function $f : A \to B$ and $b \in B$, we write $f \equiv b$ if $f(a) = b$ for all $a \in A$. Finally, for $x, y \in \mathbb{R}$ we write $x \wedge y := \min(x, y)$ and $\mathsf{x} \vee y := \max(x, y)$.

1.3. Electrical networks and random walks on graphs

1.3.1. Discrete Markov chains. With the exception of limit processes, almost all the stochastic processes that occur in this thesis are discrete-time Markov chains on finite or countably infinite state spaces. Let $(S, \mathcal{A})$ be a measurable space with finite or countably infinite S. For a stochastic process $(Y_n)_{n\in\mathbb{N}_0}$ taking values in $S^{\mathbb{N}_0}$, we write P^v for the law of $(Y_n)_{n\in\mathbb{N}_0}$ starting at $Y_0 = v$, or, more generally, we write P^ν for the law of $(Y_n)_{n\in\mathbb{N}_0}$ with initial distribution ν, where ν is a probability measure on $(S, \mathcal{A})$. The function P^ν is a probability measure on $(S^{\mathbb{N}_0}, \mathcal{F})$, where $\mathcal{F}$ is the product σ-algebra.
A (time-homogeneous) *Markov chain* $(Y_n)_{n\in\mathbb{N}_0}$ with state space S is a stochastic process taking values in $S^{\mathbb{N}_0}$ such that for any $n \in \mathbb{N}$ and $v, v_0, \dots, v_n \in S$ with $P^{v_0}(Y_1 = v_1, \dots, Y_n = v_n) > 0$, we have

$$P^{v_0}(Y_{n+1} = v \mid Y_1 = v_1, \dots, Y_n = v_n) = P^{v_n}(Y_1 = v). \tag{1.3.1}$$

Equation (1.3.1) is called the *Markov property.* As a consequence of the Markov property, a Markov chain is fully characterised by its initial distribution and its *transition matrix* (or *transition probabilities*)

$$p(u, v) := P^u(Y_1 = v),$$

where $u, v \in S$. Given a transition matrix p, a probability measure π on $(S, \mathcal{A})$ is called *stationary distribution* with respect to p if the Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ with initial distribution π satisfies

$$P^\pi(Y_1 \in \cdot) = \pi(\cdot).$$

That is, starting with initial distribution π, the law of Y_n is π for all $n \in \mathbb{N}$. A Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ with stationary distribution π is called *reversible* if π and the transition matrix p satisfy the *detailed balance equations*

$$\pi(u)p(u, v) = \pi(v)p(v, u)$$

for all $u, v \in S$.
A Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ is called *irreducible* if for all states $u, v \in S$, there exists an $n \in \mathbb{N}$ such that $P^u(Y_n = v) > 0$. That is, if it can transition between any two given states in a finite number of steps with positive probability. The *period* of a state $v \in S$ is defined as $\mathsf{gcd}(\{n \in \mathbb{N} : P^v(Y_n = v) > 0\})$, where $\mathsf{gcd}(A)$ denotes the greatest common divisor of a set $A \subseteq \mathbb{N}$. A Markov chain is called *aperiodic* if all of its states have period 1. For irreducible and aperiodic Markov chains there exists the following convergence theorem.

LEMMA 1.3.1 (Convergence Theorem, e. g. Theorem 4.9 in [**33**]). *Suppose* $|S| < \infty$. *Let* $(Y_n)_{n\in\mathbb{N}_0}$ *be an irreducible, aperiodic Markov chain with state space* $(S, \mathcal{A})$ *and stationary distribution* π. *Then there exist constants* $\gamma \in (0, 1)$ *and* $C > 0$ *such that for all* $n \in \mathbb{N}$

$$\max_{u\in S} \|P^u(Y_n \in \cdot) - \pi(\cdot)\|_{TV} \le C\gamma^n,$$

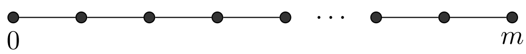
where $\|\mu - \nu\|_{TV} := \max_{A\in\mathcal{A}} |\mu(A) - \nu(A)|$ *denotes the total variation distance between two probability measures* μ *and* ν *on* $(S, \mathcal{A})$.

1.3.2. Electrical networks. For most of this thesis, we will in particular deal with weighted random walks on directed or undirected graphs. In the following section, we give a short overview of weighted random walks and list some of their properties. Good references for the topic can be found in [**33**], [**39**] and [**1**].
Let $G = (V, E)$ be a directed or undirected graph with countable vertex set V and edge set $E \subseteq V \times V$. Two vertices (states) $u, v \in V$ are called *neighbouring*, in short $u \sim v$, if there exists an edge $\langle u, v \rangle \in E$ connecting u and v. Further let $c : E \to [0, \infty)$ be a weight function for the edges of the graph. For each edge $e \in E$, we call $c(e)$ the *conductance* of e and its reciprocal $r(e) := 1/c(e)$ is called *resistance* of e, with $r(e) := \infty$ if $c(e) = 0$. The pair (G, c) of a graph G and conductances c is named a *network*. Given conductances c and under the assumption that $c(u) := \sum_{v:v\sim u} c(\langle u, v \rangle) \in (0, \infty)$ for all $u \in V$, we can define the transition matrix of a random walk $(Y_n)_{n\in\mathbb{N}_0}$ on V by demanding that for $u \sim v$, the transition probability from u to v is proportional to $c(\langle u, v \rangle)$,

$$p(u, v) := \frac{c(\langle u, v \rangle)}{c(u)}.$$

We say $(Y_n)_{n\in\mathbb{N}_0}$ is *induced by the conductances* c or the *weighted random walk* on G (with conductances c). The random walk $(Y_n)_{n\in\mathbb{N}_0}$ can be thought of as the trajectory of a particle that is dropped onto a vertex of G and then proceeds to hop along the vertices of the graph. For the particle, the set of neighbouring vertices of its position constitutes the set of possible directions of a hop, with the conductances of the corresponding edges denoting their likeliness (up to normalisation).
In the case that the graph G is undirected, for $u \sim v$ we write $c(u, v) := c(\langle u, v \rangle) = c(\langle v, u \rangle)$ and vice versa for the resistance r. From here on, we throughout assume that $c(u) \in (0, \infty)$ for all $u \in V$.

Example 1.3.2 (The gambler's ruin). A classic example of a weighted random walk is the following: A gambler wants to make a fortune of $m \in \mathbb{N}$. Therefor, he gathers all of his current capital of value $k \in \mathbb{N}$, sits down at his favourite game of chance and repeatedly plays. To keep things interesting, we suppose that $k < m$. In each round, the gambler invests 1 unit, and with a probability of $p \in (0, 1)$, he gets 2 in return (thus making a net win of 1). With probability $q := 1 - p$, he loses the round and makes a net loss of 1. The gambler is disciplined enough that he stops playing either when the value of his fortune reaches the goal of m, or when he faces bancruptcy, that is, when his fortune reaches 0. What is the gambler's ruin probability?
We can describe the evolution of the gambler's fortune with the following weighted random walk on the *line graph* $\{0, 1, \dots, m\}$, that is on the nearest-neighbour graph $G_m = (V_m, E_m)$ with vertex set $V_m := \{0, \dots, m\}$ and unoriented edges $E_m := \{\langle k, k+1 \rangle : k \in \{0, \dots, m-1\}\}$.



Let $\beta := p/q > 0$. For each edge $e = \langle k, k+1 \rangle$, $k \in \{0, \dots, m-1\}$, we define its conductance by $c(e) := \beta^k$, and define $(S_n)_{n\in\mathbb{N}_0}$ as the weighted random walk on G_m, starting at $k \in \{1, \dots, m-1\}$. Then, $(S_n)_{n\in\mathbb{N}_0}$ moves to the right with probability p, moves left with probability q and can therefore be used to describe the (initial) evolution of the gambler's fortune. Doing so, the gambler's fate depends on which of the stopping times $\sigma_0 := \inf\{j \in \mathbb{N} : S_j = 0\}$ and $\sigma_m := \inf\{j \in \mathbb{N} : S_j = m\}$ occurs first.

Reversible Markov chains and weighted random walks on networks are related as follows. On the one hand, every weighted random walk $(Y_n)_{n\in\mathbb{N}_0}$ on a finite, undirected network (G, c), that is a network whose graph $G = (V, E)$ only has finitely many vertices and whose edges are unoriented, is reversible: For $v \in V$, define $\pi(v) := c(v)/c_G$, where $c_G := \sum_{v\in V} c(v)$. Then, if we interpret

π as a probability measure on $(V, \mathcal{P}(V))$, where $\mathcal{P}(V)$ denotes the power set of V, by setting $\pi(\{v\}) := \pi(v)$, this is the stationary distribution of $(Y_n)_{n\in\mathbb{N}_0}$ and satisfies the detailed balance equations:

$$\pi(u)p(u,v) = \frac{c(u)}{c_G}\frac{c(u,v)}{c(u)} = \frac{c(v)}{c_G}\frac{c(v,u)}{c(v)} = \pi(v)p(v,u)$$

for all $u, v \in V$ with $u \sim v$. Hence $(Y_n)_{n\in\mathbb{N}_0}$ is reversible with respect to π.
On the other hand, given a reversible Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ with finite state space, we can construct a network such that the weighted random walk thereon coincides with $(Y_n)_{n\in\mathbb{N}_0}$ in law. Therefor, take the states of the Markov chain as vertices, and connect two vertices $u, v \in V$ with an edge if and only if $p(u,v) > 0$ where p is the transition matrix of $(Y_n)_{n\in\mathbb{N}_0}$. As conductances, for $u \sim v$ we set $c(u,v) := \pi(u)p(u,v)$ where π is the stationary distribution of $(Y_n)_{n\in\mathbb{N}_0}$.

1.3.3. Harmonicity, voltage and current. For now, assume we are given a finite network (G, c), where $G = (V, E)$ is an undirected graph, and a particle that travels along the edges of G according to the weighted random walk $(Y_n)_{n\in\mathbb{N}_0}$ thereon. For $A \subseteq V$, let $\sigma_A := \inf\{l \in \mathbb{N}_0 : Y_l \in A\}$ be the *first hitting time* of A, where we write σ_a if $A = \{a\}$ consists of a single vertex $a \in V$. A function $F : V \to \mathbb{R}$ is called *harmonic* at $u \in V$ if

$$F(u) = \sum_{v: v\sim u} p(u,v)F(v).$$

An example of such a function is the following. Given two disjoint sets $A, Z \subseteq V$, look at the probability that the particle visits A before Z as a function of the particle's starting point. That is, define $f : V \to [0,1]$ via

$$f(u) := P^u(\sigma_A < \sigma_Z).$$

Clearly, $f(u) = 1$ for $u \in A$ and $f(u) = 0$ for $u \in Z$. Due to the Markov property, the function f further is harmonic at all vertices $u \in V \setminus (A \cup Z)$:

$$f(u) = \sum_{v: v\sim u} P^u(Y_1 = v)P^u(\sigma_A < \sigma_Z \mid Y_1 = v) = \sum_{v: v\sim u} p(u,v)f(v).$$

Among other things, harmonic functions satisfy the following properties:

LEMMA 1.3.3 (Maximum and uniqueness principle, e. g. [**39**, p. 20]). *Let V be finite or countable and $f, g : V \to \mathbb{R}$.*

1. *If f is harmonic at all states of some subset $W \subset V$ and the supremum of f is achieved at some vertex $w \in W$, then f is constant on all states of the connected component of w in $(W, E(W))$, where $E(W)$ consists of all edges $\langle w_1, w_2\rangle \in E$ with $w_1, w_2 \in W$.*
2. *Suppose G is connected and W is a finite subset of V. If f, g are both harmonic on W and $f(v) = g(v)$ for all $v \notin W$, then $f = g$.*

For finite graphs, computing the values of a harmonic function amounts to the solution of a (finite) system of linear equations. In general, however, this is non-trivial.
The connection between weighted random walks and electrical networks can be expanded as follows. Given disjoint subsets $A, Z \subset V$, we call a function $\mathsf{v} : V \to \mathbb{R}$ such that v is harmonic for all $u \in V \setminus (A \cup Z)$ a *voltage*. As a consequence of the uniqueness principle, a voltage is uniquely determined by its boundary values $\mathsf{v}|_A$ and $\mathsf{v}|_Z$.
A *flow* θ is a mapping $\theta : V \times V \to \mathbb{R}$ such that $\theta(u,v) = 0$ for all $u, v \in V$ with $\langle u, v\rangle \notin E$ and $\theta(u,v) = -\theta(v,u)$ for $u, v \in V$ with $u \sim v$. For a flow θ, the *divergence* of θ at $u \in V$ is

$$\text{div}\theta(u) := \sum_{v: v\sim u} \theta(u,v).$$

Given disjoint subsets $A, Z \subset V$, a *flow between A and Z* is defined as a flow θ such that $\mathrm{div}\theta(u) = 0$ for all $u \notin A \cup Z$. A flow between a singleton $A = \{a\}$ and Z is called a *flow from a to Z* if $\mathrm{div}\theta(a) \geq 0$. If $\mathrm{div}\theta(a) = 1$, a flow θ from a to Z is called *unit flow*.
The *energy* $\mathcal{E}(\theta)$ of a flow θ is defined as

$$\mathcal{E}(\theta) := \sum_{e \in E} \theta(e)^2 r(e),$$

where $\theta(e)^2 := \theta(v, w)^2$ if $e = \langle v, w \rangle$.
Given a voltage v between sets A and Z, a *current* or *current flow* i is a flow between A and Z such that

$$\mathsf{i}(u, v) := c(u, v)(\mathsf{v}(u) - \mathsf{v}(v))$$

for all pairs u, v of neighbouring vertices of V. Note that by definition, $\mathsf{i}(u, v) = -\mathsf{i}(v, u)$, and for all $u \in V$ such that v is harmonic at u, we have $\mathrm{div}\mathsf{i}(u) = 0$, hence i indeed is a flow between A and Z.
In the language of electrical networks, the property $\mathrm{div}\mathsf{i}(u) = 0$ for $u \notin A \cup Z$ is called *Kirchhoff's (node) law*, and the relation

$$\mathsf{v}(u) - \mathsf{v}(v) = \mathsf{i}(u, v) r(u, v),$$

where $u, v \in V$ with $u \sim v$, which immediately follows from the definition of i, is referred to as *Ohm's law*.

1.3.4. Effective conductance. Suppose the network is such that there are two distinctive regions of the graph, characterised by disjoint subsets A and Z of V. When the particle whose trajectory is described by $(Y_n)_{n \in \mathbb{N}}$ is initially placed at a vertex that belongs to set A, we are interested in the probability that - as time evolves - the particle visits a site of Z before returning to a site of A. That is, we want to find the *escape probability* $P^a(\sigma_A^+ > \sigma_Z)$, where $a \in A$ and $\sigma_A^+ := \inf\{l \in \mathbb{N} : Y_l \in A\}$ is the *first return time* of A. In particular, we are interested in the escape probability from a singleton $A = \{a\}$. To compute this, impose a voltage of $\mathsf{v}(a) = 1$ at a and 0 on Z. It follows from harmonicity of the function $u \mapsto P^u(\sigma_a < \sigma_Z)$ on $V \setminus (Z \cup \{a\})$ and the uniqueness principle that

$$P^u(\sigma_a < \sigma_Z) = \frac{\mathsf{v}(u)}{\mathsf{v}(a)} \tag{1.3.2}$$

for all $u \notin Z \cup \{a\}$. In conjunction with a one-step evolution of the weighted random walk, this leads to

$$P^a\big(\sigma_a^+ > \sigma_Z\big) = \frac{\sum_{u:u \sim a} \mathsf{i}(a, u)}{\mathsf{v}(a) c(a)} =: \frac{\mathcal{C}(a \leftrightarrow Z)}{c(a)}$$

where i is the current flow associated with v. The expression $\mathcal{C}(a \leftrightarrow Z)$ is called the *effective conductance* between a and Z. Its reciprocal $\mathcal{R}(a \leftrightarrow Z)$ is called the *effective resistance* between a and Z, with $\mathcal{R}(a \leftrightarrow Z) := \infty$ in case $\mathcal{C}(a \leftrightarrow Z) = 0$. The effective conductance is the net amount of current flowing into the graph at a when we impose a unit voltage at a. Therefore, if we interpret Z as a single vertex and the whole graph between a and Z as a single edge between a and Z, the effective conductance is the conductance of this single edge.

1.3.5. Network reduction. Suppose we are interested in the escape probability between two single vertices $a, z \in V$ of the graph. At the moment, it remains unclear whether we can actually calculate effective conductance and resistance between a and z without explicitly computing a voltage function and the associated current flow. The most straightforward alternative to brute-force calculation is to perform a pre-processing of the graph before computing v. That is, there exist local transformations of the network such that either the vertex or edge number is reduced but the value of $\mathcal{C}(a \leftrightarrow z)$ remains unchanged in the reduced graph. The most common network reductions, which are mostly motivated by physical laws, are the following.

The series law. If, in an electrical circuit, two resistances are series connected, they can as well be viewed as one. When computing the effective conductance, this works as well: If $v \in V \setminus \{a, z\}$ is a vertex of degree 2, that is, if v has exactly two neighbouring vertices u and w, then we can replace $\langle u, v\rangle$ and $\langle v, w\rangle$ by a single edge $\langle u, w\rangle$ from u to w whose resistance satisfies

$$r(\langle u, w\rangle) := r(u, v) + r(v, w)$$

and remove v from the graph. In this case, the voltages at each remaining vertex are unchanged and the current from u to w is given by $\mathsf{i}(u, v)$.

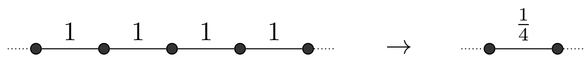


FIGURE 3. Network reduction using the series law. Instead of a series of four vertices with conductance 1 each, we are left with one edge of conductance 1/4.

In mathematical terms, this amounts to setting up a new network (G', c') where $G' = (V', E')$ with $V' := V \setminus \{v\}$ and $E' := (E \setminus \{\langle u, v\rangle, \langle v, w\rangle\}) \cup \{\langle u, w\rangle\}$. As conductances in the reduced network (G', c'), we take $c'(e) := c(e)$ for all $e \neq \langle u, w\rangle$ and $c'(u, w) := 1/(r(u, v) + r(v, w))$. If we define functions v' on V' and i' on $E' \times E'$ via

$$\mathsf{v}'(x) := \mathsf{v}(x), \quad \mathsf{i}'(x, y) := \mathsf{i}(x, y), \quad \mathsf{i}'(u, w) := \mathsf{i}(u, v),$$

where $x, y \in V'$ are such that $(x, y) \neq (u, w)$, then we can check that v' is a voltage and i' is a current function on V' and $E' \times E'$, respectively. The voltage v' obeys the same boundary conditions as v.

The parallel law. In an electrical circuit, if there are multiple conductors in parallel arrangement, they can as well be viewed as one. The mathematical analogy of this is that if there are multiple edges $e_1, \ldots, e_n$ between two vertices u and v, we can replace them with a single edge $\langle u, v\rangle$ with conductance

$$c(\langle u, v\rangle) := c(e_1) + \ldots + c(e_n).$$

As for the series law, doing so does not affect the voltages or currents except for the current along the edge $\langle u, v\rangle$, whose value is given by $\mathsf{i}(u, v) = \mathsf{i}(e_1) + \ldots + \mathsf{i}(e_n)$. Again, this has to be understood in the sense that the values of v and i still define a voltage and current function, respectively, in the reduced graph.

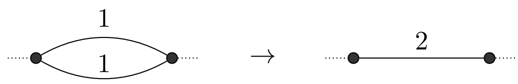


FIGURE 4. Network reduction using the series law. We replace two edges, each of conductance 1, with an edge of conductance 2.

The star-triangle law. The following law is only listed for the sake of completeness. With suitable choice of conductances, the following two local configurations of an electrical network are equivalent.

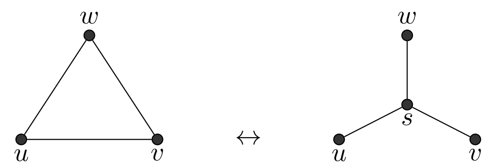


FIGURE 5. The appropriate local shape for the star-triangle law.

The proper choice of conductances is implicitly determined by setting

$$\gamma := \frac{r(u,v)+r(v,w)+r(u,w)}{r(u,v)r(v,w)r(u,w)} = \frac{c(u,s)c(v,s)c(w,s)}{c(u,s)+c(v,s)+c(w,s)},$$

and demanding that

$$c(s,u)c(v,w) = c(s,v)c(u,w) = c(s,w)c(u,v) = \gamma.$$

Quality-of-life reductions. There are further network reductions which are more straightforward but nevertheless important. First, if the voltage value at two vertices $u, v \in V \setminus \{a, z\}$ coincides, we can merge them. That is, we can replace them by a new vertex w which we then connect to all edges that formerly were adjacent to u or v. Second, isolated vertices, that is vertices $u \in V$ with exactly one adjacent edge $\langle u, v\rangle$, can safely be erased together with their adjacent edge. This is due to the fact that their voltage value coincides with that of v due to harmonicity. The same can be done with loop edges, that is edges $\langle u, u\rangle$ whose starting and endpoint coincide.

EXAMPLE 1.3.4 (The gambler's ruin revisited). We can compute the ruin probability of the gambler in Example 1.3.2 using network reduction. In the setting of Example 1.3.2, write P^k for the law of $(S_n)_{n\in\mathbb{N}_0}$ starting at $S_0 = k$. If we apply a unit voltage to node 0 and and zero voltage to node m, then it follows from (1.3.2) that

$$P^k(\sigma_0 < \sigma_m) = \mathsf{v}(k)$$

where v is the voltage function on V_m with $\mathsf{v}(0) = 1$, $\mathsf{v}(m) = 0$. Now, we apply the series law between 0 and k and between k and m. More precisely, we apply it at each of the vertices $1, \dots, k-1$ and $k+1, \dots, m-1$. This reduces the graph to the vertices $\{0, k, m\}$ and the edges $\{\langle 0, k\rangle, \langle k, m\rangle\}$. The resistances of the remaining edges are given by

$$r(\langle 0,k\rangle) = \sum_{l=0}^{k-1} \beta^{-l} = \begin{cases} k & \text{for } p = \frac{1}{2}, \\ \frac{1-\beta^{-k}}{1-\beta^{-1}} & \text{for } p \neq \frac{1}{2} \end{cases}$$

and

$$r(\langle k,m\rangle) = \sum_{l=k}^{m-1} \beta^{-l} = \begin{cases} m-k & \text{for } p = \frac{1}{2}, \\ \frac{\beta^{-k}-\beta^{-m}}{1-\beta^{-1}} & \text{for } p \neq \frac{1}{2}. \end{cases}$$

Thus, since the computation of the ruin probability amounts to finding $\mathsf{v}(k)$, and the computation of $\mathsf{v}(k)$ has been reduced to the solution of a single linear equation for harmonicity of v at k, it follows that the ruin probability is given by

$$P^k(\sigma_0 < \sigma_m) = \begin{cases} \frac{m-k}{m} & \text{for } p = \frac{1}{2}, \\ \frac{\beta^{-k}-\beta^{-m}}{1-\beta^{-m}} & \text{for } p \neq \frac{1}{2}. \end{cases} \tag{1.3.3}$$

1.3.6. Infinite networks. We are slowly closing in on completing the collection of tools we will employ in Chapter 2. Now, suppose we are dealing with an infinite network. That is, suppose that the vertex set of the given graph is countably infinite. For simplicity, we continue to assume that the graph is undirected.

EXAMPLE 1.3.5 (β-biased random walk on $\mathbb{Z}$). A straightforward example of weighted random walk on an infinite network is given by the biased random walk on $\mathbb{Z}$. Here, the network (G, c) is given by the infinite line graph $G = (\mathbb{Z}, E)$ with nearest-neighbour edges $E := \{\langle k, k+1\rangle : k \in \mathbb{Z}\}$ and conductances $c(\langle k, k+1\rangle) := \beta^k$, where $\beta > 0$. For $\beta = 1$, the weighted random walk on (G, c) is the simple random walk on $\mathbb{Z}$, whereas for $\beta \neq 1$, it becomes biased. More precisely, with $k \in \mathbb{Z}$, for $\beta \in (0, 1)$ we have $p(k, k-1) > p(k, k+1)$, thus the walk is more likely to move left than right. For $\beta \in (1, \infty)$, this relation reverses and we have $p(k, k+1) > p(k, k-1)$, leaving the walk to be more likely to move to the right.

For later reference, we gather two well-known facts about biased random walk on $\mathbb{Z}$ which can be found, e. g., as equations (13.3) and (13.4) in [**26**]. Let $(S_n)_{n\in\mathbb{N}_0}$ be the biased random walk on $\mathbb{Z}$ starting at $S_0 = 0$ with probability $p > \frac{1}{2}$ to step right and probability $q := 1 - p$ to step left. For $k \in \mathbb{Z}$, let

$$\sigma_k^{\mathbb{Z}} := \inf\{l \geq 0 : S_l = k\}$$

and denote the expectation of $(S_n)_{n\in\mathbb{N}_0}$ starting at $S_0 = 0$ by $\mathrm{E}_{\mathbb{Z}}^0$.

LEMMA 1.3.6. *For $x > 0$, it holds that*

$$\mathrm{E}_{\mathbb{Z}}^0\big[x^{\sigma_1^{\mathbb{Z}}}\big] = \frac{1 - \sqrt{1 - 4pqx^2}}{2qx}, \quad \mathrm{E}_{\mathbb{Z}}^0\big[\sigma_1^{\mathbb{Z}}\big] = \frac{1}{1 - 2q}.$$

For completeness, we include a brief proof.

PROOF. Let $x > 0$ and $f(x) := \mathrm{E}_{\mathbb{Z}}^0\big[x^{\sigma_1^{\mathbb{Z}}}\big]$. On the one hand, the Markov property gives

$$f(x) = px + qxf(x)^2. \tag{1.3.4}$$

On the other hand, $\lim_{x\searrow 0} f(x) = 0$ due to dominated convergence. Hence, solving (1.3.4) for $f(x)$ leads to the given formula. The expectation of $\sigma_1^{\mathbb{Z}}$ follows from the derivative of the generating function. □

In some cases, it turns out useful to approximate infinite networks by series of finite networks. Let (V, E) be a graph with countably infinite vertex and at most countably infinite edge set. A sequence (V_n, E_n), $n \in \mathbb{N}$, of finite subgraphs of (V, E) is called *exhausting* if $V_n \subseteq V_{n+1}$, $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$ and $V = \cup_{n\in\mathbb{N}} V_n$ as well as $E = \cup_{n\in\mathbb{N}} E_n$.

1.3.7. Recurrence and transience. We continue with the observation of a particle that moves along the vertices of an infinite network according to the weighted random walk thereon. Naturally, we ask whether the particle will stay concentrated on a certain domain (which might grow over time), or whether there is a chance that it will at some point leave e. g. the domain of its starting point and never return.
We call a state $a \in V$ *recurrent* if $P^a(\sigma_a^+ < \infty) = 1$, and *transient* otherwise. Due to irreducibility of weighted random walk on the connected component of its starting point, recurrence and transience are properties shared by all vertices of a connected component of a graph. For notational simplicity, for the remainder of this section we assume that the underlying graph of the infinite network is connected. Technically, we assume this and further exclude the pathological case of $c(e) = 0$ for some $e \in E$. We call weighted random walk on an infinite network *recurrent* (*transient*) if one - and therefore all - of its states is recurrent (transient).

Suppose (G, c) is an infinite network with an exhausting sequence $((G'_n, c))_{n\in\mathbb{N}}$, where $G'_n = (V'_n, E'_n)$ for $n \in \mathbb{N}$. From this, we define a further sequence $((G_n, c))_{n\in\mathbb{N}}$ of finite networks as follows. For $n \in \mathbb{N}$, we identify $V \setminus V'_n$ with a single vertex z_n, convert all edges $\langle u, v\rangle$ with $u \in V'_n$, $v \notin V'_n$ to edges $\langle u, z_n\rangle$ and then reduce all parallel edges that are adjacent to z_n to single edges using the parallel law. We denote the collection of these reduced edges by $\tilde{E}_n$. Then, we set $G_n = (V_n, E_n) := (V'_n \cup \{z_n\}, E'_n \cup \tilde{E}_n)$.
Let $a \in V$. The *effective conductance* $\mathcal{C}(a \leftrightarrow \infty)$ from a to ∞ is defined as the limit of the effective conductances $\mathcal{C}(a \leftrightarrow z_n)$ (starting at n sufficiently large such that $a \in V_n$) in (G_n, c):

$$\mathcal{C}(a \leftrightarrow \infty) := \lim_{n\to\infty} \mathcal{C}(a \leftrightarrow z_n).$$

Again, we define the *effective resistance* $\mathcal{R}(a \leftrightarrow \infty)$ from a to ∞ as the reciprocal of $\mathcal{C}(a \leftrightarrow \infty)$, with $\mathcal{R}(a \leftrightarrow \infty) := \infty$ if $\mathcal{C}(a \leftrightarrow \infty) = 0$.
A *flow from a to ∞* is defined as a flow θ such that $\mathrm{div}\theta(u) = 0$ for all $u \neq a$ and $\mathrm{div}\theta(a) > 0$.
The effective conductance from a to ∞ and the energy of a flow from a to ∞ are related to recurrence and transience, respectively, of the weighted random walk on an infinite network as follows.

LEMMA 1.3.7 (Proposition 21.6 in [**33**]). *Let (G, c) be an infinite network. The following are equivalent:*

(1) The weighted random walk on the network is transient.
(2) There exists $a \in V$ such that $\mathcal{C}(a \leftrightarrow \infty) > 0$.
(3) There exists a flow θ from some node $a \in V$ to ∞ with $\mathcal{E}(\theta) < \infty$.

There are several helpful tools to compute or estimate the effective conductance between a vertex $a \in V$ and ∞.

LEMMA 1.3.8 (Thomson's principle, Theorem 9.10 in [**33**]). *Let (G, c) be an infinite network and $a \in V$. Then*

$$\mathcal{R}(a \leftrightarrow \infty) = \inf\{\mathcal{E}(\theta) : \theta \textit{ is a unit flow from } a \textit{ to } \infty\}.$$

It follows from Thomson's principle that increasing the resistance of individual edges while retaining the resistance values for the remainder of the edge set can only increase the effective resistance. More precisely, the following relation holds.

LEMMA 1.3.9 (Rayleigh's monotonicity law, Theorem 9.12 in [**33**]). *If r and r' are resistances on E such that $r(e) \leq r'(e)$ for all $e \in E$, then*

$$\mathcal{R}(a \leftrightarrow \infty; r) \leq \mathcal{R}(a \leftrightarrow \infty; r'),$$

where $\mathcal{R}(a \leftrightarrow \infty; r)$ and $\mathcal{R}(a \leftrightarrow \infty; r')$ denote the effective resistance between a and ∞ in the network using resistances r and r', respectively.

A *path* P between vertices $u, v \in V$ is a finite sequence $P = (e_1, \ldots, e_n)$ of edges $e_1 = \langle u_0, u_1\rangle, \ldots, e_n = \langle u_{n-1}, u_n\rangle \in E$ with $u_0 = u$ and $u_n = v$.

EXAMPLE 1.3.10. (Transience of biased random walk in $\mathbb{Z}$) Let $(S_n)_{n\in\mathbb{N}_0}$ be the biased random walk on $\mathbb{Z}$ with conductances $c(\langle k, k+1\rangle) := \beta^k$, where $\beta \neq 1$. Then, the origin 0 is a transient state, hence $(S_n)_{n\in\mathbb{N}_0}$ is transient. For $\beta > 1$, we can see this by sending a unit flow in the direction of the bias, that is along the infinite path $P = (e_0, e_1, \ldots)$ with $e_i := \langle i, i+1\rangle$ for all $i \in \mathbb{N}_0$. To make this precise, set $\theta(i, i+1) = 1 = -\theta(i+1, i)$ for $i \in \mathbb{N}_0$ and $\theta \equiv 0$ otherwise. The energy of this flow from 0 to ∞ is given by

$$\mathcal{E}(\theta) = 2\sum_{e\in P} r(e) = 2\sum_{i=0}^{\infty} \beta^{-i} < \infty.$$

For $\beta < 1$, we can apply the same approach with the path $P = (e'_0, e'_1, \ldots)$ where $e'_i := \langle -i, -i-1 \rangle$ for $i \in \mathbb{N}_0$.

A set $\Pi \subset E$ is called an *edge-cutset* for vertices $u, v \in V$ if every path from u to v contains an edge in Π.

LEMMA 1.3.11 (Nash-Williams inequality, Proposition 9.15 in [**33**]). *Let $T \subseteq \mathbb{N}$ and suppose $(\Pi_k)_{k\in T}$ is a family of disjoint edge-cutsets which separate vertices $a, z \in V$. Then*

$$\mathcal{R}(a \leftrightarrow z) \geq \sum_{k\in T} \Bigg(\sum_{e\in\Pi_k} c(e) \Bigg)^{-1}.$$

EXAMPLE 1.3.12 (Simple random walk on $\mathbb{Z}^d$). The simple random walk on $\mathbb{Z}^d$, that is the weighted random walk on the lattice $\mathbb{Z}^d$ with conductances $c \equiv 1$, is recurrent for $d = 1, 2$ and transient otherwise. Again, to show this it suffices to show that the origin $\mathbf{0}$ is a recurrent or transient state, respectively. In the case $d = 1$, consider the exhausting sequence that consists of the subgraphs G_k that only contain vertices in $\{-k, \ldots, k\}$, together with a vertex z_k identified with the remainder of $\mathbb{Z}$. If we define disjoint edge-cutsets $\Pi_l := \{\langle l, l+1\rangle, \langle -l, -l-1\rangle\}$, $l \in \mathbb{N}_0$, then these separate the origin and z_k for $l \leq k$, and it follows from the Nash-Williams inequality that

$$\mathcal{R}(\mathbf{0} \leftrightarrow \infty) = \lim_{k\to\infty} \mathcal{R}(\mathbf{0} \leftrightarrow z_k) \geq \lim_{k\to\infty} \sum_{l=0}^{k} \frac{1}{c(\langle l, l+1\rangle) + c(\langle -l-1, -l\rangle)} = \infty.$$

Hence, the walk is recurrent.
In $\mathbb{Z}^2$, the same strategy applies with the edge cutsets Π_k consisting of all edges that connect vertices $(\mathsf{x}(v), \mathsf{y}(v)), (\mathsf{x}(w), \mathsf{y}(w)) \in \mathbb{Z}^2$ with $\max(\mathsf{x}(v), \mathsf{y}(v)) = k$ and $\max(\mathsf{x}(w), \mathsf{y}(w)) = k+1$. Each cutset Π_k consists of $4(2k+1)$ edges, therefore we get

$$\mathcal{R}(\mathbf{0} \leftrightarrow \infty) \geq \sum_{k=0}^{\infty} \frac{1}{4(2k+1)} = \infty,$$

which implies recurrence of the walk.
In dimension $d = 3$ on the other hand, a finite energy flow from $\mathbf{0}$ to ∞ can be constructed, cf. [**33**, Ex.21.9]. Thus, simple random walk on $\mathbb{Z}^3$ is transient. As $\mathbb{Z}^3$ can be viewed as a subset of $\mathbb{Z}^d$ for $d \geq 3$, transience for all $d \geq 3$ follows from Rayleigh's monotonicity law.

CHAPTER 2

Biased random walk on a one-dimensional percolation model

For the remainder of this thesis, we focus on a toy model for biased random walk on a supercritical percolation environment. In particular, we investigate the one-dimensional conditional percolation model that was introduced by Axelson-Fisk and Häggström in their papers [**4**] and [**5**]. At the time of their writing, existence of a critical bias parameter for the biased random walk on the infinite open cluster of the percolation process in $\mathbb{Z}^d$ was still an open problem. In the model of Axelson-Fisk and Häggström, dead-end regions of the environment in the direction of the bias take the simplest possible form, and the parameter λ_c marking the transition from the ballistic to the subballistic speed regime can be given as an elementary function of the edge retention parameter p.

2.1. The percolation model

We start with a description of the model of Axelson-Fisk and Häggström, together with some basic properties of the percolation environment and the biased random walk thereon.
As the underlying graph for the percolation environment, we consider the *ladder graph* $G = (V, E)$, with vertex set $V := \mathbb{Z} \times \{0, 1\}$ and edge set $E := \{\langle u, v\rangle \in V^2 : |u - v| = 1\}$, where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^2$. If $v = (x, y) \in V$, we write $\mathsf{x}(v) = x$ and $\mathsf{y}(v) = y$, and call x and y the x- and y-coordinate of v, respectively.

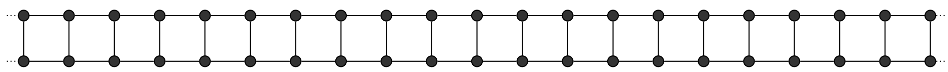


FIGURE 1. The ladder graph.

Set $p \in (0, 1)$. In a first step, we consider i.i.d. bond percolation with edge retention parameter p on G. That is, each edge $e \in E$ is retained independently of all other edges with probability p, and deleted with probability $1 - p$. As usual, we call an edge $e \in E$ *open* if it is retained and *closed* if it is deleted.
Again, the state space of the percolation process is $\Omega := \{0, 1\}^E$, which we endow with the product σ-algebra $\mathcal{F}$. The elements $\omega \in \Omega$ are called configurations. We interpret $\omega(e) = 1$ for $\omega \in \Omega$ and $e \in E$ as the edge e being open in the configuration ω.
In a naive approach, we might already try to use the standard i.i.d. bond percolation measure μ_p on $(\Omega, \mathcal{F})$ with retention parameter p to choose a configuration ω. In that case, a straightforward application of Borel-Cantelli shows that for all $p < 1$, in μ_p-almost every configuration there exist infinitely many $k \in \mathbb{Z}$ such that both of the parallel edges $\langle (k, 0), (k+1, 0)\rangle$ and $\langle (k, 1), (k+1, 1)\rangle$ are closed. This prevents the existence of an infinite cluster under μ_p for $p < 1$ and implies $p_c = 1$ for the critical percolation threshold p_c of i.i.d. bond percolation on the infinite ladder.

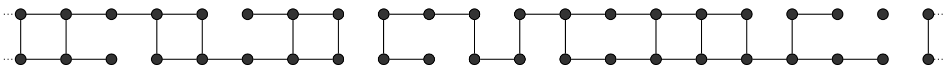


FIGURE 2. Bond percolation on the ladder graph. Due to pairs of parallel closed horizontal edges, there cannot exist an infinite open cluster stretching towards both the left and the right in x-direction.

Since the topic at hand is the speed of biased random walk on an almost surely infinite cluster, the percolation environment has to be derived under additional assumptions. These are - roughly speaking - that we condition i.i.d. bond percolation on the existence of a bi-infinite open path, which is reminiscent to the construction of the incipient infinite cluster by Kesten [**31**], but for p strictly below criticality.

Recall that a path P between vertices $u, v \in V$ is a finite sequence $P = (e_1, \dots, e_n)$ of edges $e_1 = \langle u_0, u_1 \rangle, \dots, e_n = \langle u_{n-1}, u_n \rangle \in E$ with $u_0 = u$ and $u_n = v$. The path P is called *open* if $\omega(e_k) = 1$ for $k = 1, \dots, n$.

Let Ω_{N_1,N_2} be the event that there exists an open path connecting a vertex with x-coordinate $-N_1$ to a vertex with x-coordinate N_2, and let P_{p,N_1,N_2} be the probability measure on $(\Omega, \mathcal{F})$ arising from conditioning the i.i.d. bond percolation measure μ_p with retention parameter p on the event Ω_{N_1,N_2}. Then P_{p,N_1,N_2} converges weakly as $N_1, N_2 \to \infty$ to a probability measure P_p^* on $(\Omega, \mathcal{F})$.

LEMMA 2.1.1 (Theorem 2.1 and Corollary 2.2 in [**5**]). *For any $p \in (0,1)$, as $N_1, N_2 \to \infty$, the probability measures P_{p,N_1,N_2} converge weakly to a translation invariant probability measure P_p^* on $(\Omega, \mathcal{F})$ satisfying $\mathrm{P}_p^*(\Omega^*) = 1$ where $\Omega^* = \bigcap_{N_1,N_2 \in \mathbb{N}} \Omega_{N_1,N_2}$ is the event that a bi-infinite open path exists.*

It is easily seen that P_p^*-almost surely, there is a unique infinite open cluster $\mathcal{C} \subseteq V$ consisting of all vertices $v \in V$ which are connected via open paths to vertices with arbitrary x-coordinate. We define

$$\mathrm{P}_p(\cdot) := \mathrm{P}_p^*(\cdot \mid \mathbf{0} \in \mathcal{C})$$

where $\mathbf{0} := (0,0)$ is the origin. This will serve as the law of ω.

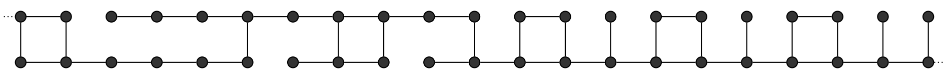


FIGURE 3. A sample of the percolation configuration according to P_p

2.1.1. General properties of the percolation environment.

From [**5**], we recall some basic properties of the conditional percolation measure P_p^*.

A function $f : \Omega \to \mathbb{R}$ is *increasing* if $f(\omega_1) \le f(\omega_2)$ for all $\omega_1, \omega_2 \in \Omega$ such that $\omega_1(e) \le \omega_2(e)$ for all $e \in E$. It follows from the FKG-inequality, e. g. [**25**, Theorem 2.4], that P_p^* *stochastically dominates* μ_p. That is, $E_{\mu_p}[f] \le E_{\mathrm{P}_p^*}[f]$ for all increasing functions f, where $E_P[\cdot]$ denotes expectation with respect to a probability measure P.

LEMMA 2.1.2 (Lemma 3.1 in [**5**]). *For any $p \in (0,1)$, the conditional percolation measure P_p^* stochastically dominates the i.i.d. bond percolation measure μ_p.*

Due to the conditional nature of the conditional percolation measure P_p, we cannot expect a direct Markov property of the environment in the sense that the probability that an edge $e \in E$ is open only depends on the state of finitely many other edges. Indeed, look at the following two configurations. Let $n \in \mathbb{N}$ and A_1 be the event that to the right of $\mathbf{0}$, the conditional percolation environment consists of two parallel lines of n open edges. More precisely, suppose that the edge $\langle (0,0),(0,1)\rangle$ and all horizontal edges $\langle (i,0),(i+1,0)\rangle$, $\langle (i,1),(i+1,1)\rangle$, $i = 0,\dots,n$ are open, and that all vertical edges $\langle (i,0),(i,1)\rangle$, $i = 1,\dots,n$ are closed. On the other hand, let A_2 be the same configuration except that the edge $\langle (0,0),(1,0)\rangle$ is closed in A_2. Both events have positive probability under P_p but, given A_1, the edge $\langle (n,1),(n+1,1)\rangle$ can be closed with positive probability whereas given A_2, the infinite path must include the edge $\langle (n,1),(n+1,1)\rangle$ hence the edge must almost surely be open under P_p.

FIGURE 4. Visualisations of the events A_1 (left) and A_2 (right). In the right case, the dotted edge must be open in order to have an infinite open path.

However, with a small amount of additional information, a Markov property does hold for the environment under P^*_p. For $i \in \mathbb{Z}$, let $E^{i,-}$ be the set of all edges $\langle u,v\rangle$ such that $\mathsf{x}(u) \le i$ and $\mathsf{x}(v) \le i$. We say that a vertex $v \in V$ with $\mathsf{x}(v) = i$ is *backwards-communicating* if there exists an infinite open self-avoiding path $P = (e_1, e_2, \dots)$ starting at v such that $e_k \in E^{i,-}$ for all $k \in \mathbb{N}$. Now, we track which of the vertices $(k,0)$ and $(k,1)$, $k \in \mathbb{Z}$ are backwards-communicating and view this as a process in time k. That is, we define the process $(\mathbb{T}_k)_{k\in\mathbb{Z}}$ taking values in $\{00, 01, 10, 11\}$ as follows:

$$\mathbb{T}_k = \begin{cases} 00 & \text{if neither } (0,k) \text{ nor } (1,k) \text{ are backwards-communicating,} \\ 10 & \text{if } (0,k) \text{ is backwards-communicating, but } (1,k) \text{ is not,} \\ 01 & \text{if } (0,k) \text{ is not backwards-communicating, but } (1,k) \text{ is,} \\ 11 & \text{if both } (0,k) \text{ and } (1,k) \text{ are backwards-communicating.} \end{cases}$$

Note that due to $\mathrm{P}^*_p(\Omega^*) = 1$, we have $\mathrm{P}^*_p(\mathbb{T}_k = 00) = 0$ for all $k \in \mathbb{Z}$, but the corresponding state is listed for completeness. The process $(\mathbb{T}_k)_{k\in\mathbb{Z}}$ is a time-homogeneous Markov chain under P^*_p.

LEMMA 2.1.3 (Theorem 3.1 in [**5**]). *Under* P^*_p*, the process* $(\mathbb{T}_k)_{k\in\mathbb{Z}}$ *is a time-homogeneous Markov chain.*

Axelson-Fisk and Häggström also explicitly computed the transition probabilities of the process $(\mathbb{T}_k)_{k\in\mathbb{Z}}$, cf. [**5**, p. 1111-1112]. Writing $p_{\mathsf{ab},\mathsf{cd}} := \mathrm{P}^*_p(\mathbb{T}_1 = \mathsf{cd} \mid \mathbb{T}_0 = \mathsf{ab})$, they can be written as

$$\begin{pmatrix} p_{01,01} & p_{01,10} & p_{01,11} \\ p_{10,01} & p_{10,10} & p_{10,11} \\ p_{11,01} & p_{11,10} & p_{11,11} \end{pmatrix} = \begin{pmatrix} 1 - p_{01,11} & 0 & p_{01,11} \\ 0 & 1 - p_{01,11} & p_{01,11} \\ p_{11,01} & p_{11,01} & 1 - 2p_{11,01} \end{pmatrix}, \tag{2.1.1}$$

where

$$p_{01,11} = \frac{1}{2p}\big(2p^2 - 1 + \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\big)$$

and

$$p_{11,01} = \frac{1}{4(1-p)}\big(2(1-p) - (3-2p)\big(1 + 2p - 2p^2 - \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\big)\big).$$

As an irreducible, aperiodic Markov chain taking values in $\{01, 10, 11\}$, the law of $\mathbb{T}_k$ converges towards a stationary distribution π whose values are given as elementary functions of p, cf. [**5**, p. 1112],

$$\begin{pmatrix}\pi(01) & \pi(10) & \pi(11)\end{pmatrix} = \begin{pmatrix}\frac{p_{11,01}}{2p_{11,01}+p_{01,11}} & \frac{p_{11,01}}{2p_{11,01}+p_{01,11}} & \frac{p_{01,11}}{2p_{11,01}+p_{01,11}}\end{pmatrix}.$$

In the limit $p \to 0$, the states 01 and 10 have probability $1/2$ under π, respectively. Roughly speaking, this can be interpreted in the sense that for small values of p, the percolation environment locally looks like the infinite line graph $\mathbb{Z}$.
Given $(\mathbb{T}_k)_{k\in\mathbb{Z}}$, the random environment ω can be drawn as i.i.d. bond percolation conditioned on being compatible with $(\mathbb{T}_k)_{k\in\mathbb{Z}}$. To make this precise, we divide the edge set E into the triplets $E^i := E^{i,-} \setminus E^{i-1,-}$, $i \in \mathbb{Z}$. Note that the values of $\mathbb{T}_{i-1}$ and $\omega(E^i)$ determine the value of $\mathbb{T}_i$.
Let $\eta \in \{0,1\}^{E^i}$. Given $\mathbb{T}_{i-1} = \mathsf{ab}$, $\mathbb{T}_i = \mathsf{cd}$, where $\mathsf{ab}, \mathsf{cd} \in \{01, 10, 11\}$, we say η is $\mathbb{T}_{i-1}$-$\mathbb{T}_i$-*compatible* if $\mathbb{T}_{i-1} = \mathsf{ab}$ and $\omega(E^i) = \eta$ imply $\mathbb{T}_i = \mathsf{cd}$. We further define the probability measure $\mathrm{P}^*_{p,i,\mathbb{T}_{i-1},\mathbb{T}_i}$ on E^i via

$$\mathrm{P}^*_{p,i,\mathbb{T}_{i-1},\mathbb{T}_i}(\eta) = \frac{\mathbb{1}_{\{\eta \text{ is } \mathbb{T}_{i-1}\text{-}\mathbb{T}_i\text{-compatible}\}}}{Z_{p,i,\mathbb{T}_{i-1},\mathbb{T}_i}} \prod_{e\in E^i} p^{\eta(e)}(1-p)^{1-\eta(e)},$$

where $\eta \in \{0,1\}^{E^i}$ and $Z_{p,i,\mathbb{T}_{i-1},\mathbb{T}_i}$ is a normalizing constant.

LEMMA 2.1.4 (Theorem 3.2 in [**5**]). *The conditional distribution of ω under P^*_p given $(\mathbb{T}_k)_{k\in\mathbb{Z}}$ is*

$$\prod_{i\in\mathbb{Z}} \mathrm{P}^*_{p,i,\mathbb{T}_{i-1},\mathbb{T}_i}.$$

It follows that we can construct the conditional percolation environment under P^*_p as the outcome of a Markov process as follows. Given $\mathbb{T}_k$, $k \in \mathbb{Z}$, we first draw the value of $\mathbb{T}_{k+1}$ according to the law of $\mathbb{T}_{k+1}$ given $\mathbb{T}_k$. Then, we draw $\omega(E^k)$ according to $\mathrm{P}^*_{p,k,\mathbb{T}_{k-1},\mathbb{T}_k}$ and so forth. The Markov property carries over to the environment chosen according to P_p since conditioning on the event $\{\mathbf{0} \in \mathcal{C}\}$ only affects the law of $\mathrm{P}^*_{p,0,\mathbb{T}_{-1},\mathbb{T}_0}$.
Unfortunately, there exists no canonical coupling of the P^*_p's as in the i.i.d. bond percolation case such that $\mathrm{P}^*_{p_1}$ stochastically dominates $\mathrm{P}^*_{p_2}$ for all $p_1 \geq p_2$.

LEMMA 2.1.5 (Proposition 5.1 in [**5**]). *For any fixed $p \in (0,1)$, there exists $\epsilon \in (0,p)$ such that P^*_p does not stochastically dominate $\mathrm{P}^*_{p'}$ for all $p' \in (0,\epsilon)$.*

This is due to the aforementioned fact that for $p \to 0$, the environment basically looks like an infinite line, where the infinite line with x-coordinate 0 and the infinite line with x-coordinate 1 are equally likely. That is, for any fixed $n \in \mathbb{N}$, in the limit as $p \to 0$, the probability that all edges $\langle(i,0),(i+1,0)\rangle$, $i = 0,\ldots,n$ are open becomes $1/2$. On the other hand, for fixed $p > 0$, the probability of this event vanishes as $n \to \infty$.
However, the edge density of the graph, that is the average number of open edges in E^i, $i \in \mathbb{Z}$ does increase with p. For $p \in (0,1)$, define

$$\vartheta(p) := \sum_{e\in E^0} \mathrm{P}^*_p(\omega(e) = 1).$$

LEMMA 2.1.6 (Theorem 5.1 in [**5**]). *For $0 < p_1 < p_2 < 1$, we have $\vartheta(p_1) \leq \vartheta(p_2)$.*

2.2. Random walk on the conditional percolation model

Henceforth, we fix the parameter $p \in (0,1)$ associated with edge retention. Almost all constants and objects defined below will depend on p, but this will not always figure in the notation. Given a configuration $\omega \in \{0,1\}^E$ chosen according to P_p, we define a random walk on G with bias $\lambda \in \mathbb{R}$ as follows. Let the conductances $(c(e))_{e\in E}$ be defined via

$$c(\langle u,v\rangle) := e^{\lambda(\mathsf{x}(u)+\mathsf{x}(v))}, \quad \langle u,v\rangle \in E.$$

Then $(Y_n)_{n\in\mathbb{N}_0}$ is defined as the lazy random walk on the infinite cluster $\mathcal{C}$ induced by the conductances c starting at $Y_0 := \mathbf{0}$. That is, when at $u \in V$ the walk attempts to move to a neighbour $v \in V$ in G with probability proportional to $c(\langle u,v\rangle)$. If $Y_n = u$, the direction Y_{n+1}^{cand} of the attempted step is

$$Y_{n+1}^{\mathrm{cand}} = \begin{cases} (\mathsf{x}(u)+1,\mathsf{y}(u)) & \text{with probability } \frac{e^{\lambda}}{e^{\lambda}+1+e^{-\lambda}}, \\ (\mathsf{x}(u),1-\mathsf{y}(u)) & \text{with probability } \frac{1}{e^{\lambda}+1+e^{-\lambda}}, \\ (\mathsf{x}(u)-1,\mathsf{y}(u)) & \text{with probability } \frac{e^{-\lambda}}{e^{\lambda}+1+e^{-\lambda}}. \end{cases}$$

Then, the step is actually performed if the corresponding edge is open, $\omega(\langle u,v\rangle) = 1$. Otherwise, the walk stays put.

$$Y_{n+1} = \begin{cases} Y_{n+1}^{\mathrm{cand}} & \text{if } \omega(\langle Y_n, Y_{n+1}^{\mathrm{cand}}\rangle) = 1, \\ Y_n & \text{otherwise.} \end{cases}$$

We denote the law of $(Y_n)_{n\in\mathbb{N}_0}$ on $(V^{\mathbb{N}_0},\mathcal{G})$ by $P_{\omega,\lambda}$, where $\mathcal{G}$ is the product σ-algebra on $V^{\mathbb{N}_0}$. Further, we write $P_{\omega,\lambda}^{v}$ for the law of the Markov chain with the same transition probabilities but with start at $v \in V$. Due to the symmetry of the law of ω, we only consider the case $\lambda > 0$. As before, the distribution $P_{\omega,\lambda}$ is referred to as the *quenched* law of $(Y_n)_{n\in\mathbb{N}_0}$, that is, the law of $(Y_n)_{n\in\mathbb{N}_0}$ given ω. The corresponding *annealed* law $\mathbb{P}$ is obtained by averaging the quenched laws over $\omega \in \Omega$ using P_p. That is, we define the probability measure $\mathbb{P}$ on $\{0,1\}^E \times V^{\mathbb{N}_0}$ by setting, for $A \in \mathcal{F}, B \in \mathcal{G}$,

$$\mathbb{P}(A\times B) := \int_A P_{\omega,\lambda}(B)\,\mathrm{P}_p(\mathrm{d}\omega). \tag{2.2.1}$$

Notice that $\mathbb{P}$ depends on p and λ even though both parameters do not figure in the notation.

2.2.1. Basic properties of the percolation model and the random walk. We recall some properties of the biased random walk on the conditional percolation model from [**4**].
Using electrical analysis, it can be checked that for $\lambda > 0$, for P_p-almost every configuration ω the walk is transient under $P_{\omega,\lambda}$. To do so, it suffices to construct a unit current from $\mathbf{0}$ to ∞ with finite energy.

Lemma 2.2.1 (Proposition 3.1 in [**4**]). *Fix $\lambda > 0$. The walk $(Y_n)_{\in\mathbb{N}_0}$ is $\mathbb{P}$-almost surely transient.*

We give a short sketch of the proof as we will resort to the same type of argument later in the proof of Lemma 2.5.8.

Proof(Sketch). From the Markov property of the environment, it follows that for P_p-almost every ω, there exists a vertical edge with negative x-coordinate that is open under ω. Let $m \in \mathbb{N}$ be the smallest natural number such that the vertical edge $\langle(-m,0),(-m,1)\rangle$ is open in ω. In the subgraph of G whose edge set consists of those edges that are open in ω, there either exists an infinite open self-avoiding path $P = (e_1, e_2, \ldots)$ connecting $\mathbf{0}$ with ∞ that never backtracks in the sense that the sequence of the x-coordinates of the vertices of the path is nondecreasing, or there exists such a path $\hat{P} = (\hat{e}_1, \hat{e}_2, \ldots)$ starting from $(-m,0)$ and all edges $\langle(-k,0),(-k+1,0)\rangle$, $k = 1,\ldots,m$ are open in ω. In the first case, define a flow θ from $\mathbf{0}$ to ∞

by pushing a unit current along P. In the second case, define a flow θ' from $\mathbf{0}$ to ∞ by pushing a unit current along the path $P' = (\langle(0,0),(-1,0)\rangle,\dots,\langle(-m+1,0),(-m,0)\rangle,\hat{e}_1,\hat{e}_2,\dots)$. The energy of either of these flows is bounded by

$$\mathcal{E}(\theta') = \sum_{e\in P'} \theta(e)^2 r(e) \leq e^{2\lambda m} + \sum_{k=1}^{m} e^{(2k-1)\lambda} + \sum_{k=-m}^{\infty} e^{-2\lambda k} \tag{2.2.2}$$

which is finite if and only if $\lambda > 0$. This implies transience of the walk $(Y_n)_{n\in\mathbb{N}_0}$ under $P_{\omega,\lambda}$ for P_p-almost every ω. Hence, the walk $(Y_n)_{n\in\mathbb{N}}$ is transient under $\mathbb{P}$. $\square$

The linear speed limit of the walk exhibits a phase transition from positive to zero speed at a critical value of the bias parameter λ. To be more precise, define

$$X_n := \mathsf{x}(Y_n).$$

Then the following result holds.

PROPOSITION 2.2.2 (Theorem 3.2 in [**4**]). *Fix $\lambda > 0$. The walk $(Y_n)_{n\in\mathbb{N}_0}$ satisfies $\lim_{n\to\infty} \frac{X_n}{n} = \overline{\mathrm{v}}(\lambda)$ $\mathbb{P}$-a.s. with*

$$\overline{\mathrm{v}}(\lambda) = \begin{cases} > 0 & \text{for } \lambda \in (0,\lambda_{\mathrm{c}}), \\ = 0 & \text{for } \lambda \geq \lambda_{\mathrm{c}} \end{cases}$$

where $\lambda_{\mathrm{c}} = \frac{1}{2}\log\left(2/\left(1+2p-2p^2-\sqrt{1+4p^2-8p^3+4p^4}\right)\right)$.

As indicated in Section 1.2, existence of a critical value for the bias has been proven in similar models, e. g. in [**38**] for biased random walks on Galton-Watson trees and in [**22**] for biased random walk on the supercritical percolation cluster in $\mathbb{Z}^d$. In contrast to the results for biased random walks on Galton-Watson trees or the supercritical percolation cluster in $\mathbb{Z}^d$, in the present setting the value of λ_{c} is given as an elementary function of p.
Paraphrasing the argument in [**4**], the phase transition of $\overline{\mathrm{v}}$ is triggered by the following. Suppose the walk $(Y_n)_{n\in\mathbb{N}_0}$ reaches the rightmost node of a dead-end in the direction of the bias. Furthermore, suppose the dead-end is such that in order to leave, the walk requires m backtracking steps. Now, look at the time spent by $(Y_n)_{n\in\mathbb{N}_0}$ in this trap. Along the same lines as the argument that leads to (2.2.2), Axelson-Fisk and Häggström show that the quenched expectation of the occupation time of such a trap is of order $e^{2\lambda m}$. On the other hand, it turns out that the probability to generate this type of trap is proportional to $e^{-2\lambda_{\mathrm{c}} m}$. It follows that the annealed expectation of the time spent in a dead end of unspecified length is of order $\sum_{m=1}^{\infty} e^{2\lambda m} e^{-2\lambda_{\mathrm{c}} m}$, which is finite if $\lambda < \lambda_{\mathrm{c}}$, and infinite otherwise. Axelson-Fisk and Häggström then show that this observation carries over to the linear speed limit of the walk.
We organise the further investigation of the properties of the biased random walk $(Y_n)_{n\in\mathbb{N}_0}$ in bottom-up fashion as follows. First, we introduce a decomposition of the percolation cluster at regeneration points in the upcoming Section 2.3. Using these regenerations, the trajectory of $(Y_n)_{n\in\mathbb{N}_0}$ and the environment can simultaneously be decomposed into i.i.d. segments. The most important result of this section is Proposition 2.3.5 which gives fine estimates for the tails of the time spent by $(Y_n)_{n\in\mathbb{N}_0}$ between regenerations.
To prove these tail estimates, it is necessary to have decent estimates for the time spent by the walk in traps. Therefore, we first give a detailed description of traps in Section 2.4 before proving the tail estimate over the course of Section 2.5. After that, we proceed with our main results about the asymptotic behaviour of $(Y_n)_{n\in\mathbb{N}_0}$ in Section 2.6.

2.3. Regeneration in the one-dimensional percolation environment

For the in-depth analysis of the random walk, we use a decomposition of the percolation cluster at regeneration points from [**23**]. Regeneration points are defined in two steps. Given a configuration $\omega \in \Omega$, a vertex $v = (\mathsf{x}(v), 0) \in V$ is called a *pre-regeneration point* if $v \in \mathcal{C}$ and $(\mathsf{x}(v), 1)$ is an isolated vertex in ω, that is, all three edges adjacent to $(\mathsf{x}(v), 1)$ are closed in ω.
As a consequence of Lemma 2.1.3 and Lemma 2.1.4, it follows that there exist infinitely many pre-regeneration points.

LEMMA 2.3.1 (Lemma 5.1 and Corollary 5.2 in [**4**]). *With* P_p*-probability one, there exist infinitely many pre-regeneration points both left and right of the origin.*

We enumerate the pre-regeneration points in ω by $\ldots, R_{-1}^{\mathrm{pre}}, R_0^{\mathrm{pre}}, R_1^{\mathrm{pre}}, \ldots$ such that $\mathsf{x}(R_{-1}^{\mathrm{pre}}) < 0 \leq \mathsf{x}(R_0^{\mathrm{pre}})$ and $\mathsf{x}(R_n^{\mathrm{pre}}) < \mathsf{x}(R_{n+1}^{\mathrm{pre}})$ for all $n \in \mathbb{Z}$.

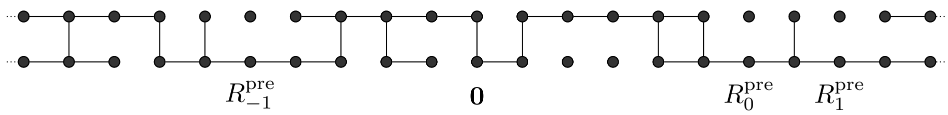


FIGURE 5. Pre-regeneration points close to the origin

The pre-regeneration points can be used to decompose the percolation cluster into independent pieces. For $a, b \in \mathbb{Z}$, we denote the subgraph of ω with vertex set $V_{[a,b)} := \{v \in V : a \leq \mathsf{x}(v) \leq b\}$ and edge set $E_{[a,b)} := \{e = \langle u, v\rangle \in E : \, u, v \in V_{[a,b)}, \, \mathsf{x}(u) \wedge \mathsf{x}(v) < b, \, \omega(e) = 1\}$ by $[a, b)$ and call $[a, b)$ a *piece* or *block* (of ω). We then define

$$\omega_n := [\mathsf{x}(R_{n-1}^{\mathrm{pre}}), \mathsf{x}(R_n^{\mathrm{pre}})), \quad n \in \mathbb{Z}.$$

Using this definition, we may introduce the cycle-stationary percolation law P_p°.

DEFINITION 2.3.2. The *cycle-stationary percolation law* P_p° is defined to be the unique probability measure on $(\Omega, \mathcal{F})$ such that the cycles ω_n, $n \in \mathbb{Z}$ are i.i.d. under P_p° with each ω_n having the same law under P_p° as ω_1 under P_p^{*}, and such that $R_0^{\mathrm{pre}} = \mathbf{0}$.

We write $\mathbb{P}^{\circ}$ for the annealed law of the biased random walk and the percolation configuration when the latter is drawn using P_p° instead of P_p. To be more precise, $\mathbb{P}^{\circ}$ is defined as $\mathbb{P}$ in (2.2.1), but with P_p replaced by P_p°.

DEFINITION 2.3.3. We call a $v \in V$ *regeneration point* if
1. it is a pre-regeneration point and
2. the random walk $(Y_n)_{n \in \mathbb{N}_0}$ visits v exactly once.

It follows from the discussion in Section 4 of [**23**] that there are infinitely many regeneration points to the right of $\mathbf{0}$. We set $R_0 := \mathbf{0}$ and, for $n \in \mathbb{N}$, define R_n to be the first regeneration point to the right of R_{n-1}. Thus, $\rho_n < \rho_{n+1}$ for all $n \in \mathbb{N}_0$ where $\rho_n := \mathsf{x}(R_n)$, $n \in \mathbb{N}_0$. Furthermore, let $\tau_0 := 0$ and

$$\tau_n := \inf\{k \in \mathbb{N}_0 : Y_k = R_n\}, \qquad n \in \mathbb{N}.$$

For $n \geq 1$, τ_n is the unique time at which the nth regeneration point R_n is visited by the walk $(Y_k)_{k \in \mathbb{N}_0}$. In particular, $0 = \tau_0 < \tau_1 < \ldots$. We will call τ_n the nth *regeneration time*. Below, we write

$$\alpha := \lambda_{\mathrm{c}}/\lambda.$$

The following assertions are known from [**23**] about the regeneration times and points.

LEMMA 2.3.4 (Lemmas 4.1 and 4.2, Proposition 4.3 in [**23**]). *Fix $\lambda > 0$.*

(a) Under $\mathbb{P}$, the pairs $(\tau_{n+1}-\tau_n, \rho_{n+1}-\rho_n)$, $n \in \mathbb{N}$ are i.i.d. and independent of (τ_1, ρ_1), and

$$\mathbb{P}((\tau_2-\tau_1, \rho_2-\rho_1) \in \cdot) = \mathbb{P}^\circ((\tau_1, \rho_1) \in \cdot \mid Y_n \neq \mathbf{0} \text{ for all } n \geq 1).$$

(b) There exists some $\delta > 0$ such that $\mathbb{E}[e^{\delta(\rho_2-\rho_1)}] < \infty$.

(c) It holds that $\mathbb{E}[(\tau_2 - \tau_1)^\kappa] < \infty$ if and only if $\kappa < \alpha = \lambda_c/\lambda$.

(d) The ballistic speed satisfies $\overline{\mathrm{v}}(\lambda) = \mathbb{E}[\rho_2 - \rho_1]/\mathbb{E}[\tau_2 - \tau_1]$.

It should be noted that according to Lemma 2.3.4(d), the quantity $\overline{\mathrm{v}}$ fits the elementary description of speed as a fraction of displacement over time since it can be written as the ratio of the average distance traveled between regenerations of the random walk to the average time spent between those regenerations.
Lemma 2.3.4(c) indicates that $\mathbb{P}(\tau_2 - \tau_1 \geq n)$ is roughly of the order $n^{-\alpha}$ as $n \to \infty$. We give a more precise statement in the following proposition.

PROPOSITION 2.3.5. *For any $\lambda > \log(2)/2$, in particular for $\lambda \geq \lambda_c/2$, there exist constants $0 < c \leq d < \infty$ (depending on p and λ) such that, for all $n \in \mathbb{N}$,*

$$cn^{-\alpha} \leq \mathbb{P}(\tau_2 - \tau_1 \geq n) \leq dn^{-\alpha}$$

and

$$cn^{-\alpha} \leq \mathbb{P}(\tau_1 \geq n) \leq dn^{-\alpha} \log n.$$

We will prove Proposition 2.3.5 over the course of Section 2.5.

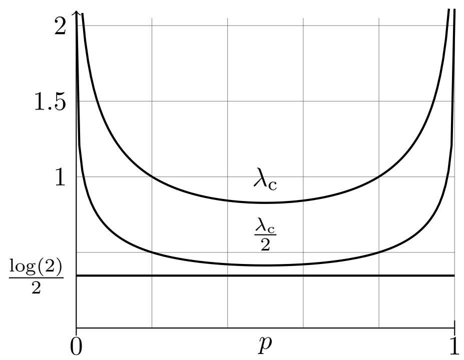


FIGURE 6. The figure shows λ_c and $\lambda_c/2$ as functions of p. Proposition 2.3.5 giving precise tail asymptotics for the regeneration times applies for $\lambda > \log(2)/2$, which is strictly smaller than $\lambda_c/2$ for any $p \in (0, 1)$

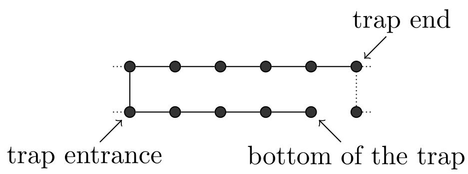
2.4. Traps

As for biased random walk on Galton-Watson trees or on the supercritical percolation cluster in $\mathbb{Z}^d$, the slowdown of biased random walk in the one-dimensional percolation model considered here is due to traps. These are dead-end regions stretching in the direction of the bias. For conditional percolation on the ladder graph, this boils down to parallel finite open horizontal line segments with no vertical connections.
To give a formal definition of a trap, we introduce some notation. For a vertex $u \in V$, we write u' for $(\mathsf{x}(u), 1 - \mathsf{y}(u))$. Further, if $e = \langle u, v \rangle \in E$, we let $e' := \langle u', v' \rangle$. In particular, $e = e'$ if e is a vertical edge, and e' is the horizontal edge parallel to e if e is a horizontal edge.

Now we define a *trap* (in ω) to be an open path $P = (e_1, \dots, e_m)$ of length $m \in \mathbb{N}$ with edges $e_1 = \langle u_0, u_1 \rangle, \dots, e_m = \langle u_{m-1}, u_m \rangle \in E$ such that

1. $\mathsf{x}(u_k) = \mathsf{x}(u_{k-1}) + 1$ and $\mathsf{y}(u_k) = \mathsf{y}(u_{k-1})$ for $k = 1, \dots, m$;
2. the edges $\langle u_0, u_0' \rangle$ and e_k', $k = 1, \dots, m$ are open (in ω);
3. the edge $\langle u_m, u_{m+1} \rangle$ is closed (in ω) where $u_{m+1} = (\mathsf{x}(u_m) + 1, \mathsf{y}(u_m))$;
4. all vertical edges $\langle u_k, u_k' \rangle$ for $k = 1, \dots, m$ are closed (in ω).

Here, m is called the *length of the trap*, u_0 is called the *trap entrance* and u_m is called the *bottom of the trap*. The piece $[\mathsf{x}(u_0), \mathsf{x}(u_{m+1}))$ is called (the corresponding) *trap piece*. The figure below shows the trap piece of a trap of length 4.



We define the backbone $\mathcal{B}$ to be the subgraph of the infinite cluster $\mathcal{C}$ obtained by deleting from $\mathcal{C}$ all edges and all vertices in traps except the trap entrance vertices. Clearly, $\mathcal{B}$ is connected and contains all pre-regeneration points.

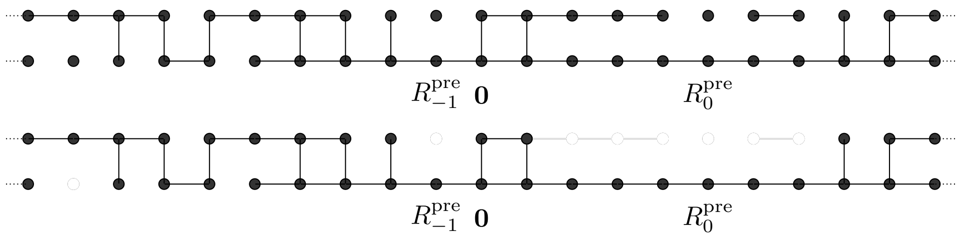


FIGURE 7. The original percolation configuration (above) and the backbone (below)

Due to the Markovian structure of the percolation process under P_p, there are infinitely many traps both to the left and to the right of the origin $\mathbf{0}$. Let T_n, $n \in \mathbb{Z}$ be an enumeration of all trap pieces such that T_n is strictly to the left of T_{n+1} for each $n \in \mathbb{Z}$ and that T_1 is the trap piece with minimal nonnegative x-coordinate of the trap entrance. Denoting the length of the trap in the trap piece T_n by ℓ_n, the following result holds.

LEMMA 2.4.1 (Lemma 3.5 in [**23**]). *(a) Under* P_p, $(\ell_n)_{n \neq 0}$ *is a family of i.i.d. positive random variables independent of* ℓ_0 *with* $\mathrm{P}_p(\ell_1 = m) = (e^{2\lambda_c} - 1)e^{-2\lambda_c m}$, $m \in \mathbb{N}$.
(b) There is a constant $\chi(p)$ *such that* $\mathrm{P}_p(\ell_0 = m) \le \chi(p) m e^{-2\lambda_c m}$, $m \in \mathbb{N}$.

This is a consequence of the Markovian structure of the percolation process under P_p^*, of the fact that the transition probabilities of the process $(\mathbb{T}_k)_{k \in \mathbb{Z}}$ tracking the backwards-communication of pairs of parallel vertices can be written in terms of elementary functions of p in the form of

(2.1.1), and of Lemma 2.1.4. Under P_p^*, given $\mathbb{T}_0 = 11$, the event that state 11 of $(\mathbb{T}_k)_{k\in\mathbb{Z}}$ occurs m times in a row, $\mathbb{T}_1 = \ldots = \mathbb{T}_m = 11$, in combination with the event that all horizontal edges $\langle(k,0),(k+1,0)\rangle$, $\langle(k,1),(k+1,1)\rangle$, $k = 1,\ldots,m$ are open and all vertical edges $\langle(k,0),(k,1)\rangle$, $k = 1,\ldots,m$ are closed, precisely has probability $e^{-2\lambda_c m}$.

2.4.1. Biased random walk on a line segment. An excursion of the random walk $(Y_n)_{n\in\mathbb{N}_0}$ into a fixed trap of length m can be identified with an excursion of a biased random walk on the line graph $\{0,1,\ldots,m\}$ introduced in Example 1.3.2, where m is the length of the trap. Let

$$p_\lambda := \tfrac{e^\lambda}{e^{-\lambda}+e^\lambda}, \quad q_\lambda := 1-p_\lambda, \quad \text{and} \quad \gamma := \tfrac{q_\lambda}{p_\lambda} = e^{-2\lambda}.$$

We write $P_{m,\lambda}^k$ for the law of a biased random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\{0,\ldots,m\}$ starting at $k \in \{0,\ldots,m\}$, moving to the right with probability p_λ and moving left with probability q_λ from any vertex other than $0, m$. The origin 0 is supposed to be absorbing and at m the walk stays put with probability p_λ and moves left with probability q_λ. We write $E_{m,\lambda}^k$ for the corresponding expectation. We drop the superscript k, both in $P_{m,\lambda}^k$ as well as $E_{m,\lambda}^k$, if $k = 1$.
For $k \in \{0,\ldots,m\}$ we write $\sigma_k := \inf\{j \in \mathbb{N}_0 : S_j = k\}$ and $\sigma_k^+ := \inf\{j \in \mathbb{N} : S_j = k\}$ for the hitting time and the first return time of node k under $(S_n)_{n\in\mathbb{N}_0}$, respectively. For $k,l \in \{0,\ldots,m\}$, we write $\sigma_{k\to l} = \inf\{j \in \mathbb{N}_0 : S_j = l\}$ on $\{S_0 = k\}$ for the time it takes until the walk $(S_n)_{n\in\mathbb{N}_0}$ visits l for the first time when starting at k. Let $\mathsf{e}_m := P_{m,\lambda}^m(\sigma_0^+ < \sigma_m^+)$ be the escape probability from the rightmost node in the trap to the trap entrance without a rebound to the rightmost node in the trap. By the gambler's ruin formula, Equation (1.3.3) with $\beta := \gamma^{-1}$, this is

$$\mathsf{e}_m = P_{m,\lambda}^m(\sigma_0^+ < \sigma_m^+) = q_\lambda \frac{\gamma^{m-1}-\gamma^m}{1-\gamma^m} = \gamma^m p_\lambda \frac{1-\gamma}{1-\gamma^m}. \tag{2.4.1}$$

2.5. Tail estimate for regeneration times

Over the course of this section, we prove the tail estimate for regeneration times, Proposition 2.3.5.

2.5.1. The proof of the lower bound. The analysis of traps from the previous section almost immediately results in a proof of the lower bound in Proposition 2.3.5.

LEMMA 2.5.1. *There exists some $c > 0$ such that, for all $n \in \mathbb{N}$,*

$$\mathbb{P}(\tau_2 - \tau_1 \geq n) \geq cn^{-\alpha} \quad \textit{and} \quad \mathbb{P}(\tau_1 \geq n) \geq cn^{-\alpha}$$

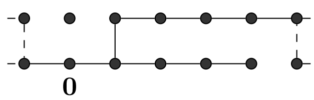
PROOF. According to Lemma 2.3.4, we find

$$\mathbb{P}(\tau_2-\tau_1 \geq n) = \mathbb{P}^\circ(\tau_1 \geq n \mid Y_k \neq \mathbf{0} \text{ for all } k \geq 1) \geq \mathbb{P}^\circ(\tau_1 \geq n,\, Y_k \neq \mathbf{0} \text{ for all } k \geq 1).$$

On the other hand, as $\mathbb{P}(R_0^{\mathrm{pre}} = \mathbf{0}) > 0$, we can safely write

$$\begin{aligned}\mathbb{P}(\tau_1 \geq n) &\geq \mathbb{P}(R_0^{\mathrm{pre}} = \mathbf{0})\mathbb{P}(\tau_1 \geq n, Y_k \neq \mathbf{0} \text{ for all } k \geq 1 \mid R_0^{\mathrm{pre}} = \mathbf{0}).\\ &= \mathbb{P}(R_0^{\mathrm{pre}} = \mathbf{0})\mathbb{P}^\circ(\tau_1 \geq n, Y_k \neq \mathbf{0} \text{ for all } k \geq 1).\end{aligned}$$

We therefore provide a lower bound for $\mathbb{P}^\circ(\tau_1 \geq n,\, Y_k \neq \mathbf{0}$ for all $k \geq 1)$. Under P_p°, there is a pre-regeneration point at $\mathbf{0}$ as depicted in the figure below.



Given there is a pre-regeneration point at $\mathbf{0}$ (as is always the case under P_p°), the law of the percolation cluster to the right of the origin under P_p and P_p° coincides since the ω_n, $n \in \mathbb{N}$ have the same law under P_p and P_p°. We may thus argue as on p. 3404 of [**4**] to conclude that the probability that directly to the right of the origin, there is a trap of length m as in the picture above is $\gamma(p)e^{-2\lambda_c m}$ for some constant $\gamma(p) \in (0,1)$.
We write T for the time spent on the first excursion of $(Y_n)_{n\in\mathbb{N}_0}$ into the trap right of the origin. We have

$$\mathbb{P}^\circ(\tau_1 \geq n, Y_k \neq \mathbf{0} \text{ for all } k \geq 1) \geq \mathbb{P}^\circ(T \geq n, \text{there is a trap directly to the right of the origin}).$$

Typically, after entering the trap the walk drifts towards the bottom of the trap and then requires a geometric number of trials to leave again. It follows from the gambler's ruin formula (1.3.3) that for all m, hitting the bottom before leaving the trap has positive probability bounded from below:

$$P^1_{m,\lambda}(\sigma_m < \sigma_0) = \frac{1-\gamma^1}{1-\gamma^m} > 1-\gamma > 0.$$

The probability of leaving the trap from the bottom without rebound to the bottom is e_m. In order to visit the trap in the situation as depicted above, two steps to the right at the start suffice. Thus we get

$$\begin{aligned}\mathbb{P}^\circ(\tau_1 \geq n,\, Y_k \neq \mathbf{0} \text{ for all } k \geq 1) &\geq \Big(\frac{e^\lambda}{e^\lambda + 1 + e^{-\lambda}}\Big)^2 \sum_{m=2}^\infty \gamma(p) e^{-2\lambda_c m} P^1_{m,\lambda}(T \geq n, \sigma_m < \sigma_0) \\ &\geq \frac{(1-\gamma)e^{2\lambda}\gamma(p)}{(e^\lambda+1+e^{-\lambda})^2} \sum_{m=2}^\infty e^{-2\lambda_c m}(1-\mathsf{e}_m)^{n-1},\end{aligned}$$

Restricting this sum to the term of order $\hat{x} := \frac{\log n}{|\log \gamma|}$ leads to

$$\begin{aligned}\mathbb{P}^\circ(\tau_1 \geq n,\, Y_k \neq \mathbf{0} \text{ for all } k \geq 1) &\geq \frac{(1-\gamma)e^{2\lambda}\gamma(p)}{(e^\lambda+1+e^{-\lambda})^2} e^{-2\lambda_c \lfloor \hat{x} \rfloor}(1-\mathsf{e}_{\lfloor \hat{x} \rfloor})^{n-1} \\ &\geq \frac{(e^{2\lambda}-1)\gamma(p)}{(e^\lambda+1+e^{-\lambda})^2} e^{-2\lambda_c \hat{x}}(1-\mathsf{e}_{\hat{x}-1})^{n-1} \\ &= n^{-\alpha} \frac{(e^{2\lambda}-1)\gamma(p)}{(e^\lambda+1+e^{-\lambda})^2} \exp(-p_\lambda(e^{2\lambda}-1))(1+o_n(1)).\end{aligned}$$

□

2.5.2. Sketch of the proof of the upper bound and preliminaries. To prove the upper bound in Proposition 2.3.5, we can use the same approach as in the proof of the lower bound - at least to prove a tail estimate for the time spent in a single trap. The overall proof of the upper bound will proceed along the following steps.

1. We divide the time $\tau_2 - \tau_1$ spent between the visit to the first and the second regeneration point, respectively, as follows.
$$\tau_2 - \tau_1 = (\tau_2 - \tau_1)^{\mathcal{B}} + (\tau_2 - \tau_1)^{\text{traps}}$$
where $(\tau_2 - \tau_1)^{\mathcal{B}}$ and $(\tau_2 - \tau_1)^{\text{traps}}$ are the times spent in the backbone and in traps, respectively, during the time interval $[\tau_1, \tau_2)$.
2. The term $(\tau_2 - \tau_1)^{\mathcal{B}}$ has sufficiently high moments to be neglected.
3. The term $(\tau_2 - \tau_1)^{\text{traps}}$ can be written as the sum of the occupation times of the distinct traps in the block $[\rho_1, \rho_2)$. In the annealed case, the occupation times can be bounded by identically distributed - albeit not independent - random variables that offer the tail estimate of Proposition 2.3.5 up to lower order terms.

4. Using a taylor-made coupling, we can further estimate these by a family of i.i.d. random variables with the same (up to a constant) tail probabilities.

To estimate the time spent on bottom-to-bottom excursions uniformly in the trap length, let $(S'_n)_{n\in\mathbb{N}_0}$ be a biased random walk on $\mathbb{Z}$ that mimics the steps of $(S_n)_{n\in\mathbb{N}_0}$ without staying put. More precisely, set $S'_0 := 0$ and for $n < \sigma_0$, let

$$\begin{aligned} S'_{n+1} &= S'_n + 1 \qquad \text{if } S_{n+1} = S_n + 1 \text{ or } S_{n+1} = S_n = m, \\ S'_{n+1} &= S'_n - 1 \qquad \text{if } S_{n+1} = S_n - 1. \end{aligned}$$

After $(S_n)_{n\in\mathbb{N}_0}$ hits the absorbing state 0, we let $(S'_n)_{n\in\mathbb{N}_0}$ move along as the usual biased random walk on $\mathbb{Z}$ with probability p_λ to jump right. For $z \in \mathbb{Z}$, write $P^z_{\mathbb{Z},\lambda}$ and $\mathrm{E}^z_{\mathbb{Z},\lambda}$ for the law of $(S'_n)_{n\in\mathbb{N}_0}$ starting at $S'_0 = z$ and the corresponding expectation, respectively. For $k \in \mathbb{Z}$, set

$$\sigma^{\mathbb{Z}}_k := \inf\{l \in \mathbb{N}_0 : S'_l = k\}.$$

The second step in the outline given above comes as a consequence of the following Lemma.

LEMMA 2.5.2 (Lemma 7.5 in [**23**]). *For any $\kappa > 0$, we have $\mathbb{E}[((\tau_2 - \tau_1)^{\mathcal{B}})^\kappa] < \infty$.*

This and Markov's inequality imply the following result.

LEMMA 2.5.3. *It holds that $\mathbb{P}((\tau_2 - \tau_1)^{\mathcal{B}} > n) = o(n^{-\alpha})$ as $n \to \infty$.*

To obtain an upper bound on $\mathbb{P}(\tau_2 - \tau_1 > n)$, we thus need to consider the time spent in traps. We write $(\tau_2 - \tau_1)^{\text{traps}}$ as

$$(\tau_2 - \tau_1)^{\text{traps}} = \sum_{i=1}^{T} \sum_{j=1}^{V_i} T_{ij},$$

where T is the number of traps in $[\rho_1, \rho_2)$, V_i is the number of visits in the ith trap in $[\rho_1, \rho_2)$ and T_{ij} is the time $(Y_n)_{n\in\mathbb{N}_0}$ spends during the jth excursion into the ith trap in $[\rho_1, \rho_2)$.

2.5.3. Tail estimates for the time spent in a single trap. If we fix a percolation environment ω, the time spent in a single trap of length m can be split into the time spent on bottom-to-bottom excursions and the time spent to reach or leave the bottom without a rebound to the left- or rightmost, respectively, node of the trap. This leads to the following result for a fixed number of excursions into a single trap.

LEMMA 2.5.4. *Let $(S_{n,j})_{n\in\mathbb{N}_0}$, $j \in \mathbb{N}$ be i.i.d. copies of $(S_n)_{n\in\mathbb{N}_0}$ starting at 1. Further, let $T^{\text{qu,a}}_{ij}$ be the absorption time at 0 of the walk $(S_{n,j})_{n\in\mathbb{N}_0}$, $j \in \mathbb{N}$. Let $R := E^0_{\mathbb{Z},\lambda}[\sigma^{\mathbb{Z}}_1] = \frac{1}{1-2q_\lambda}$. Then, for any $l \in \mathbb{N}$, there exist independent $Z_1, ..., Z_l \sim$ geom(e_m) and $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned} P_{m,\lambda}\Bigg(\sum_{j=1}^{l} T^{\text{qu,a}}_{ij} \geq n\Bigg) \leq{}& 2P_{m,\lambda}\Bigg(\sum_{j=1}^{l} Z_j \geq \frac{n}{4R}\Bigg) \\ &+ 3l \max\big\{P^1_{m,\lambda}\big(\sigma_{1\to 0} \geq \tfrac{n}{6l}, \sigma_0 < \sigma_m\big), P^1_{m,\lambda}\big(\sigma_{1\to m} \geq \tfrac{n}{6l}, \sigma_m < \sigma_0\big), \\ &\qquad P^m_{m,\lambda}\big(\sigma_{m\to 0} \geq \tfrac{n}{6l}, \sigma_0 < \sigma^+_m\big)\big\}. \end{aligned}$$

PROOF. Let $Z^{(j)}$ be the number of returns to m of $(S_{n,j})_{n\in\mathbb{N}_0}$ before absorption. For completeness, we define $Z^{(j)} := 0$ on the event where $(S_{n,j})_{n\in\mathbb{N}_0}$ visits m at most once. By the strong Markov property, $P_{m,\lambda}(Z^{(j)} = k) = P^1_{m,\lambda}(\sigma_m < \sigma_0)(1 - \mathsf{e}_m)^k \mathsf{e}_m$ for $k \in \mathbb{N}$ and $P_{m,\lambda}(Z^{(j)} = 0) = P^1_{m,\lambda}(\sigma_0 < \sigma_m) + P^1_{m,\lambda}(\sigma_m < \sigma_0)\mathsf{e}_m$. We write $\tilde{T}_{jk}$, $k = 1, \ldots, Z^{(j)}$ for the durations of consecutive excursions of $(S_{n,j})_{n\in\mathbb{N}_0}$ from m to m, and let $\tilde{T}_{jk}$, $k > Z^{(j)}$, be a family of i.i.d. random variables distributed as the duration of an excursion of $(S_n)_{n\in\mathbb{N}_0}$ from m to m

conditioned on the event $\{\sigma_m^+ < \sigma_0\}$. When starting at 1, the walk $(S_n)_{n\in\mathbb{N}_0}$ either hits the absorbing state 0 before reaching the trap bottom, or hits the bottom, does a geometric number of bottom-to-bottom excursions, and then gets absorbed. We have

$$\begin{aligned}
P_{m,\lambda}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} \geq n\Big) &= P_{m,\lambda}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} \geq n, \Big|\sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} - \sum_{j=1}^{l}\sum_{k=1}^{Z^{(j)}} \tilde{T}_{jk}\Big| \leq \frac{n}{2}\Big) \\
&\quad + P_{m,\lambda}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} \geq n, \Big|\sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} - \sum_{j=1}^{l}\sum_{k=1}^{Z^{(j)}} \tilde{T}_{jk}\Big| > \frac{n}{2}\Big) \\
&\leq P_{m,\lambda}\Big(\sum_{j=1}^{l}\sum_{k=1}^{Z^{(j)}} \tilde{T}_{jk} \geq \frac{n}{2}\Big) \\
&\quad + 3l \max\big\{P_{m,\lambda}^{1}\big(\sigma_{1\to 0} \geq \tfrac{n}{6l}, \sigma_0 < \sigma_m\big), P_{m,\lambda}^{1}\big(\sigma_{1\to m} \geq \tfrac{n}{6l}, \sigma_m < \sigma_0\big), \\
&\qquad\qquad P_{m,\lambda}^{m}\big(\sigma_{m\to 0} \geq \tfrac{n}{6l}, \sigma_0 < \sigma_m^+\big)\big\}.
\end{aligned}$$

We can safely replace $Z^{(j)}, j = 1, ..., l$ by an independent family of i.i.d. random variables Z_j with law geom(e_m) under $P_{m,\lambda}$. As $\tilde{T}_{jk}, j = 1, ..., l, k \in \mathbb{N}$ are nonnegative and i.i.d., we have

$$\begin{aligned}
P_{m,\lambda}\Big(\sum_{j=1}^{l} Z_j < n\Big) &= P_{m,\lambda}\Big(\sum_{j=1}^{l} Z_j < n, \sum_{k=1}^{n} \tilde{T}_{1k} \geq 2Rn\Big) + P_{m,\lambda}\Big(\sum_{j=1}^{l} Z_j < n, \sum_{k=1}^{n} \tilde{T}_{1k} < 2Rn\Big) \\
&\leq P_{m,\lambda}\Big(\sum_{k=1}^{n} \tilde{T}_{1k} \geq 2Rn\Big) + P_{m,\lambda}\Big(\sum_{k=1}^{Z_1+\ldots+Z_l} \tilde{T}_{1k} < 2Rn\Big) \\
&= P_{m,\lambda}\Big(\sum_{k=1}^{n} \tilde{T}_{1k} \geq 2Rn\Big) + P_{m,\lambda}\Big(\sum_{j=1}^{l}\sum_{k=1}^{Z_j} \tilde{T}_{jk} < 2Rn\Big).
\end{aligned}$$

This implies

$$P_{m,\lambda}\Big(\sum_{j=1}^{l}\sum_{k=1}^{Z_j} \tilde{T}_{jk} \geq 2Rn\Big) \leq P_{m,\lambda}\Big(\sum_{j=1}^{l} Z_j \geq n\Big) + P_{m,\lambda}\Big(\sum_{k=1}^{n} \tilde{T}_{1k} \geq 2Rn\Big). \tag{2.5.1}$$

Using Markov's inequality, the Markov property, stochastic domination and Lemma 1.3.6, for $\mu > 0$, we have

$$\begin{aligned}
P_{m,\lambda}\Big(\sum_{k=1}^{n} \tilde{T}_{1k} \geq 2Rn\Big) &\leq e^{-2\mu Rn} E_{m,\lambda}^{m}\big[e^{\mu\sigma_m^+} \big| \sigma_m^+ < \sigma_0\big]^n \leq e^{-2\mu Rn} E_{\mathbb{Z},\lambda}^{0}\big[e^{\mu\sigma_1^{\mathbb{Z}}}\big]^n \\
&= e^{-2\mu Rn}\Big(\frac{1-\sqrt{1-4p_\lambda q_\lambda e^{2\mu}}}{2q_\lambda e^{\mu}}\Big)^n.
\end{aligned}$$

The function $f : \big[0, \frac{1}{2}\log\big(\frac{1}{4p_\lambda q_\lambda}\big)\big] \to \mathbb{R}$ given by

$$f(\mu) := e^{-2\mu R}\frac{1-\sqrt{1-4p_\lambda q_\lambda e^{2\mu}}}{2q_\lambda e^{\mu}}$$

is differentiable and satisfies

$$f(0) = \frac{1-(1-2q_\lambda)}{2q_\lambda} = 1, \qquad f'(0) = \frac{-1}{1-2q_\lambda} < 0.$$

Hence, there exists $\hat{\mu} > 0$ with $f(\hat{\mu}) < 1$, and

$$P_{m,\lambda}\bigg(\sum_{k=1}^{n} \tilde{T}_{1k} \geq 2Rn\bigg) \leq \bigg(\frac{f(\hat{\mu})}{1-\mathsf{e}_m}\bigg)^n \cdot P_{m,\lambda}(Z_1 \geq n).$$

As $\mathsf{e}_m \to 0$ for $m \to \infty$, there exists m_0 such that $\frac{f(\hat{\mu})}{1-\mathsf{e}_m} < 1$ for all $m \geq m_0$. This and (2.5.1) lead to

$$\begin{aligned} P_{m,\lambda}\bigg(\sum_{j=1}^{l}\sum_{k=1}^{Z_j} \tilde{T}_{jk} \geq 2Rn\bigg) &\leq P_{m,\lambda}\bigg(\sum_{j=1}^{l} Z_j \geq n\bigg) + \bigg(\frac{f(\hat{\mu})}{1-\mathsf{e}_{m_0}}\bigg)^n P_{m,\lambda}(Z_1 \geq n) \\ &\leq 2P_{m,\lambda}\bigg(\sum_{j=1}^{l} Z_j \geq n\bigg) \end{aligned}$$

for $m \geq m_0$. □

Lemma 2.5.4 can be adapted to the case where the random walk is allowed to take lazy steps. Let $(S_n^{\text{lazy}})_{n\in\mathbb{N}_0}$ be the lazy biased random walk on the line graph $\{0, 1, \ldots, m\}$ that moves to the right with probability $e^{\lambda}/(e^{\lambda}+1+e^{-\lambda})$, to the left with probability $e^{-\lambda}/(e^{\lambda}+1+e^{-\lambda})$ and stays put with probability $1/(e^{\lambda}+1+e^{-\lambda})$ from any vertex other than $0, m$. The origin 0 is again supposed to be absorbing and at m, the walk stays put with probability $(e^{\lambda}+1)/(e^{\lambda}+1+e^{-\lambda})$ and moves left with probability $e^{-\lambda}/(e^{\lambda}+1+e^{-\lambda})$. Slightly abusing notation, we again write $P_{m,\lambda}$ for the law of $(S_n^{\text{lazy}})_{n\in\mathbb{N}_0}$ starting at $S_0^{\text{lazy}} = 1$, and $E_{m,\lambda}$ for the corresponding expectation.

Lemma 2.5.5. *Let $(S_{n,j}^{\text{lazy}})_{n\in\mathbb{N}_0}$, $j \in \mathbb{N}$ be i.i.d. copies of $(S_n^{\text{lazy}})_{n\in\mathbb{N}_0}$ starting at 1. Further, let T_{ij}^{qu} be the absorption time at 0 of the walk $(S_{n,j}^{\text{lazy}})_{n\in\mathbb{N}_0}$, $j \in \mathbb{N}$. Let $R := E^0_{\mathbb{Z},\lambda}[\sigma_1^{\mathbb{Z}}] = \frac{1}{1-2q_\lambda}$ and $r_\lambda > e^{2\lambda}+e^{\lambda}$. Then, for any $l \in \mathbb{N}$, there exist independent $Z_1, \ldots, Z_l \sim$ geom(e_m) and $m_1 \in \mathbb{N}$ such that, for $m \geq m_0 \vee m_1$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned} P_{m,\lambda}\bigg(\sum_{j=1}^{l} T_{ij}^{\text{qu}} \geq n\bigg) &\leq 3P_{m,\lambda}\bigg(\sum_{j=1}^{l} Z_j \geq \frac{n}{4r_\lambda R}\bigg) \\ &\quad + 3l \max\big\{P^1_{m,\lambda}\big(\sigma_{1\to 0} \geq \tfrac{n}{6lr_\lambda}, \sigma_0 < \sigma_m\big), P^1_{m,\lambda}\big(\sigma_{1\to m} \geq \tfrac{n}{6lr_\lambda}, \sigma_m < \sigma_0\big), \\ &\qquad P^m_{m,\lambda}\big(\sigma_{m\to 0} \geq \tfrac{n}{6lr_\lambda}, \sigma_0 < \sigma_m^+\big)\big\}. \end{aligned}$$

Proof. We have

$$\sum_{j=1}^{l} T_{ij}^{\text{qu}} \overset{\text{law}}{=} \sum_{j=1}^{l} \sum_{k=1}^{T_{ij}^{\text{qu,a}}} \tilde{Z}_{k,j},$$

where $T_{ij}^{\text{qu,a}}$, $j \in \mathbb{N}$ are as in Lemma 2.5.4, and $\tilde{Z}_{k,j}$, $k, j \in \mathbb{N}$ are independent random variables distributed as the number of times the walk $(S_{n,j}^{\text{lazy}})_{n\in\mathbb{N}_0}$ stays put before it changes its position for the kth time. Since the probability for $(S_{n,j}^{\text{lazy}})_{n\in\mathbb{N}_0}$ to change its position at any vertex other than the absorbing state 0 is bounded from below by $\tilde{p} := e^{-\lambda}/(e^{\lambda}+1+e^{-\lambda})$, we have $\tilde{Z}_{k,j} \preccurlyeq Z_{k,j}$ where $Z_{k,j}$, $k, j \in \mathbb{N}$ is a family of i.i.d. geometric random variables with success probability $\tilde{p}$. Notice that $E_{m,\lambda}[Z_{1,1}] = (1-\tilde{p})/\tilde{p} = e^{2\lambda}+e^{\lambda} > 2$. Choose $r_\lambda > e^{2\lambda}+e^{\lambda}$. Then, as the $Z_{k,j}$,

$k, j \in \mathbb{N}$ are nonnegative and i.i.d., we find

$$P_{m,\lambda}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu}} \geq n\Big) \leq P_{m,\lambda}\Big(\sum_{j=1}^{l}\sum_{k=1}^{T_{ij}^{\mathrm{qu,a}}} Z_{k,j} \geq n\Big)$$

$$= P_{m,\lambda}\Big(\sum_{j=1}^{l}\sum_{k=1}^{T_{ij}^{\mathrm{qu,a}}} Z_{k,j} \geq n, \sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} > \Big\lfloor \frac{n}{r_\lambda}\Big\rfloor\Big) + P_{m,\lambda}\Big(\sum_{j=1}^{l}\sum_{k=1}^{T_{ij}^{\mathrm{qu,a}}} Z_{k,j} \geq n, \sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} \leq \Big\lfloor \frac{n}{r_\lambda}\Big\rfloor\Big)$$

$$\leq P_{m,\lambda}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu,a}} > \Big\lfloor \frac{n}{r_\lambda}\Big\rfloor\Big) + P_{m,\lambda}\Big(\sum_{k=1}^{\lfloor \frac{n}{r_\lambda}\rfloor} Z_{k,1} \geq n\Big).$$

Standard large deviation estimates yield that $P_{m,\lambda}(\sum_{k=1}^{\lfloor n/r_\lambda\rfloor} Z_{k,1} \geq n)$ decays exponentially fast as $n \to \infty$ (with a rate which is independent of m). Hence, as $\mathsf{e}_m \to 0$ for $m \to \infty$, there exists $m_1 = m_1(\lambda) \in \mathbb{N}$ such that for all $m \geq m_1$

$$P_{m,\lambda}\Bigg(\sum_{k=1}^{\lfloor \frac{n}{r_\lambda}\rfloor} Z_{k,1} \geq n\Bigg) \leq (1-\mathsf{e}_m)^{\lceil \frac{n}{4r_\lambda R}\rceil} = P_{m,\lambda}\Big(Z_1 \geq \frac{n}{4r_\lambda R}\Big).$$

The remainder of the proof now follows from Lemma 2.5.4. □

In the annealed case, Lemma 2.5.5 translates to a tail probability of basically order $n^{-\alpha}$ (given the trap is actually seen).

LEMMA 2.5.6. *Let R, r_λ, m_0, m_1 be as in Lemma 2.5.5 and $\mu > 0$ be such that $E_{\mathbb{Z},\lambda}^{0}\big[e^{\mu\sigma_1^{\mathbb{Z}}}\big] < \infty$. Further, let T_{ij}^{ann}, $i \in \mathbb{Z}$, $j \in \mathbb{N}$ be a family of random variables which are independent given ω and with T_{ij}^{ann} given ω being distributed as the hitting time of the entrance of the trap in T_i by $(Y_n)_{n\in\mathbb{N}_0}$ under $P_{\omega,\lambda}$ when $(Y_n)_{n\in\mathbb{N}_0}$ starts at the right neighbor of the trap entrance. Then*

$$\mathbb{P}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n, \ell_i \geq m_0 \vee m_1\Big) \leq \begin{cases} c_1 l^{\alpha+1} n^{-\alpha} + c_2 l e^{-\mu\frac{n}{6lr_\lambda}}, & \text{for } i \neq 0, \\ c_1' l^{\alpha+1} n^{-\alpha}\log n + c_2' l e^{-\mu\frac{n}{6lr_\lambda}} & \text{for } i = 0, \end{cases}$$

where $c_1 = c_1(p,\lambda), c_2 = c_2(p,\lambda), c_1' = c_1'(p,\lambda), c_2' = c_2'(p,\lambda)$ are positive, finite constants neither depending on n nor l.

PROOF. Using Lemmas 2.4.1 and 2.5.5, we can estimate $\mathbb{P}\big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n, \ell_i \geq m_0 \vee m_1\big)$ using independent $Z_1, ..., Z_l \sim \mathrm{geom}(\mathsf{e}_m)$ and T_{ij}^{qu}, $j = 1, \dots, l$, r_λ and R as defined in Lemma 2.5.5 by

$$\mathbb{P}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n, \ell_i \geq m_0 \vee m_1\Big) = \sum_{m=m_0\vee m_1}^{\infty} \mathrm{P}_p(\ell_i = m) P_{m,\lambda}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu}} \geq n\Big)$$

$$\leq 3 \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_{\mathrm{c}} m} P_{m,\lambda}\Big(\sum_{j=1}^{l} Z_j \geq \frac{n}{4r_\lambda R}\Big)$$

$$+ 3l \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_{\mathrm{c}} m} \max\big\{P_{m,\lambda}^{1}\big(\sigma_{1\to 0} \geq \tfrac{n}{6lr_\lambda}, \sigma_0 < \sigma_m\big), P_{m,\lambda}^{1}\big(\sigma_{1\to m} \geq \tfrac{n}{6lr_\lambda}, \sigma_m < \sigma_0\big),$$

(2.5.2)

$$P_{m,\lambda}^{m}\big(\sigma_{m\to 0} \geq \tfrac{n}{6lr_\lambda}, \sigma_0 < \sigma_m^{+}\big)\big\},$$

where $\alpha_i(m) := (e^{2\lambda_c} - 1)$ for $i \neq 0$ and $\alpha_0(m) := \chi(p)m$. We consider the second series first. For $y \in \{0, ..., m\}$ we write

$$h(y) := P^y_{m,\lambda}(\sigma_0 < \sigma_m).$$

Due to the Gambler's ruin formula we have $h(y) = \frac{\gamma^y - \gamma^m}{1-\gamma^m}$. An excursion of $(S_n)_{n\in\mathbb{N}_0}$ starting from either 1 or m to the origin 0 conditioned on $\sigma_0 < \sigma_m^+$ has the transition probabilities

$$P^y_{m,\lambda}(S_1 = z \mid \sigma_0 < \sigma_m^+) = \frac{h(z)}{h(y)} p(y, z),$$

where $y \in \{1, ..., m-1\}$, $z \in \{0, ..., m\}$ and $p(y, z) := P^y_{m,\lambda}(S_1 = z)$. For $y \in \{1, ..., m-1\}$ this implies

$$\frac{P^y_{m,\lambda}(S_1 = y+1 \mid \sigma_0 < \sigma_m^+)}{P^y_{m,\lambda}(S_1 = y-1 \mid \sigma_0 < \sigma_m^+)} = \frac{h(y+1)}{h(y-1)} \frac{p(y, y+1)}{p(y, y-1)} < \gamma,$$

whereas

$$\frac{P^m_{m,\lambda}(S_1 = m \mid \sigma_0 < \sigma_m^+)}{P^m_{m,\lambda}(S_1 = m-1 \mid \sigma_0 < \sigma_m^+)} = 0 < \gamma.$$

In other words, conditioned on $\sigma_0 < \sigma_m^+$, the walk $(S_n)_{n\in\mathbb{N}_0}$ drifts towards to the left at least as strong as the unconditioned walk drifts towards the right. Estimating all three quantities in the max-term by corresponding quantities for $(S'_n)_{n\in\mathbb{N}_0}$, the biased random walk on $\mathbb{Z}$, we get

$$\begin{aligned}
&\max\big\{P^1_{m,\lambda}\big(\sigma_{1\to 0} \geq \tfrac{n}{6lr_\lambda}, \sigma_0 < \sigma_m\big), P^1_{m,\lambda}\big(\sigma_{1\to m} \geq \tfrac{n}{6lr_\lambda}, \sigma_m < \sigma_0\big), P^m_{m,\lambda}\big(\sigma_{m\to 0} \geq \tfrac{n}{6lr_\lambda}, \sigma_0 < \sigma_m^+\big)\big\} \\
&\quad \leq \max\big\{P^0_{\mathbb{Z},\lambda}\big(\sigma_1^{\mathbb{Z}} \geq \tfrac{n}{6lr_\lambda}\big), P^1_{\mathbb{Z},\lambda}\big(\sigma_m^{\mathbb{Z}} \geq \tfrac{n}{6lr_\lambda}\big), P^0_{\mathbb{Z},\lambda}\big(\sigma_m^{\mathbb{Z}} \geq \tfrac{n}{6lr_\lambda}\big)\big\} \\
&\quad = P^0_{\mathbb{Z},\lambda}\big(\sigma_m^{\mathbb{Z}} \geq \tfrac{n}{6lr_\lambda}\big).
\end{aligned}$$

Using Markov's inequality and Lemma 1.3.6, we get that for $\mu > 0$ with $E^0_{\mathbb{Z},\lambda}\big[e^{\mu\sigma_1^{\mathbb{Z}}}\big] < \infty$,

$$\begin{aligned}
&3l \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m} P^0_{\mathbb{Z},\lambda}\big(\sigma_m^{\mathbb{Z}} \geq \tfrac{n}{6lr_\lambda}\big) \\
&\qquad \leq 3l e^{-\mu\frac{n}{6lr_\lambda}} \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m} E^0_{\mathbb{Z},\lambda}\big[e^{\mu\sigma_1^{\mathbb{Z}}}\big]^m \\
&\qquad = 3l e^{-\mu\frac{n}{6lr_\lambda}} \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m} \left(\frac{1 - \sqrt{1 - 4p_\lambda q_\lambda e^{2\mu}}}{2q_\lambda e^{\mu}}\right)^m.
\end{aligned}$$

The latter series is finite. To see this, notice that if $\lambda < \lambda_c$, we have $e^{-2\lambda_c} < e^{-2\lambda} = \frac{q_\lambda}{p_\lambda}$ and thus

$$e^{-2\lambda_c} \frac{1 - \sqrt{1 - 4p_\lambda q_\lambda e^{2\mu}}}{2q_\lambda e^{\mu}} < 1$$

as $1 - \sqrt{1 - 4p_\lambda q_\lambda e^{2\mu}} \leq 1$ and $2p_\lambda e^{\mu} > 1$. If on the other hand $\lambda \geq \lambda_c$, we have $E^0_{\mathbb{Z},\lambda}[e^{\mu\sigma_1^{\mathbb{Z}}}] \leq E^0_{\mathbb{Z},\lambda_c}[e^{\mu\sigma_1^{\mathbb{Z}}}]$ and the series converges using the same argument.

For the first series on the right-hand side of (2.5.2), we use the union bound to get

$$\begin{aligned}
3 \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m} P_{m,\lambda}\left(\sum_{j=1}^{l} Z_j \geq \frac{n}{4r_\lambda R}\right) &\leq 3l \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m} P_{m,\lambda}\left(Z_1 \geq \frac{n}{4r_\lambda R l}\right) \\
&= 3l \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m} (1 - \mathsf{e}_m)^{\lceil \frac{n}{4r_\lambda R l} \rceil}.
\end{aligned}$$

We set $n_0 := \lceil \frac{n}{4r_\lambda Rl} \rceil$. Since $\mathsf{e}_m \geq (p_\lambda - q_\lambda)\gamma^m$, we have

$$3l \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m}(1-\mathsf{e}_m)^{n_0} \leq 3l \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m}(1-(p_\lambda - q_\lambda)\gamma^m)^{n_0}.$$

Let $t \in \mathbb{N}_0$ be such that $(p_\lambda - q_\lambda)\gamma^{-t} \leq 1 < (p_\lambda - q_\lambda)\gamma^{-(t+1)}$. Then

$$\begin{aligned}
&3l \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m) e^{-2\lambda_c m}(1-(p_\lambda - q_\lambda)\gamma^m)^{n_0} \\
&\qquad \leq 3le^{2\lambda_c t} \sum_{m=m_0\vee m_1}^{\infty} \alpha_i(m+t) e^{-2\lambda_c (m+t)}\Big(1 - \frac{p_\lambda - q_\lambda}{\gamma^t}\gamma^{m+t}\Big)^{n_0} \\
&\qquad \leq 3le^{2\lambda_c t} \sum_{m=0}^{\infty} \alpha_i(m) e^{-2\lambda_c m}\Big(1 - \frac{p_\lambda - q_\lambda}{\gamma^t}\gamma^{m}\Big)^{n_0} \\
&\qquad = 3le^{2\lambda_c t} \sum_{m=0}^{\infty} \alpha_i(m) e^{-2\lambda_c m}\sum_{j=0}^{n_0}\binom{n_0}{j}\Big(-\frac{p_\lambda - q_\lambda}{\gamma^t}\gamma^{m}\Big)^{j} 1^{n_0-j} \\
&\qquad = 3le^{2\lambda_c t} \sum_{j=0}^{n_0}\binom{n_0}{j}(-1)^j\Big(\frac{p_\lambda - q_\lambda}{\gamma^t}\Big)^{j} \sum_{m=0}^{\infty} \alpha_i(m)\gamma^{\alpha m + jm} \\
&\qquad = \begin{cases} 3le^{2\lambda_c t}(e^{2\lambda_c}-1)\sum_{j=0}^{n_0}\binom{n_0}{j}(-1)^j\big(\frac{p_\lambda - q_\lambda}{\gamma^t}\big)^j \frac{1}{1-\gamma^{\alpha+j}} & \text{if } i \neq 0, \\ 3le^{2\lambda_c t}\chi(p)\sum_{j=0}^{n_0}\binom{n_0}{j}(-1)^j\big(\frac{p_\lambda - q_\lambda}{\gamma^t}\big)^j \frac{\gamma^{\alpha+j}}{(1-\gamma^{\alpha+j})^2} & \text{if } i = 0. \end{cases}
\end{aligned}$$

To find the asymptotic behavior of the two expressions in the Lemma, we apply residue calculus. Define the complex function ϕ via $\phi(z) := \frac{(p_\lambda - q_\lambda)^z}{\gamma^{tz}(1-\gamma^{\alpha+z})}$ for $z \in \mathbb{C}$. Then ϕ is holomorphic in $\mathbb{C}$ except at the poles $z_k := \frac{2\pi \mathrm{i} k}{\log\gamma} - \alpha$, $k \in \mathbb{Z}$. Moreover, by the choice of t, $|\phi(z)|$ remains bounded as $|\mathrm{Re}(z)| \to \infty$. Consequently, Theorem 2(i) in [**21**] applies and gives

$$\sum_{j=0}^{n_0}\binom{n_0}{j}(-1)^j\Big(\frac{p_\lambda - q_\lambda}{\gamma^t}\Big)^{j}\frac{1}{1-\gamma^{\alpha+j}} = (-1)^{n_0+1}\sum_{k\in\mathbb{Z}}\operatorname*{Res}_{z=z_k}\Big(\frac{1}{1-\gamma^{\alpha+z}}\frac{(p_\lambda - q_\lambda)^z n_0!}{\gamma^{tz}z(z-1)\dots(z-n_0)}\Big).$$

Along the lines of Example 3 in [**21**], we get

$$\begin{aligned}
\sum_{j=0}^{n_0}\binom{n_0}{j}(-1)^j\Big(\frac{p_\lambda - q_\lambda}{\gamma^t}\Big)^{j}\frac{1}{1-\gamma^{\alpha+j}} &= \Big(-\frac{1}{\log\gamma}\Big)\sum_{k\in\mathbb{Z}}\frac{\Gamma(n_0+1)}{\Gamma(n_0+1-z_k)}\Gamma(-z_k)\Big(\frac{p_\lambda - q_\lambda}{\gamma^t}\Big)^{z_k} \\
&= \frac{1}{2\lambda}n_0^{-\alpha}\sum_{k\in\mathbb{Z}} n_0^{\frac{2\pi \mathrm{i}k}{\log\gamma}}\frac{\Gamma(n_0+1)n_0^{-z_k}}{\Gamma(n_0+1-z_k)}\Gamma(-z_k)\Big(\frac{p_\lambda - q_\lambda}{\gamma^t}\Big)^{z_k} \\
&\leq \frac{\gamma^{t\alpha}(4r_\lambda R)^\alpha l^\alpha}{2\lambda(p_\lambda - q_\lambda)^\alpha}n^{-\alpha}\sum_{k\in\mathbb{Z}} e^{2\pi \mathrm{i}k(\log_\gamma((p_\lambda - q_\lambda)n_0)-t)}\frac{\Gamma(n_0+1)n_0^{-z_k}}{\Gamma(n_0+1-z_k)}\Gamma(-z_k) \\
&= \frac{e^{-2\lambda_c t}(4r_\lambda R)^\alpha l^\alpha}{2\lambda(p_\lambda - q_\lambda)^\alpha}n^{-\alpha}\sum_{k\in\mathbb{Z}} e^{2\pi \mathrm{i}k\log_\gamma((p_\lambda - q_\lambda)n_0)}\frac{\Gamma(n_0+1)n_0^{-z_k}}{\Gamma(n_0+1-z_k)}\Gamma(-z_k) \qquad (2.5.3)
\end{aligned}$$

where Γ is the complex gamma function. From Stirling's formula, e.g. [**2**, Theorem 1.4.2], we know that

$$\log\Gamma(z) = \frac{1}{2}\log(2\pi) + \Big(z - \frac{1}{2}\Big)\log z - z + R(z)$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$ where log is the branch of the complex logarithm, defined on $\mathbb{C} \setminus (-\infty, 0]$, with $\log x \in \mathbb{R}$ for all $x > 0$ and where $R(z)$ satisfies $|R(z)| \leq \frac{c}{|z|}$ for some constant $c > 0$. Hence,

$$\begin{aligned}\frac{\Gamma(n_0+1)n_0^{-z_k}}{\Gamma(n_0+1-z_k)} &= \exp\Bigg(\big(n_0+\tfrac{1}{2}\big)\log(n_0+1) - (n_0+1) + R(n_0+1) - z_k \log n_0 \\ &\qquad - \Big(\big(n_0+\tfrac{1}{2}-z_k\big)\log(n_0+1-z_k) - (n_0+1-z_k) + R(n_0+1-z_k)\Big)\Bigg) \\ &= \Big(\frac{n_0+1-z_k}{n_0}\Big)^{z_k} \Big(\frac{n_0+1}{n_0+1-z_k}\Big)^{n_0+\frac{1}{2}} e^{-z_k} e^{R(n_0+1)-R(n_0+1-z_k)}.\end{aligned}$$

In this product, the first and second factors are bounded in absolute value by 1, the third by e^{α}, and the fourth by e^{2c}. Using Corollary 1.4.4 in [**2**], we conclude that $|\Gamma(-z_k)| \to 0$ exponentially fast as $|k| \to \infty$ and that the bi-infinite series in (2.5.3) is finite and can be bounded by a finite constant c_1 that neither depends on n nor l.
For $i = 0$, we again use Theorem 2(i) in [**21**] and find

$$\sum_{j=0}^{n_0} \binom{n_0}{j} (-1)^j \Big(\frac{p_\lambda - q_\lambda}{\gamma^t}\Big)^j \frac{\gamma^{\alpha+j}}{(1-\gamma^{\alpha+j})^2} = (-1)^{n_0+1} \sum_{k\in\mathbb{Z}} \operatorname*{Res}_{z=z_k} \Big(\frac{\gamma^{\alpha+z}}{(1-\gamma^{\alpha+z})^2} \frac{(p_\lambda-q_\lambda)^z n_0!}{\gamma^{tz} z(z-1)\dots(z-n_0)}\Big)$$

with $z_k = \frac{2\pi i k}{\log \gamma} - \alpha$ as above. Evaluating the residues leads to

$$\begin{aligned}&\sum_{j=0}^{n_0} \binom{n_0}{j} (-1)^j \Big(\frac{p_\lambda - q_\lambda}{\gamma^t}\Big)^j \frac{\gamma^{\alpha+j}}{(1-\gamma^{\alpha+j})^2} \\ &= (-1)^{n_0+1} \sum_{k\in\mathbb{Z}} \frac{n_0!}{(\log\gamma)^2 z_k(z_k-1)\dots(z_k-n_0)} \Big(\log\Big(\frac{p_\lambda-q_\lambda}{\gamma^t}\Big) + \sum_{j=0}^{n_0} \frac{1}{j-z_k}\Big)\Big(\frac{p_\lambda-q_\lambda}{\gamma^t}\Big)^{z_k} \\ &= \frac{e^{-2\lambda_c t}}{4\lambda^2(p_\lambda-q_\lambda)^\alpha} n_0^{-\alpha} \sum_{k\in\mathbb{Z}} e^{2\pi i k \log_\gamma((p_\lambda-q_\lambda)n_0)} \frac{\Gamma(n_0+1)n_0^{-z_k}}{\Gamma(n_0+1-z_k)} \Gamma(-z_k) \Big(\log\Big(\frac{p_\lambda-q_\lambda}{\gamma^t}\Big) + \sum_{j=0}^{n_0}\frac{1}{j-z_k}\Big) \\ &\leq \frac{e^{-2\lambda_c t}(4r_\lambda R)^\alpha}{4\lambda^2(p_\lambda-q_\lambda)^\alpha} l^\alpha n^{-\alpha} \sum_{k\in\mathbb{Z}} e^{2\pi i k \log_\gamma((p_\lambda-q_\lambda)n_0)} \frac{\Gamma(n_0+1)n_0^{-z_k}}{\Gamma(n_0+1-z_k)} \Gamma(-z_k) \sum_{j=0}^{n_0} \frac{1}{j-z_k}.\end{aligned}$$

Along the same lines as above, we can show that this bi-infinite series has finite value and the whole term can be bounded by $c_1' l^\alpha n^{-\alpha} \log n$, where $c_1' \in (0,\infty)$ does not depend on n or l. ☐

2.5.4. Coupling of the biased random walk with a biased random walk on the backbone. As the times spent in different traps are not independent, further work is needed to transfer the tail estimate for the time spent in a single trap to the time spent in the possibly several traps inside a block $[\rho_i, \rho_{i+1})$. Therefore, we introduce a random walk on a subgraph ω^{p} of the initial environment ω as follows. We take the initial graph ω sampled according to P_p or P_p° and modify it as follows. For each trap $P = (e_1, \dots, e_m)$ in ω with trap entrance u_0 and edges $e_1 = \langle u_0, u_1\rangle, \dots, e_m = \langle u_{m-1}, u_m\rangle$, we delete the edges $e_1, \dots, e_m$ from ω and also the vertices $u_1, \dots, u_m$. We further delete the opposite vertices $u_1', \dots, u_m'$ and replace the parallel edges $e_1', \dots, e_m', \langle u_m', u_m' + (1,0)\rangle$ with a single edge connecting u_0' and $u_m' + (1,0)$ with resistance given by the sum of the resistances of the single edges. We shall call the vertex u_0' opposite the former trap entrance an *obstacle*. Should this procedure lead to the deletion of $\mathbf{0}$, we assign x-coordinate 0 in ω^{p} to the obstacle that replaced the trap piece which contained $\mathbf{0}$ in ω. In this way, we also obtain new conductances c^{s} on ω^{p}.

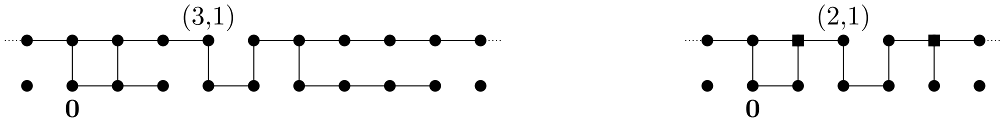


FIGURE 8. Comparison of ω (left) and the resulting ω^{p} (right). Normal vertices are drawn as filled circles, the obstacles as filled boxes.

By the series law, the corresponding resistances r^{s} between the first obstacle v to the right of $\mathbf{0}$ that replaces a trap piece covering x-level k to $k+m+1$ and its neighbors u to the left and w to the right satisfy

$$r^{\mathrm{s}}(\langle u,v\rangle) = r(\langle u,v\rangle) = e^{-\lambda(\mathsf{x}(u)+\mathsf{x}(v))} = e^{-\lambda(2k-1)}$$

and

$$r^{\mathrm{s}}(\langle v,w\rangle) = \sum_{j=k}^{k+m} r(\langle j,\mathsf{y}(v)\rangle,\langle j+1,\mathsf{y}(v)\rangle) = \sum_{j=k}^{k+m} e^{-\lambda(2j+1)} = e^{-\lambda(2k+1)}\frac{1-e^{-2\lambda(m+1)}}{1-e^{-2\lambda}}.$$

Based on this, we define the *pruned random walk* as the lazy random walk $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ on ω^{p} with transition probabilities proportional to the conductances

$$c^{\mathrm{p}}(\langle u,v\rangle) = e^{\lambda(\mathsf{x}(u)+\mathsf{x}(v))}\cdot(1-e^{-2\lambda})^{p(v)}$$

where $\mathsf{x}(u)\le\mathsf{x}(v)$ and $p(v)$ is the number of obstacles with x-coordinate $\in[0,\mathsf{x}(v))$. More precisely, if $Y_n^{\mathrm{p}}=u$, then the walk attempts to step from u to v with probability proportional to $c^{\mathrm{p}}(\langle u,v\rangle)$. If the edge between u and v is present in ω^{p}, then the step is actually performed, otherwise the walk stays put.

Roughly speaking, $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ is the lazy random walk on the non-trap pieces of ω when all traps are set to have infinite length. Intuitively, as the traps in ω have finite lengths, the embedding of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ into ω will lag behind the random walk $(Y_n)_{n\in\mathbb{N}_0}$. Regenerations of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ also amount to regenerations of $(Y_n)_{n\in\mathbb{N}_0}$ without implications on the lengths of the traps in the underlying piece of ω. Furthermore, $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ can be used to bound the number of visits to any trap by a quantity independent of the trap lengths, thus greatly reducing the difficulties in transforming the estimate of Lemma 2.5.6 to an estimate for the time spent in the whole block $[\rho_i,\rho_{i+1})$ in ω. To make this precise, we give a coupling of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ with the described properties. Technically, the coupling is such that we obtain processes with the same distributions as $(Y_n)_{n\in\mathbb{N}_0}$ and $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ and the desired properties, but we shall again refer to them as $(Y_n)_{n\in\mathbb{N}_0}$ and $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$, respectively, once equality of the corresponding laws is established.

First, let $(O_i)_{i\in\mathbb{Z}}$ be an enumeration of the obstacles in ω^{p} such that $\ldots<\mathsf{x}(O_{-1})<0\le\mathsf{x}(O_0)<\mathsf{x}(O_1)<\ldots$. Starting from ω^{p}, take an independent family $(L_i)_{i\in\mathbb{Z}}$ of random variables, with $(L_i)_{i\neq 0}$ independent of ω. We re-insert at O_i a trap piece with a trap of length L_i. Here, we let L_i have the same distribution as ℓ_i for $i\neq 0$. For $i=0$, let the law of L_0 given $\mathsf{x}(O_0)>0$ be the law of ℓ_1. Further notice that if $\mathsf{x}(O_0)=0$, then, by the definition of T_0 and T_1, either $\mathbf{0}$ is one of the two leftmost vertices in T_1 or $\mathbf{0}\in\mathrm{int}(T_0)$ which consists of all vertices from T_0 except the two leftmost and the two rightmost vertices. Thus, we define the law of L_0 given $\mathsf{x}(O_0)=0$ by

$$\mathrm{P}_p(\mathbf{0}\in T_1\mid\mathbf{0}\in T_1\cup\mathrm{int}(T_0))\mathrm{P}_p(\ell_1\in\cdot)+\mathrm{P}_p(\mathbf{0}\in\mathrm{int}(T_0)\mid\mathbf{0}\in T_1\cup\mathrm{int}(T_0))\mathrm{P}_p(\ell_0\in\cdot\mid\mathbf{0}\in\mathrm{int}(T_0)).$$

In other words, we toss a coin with probability $\mathrm{P}_p(\mathbf{0}\in T_1\mid\mathbf{0}\in T_1\cup\mathrm{int}(T_0))$ for heads. If the coin comes up heads, we sample the value of L_0 using an independent copy of ℓ_1 (under P_p). If the coin comes up tails, we sample the value of L_0 using an independent copy of ℓ_0 (under P_p given that $\mathbf{0}\in\mathrm{int}(T_0)$, this random variable satisfies the bound in Lemma 2.4.1(b)). Additionally,

if the coin comes up tails, we shift horizontally by a value $k \in \{1, \dots, L_0\}$ according to the distribution under P_p of the position of $\mathbf{0}$ in T_0 given $\mathbf{0} \in \mathrm{int}(T_0)$. This gives a new configuration $\tilde{\omega}$. By construction, $\tilde{\omega} \stackrel{\text{law}}{=} \omega$.
Slightly abusing notation, we write ω^{p} for both ω^{p} and the subset of $\tilde{\omega}$ corresponding to it. We further write $V(\omega^{\mathrm{p}})$ and $V(\tilde{\omega})$ for the corresponding vertex sets. Consequently, we write $u = v$ for vertices $u \in V(\omega^{\mathrm{p}})$, $v \in V(\tilde{\omega})$ if v is the node in $\tilde{\omega}$ corresponding to u in ω^{p}. Given ω^{p} and $\tilde{\omega}$, we define a random walk $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ on $V(\omega^{\mathrm{p}}) \times V(\tilde{\omega}) \times \{-1, 0, 1\}$, where the first and second component (up to random waiting times) behave like $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$, respectively, and the third component exclusively acts as a memory of the directions taken at certain nodes. This is to ensure that $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ is a Markov chain.
At each time $n \in \mathbb{N}_0$, first a candidate $\mathcal{Y}_{n+1}^{\mathrm{cand}} = (\mathcal{Y}_{n+1,1}^{\mathrm{cand}}, \mathcal{Y}_{n+1,2}^{\mathrm{cand}}, \mathcal{Y}_{n+1,3}^{\mathrm{cand}})$ for the next step is chosen and afterwards the chosen step is taken only if the corresponding edges in ω^{p} or $\tilde{\omega}$, respectively, are open:

$$\mathcal{Y}_{n+1,1} = \begin{cases} \mathcal{Y}_{n+1,1}^{\mathrm{cand}} & \text{if } \omega^{\mathrm{p}}(\langle \mathcal{Y}_{n,1}, \mathcal{Y}_{n+1,1}^{\mathrm{cand}} \rangle) = 1, \\ \mathcal{Y}_{n,1} & \text{otherwise,} \end{cases} \qquad \mathcal{Y}_{n+1,2} = \begin{cases} \mathcal{Y}_{n+1,2}^{\mathrm{cand}} & \text{if } \tilde{\omega}(\langle \mathcal{Y}_{n,1} \mathcal{Y}_{n+1,1}^{\mathrm{cand}} \rangle) = 1, \\ \mathcal{Y}_{n,2} & \text{otherwise} \end{cases}$$

and $\mathcal{Y}_{n+1,3} = \mathcal{Y}_{n+1,3}^{\mathrm{cand}}$.
We start at $\mathcal{Y}_0 = (\mathbf{0}, \mathbf{0}, 0)$ and give the transition matrix of $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ in a case-by-case description depending on the position $(u, v, w) \in V(\omega^{\mathrm{p}}) \times V(\tilde{\omega}) \times \{-1, 0, 1\}$ at time n.

(1) If $u = v$ when regarding ω^{p} as a subset of $\tilde{\omega}$, and if $u \neq O_i$ for all $i \in \mathbb{Z}$, we let $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ attempt to do exactly the same steps in its first two components. In that case

$$\mathcal{Y}_{n+1}^{\mathrm{cand}} = \begin{cases} (u + (1,0), v + (1,0), 0) & \text{with probability } \frac{e^{\lambda}}{e^{\lambda}+1+e^{-\lambda}}, \\ (u - (1,0), v - (1,0), 0) & \text{with probability } \frac{e^{-\lambda}}{e^{\lambda}+1+e^{-\lambda}}, \\ (u', v', 0) & \text{with probability } \frac{1}{e^{\lambda}+1+e^{-\lambda}}. \end{cases}$$

Note that if v is a trap entrance in $\tilde{\omega}$, a step to the right by $(\mathcal{Y}_{n+1,1}^{\mathrm{cand}}, \mathcal{Y}_{n+1,2}^{\mathrm{cand}})$ induces a lazy step of $(\mathcal{Y}_{k,1})_{k\in\mathbb{N}_0}$ whereas $(\mathcal{Y}_{k,2})_{k\in\mathbb{N}_0}$ moves into the trap. In that case, as will be described in detail below, $(\mathcal{Y}_{k,2})_{k\in\mathbb{N}_0}$ will make an excursion into the trap afterwards whereas $(\mathcal{Y}_{k,1})_{k\in\mathbb{N}_0}$ will stay put in u until $(\mathcal{Y}_{k,2})_{k\in\mathbb{N}_0}$ returns to the trap entrance v. Similarly, when a step of $(\mathcal{Y}_{k,1})_{k\in\mathbb{N}_0}$ to the left means moving to an obstacle, $(\mathcal{Y}_{k,2})_{k\in\mathbb{N}_0}$ will then step onto a backbone node in $\tilde{\omega} \setminus \omega^{\mathrm{p}}$. In this case $(\mathcal{Y}_{k,1})_{k\in\mathbb{N}_0}$ will also stay put until $(\mathcal{Y}_{k,2})_{k\in\mathbb{N}_0}$ reaches a node in $\tilde{\omega} \cap \omega^{\mathrm{p}}$.

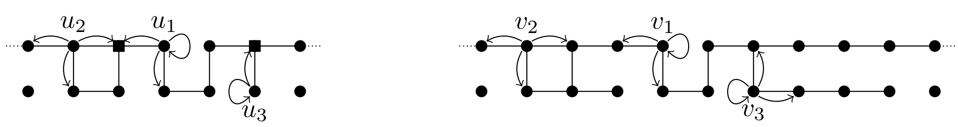


FIGURE 9. The figure shows possible transitions on non-obstacle backbone-nodes from (u_1, v_1), (u_2, v_2) and (u_3, v_3), where u_j in ω^{p} 'equals' v_j in $\tilde{\omega}$.

(2) If $u = v$, but $u = O_i$ for some $i \in \mathbb{Z}$, then the step in the first component is taken according to the conductances c^{p}. The second component mimics this, but with the additional option to move right even if the first component does not. This is to adjust the transition probabilities of the second component to match those of $(Y_n)_{n\in\mathbb{N}_0}$. If the first component moves right, we demand that the second component leaves the coming trap piece at the right end, which we encode in the third component. Since we further want the walk in the second component to have the same law as $(Y_n)_{n\in\mathbb{N}_0}$, we have to make sure that in total, it leaves the trap piece at the right

resp. left end with the correct probability. These restrictions lead to a system of linear equations for the transition probabilities whose solution is given as follows.

$$\mathcal{Y}_{n+1}^{\mathrm{cand}} = \begin{cases} (u+(1,0), v+(1,0), 1) & \text{with probability } \frac{e^{\lambda}(1-e^{-2\lambda})}{e^{\lambda}(1-e^{-2\lambda})+1+e^{-\lambda}} = \frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+1}, \\ (u-(1,0), v-(1,0), 0) & \text{with probability } \frac{e^{-\lambda}}{e^{\lambda}+1+e^{-\lambda}}, \\ (u', v', 0) & \text{with probability } \frac{1}{e^{\lambda}+1+e^{-\lambda}}, \\ (u-(1,0), v+(1,0), 1) & \text{with probability } \frac{e^{-\lambda}}{1+e^{-\lambda}}\big(\mathsf{e}'_{L_i+1} - \frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+1}\big), \\ (u-(1,0), v+(1,0), -1) & \text{with probability } e^{-\lambda}\big(\frac{1}{1+e^{-\lambda}} - \frac{1}{e^{\lambda}+1+e^{-\lambda}} - \frac{1}{1+e^{-\lambda}}\mathsf{e}'_{L_i+1}\big), \\ (u', v+(1,0), 1) & \text{with probability } \frac{1}{1+e^{-\lambda}}\big(\mathsf{e}'_{L_i+1} - \frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+1}\big), \\ (u', v+(1,0), -1) & \text{with probability } \frac{1}{1+e^{-\lambda}} - \frac{1}{e^{\lambda}+1+e^{-\lambda}} - \frac{1}{1+e^{-\lambda}}\mathsf{e}'_{L_i+1}, \end{cases}$$

where L_i is the length of the trap right of v and

$$\mathsf{e}'_m := \frac{e^{\lambda}}{e^{\lambda}+1+e^{-\lambda}} P^1_{m,\lambda}(\sigma_m < \sigma_0) = \frac{e^{\lambda}}{e^{\lambda}+1+e^{-\lambda}} \frac{1-e^{-2\lambda}}{1-e^{-2\lambda m}}$$

is the probability that the biased random walk $(S'_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ starting from 0 first makes a step to the right and then hits m before 0.

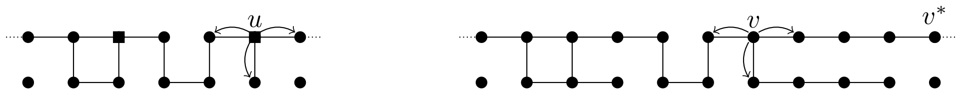


FIGURE 10. Transitions from obstacles. Depending on the value of $\mathcal{Y}_{n+1,3}$, after a step to the right it is already determined whether the random walk on $\tilde{\omega}$ hits the boundary of the trap piece at v or v^*.

(3) If v is in the interior of the backbone part of a trap piece in $\tilde{\omega}$ (and thus not in ω^{p}), then we write L_v for the length of the corresponding trap. In this case, the first component of $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ stays put while the second component moves in the trap piece with transition probabilities according to the biased random walk $(Y_n)_{n\in\mathbb{N}_0}$, possibly conditioned on the event that the boundary of the trap piece is first hit at the left- or rightmost end, respectively. Let $p_{k,0}$, $p_{k,-1}$, $p_{k,1}$ be the transition matrices of the lazy biased random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\{0,...,k\}$ (which steps to the right, steps to the left or stays put with probability proportional to e^{λ}, $e^{-\lambda}$ and 1, respectively) and the lazy biased random walk on $\{0,...,k\}$ conditioned on $\{\sigma_0 < \sigma_k\}$ resp. $\{\sigma_0 > \sigma_k\}$, where $\sigma_j := \inf\{n \in \mathbb{N}_0 : S_n = j\}$. Then we set

$$\mathcal{Y}_{n+1}^{\mathrm{cand}} = \begin{cases} (u, v+(1,0), w) & \text{with probability } p_{L_v+1,w}(\mathsf{x}_v, \mathsf{x}_v+1), \\ (u, v-(1,0), w) & \text{with probability } p_{L_v+1,w}(\mathsf{x}_v, \mathsf{x}_v-1), \\ (u, v', w) & \text{with probability } p_{L_v+1,w}(\mathsf{x}_v, \mathsf{x}_v), \end{cases}$$

where $\mathsf{x}_v \in \{1,...,L_v\}$ is the relative horizontal position of v in the trap piece.

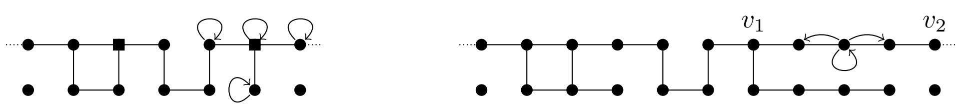


FIGURE 11. Transitions in the backbone part of trap pieces. If $\mathcal{Y}_{n,3} \in \{-1,1\}$, then it is predetermined that the walk hits the boundary of the trap piece at v_1 or v_2, respectively.

(4) If v is a trap node in $\tilde{\omega}$, the first component of $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ stays put while the second component moves inside the trap with transition probabilities according to the biased random walk $(Y_n)_{n\in\mathbb{N}_0}$. That is,

$$\mathcal{Y}_{n+1}^{\text{cand}} = \begin{cases} (u, v+(1,0), 0) & \text{with probability } \frac{e^{\lambda}}{e^{\lambda}+1+e^{-\lambda}}, \\ (u, v-(1,0), 0) & \text{with probability } \frac{e^{-\lambda}}{e^{\lambda}+1+e^{-\lambda}}, \\ (u, v', 0) & \text{with probability } \frac{1}{e^{\lambda}+1+e^{-\lambda}}. \end{cases}$$

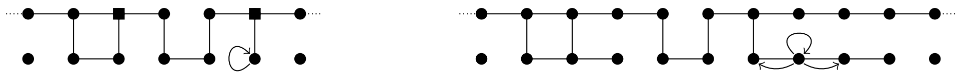


FIGURE 12. Transitions in the dead end part of trap pieces

(5) Finally, when $v \in \tilde{\omega} \cap \omega^{\mathrm{p}}$, but the positions of the two components of $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ do not correspond, the second component stays put, while the first component moves with transition probabilities given by the conductances c^{p}:

$$\mathcal{Y}_{n+1}^{\text{cand}} = \begin{cases} (u+(1,0), v, 0) & \text{with probability proportional to } c^{\mathrm{p}}(\langle u, u+(1,0)\rangle), \\ (u-(1,0), v, 0) & \text{with probability proportional to } c^{\mathrm{p}}(\langle u, u-(1,0)\rangle), \\ (u', v, 0) & \text{with probability proportional to } c^{\mathrm{p}}(\langle u, u'\rangle). \end{cases}$$

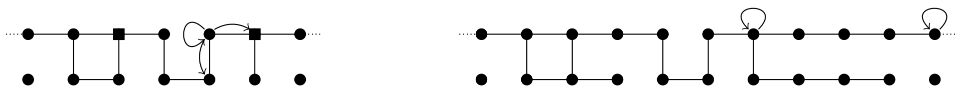


FIGURE 13. Transitions on the backbone when coordinates do not coincide. In this case, the walk on $\tilde{\omega}$ waits at a trap end or a vertex opposite a trap entrance. This vertex must be passed by the walk on ω^{p} provided that this walk is transient to the right. The walk on $\tilde{\omega}$ pauses until the walk on ω^{p} hits its position.

We write P_p' for the distribution of the environment $(\omega^{\mathrm{p}}, \tilde{\omega})$ and $P'_{\omega^{\mathrm{p}},\tilde{\omega},\lambda}$ for the quenched law of $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$ as described above. With these, we define a measure $\mathbb{P}'$ on $(\{0,1\}^E \times \{0,1\}^E) \times (V^2 \times \{-1,0,1\})^{\mathbb{N}_0}$, endowed with the product σ-Algebra, by

$$\mathbb{P}'(A \times B) := \int_A P'_{\omega^{\mathrm{p}},\tilde{\omega},\lambda}(B)\, \mathrm{P}_p'(\mathrm{d}(\omega^{\mathrm{p}}, \tilde{\omega})).$$

Sometimes, the walks on ω^{p} and $\tilde{\omega}$ are at different positions (when ω^{p} is embedded in $\tilde{\omega}$). Then, depending on the particular situation, one of the walks waits while the other moves until they meet again. The times at which each of the walks moves without being forced to hold as described above are collected in the following sets:

$$N_1 := \{n \in \mathbb{N}_0 : \mathcal{Y}_{n,2} \text{ is at a vertex in } \tilde{\omega} \text{ corresponding to a vertex in } \omega^{\mathrm{p}}\},$$
$$N_2 := \{n \in \mathbb{N}_0 : \mathcal{Y}_{n,1} = \mathcal{Y}_{n,2}\} \cup \{n \in \mathbb{N}_0 : \mathcal{Y}_{n,2} \text{ is in the interior of a trap piece}\}.$$

Let $(s_{1,k})_{k\in\mathbb{N}}$ resp. $(s_{2,k})_{k\in\mathbb{N}}$ be enumerations of N_1 resp. N_2 in ascending order. Then the following processes coincide in law with $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$, respectively. More precisely, with

$$(\mathcal{Y}_n^{\mathrm{p}})_{n\in\mathbb{N}_0} := (\mathcal{Y}_{1,s_{1,n}})_{n\in\mathbb{N}_0}, \quad (\tilde{Y}_n)_{n\in\mathbb{N}_0} := (\mathcal{Y}_{2,s_{2,n}})_{n\in\mathbb{N}_0}$$

the following Lemma holds.

LEMMA 2.5.7. *We have*

$$(\mathcal{Y}_n^{\mathrm{p}})_{n\in\mathbb{N}_0} \overset{\text{law}}{=} (Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}, \quad (\tilde{Y}_n)_{n\in\mathbb{N}_0} \overset{\text{law}}{=} (Y_n)_{n\in\mathbb{N}_0}.$$

PROOF. Since $(\mathcal{Y}_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ and $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ are defined on the same environment, and the environments of $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ are identically distributed by construction, it suffices to check the quenched transition probabilities of $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ and $(\mathcal{Y}_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$, respectively. One can check that the transition probabilities of $(\mathcal{Y}_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ coincide with those of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$, thus the equality in law of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ and $(\mathcal{Y}_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ follows from the Markov property of $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$. For $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$, at most nodes this is also obvious except for transitions at obstacles and inside trap pieces. However, it suffices to show that on obstacles, steps into the different directions are taken with the correct probability and that excursions on the following trap pieces end on the left resp. right end with the correct probability, i.e., that $(\mathcal{Y}_{n,3})_{n\in\mathbb{N}_0}$ takes value -1 or 1 with the correct probability. This amounts to a system of linear equations which is solved by the transition probabilities defined under (2). The result now also follows from the Markov property of $(\mathcal{Y}_n)_{n\in\mathbb{N}_0}$. ☐

From now on, all results concerning $(Y_n)_{n\in\mathbb{N}_0}$ will be discussed in terms of the process $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ under $\mathbb{P}'$. To ease notation, we shall write $(Y_n)_{n\in\mathbb{N}_0}$ and $\mathbb{P}$ for $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ and $\mathbb{P}'$, respectively. We shall also write ℓ_i though technically referring to L_i. Consequently, we will not distinguish between $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ and $(\mathcal{Y}_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ nor between ω and $\tilde{\omega}$. Also, we write $P_{\omega^{\mathrm{p}},\lambda}$ for the quenched law of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$.

LEMMA 2.5.8. *For $\lambda > \lambda^* := \frac{\log(2)}{2}$, in particular for $\lambda \geq \frac{\lambda_c}{2}$, it holds that* $\lim_{n\to\infty} \mathsf{x}(Y_n^{\mathrm{p}}) = \infty$ *almost surely under $P_{\omega^{\mathrm{p}},\lambda}$.*

The proof of the Lemma is very similar to that of Proposition 3.1 in [**4**].

PROOF. It is sufficient to show that $\mathbf{0}$ is a transient state for the biased random walk on $V(\omega^{\mathrm{p}})$. We use electrical network theory. Write $\mathcal{R}^{\mathrm{p}}(\mathbf{0} \leftrightarrow \infty)$ for the effective resistance between $\mathbf{0}$ and $+\infty$ in the random conductance model on ω^{p} with conductances $c^{\mathrm{p}}(e)$ for $e \in E$ with $\omega^{\mathrm{p}}(e) = 1$. Using Thomson's Principle, we infer

$$\mathcal{R}^{\mathrm{p}}(\mathbf{0} \leftrightarrow \infty) \leq \mathcal{E}^{\mathrm{p}}(\theta)$$

for all unit flows θ from $\mathbf{0}$ to ∞ where $\mathcal{E}^{\mathrm{p}}(\theta)$ is the energy of the flow θ. The energy of the flow θ is $\mathcal{E}^{\mathrm{p}}(\theta) = \sum_{e:\omega^{\mathrm{p}}(e)=1} \theta(e)^2/c^{\mathrm{p}}(e)$ where $\theta(e)^2 = \theta(v,w)^2$ if $e = \langle v,w\rangle$. Since there are no traps in ω^{p}, there exists an infinite open self-avoiding path $P = (e_1, e_2, \ldots)$ connecting $\mathbf{0}$ with ∞. This path never backtracks in the sense that the sequence of x-coordinates of the vertices on this path is nondecreasing. Now define a flow θ from $\mathbf{0}$ to ∞ by pushing a unit current through P. More precisely, if $e_n = \langle u_{n-1}, u_n\rangle$ with $u_0 := \mathbf{0}$, then let $\theta(u_{n-1}, u_n) = 1 = -\theta(u_n, u_{n-1})$ for all $n \in \mathbb{N}$ and $\theta(v,w) = 0$ whenever $\langle v,w\rangle$ is not on the path P. For every x-level $n \in \mathbb{N}_0$ there is at most one edge e in P connecting the two vertices with x-value n. The resistance of this edge is bounded by $r^{\mathrm{p}}(e) \leq e^{-2\lambda n}(1-e^{-2\lambda})^{-p(n)}$ where $p(n)$ is the number of obstacles with x-value $< n$. There are at most n such obstacles. Therefore, $r^{\mathrm{p}}(e) \leq e^{-2\lambda n}(1 - e^{-2\lambda})^{-n}$. Further, for every $n \in \mathbb{N}$, there is exactly one edge on P leading from a vertex with x-value $n-1$ to x-value n. The resistance of this edge is bounded by $r^{\mathrm{p}}(e) \leq e^{-\lambda(2n-1)}(1-e^{-2\lambda})^{-p(n)} \leq e^{-\lambda(2n-1)}(1-e^{-2\lambda})^{-n}$. Consequently, the energy $\mathcal{E}^{\mathrm{p}}(\theta)$ is bounded by

$$\mathcal{E}^{\mathrm{p}}(\theta) = \sum_{e\in P} \theta(e)^2 r^{\mathrm{p}}(e) \leq 1 + \sum_{n=1}^{\infty} (e^{-\lambda(2n-1)} + e^{-2\lambda n})(1-e^{-2\lambda})^{-n} \leq 1 + 2e^{\lambda} \sum_{n=1}^{\infty} \Big(\frac{e^{-2\lambda}}{1-e^{-2\lambda}}\Big)^n.$$

The latter series is finite iff $\frac{e^{-2\lambda}}{1-e^{-2\lambda}} < 1$ or, equivalently, $\lambda > \frac{\log(2)}{2} =: \lambda^*$. Comparing this with $\lambda_c/2$, for which we have an explicit formula in terms of p given in Proposition 2.2.2 with unique minimizer $p = 1/2$, we have

$$\frac{\lambda_c}{2} \geq \frac{\lambda_c(1/2)}{2} = \frac{1}{4}\log\left(\frac{4}{3-\sqrt{5}}\right) = \frac{1}{2}\log\left(\frac{2}{\sqrt{3-\sqrt{5}}}\right) > \frac{\log(2)}{2} = \lambda^*.$$

□

It also follows from the proof of Lemma 2.5.8 that for $u \in \omega^{\mathrm{p}}$ and $\lambda \geq \lambda_c/2$, the escape probability at u, that is the probabilty to leave u and never return, is uniformly bounded from below. For $u \in \omega^{\mathrm{p}}$, let $\sigma_u^{\mathrm{p}} := \inf\{n \in \mathbb{N} : Y_n^{\mathrm{p}} = u\}$. Also let $\mathcal{R}^{\mathrm{p}}(u \leftrightarrow \infty)$ and $c^{\mathrm{p}}(u)$ be the effective resistance between u and $+\infty$ and the sum of conductances of all incident edges at u, respectively, in the random conductance model on ω^{p} with conductances $c^{\mathrm{p}}(e)$ for $e \in E$ with $\omega^{\mathrm{p}}(e) = 1$. Then pushing a unit current from u to $+\infty$ as in the proof of Lemma 2.5.8, we get

$$\begin{aligned} P_{\omega,\lambda}^u(\sigma_u^{\mathrm{p}} = \infty) &= \frac{1}{c^{\mathrm{p}}(u)\mathcal{R}^{\mathrm{p}}(u \leftrightarrow \infty)} \\ &\geq \frac{1}{3e^{(2\mathsf{x}(u)+1)\lambda}(1-e^{-2\lambda})^{p(u)}e^{-2\lambda\mathsf{x}(u)}(1-e^{-2\lambda})^{-p(u)}\left(1+2e^{2\lambda}\sum_{n=1}^{\infty}\left(\frac{e^{-2\lambda}}{1-e^{-2\lambda}}\right)^n\right)} \\ &= \frac{1}{3e^{\lambda}\left(1+2e^{2\lambda}\sum_{n=1}^{\infty}\left(\frac{e^{-2\lambda}}{1-e^{-2\lambda}}\right)^n\right)} > 0. \end{aligned} \tag{2.5.4}$$

Let $R_1^{\mathrm{p}}, R_2^{\mathrm{p}}, \ldots$ be an enumeration from left to right of the pre-regeneration points in ω^{p} which are visited exactly once by $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$. Further, let $\rho_0^{\mathrm{p}} = 0$ and $\rho_n^{\mathrm{p}} := \mathsf{x}(R_n^{\mathrm{p}})$ for $n \in \mathbb{N}$. Finally, for $n \in \mathbb{N}$, let τ_n^{p} be the unique time k with $\mathsf{x}(Y_k^{\mathrm{p}}) = \rho_n^{\mathrm{p}}$. We refer to the R_n^{p}'s and τ_n^{p}'s as regeneration points and times, respectively, of the pruned walk.

LEMMA 2.5.9. *With $\mathbb{P}$-probability 1, there exist infinitely many regeneration points of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$.*

PROOF. This can be proven along exactly the same lines as for $(Y_n)_{n\in\mathbb{N}_0}$ in [**4**, Lemma 5.1], as the argument there only relies on a uniform lower bound on the escape probability at any pre-regeneration point u. Here, (2.5.4) gives this estimate. □

LEMMA 2.5.10. *Let $\lambda > \lambda^*$. Then there exists $\delta > 0$ such that*

$$\mathbb{E}^\circ\left[e^{\delta(\rho_1^{\mathrm{p}} - \min_{j\in\mathbb{N}} \mathsf{x}(Y_j^{\mathrm{p}}))}\right] < \infty.$$

Furthermore, $\mathbb{E}^\circ[(\tau_1^{\mathrm{p}})^\kappa] < \infty$ for any $\kappa > 0$.

Both statements still hold true when $\mathbb{E}^\circ$ is replaced by $\mathbb{E}$.

PROOF. This is proven along the same lines as Lemmas 6.3, 6.4 and 6.5 in [**23**]. We set

$$p_{\mathrm{esc}}^{\mathrm{p}} := \frac{1}{3e^{\lambda}\left(1+2e^{2\lambda}\sum_{n=1}^{\infty}\left(\frac{e^{-2\lambda}}{1-e^{-2\lambda}}\right)^n\right)}$$

as the uniform lower bound from equation (2.5.4) on the escape probability at any vertex $u \in \omega^{\mathrm{p}}$ in the random conductance model on ω^{p} with conductances $c^{\mathrm{p}}(e)$ for $e \in E$ with $\omega^{\mathrm{p}}(e) = 1$. Furthermore, we define $F_0 := E_0 := M_0 := 0$, $D((y_n)_{\in\mathbb{N}_0}) := \inf\{k \in \mathbb{N} : y_k = y_0\}$, where $\inf\emptyset := \infty$, and

$$\begin{aligned} F_k &:= \inf\{j \in \mathbb{N}_0 : Y_j^{\mathrm{p}} \in \mathcal{R}^{\mathrm{pre}}, \mathsf{x}(Y_j^{\mathrm{p}}) > M_{k-1}\}, \\ E_k &:= D((Y_{F_k+j}^{\mathrm{p}})_{j\in\mathbb{N}_0}), \\ M_k &:= \sup\{\mathsf{x}(Y_j^{\mathrm{p}}) : 0 \leq j < E_k\}. \end{aligned}$$

With $K := \inf\{k \in \mathbb{N} : F_k < \infty, E_k = \infty\}$, we have $F_K = \tau_1^{\mathrm{p}}$ and $\mathsf{x}(Y^{\mathrm{p}}_{F_K}) = \rho_1^{\mathrm{p}}$. Let $r > 0$. Then

$$\mathbb{P}^\circ(\rho_1^{\mathrm{p}} \geq k) = \mathbb{P}^\circ\Big(\sum_{l=1}^{K} \mathsf{x}(Y^{\mathrm{p}}_{F_l}) - \mathsf{x}(Y^{\mathrm{p}}_{F_{l-1}}) \geq k\Big)$$

$$\leq \mathbb{P}^\circ(K \geq k/r) + \mathbb{P}^\circ\Big(\sum_{l=1}^{\lfloor k/r\rfloor} (\mathsf{x}(Y^{\mathrm{p}}_{F_l}) - \mathsf{x}(Y^{\mathrm{p}}_{F_{l-1}}))\mathbb{1}_{\{F_l<\infty\}} \geq k\Big). \tag{2.5.5}$$

By definition of $p^{\mathrm{p}}_{\mathrm{esc}}$, we have

$$\mathbb{P}^\circ(K \geq n) \leq (1 - p^{\mathrm{p}}_{esc})^{n-1}, \quad n \in \mathbb{N}$$

and thus the first term in (2.5.5) can be bounded by $(1 - p^{\mathrm{p}}_{\mathrm{esc}})^{\lfloor k/r\rfloor}$. Using Markov's inequality, for $\mu > 0$ we further get that

$$\mathbb{P}^\circ\Big(\sum_{l=1}^{\lfloor k/r\rfloor} (\mathsf{x}(Y^{\mathrm{p}}_{F_l}) - \mathsf{x}(Y^{\mathrm{p}}_{F_{l-1}}))\mathbb{1}_{\{F_l<\infty\}} \geq k\Big) \leq e^{-\mu k}\mathbb{E}^\circ\Big[\exp\Big(\mu \sum_{l=1}^{\lfloor k/r\rfloor} (\mathsf{x}(Y^{\mathrm{p}}_{F_l}) - \mathsf{x}(Y^{\mathrm{p}}_{F_{l-1}}))\mathbb{1}_{\{F_l<\infty\}}\Big)\Big]$$

$$= e^{-\mu k}\mathbb{E}^\circ\Big[e^{\mu \mathsf{x}(Y^{\mathrm{p}}_{F_1})\mathbb{1}_{\{F_1<\infty\}}}\Big]^{\lfloor k/r\rfloor}$$

$$\leq e^{-\mu k}\mathbb{E}^\circ\Big[e^{\mu \mathsf{x}(Y^{\mathrm{p}}_{F_1})\mathbb{1}_{\{F_1<\infty\}}}\Big]^{k/r}.$$

For $m \in \mathbb{N}$, on $\{F_1 < \infty\}$ we have

$$\mathbb{P}^\circ(\mathsf{x}(Y^{\mathrm{p}}_{F_1}) > 2m) = \mathbb{P}^\circ(\mathsf{x}(Y^{\mathrm{p}}_{F_1}) > 2m, M_1 \geq m) + \mathbb{P}^\circ(\mathsf{x}(Y^{\mathrm{p}}_{F_1}) > 2m, M_1 < m). \tag{2.5.6}$$

For the first term in (2.5.6), we can use electrical analysis to show that this is bounded by a term of order $(e^{-2\lambda}/(1 - e^{-2\lambda}))^m$: Let $u \in \omega^{\mathrm{p}}$ with $\mathsf{x}(u) = l$, and define $s_k := \inf\{n \in \mathbb{N} : \mathsf{x}(Y^{\mathrm{p}}_n) = k\}$ for $k \in \mathbb{Z}$. For $k < l < n$, it follows from formula (4) in **[11]** that

$$P^u_{\omega^{\mathrm{p}},\lambda}(s_k < s_n) \leq \frac{\mathcal{R}^{\mathrm{p}}(u \leftrightarrow \{(n, 0), (n, 1)\})}{\mathcal{R}^{\mathrm{p}}(u \leftrightarrow \{(k, 0), (k, 1)\})},$$

where $\mathcal{R}^{\mathrm{p}}(u \leftrightarrow A)$ for a set A of vertices in ω^{p} denotes the effective resistance between u and A in the random conductance model on ω^{p} with conductances $c^{\mathrm{p}}(e)$ for $e \in E$ with $\omega^{\mathrm{p}}(e) = 1$. As $u \in \omega^{\mathrm{p}}$, there exists a self-avoiding path P from u to $\{(n, 0), (n, 1)\}$ such that the sequence of x-coordinates in this path is nondecreasing. Sending a unit flow from u to $\{(n, 0), (n, 1)\}$ along the path P as in the proof of Lemma 2.5.8, we have

$$\mathcal{R}^{\mathrm{p}}(u \leftrightarrow \{(n, 0), (n, 1)\}) \leq \sum_{e\in P} r(e) \leq \sum_{j=l}^{n} 2e^{-(2j-1)\lambda}(1 - e^{-2\lambda})^{-j}$$

$$= 2e^{\lambda}\Big(\frac{e^{-2\lambda}}{1 - e^{-2\lambda}}\Big)^l \frac{1 - \big(\frac{e^{-2\lambda}}{1-e^{-2\lambda}}\big)^{n-l+1}}{1 - \frac{e^{-2\lambda}}{1-e^{-2\lambda}}}.$$

On the other hand, the sets $\{\langle (j-1, i), (j, i)\rangle, i = 0, 1\}$, $j = k+1, \ldots, l$ form disjoint edge-cutsets for u and $\{(k, 0), (k, 1)\}$. Hence we can use the Nash-Williams inequality to get

$$\mathcal{R}^{\mathrm{p}}(u \leftrightarrow \{(k, 0), (k, 1)\}) \geq \sum_{j=k+1}^{l} \big(2e^{-(2j-1)\lambda}\big) = 2e^{\lambda}e^{-2(k+1)}\frac{1 - e^{-2(l-k)\lambda}}{1 - e^{-2\lambda}}.$$

Combining these two estimates in particular implies that

$$P^u_{\omega^{\mathrm{p}},\lambda}(s_0 < \infty) \leq \Big(\frac{e^{-2\lambda}}{1 - e^{-2\lambda}}\Big)^{\mathsf{x}(u)} \frac{1 - e^{-2\lambda}}{e^{-2\lambda}(1 - e^{-2\lambda})}, \tag{2.5.7}$$

uniformly for all $u \in \omega^{\mathrm{p}}$ with $\mathsf{x}(u) > 0$.

For the second term in (2.5.6), note that $\mathsf{x}(Y^{\mathrm{p}}_{F_1}) > 2m$ and $M_1 < m$ imply that there exist no pre-regeneration points in the subgraph of ω^{p} that is formed by the vertices with x-coordinate between m and $2m$. Due to the Markovian structure of the environment, this event has a probability that decays exponentially in m.
Therefor, both terms on the right-hand side of (2.5.6) can be bounded by a term that decays exponentially in m. With suitable choice of r, this implies the existence of exponential moments of ρ^{p}_1.
Along the same lines that led to (2.5.7), it follows from electrical analysis that

$$P_{\omega^{\mathrm{p}},\lambda}(\min_{l\in\mathbb{N}} \mathsf{x}(Y^{\mathrm{p}}_l) = -k) \leq (e^{-2\lambda}/(1-e^{-2\lambda}))^k,$$

hence $-\min_{l\in\mathbb{N}} \mathsf{x}(Y^{\mathrm{p}}_l)$ also has exponential moments. The first result thus follows from the Cauchy-Schwarz inequality.
To prove the existence of power moments of τ^{p}_1, for $v \in \omega^{\mathrm{p}}$ we write $N(v) := \#\{k \in \mathbb{N}_0 : Y^{\mathrm{p}}_k = v\}$ for the number of times v is visited by $(Y^{\mathrm{p}}_n)_{n\in\mathbb{N}_0}$. We have

$$\begin{aligned}
\mathbb{E}^\circ[(\tau^{\mathrm{p}}_1)^\kappa] &\leq 2^{\kappa-1}\Big(\mathbb{E}^\circ\Big[\Big(\sum_{\substack{v\in\omega^{\mathrm{p}}:\\ \mathsf{x}(v)<0}} N(v)\Big)^\kappa\Big] + \mathbb{E}^\circ\Big[\Big(\sum_{\substack{v\in\omega^{\mathrm{p}}:\\ 0\leq\mathsf{x}(v)<\rho^{\mathrm{p}}_1}} N(v)\Big)^\kappa\Big]\Big)\\
&= 2^{\kappa-1}\Big(\sum_{k=1}^\infty \mathbb{E}^\circ\Big[\mathbb{1}_{\{\min_{l\in\mathbb{N}}\mathsf{x}(Y^{\mathrm{p}}_l)=-k\}}(2k)^\kappa\Big(\frac{1}{2k}\sum_{\substack{v\in\omega^{\mathrm{p}}:\\ -k\leq\mathsf{x}(v)<0}} N(v)\Big)^\kappa\Big]\\
&\qquad + \sum_{n=1}^\infty \mathbb{E}^\circ\Big[\mathbb{1}_{\{\rho^{\mathrm{p}}_1=n\}}(2n)^\kappa\Big(\frac{1}{2n}\sum_{\substack{v\in\omega^{\mathrm{p}}:\\ 0\leq\mathsf{x}(v)<n}} N(v)\Big)^\kappa\Big]\Big)\\
&\leq 2^{\kappa-1}\Big(\sum_{k=1}^\infty \mathbb{E}^\circ\Big[\mathbb{1}_{\{\min_{l\in\mathbb{N}}\mathsf{x}(Y^{\mathrm{p}}_l)=-k\}}(2k)^{\kappa-1}\sum_{\substack{v\in\omega^{\mathrm{p}}:\\ -k\leq\mathsf{x}(v)<0}} N(v)^\kappa\Big]\\
&\qquad + \sum_{n=1}^\infty \mathbb{E}^\circ\Big[\mathbb{1}_{\{\rho^{\mathrm{p}}_1=n\}}(2n)^{\kappa-1}\sum_{\substack{v\in\omega^{\mathrm{p}}:\\ 0\leq\mathsf{x}(v)<n}} N(v)^\kappa\Big]\Big)\\
&\leq 2^{\kappa-1}\Big(\sum_{k=1}^\infty \mathbb{E}^\circ\big[\mathbb{1}_{\{\min_{l\in\mathbb{N}}\mathsf{x}(Y^{\mathrm{p}}_l)=-k\}}(2k)^{2(\kappa-1)}\big]^{1/2}\Big(\sum_{\substack{v\in V:\\ 0\leq\mathsf{x}(v)<n}} \mathbb{E}^\circ\big[\mathbb{1}_{\{v\in\omega^{\mathrm{p}}\}}N(v)^{2\kappa}\big]\Big)^{1/2}\\
&\qquad + \sum_{n=1}^\infty \mathbb{E}^\circ\big[\mathbb{1}_{\{\rho^{\mathrm{p}}_1=n\}}(2n)^{2(\kappa-1)}\big]^{1/2}\Big(\sum_{\substack{v\in V:\\ 0\leq\mathsf{x}(v)<n}} \mathbb{E}^\circ\big[\mathbb{1}_{\{v\in\omega^{\mathrm{p}}\}}N(v)^{2\kappa}\big]\Big)^{1/2}\Big),
\end{aligned}$$

where we have used convexity of the function $x \mapsto x^\kappa$ on $[0,\infty)$ in the second-to-last and the Cauchy-Schwarz inequality in the last inequality. As the estimate of the escape probability in (2.5.4) is uniform in ω^{p}, for any $v \in \omega^{\mathrm{p}}$ we can estimate $N(v)$ by a geometric random variable with success parameter $p^{\mathrm{p}}_{\mathrm{esc}}$. This leads to

$$\sum_{\substack{v\in V:\\ -k\leq\mathsf{x}(v)<0}} \mathbb{E}^\circ\big[\mathbb{1}_{\{v\in\omega^{\mathrm{p}}\}}N(v)^{2\kappa}\big] \leq 2k\mathbb{E}^\circ[N^{2\kappa}] < \infty,$$

where $N \sim \mathrm{geom}(p^{\mathrm{p}}_{\mathrm{esc}})$, and vice versa

$$\sum_{\substack{v\in V:\\ 0\leq\mathsf{x}(v)<n}} \mathbb{E}^\circ\big[\mathbb{1}_{\{v\in\omega^{\mathrm{p}}\}}N(v)^{2\kappa}\big] \leq 2n\mathbb{E}^\circ\big[N^{2\kappa}\big] < \infty.$$

In conjunction with the existence of exponential moments for ρ_1^{p} and $-\min_{l\in\mathbb{N}} \mathsf{x}(Y_l^{\mathrm{p}})$, this implies the second result. $\square$

2.5.5. The proof of the upper bound. We can now conclude the proof of Proposition 2.3.5.

Proof of Proposition 2.3.5. For each $n \in \mathbb{N}$, we have

$$\mathbb{P}(\tau_2 - \tau_1 \geq n) \leq \mathbb{P}\big((\tau_2 - \tau_1)^{\mathcal{B}} \geq n/2\big) + \mathbb{P}\big((\tau_2 - \tau_1)^{\mathrm{traps}} \geq n/2\big).$$

The time spent on the backbone can be neglected due to Lemma 2.5.3. We now estimate the time spent in traps. From Lemma 2.3.4, we infer

$$\mathbb{P}\big((\tau_2 - \tau_1)^{\mathrm{traps}} \geq n\big) = \mathbb{P}^{\circ}\big(\tau_1^{\mathrm{traps}} \geq n \mid X_k \geq 1 \text{ for all } k \in \mathbb{N}\big).$$

If $\mathbf{0}$ is a pre-regeneration point (or just connected to $+\infty$ via a path that does not visit vertices with x-coordinate strictly smaller than 0), the argument that leads to (24) in [**4**] gives

$$P_{\omega,\lambda}(Y_n \neq \mathbf{0} \text{ for all } n \in \mathbb{N}) \geq \frac{(\sum_{k=0}^{\infty} e^{-\lambda k})^{-1}}{e^{\lambda} + 1 + e^{-\lambda}} = \frac{1 - e^{-\lambda}}{e^{\lambda} + 1 + e^{-\lambda}} =: p_{\mathrm{esc}}.$$

Integration with respect to P_p° gives

$$p_{\mathrm{esc}} \leq \mathbb{P}^{\circ}(Y_n \neq \mathbf{0} \text{ for all } n \geq 1) \leq 1.$$

Notice that the same bound holds when $\mathbb{P}^{\circ}$ is replaced by $\mathbb{P}$. Thus

$$\mathbb{P}^{\circ}\big(\tau_1^{\mathrm{traps}} \geq n \mid X_k \geq 1 \text{ for all } k \in \mathbb{N}\big) \leq \frac{1}{p_{\mathrm{esc}}} \mathbb{P}^{\circ}\big(\tau_1^{\mathrm{traps}} \geq n, X_k \geq 1 \text{ for all } k \in \mathbb{N}\big).$$

Analogously, when estimating $\mathbb{P}(\tau_1 \geq n)$, the time spent on the backbone can be neglected by Lemma 2.5.10, so that it suffices to bound $\mathbb{P}(\tau_1^{\mathrm{traps}} \geq n)$ in this case. We shall only estimate $\mathbb{P}^{\circ}(\tau_1^{\mathrm{traps}} \geq n, X_k \geq 1 \text{ for all } k \in \mathbb{N})$ as $\mathbb{P}(\tau_1^{\mathrm{traps}} \geq n)$ can be estimated similarly. To this end, we consider $(Y_n)_{n\in\mathbb{N}_0}$ and $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ as constructed in Section 2.5.4. Further, we use the family T_{ij}^{ann}, $i \in \mathbb{Z}$, $j \in \mathbb{N}$ of random variables introduced in Lemma 2.5.6. By construction, the number of times $(Y_n)_{n\in\mathbb{N}_0}$ visits any node in ω which is not in the interior of a trap piece can be bounded by the number of times $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$ visits the corresponding node in ω^{p}. This holds in particular for all trap entrances. By Lemma 2.5.9, there exist regeneration points of $(Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0}$. These also are regeneration points for $(Y_n)_{n\in\mathbb{N}_0}$. We have

$$\mathbb{P}^{\circ}\big(\tau_1^{\mathrm{traps}} \geq n, X_k \geq 1 \text{ for all } k \in \mathbb{N}\big) \leq \mathbb{P}^{\circ}\bigg(\sum_{i=1}^{T}\sum_{j=1}^{V_i} T_{ij} \geq n\bigg) \leq \mathbb{P}^{\circ}\bigg(\sum_{i=1}^{\rho_1^{\mathrm{p}}}\sum_{j=1}^{\tau_1^{\mathrm{p}}} T_{ij}^{\mathrm{ann}} \geq n\bigg),$$

where T is the number of traps in $[0, \rho_1)$, V_i is the number of visits to the ith trap, T_{ij} is the time $(Y_n)_{n\in\mathbb{N}_0}$ spends during the jth excursion into the ith trap, and $(T_{ij}^{\mathrm{ann}})_{i,j\in\mathbb{N}}$ is a family of random variables independent of $(\omega^{\mathrm{p}}, (Y_n^{\mathrm{p}})_{n\in\mathbb{N}_0})$ such that the T_{ij}^{ann}, $i, j \in \mathbb{N}$ are independent given the family $(\ell_i)_{i\in\mathbb{N}}$ with T_{ij}^{ann} being distributed as the duration of one excursion of $(Y_n)_{n\in\mathbb{N}_0}$ under $P_{\omega,\lambda}$ into a trap of length ℓ_i. Since $(\rho_1^{\mathrm{p}}, \tau_1^{\mathrm{p}})$ and $(T_{ij}^{\mathrm{ann}})_{i,j\in\mathbb{N}}$ are independent, we can write this as

$$\mathbb{P}^{\circ}\bigg(\sum_{i=1}^{\rho_1^{\mathrm{p}}}\sum_{j=1}^{\tau_1^{\mathrm{p}}} T_{ij}^{\mathrm{ann}} \geq n\bigg) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \mathbb{P}^{\circ}\bigg(\rho_1^{\mathrm{p}} = k, \tau_1^{\mathrm{p}} = l, \sum_{i=1}^{k}\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n\bigg)$$

$$= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \mathbb{P}^{\circ}\big(\rho_1^{\mathrm{p}} = k, \tau_1^{\mathrm{p}} = l\big) \cdot \mathbb{P}\bigg(\sum_{i=1}^{k}\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n\bigg). \tag{2.5.8}$$

First look at $\mathbb{P}\big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n\big)$ for fixed i and $l \in \mathbb{N}$. We write this as

$$\mathbb{P}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n\Big) = \mathbb{P}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n, \ell_i < m_0 \vee m_1\Big) + \mathbb{P}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n, \ell_i \geq m_0 \vee m_1\Big),$$

with m_0, m_1 as in Lemma 2.5.6. With $P_{m,\lambda}$ and T_{ij}^{qu}, $i,j \in \mathbb{N}$ as in Lemma 2.5.5, Markov's inequality and the convexity of $x \mapsto x^{\alpha+1}$ on $[0,\infty)$ give

$$\begin{aligned}
\mathbb{P}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n, \ell_i < m_0 \vee m_1\Big) &= \sum_{m=1}^{m_0\vee m_1 - 1} \mathrm{P}_p(\ell_i = m) P_{m,\lambda}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu}} \geq n\Big) \\
&\leq (m_0 \vee m_1) \max_{m\in\{1,\dots,m_0\vee m_1 - 1\}} E_{m,\lambda}\Big[\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{qu}}\Big)^{\alpha+1}\Big] n^{-(\alpha+1)} \\
&\leq (m_0 \vee m_1) \max_{m\in\{1,\dots,m_0\vee m_1 - 1\}} E_{m,\lambda}\Big[l^{\alpha} \sum_{j=1}^{l} (T_{ij}^{\mathrm{qu}})^{\alpha+1}\Big] n^{-(\alpha+1)} \\
&= (m_0 \vee m_1) l^{\alpha+1} n^{-(\alpha+1)} \max_{m\in\{1,\dots,m_0\vee m_1 - 1\}} E_{m,\lambda}\big[(T_{i1}^{\mathrm{qu}})^{\alpha+1}\big].
\end{aligned}$$

Let $N(k)$ be the number of times the walk $(S_n)_{n\in\mathbb{N}_0}$ visits vertex $k \in \{1,\dots,m\}$. Note that in order to describe T_{i1}^{qu}, we also need to take lazy steps into account. This means that, under $P_{m,\lambda}$, we have the following identity in law,

$$T_{i1}^{\mathrm{qu}} \overset{\mathrm{law}}{=} \sum_{k=1}^{m} \sum_{l=1}^{N(k)} (1 + Z_{k,l})$$

where $N(k)$ has distribution $\mathrm{geom}(\mathsf{e}_k)$ and the $Z_{k,l}$'s are a family of independent random variables, independent of $(N(1),\dots,N(k))$, with distribution $\mathrm{geom}\big(\frac{e^{\lambda}+e^{-\lambda}}{e^{\lambda}+1+e^{-\lambda}}\big)$ for $k = 1,\dots,m-1$, $l \in \mathbb{N}$ and $\mathrm{geom}\big(\frac{e^{-\lambda}}{e^{\lambda}+1+e^{-\lambda}}\big)$ for $k = m$, $l \in \mathbb{N}$, respectively.
Since $m < m_0 \vee m_1$ and the escape probability e_k is nonincreasing in k, we can bound e_k by $\mathsf{e}_{m_0\vee m_1}$ for all $k \in \{1,\dots,m\}$. We use this to stochastically bound $N(k)$. In combination with the convexity of $x \mapsto x^{\alpha+1}$ on $[0,\infty)$ this leads to

$$\begin{aligned}
E_{m,\lambda}\big[(T_{i1}^{\mathrm{qu}})^{\alpha+1}\big] &= E_{m,\lambda}\Big[\Big(\sum_{k=1}^{m}\sum_{l=1}^{N(k)} (1+Z_{k,l})\Big)^{\alpha+1}\Big] \leq m^{\alpha} \sum_{k=1}^{m} E_{m,\lambda}\big[N(k)^{\alpha+1}\big] E_{m,\lambda}\big[(1+Z_{k,m})^{\alpha+1}\big] \\
&\leq (m_0 \vee m_1)^{\alpha+1} E_{m,\lambda}[N^{\alpha+1}] E_{m,\lambda}[(1+Z)^{\alpha+1}]
\end{aligned}$$

where $N \sim \mathrm{geom}(\mathsf{e}_{m_0\vee m_1})$ and $Z \sim \mathrm{geom}\big(\frac{e^{-\lambda}}{e^{\lambda}+1+e^{-\lambda}}\big)$. Thus

$$\max_{m\in\{1,\dots,m_0\vee m_1 - 1\}} E_{m,\lambda}\big[(T_{i1}^{\mathrm{qu}})^{\alpha+1}\big] \leq c(m_0, m_1, \lambda) = c(\lambda)$$

for some constant $c(\lambda)$. Combining this with the estimate for $\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}}$ in the case of traps of length larger or equal to $m_0 \vee m_1$ from Lemma 2.5.6, we get that there exists $d' = d'(p,\lambda) > 0$ such that

$$\mathbb{P}\Big(\sum_{j=1}^{l} T_{ij}^{\mathrm{ann}} \geq n\Big) \leq d'\Big(l^{\alpha+1} n^{-(\alpha+1)} + l^{\alpha+1} n^{-\alpha} + l e^{-\mu \frac{n}{6 l r_\lambda}}\Big).$$

We further conclude

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{k}\sum_{j=1}^{l}T_{ij}^{\mathrm{ann}}\geq n\Big) &\leq k\mathbb{P}\Big(\sum_{j=1}^{l}T_{1j}^{\mathrm{ann}}\geq \frac{n}{k}\Big)\\ &\leq kd'\Big(l^{\alpha+1}\Big(\frac{n}{k}\Big)^{-(\alpha+1)}+l^{\alpha+1}\Big(\frac{n}{k}\Big)^{-\alpha}+le^{-\mu\frac{n}{6lr_\lambda k}}\Big)\\ &\leq k^{\alpha+2}l^{\alpha+1}d'\big(o(n^{-\alpha})+n^{-\alpha}\big)+kld'e^{-\mu\frac{n}{6lr_\lambda k}}. \qquad (2.5.9)\end{aligned}$$

Note that when estimating τ_1 under $\mathbb{P}$, all calculations using Lemma 2.5.6 involve an additional factor of $\log n$. Combining (2.5.8) and (2.5.9), we get

$$\begin{aligned}\mathbb{P}^\circ\big(\tau_1^{\mathrm{traps}}\geq n\big) \leq d'&\sum_{k,l=1}^{\infty}\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k,\tau_1^{\mathrm{p}}=l\big)k^{\alpha+2}l^{\alpha+1}n^{-\alpha}(1+o_n(1))\\ +\,d'&\sum_{k,l=1}^{\infty}\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k,\tau_1^{\mathrm{p}}=l\big)kle^{-\mu\frac{n}{6lr_\lambda k}}. \qquad (2.5.10)\end{aligned}$$

For $k,l\in\mathbb{N}$, we write

$$\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k,\tau_1^{\mathrm{p}}=l\big)=\mathbb{P}^\circ\big(\tau_1^{\mathrm{p}}=l\big)\cdot\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k\mid\tau_1^{\mathrm{p}}=l\big).$$

As the second factor vanishes for $k>l$, we get

$$\begin{aligned}\sum_{k,l=1}^{\infty}\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k,\tau_1^{\mathrm{p}}=l\big)k^{\alpha+2}l^{\alpha+1} &= \sum_{l=1}^{\infty}\mathbb{P}^\circ\big(\tau_1^{\mathrm{p}}=l\big)l^{\alpha+1}\sum_{k=1}^{l}\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k\mid\tau_1^{\mathrm{p}}=l\big)k^{\alpha+2}\\ &\leq\sum_{l=1}^{\infty}\mathbb{P}^\circ\big(\tau_1^{\mathrm{p}}=l\big)l^{2\alpha+4}.\end{aligned}$$

Hence, it follows from Lemma 2.5.10 that the first sum on the right-hand side of (2.5.10) is bounded by a constant times $n^{-\alpha}$. For τ_1 under $\mathbb{P}$, this becomes a constant times $n^{-\alpha}\log n$. It also follows from Lemma 2.5.10 and Markov's inequality that for any $\kappa>0$

$$\begin{aligned}\sum_{k,l=1}^{\infty}\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k,\tau_1^{\mathrm{p}}=l\big)kle^{-\mu\frac{n}{6lr_\lambda k}} &= \sum_{l=1}^{\infty}\mathbb{P}^\circ(\tau_1^{\mathrm{p}}=l)l\sum_{k=1}^{l}\mathbb{P}^\circ\big(\rho_1^{\mathrm{p}}=k\mid\tau_1^{\mathrm{p}}=l\big)ke^{-\mu\frac{n}{6lr_\lambda k}}\\ &\leq\sum_{l=1}^{\infty}\mathbb{P}^\circ\big(\tau_1^{\mathrm{p}}=l\big)l^3e^{-\mu\frac{n}{6l^2r_\lambda}}\leq\mathbb{E}^\circ\big[(\tau_1^{\mathrm{p}})^\kappa\big]\sum_{l=1}^{\infty}l^{-\kappa+3}e^{-\mu\frac{n}{6l^2r_\lambda}}.\end{aligned}$$

Setting $l^*:=\sqrt{\frac{\mu}{6r_\lambda(\alpha+1)}\frac{n}{\log n}}$ we get

$$\begin{aligned}\sum_{l=1}^{\infty}l^{-\kappa+3}e^{-\mu\frac{n}{6l^2r_\lambda}} &= \sum_{l\leq l^*}l^{-\kappa+3}e^{-\mu\frac{n}{6l^2r_\lambda}}+\sum_{l>l^*}l^{-\kappa+3}e^{-\mu\frac{n}{6l^2r_\lambda}}\\ &\leq e^{-\mu\frac{n}{6r_\lambda(l^*)^2}}\sum_{l=1}^{\infty}l^{-\kappa+3}+(l^*)^{\frac{-\kappa+3}{2}}\sum_{l=1}^{\infty}l^{\frac{-\kappa+3}{2}}=o(n^{-\alpha})\end{aligned}$$

for sufficiently large κ. $\square$

2.6. Displacement results

The main results of this thesis concern the speed of biased random walk in the sub-ballistic regime. If the bias is critical ($\lambda = \lambda_c$), we see that X_n is of order $n/\log n$. This is in alignment with simulation results for biased random walk on the infinite cluster of supercritical bond percolation in $\mathbb{Z}^d$ in [**19**].

THEOREM 2.6.1. *In the case* $\lambda = \lambda_c$, *there exist constants* $0 < a < b < \infty$ *such that*

$$\lim_{n\to\infty} \mathbb{P}\big(\tfrac{X_n}{n/\log n} \in [a,b]\big) = 1.$$

The proof of Theorem 2.6.1 is based on the fine estimates for the tails of $\tau_2 - \tau_1$ in Proposition 2.3.5. Less accurate estimates for the tails of the regeneration times which were derived in [**23**] revealed a second phase transition at $\lambda = \lambda_c/2$. Namely, a central limit theorem for $(X_n)_{n\in\mathbb{N}_0}$ with square-root scaling holds if and only if $\lambda < \lambda_c/2$. More precisely, let us denote a standard one-dimensional Brownian motion starting at the origin by $(B(t))_{t\in[0,1]}$, and define

$$B_n(t) := \frac{1}{\sqrt{n}}(X_{\lfloor nt\rfloor} - \lfloor nt\rfloor \overline{\mathrm{v}}), \quad 0 \le t \le 1.$$

The processes $(B_n(t))_{t\in[0,1]}$ and $(B(t))_{t\in[0,1]}$ take values in the Skorohod space $D[0,1]$ of real-valued functions that are right-continuous with finite left limits.

PROPOSITION 2.6.2 (Theorem 2.6 in [**23**]). *Let* $\lambda \in (0, \lambda_c/2)$. *Then there is a constant* $\sigma = \sigma(p,\lambda) \in (0,\infty)$ *such that*

$$(B_n(t))_{t\in[0,1]} \xrightarrow{\mathrm{d}} (\sigma B(t))_{t\in[0,1]} \quad \textit{under } \mathbb{P}, \tag{2.6.1}$$

If $\lambda \ge \lambda_c/2$, *then* (2.6.1) *fails to hold.*

With the finer tail estimates derived in Proposition 2.3.5, we can determine the fluctuations of $(X_n)_{n\in\mathbb{N}_0}$ in the remaining parameter range $\lambda \in [\lambda_c/2, \infty)$.

THEOREM 2.6.3. *Suppose that* $\lambda \ge \lambda_c/2$, $\lambda \ne \lambda_c$.
(a) Let $\lambda = \lambda_c/2$, *i.e.,* $\alpha = 2$. *Then the laws of* $\big(\frac{X_n - n\overline{\mathrm{v}}}{\sqrt{n\log n}}\big)_{n\ge 2}$ *under* $\mathbb{P}$ *are tight.*
(b) Let $\lambda \in (\lambda_c/2, \lambda_c)$, *i.e.,* $\alpha \in (1,2)$. *Then the laws of* $\big(\frac{X_n - n\overline{\mathrm{v}}}{n^{1/\alpha}}\big)_{n\in\mathbb{N}}$ *under* $\mathbb{P}$ *are tight.*
(c) Let $\lambda > \lambda_c$, *i.e.,* $\alpha \in (0,1)$. *Then the laws of* $\big(\frac{X_n}{n^\alpha}\big)_{n\in\mathbb{N}}$ *under* $\mathbb{P}$ *are tight.*

In all three cases overed by Theorem 2.6.3, it is doubtful whether tightness can be strenghtened to convergence in distribution due to a lack of regular variation of the tails of the regeneration times, see Lemma 2.5.6 and the proof thereof. Numerical simulations in the case of supercritical bond percolation on $\mathbb{Z}^d$ also indicate cyclic behaviour, cf. [**43**]. Instead, only convergence along certain subsequences as found for biased random walk on Galton-Watson trees can be expected, cf. [**8**] and [**14**].
From Lemma 2.3.4, we infer that the τ_n, $n \in \mathbb{N}$ are the points of a delayed renewal process on the integers. The corresponding *renewal counting process* and *first passage times*, we denote by

$$k(n) := \max\big\{k \in \mathbb{N}_0 : \tau_k \le n\big\} \quad \text{and} \quad \nu(n) := k(n) + 1,$$

respectively, where $n \in \mathbb{N}_0$. Notice that $k(n) = \max\{k \in \mathbb{N}_0 : \rho_k \le X_n\}$, $n \in \mathbb{N}_0$.
To infer Theorems 2.6.1 and 2.6.3 from Proposition 2.3.5, we shall choose a sequence $(\xi_k)_{k\in\mathbb{N}}$ of independent random variables of whom ξ_k, $k \ge 2$ are i.i.d. with $\tau_2 - \tau_1 \preccurlyeq \xi_2$ and $\mathbb{P}(\xi_2 > n) \sim dn^{-\alpha}$ as $n \to \infty$ (where d is chosen as in Proposition 2.3.5). Then the law of ξ_2 is in the (normal) domain of attraction of an α-stable law. From general theory it then follows that, after a suitable renormalisation, the first passage times $\nu_\xi(t) := \inf\big\{k \in \mathbb{N} : \sum_{i=1}^k \xi_i > t\big\}$ converge in distribution as $t \to \infty$. This will imply tightness of the first passage times $\nu(n)$ with the

same renormalisation. From this, we shall derive the dual results for X_n which translate into the statements of Theorems 2.6.1 and 2.6.3. We begin with the proof of the results in the subballistic regimes.

Proof of Theorem 2.6.1 and Theorem 2.6.3(c). Suppose that $\lambda \geq \lambda_c$ so that $\alpha \in (0,1]$. Let $a_n := n^\alpha$ if $\alpha \in (0,1)$ and $a_n := n/\log n$ if $\alpha = 1$. For $n \in \mathbb{N}$, we have

$$\frac{\rho_{k(n)}}{a_n} \leq \frac{X_n}{a_n} \leq \frac{\rho_{\nu(n)}}{a_n} = \frac{\rho_{\nu(n)}}{\nu(n)} \frac{\nu(n)}{a_n}. \tag{2.6.2}$$

Since $\nu(n) \to \infty$ $\mathbb{P}$-a.s. as $n \to \infty$, Lemma 2.3.4 and the strong law of large numbers imply

$$\frac{\rho_{\nu(n)}}{\nu(n)} = \frac{1}{\nu(n)} \sum_{k=1}^{\nu(n)} (\rho_k - \rho_{k-1}) \to \mathbb{E}[\rho_2 - \rho_1] \quad \mathbb{P}\text{-a.s.}$$

Using Proposition 2.3.5, we can find independent random variables η_k, $k \in \mathbb{N}$ and ξ_k, $k \in \mathbb{N}$ such that $\eta_1, \eta_2, \ldots$ are i.i.d. and $\xi_2, \xi_3, \ldots$ are i.i.d. and such that $\eta_k \preccurlyeq \tau_k - \tau_{k-1} \preccurlyeq \xi_k$ for all $k \in \mathbb{N}$ and

$$\mathbb{P}(\eta_1 > n) \sim cn^{-\alpha} \quad \text{and} \quad \mathbb{P}(\xi_2 > n) \sim dn^{-\alpha} \qquad \text{as } n \to \infty.$$

Further, we may choose ξ_1 independent of $\xi_2, \xi_3, \ldots$ such that $\mathbb{P}(\xi_1 > n) \sim dn^{-\alpha} \log n$ as $n \to \infty$. We set $\nu_\eta(n) := \inf\{k \in \mathbb{N} : \sum_{i=1}^k \eta_i > n\}$ and $\nu_\xi(n) := \inf\{k \in \mathbb{N} : \sum_{i=1}^k \xi_i > n\}$. Then it holds that $\nu_\xi(n) \preccurlyeq \nu(n) \preccurlyeq \nu_\eta(n)$ for all $n \in \mathbb{N}_0$. Furthermore, Theorem 3a in [**12**] says that there is an α-stable subordinator $(Y_\alpha(t))_{t \geq 0}$ with Laplace exponent $\log \mathbb{E}[\exp(-sY_\alpha(t))] = -ts^\alpha$ for $s, t \geq 0$ such that

$$a_n^{-1} \nu_\eta(n) \xrightarrow{\mathrm{d}} c_\eta X_\alpha \quad \text{and} \quad a_n^{-1} \nu_\xi(n) \xrightarrow{\mathrm{d}} c_\xi X_\alpha \tag{2.6.3}$$

where $X_\alpha = \sup\{t \geq 0 : Y_\alpha(t) \leq 1\}$ and $0 < c_\xi \leq c_\eta < \infty$. (Notice that other than in [**12**], here we allow ξ_1 to have a distribution different than that of $\xi_2, \xi_3, \ldots$, but the contribution of the first step vanishes as $n \to \infty$.) The difference of upper and lower bound in (2.6.2) satisfies

$$\frac{\rho_{\nu(n)}}{a_n} - \frac{\rho_{k(n)}}{a_n} = \frac{\rho_{\nu(n)} - \rho_{\nu(n)-1}}{\nu(n)} \frac{\nu(n)}{a_n} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty. \tag{2.6.4}$$

Indeed, the first factor on the right-hand side converges to 0 $\mathbb{P}$-a.s. as $n \to \infty$ due to Lemma 2.3.4(b) and [**26**, Theorem 1.2.3(i)] while the family of laws corresponding to the second factor are tight by (2.6.3). Consequently, the difference in (2.6.4) converges to 0 in distribution and thus in $\mathbb{P}$-probability.

Now suppose $\alpha = 1$. Then $Y_1(t) = t$ $\mathbb{P}$-a.s. and hence $X_1 = 1$ $\mathbb{P}$-a.s. The convergence in (2.6.3) thus is in fact convergence in probability. This completes the proof of Theorem 2.6.1.

Finally, if $0 < \alpha < 1$, then (2.6.3) and $\nu_\xi(n) \preccurlyeq \nu(n) \preccurlyeq \nu_\eta(n)$ for all $n \in \mathbb{N}_0$ imply that the family of laws of $(\nu(n)/n^\alpha)_{n \in \mathbb{N}}$ is tight. From (2.6.2) and (2.6.4) we conclude that this carries over to the family of laws of $(X_n/n^\alpha)_{n \in \mathbb{N}}$. ☐

We now turn to the proof of the main results for noncritical biases.

Proof of Theorem 2.6.3. We prove (a) and (b) simultaneously. Let $a_n := n^{1/\alpha}$ in the case $\alpha \in (1,2)$ and $a_n := \sqrt{n \log n}$ if $\alpha = 2$. For $n \in \mathbb{N}$, we have

$$\frac{\rho_{k(n)} - n\overline{\mathrm{v}}}{a_n} \leq \frac{X_n - n\overline{\mathrm{v}}}{a_n} \leq \frac{\rho_{\nu(n)} - n\overline{\mathrm{v}}}{a_n}.$$

By the strong law of large numbers, $\nu(n)/n \to 1/\mathbb{E}[\tau_2 - \tau_1] \in (0, \infty)$ $\mathbb{P}$-a. s. This together with Lemma 2.3.4 and [**26**, Theorem 1.2.3(i)] implies $(\rho_{\nu(n)} - \rho_{k(n)})/a_n \to 0$ $\mathbb{P}$-a. s. On the other hand,

$$\frac{\rho_{\nu(n)} - n\overline{\mathrm{v}}}{a_n} = \frac{\rho_{\nu(n)} - \nu(n)\mathbb{E}[\rho_2 - \rho_1]}{a_n} + \frac{\nu(n)\mathbb{E}[\rho_2 - \rho_1] - n\overline{\mathrm{v}}}{a_n}.$$

The first summand converges to 0 $\mathbb{P}$-a. s. by [**26**, Theorem 1.2.3(ii)] if $\alpha \in (1, 2)$ and it converges to 0 in $\mathbb{P}$-probability by [**26**, Theorem 1.3.1] if $\alpha = 2$. It thus remains to check tightness of the family of laws of

$$\frac{\nu(n)\mathbb{E}[\rho_2 - \rho_1] - n\overline{\mathrm{v}}}{a_n} = \mathbb{E}[\rho_2 - \rho_1] \frac{\nu(n) - n/\mathbb{E}[\tau_2 - \tau_1]}{a_n}, \quad n \in \mathbb{N}.$$

For this, uniform integrability of the sequence $(a_n^{-1}(\nu(n) - n/\mathbb{E}[\tau_2 - \tau_1]))_{n\in\mathbb{N}}$ is sufficient. It thus remains to refer to Proposition A.1.1 in the appendix. □

2.6.1. Almost sure behaviour. From our fine estimates for the tails of the regeneration times of the walk we can further infer $\mathbb{P}$-almost sure properties of the walk. Namely, under suitable renormalisation, a law of iterated logarithm holds.

Usually, the notion of a *law of iterated logarithm* is tied to the following statement which can e. g. be found in [**7**, Theorem VII.31.1]. Let $A_1, A_2, \dots$ be a family of i.i.d. real-valued random variables on a probability space $(\Omega, \mathcal{A}, P)$ with mean 0 and finite variance $\sigma^2 \in (0, \infty)$. Then, P-almost surely

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n A_k}{\sqrt{2n \log\log n}} = \sigma.$$

Based on our fine estimates for the i.i.d. sequence of regeneration times $(\tau_{k+1} - \tau_k)_{k\in\mathbb{N}}$, we want to derive a similar result for $(Y_n)_{n\in\mathbb{N}_0}$. In our case, however, for $\lambda \geq \lambda_{\mathrm{c}}/2$ the second moment of the time interval $\tau_2 - \tau_1$ is infinite. Therefore, a different renormalisation scheme is required. In [**16**], for a sequence $B_1, B_2, \dots$ of i.i.d. real-valued random variables on a probability space $(\Omega, \mathcal{A}, P)$ that are distributed according to a symmetric, α-stable law where $\alpha \in (0, 2)$, it was shown that

$$\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^n B_k}{n^{1/\alpha}} \right|^{1/\log\log n} = e^{1/\alpha}$$

almost surely under P. Roughly speaking, this says that for large n, the absolute value of the partial sums of $(B_k)_{k\in\mathbb{N}}$ takes order $(n \log n)^{1/\alpha}$. We will make use of the following refinement which due to Li and Chen [**34**].

Lemma 2.6.4 (Theorems 2.3 and 2.5 in [**34**]). *let $B, B_1, B_2, \dots$ be a family of i.i.d. real-valued random variables on a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$.

(a) Let $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$. Then

$$\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^n B_k}{n^{1/\alpha}} \right|^{1/\log\log n} = e^{\beta} \quad \mathbb{P}\text{-almost surely}$$

if and only if $\beta = \inf\big\{b \in \mathbb{R} : \mathbb{E}\big[\frac{|B|^\alpha}{(\log(e\vee|B|))^{b\alpha}}\big] < \infty\big\}$.

(b) Let $\alpha \in (1, 2)$, $\beta \in \mathbb{R}$. Then

$$\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^n B_k}{n^{1/\alpha}} \right|^{1/\log\log n} = e^{\beta} \quad \mathbb{P}\text{-almost surely}$$

if and only if $\mathbb{E}(B) = 0$ *and* $\beta = \inf\big\{b \in \mathbb{R} : \mathbb{E}\big[\frac{|B|^\alpha}{(\log(e\vee|B|))^{b\alpha}}\big] < \infty\big\}$.

In conjunction with Proposition 2.3.5, this leads to the following law(s) of iterated logarithm.

Theorem 2.6.5. *Suppose that $\lambda > \lambda_{\mathrm{c}}/2$, $\lambda \neq \lambda_{\mathrm{c}}$.*

(A) Let $\lambda \in (\lambda_c/2, \lambda_c)$*, i.e.,* $\alpha \in (1,2)$*. Then*

$$\limsup_{n\to\infty} \left|\frac{X_n - n\overline{v}}{n^{1/\alpha}}\right|^{1/\log\log n} = e^{1/\alpha} \quad \mathbb{P}\text{-almost surely.}$$

(B) Let $\lambda > \lambda_c$*, i.e.,* $\alpha \in (0,1)$*. Then*

$$\liminf_{n\to\infty} \left|\frac{X_n}{n^{\alpha}}\right|^{1/\log\log n} = e^{-1} \quad \mathbb{P}\text{-almost surely.}$$

PROOF OF THEOREM 2.6.5(A). Suppose that $\alpha \in (1,2)$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{X_n - n\overline{v}}{n^{1/\alpha}} &= \frac{X_n - \nu(n)\mathbb{E}[\rho_2 - \rho_1]}{n^{1/\alpha}} + \overline{v}\frac{\nu(n)\mathbb{E}[\tau_2 - \tau_1] - n}{n^{1/\alpha}} \\ &=: A_n + \overline{v}\frac{\nu(n)\mathbb{E}[\tau_2 - \tau_1] - n}{n^{1/\alpha}}. \end{aligned} \tag{2.6.5}$$

For $a, b \in \mathbb{R}$, the triangle inequality and monotonicity of the p-norms on $\mathbb{R}^2$ imply

$$\begin{aligned} &|a+b|^{1/\log\log n} - |b|^{1/\log\log n} \\ &\quad\le \left(\left(|a|^{1/\log\log n}\right)^{\log\log n} + \left(|b|^{1/\log\log n}\right)^{\log\log n}\right)^{1/\log\log n} - |b|^{1/\log\log n} \\ &\quad\le \left\| \begin{pmatrix} |a|^{1/\log\log n} \\ |b|^{1/\log\log n} \end{pmatrix} \right\|_1 - |b|^{1/\log\log n} \\ &\quad= |a|^{1/\log\log n}, \end{aligned}$$

where $\|\cdot\|_1$ denotes the 1-norm on $\mathbb{R}^2$, $\|x\|_1 := |x_1| + |x_2|$, and vice versa

$$|a+b|^{1/\log\log n} - |b|^{1/\log\log n} \ge -|a|^{1/\log\log n}.$$

Hence

$$\left||a+b|^{1/\log\log n} - |b|^{1/\log\log n}\right| \le |a|^{1/\log\log n}. \tag{2.6.6}$$

We write A_n as

$$A_n = \frac{X_n - \rho_{\nu(n)}}{n^{1/\alpha}} + \frac{\rho_{\nu(n)} - \nu(n)\mathbb{E}[\rho_2 - \rho_1]}{n^{1/\alpha}}.$$

The second summand converges to 0 $\mathbb{P}$-a.s. by [**26**, Theorem 1.2.3(ii)]. On the other hand, the absolute value of the first summand is bounded by $(\rho_{\nu(n)} - \rho_{k(n)})/n^{1/\alpha}$. By the strong law of large numbers, $\nu(n)/n \to 1/\mathbb{E}[\tau_2 - \tau_1] \in (0,\infty)$ $\mathbb{P}$-a.s. Together with Lemma 2.3.4 and [**26**, Theorem 1.2.3(i)], this implies that $(X_n - \rho_{\nu(n)})/n^{1/\alpha} \to 0$ $\mathbb{P}$-a.s. Hence, $A_n \to 0$ $\mathbb{P}$-a.s. and as $1/\log\log n < 1$ for sufficiently large n, $|A_n|^{1/\log\log n} \to 0$ $\mathbb{P}$-a.s. In conjunction with (2.6.5) and (2.6.6), this implies

$$\limsup_{n\to\infty} \left|\frac{X_n - n\overline{v}}{n^{1/\alpha}}\right|^{1/\log\log n} = \limsup_{n\to\infty} \left|\overline{v}\frac{\nu(n)\mathbb{E}[\tau_2 - \tau_1] - n}{n^{1/\alpha}}\right|^{1/\log\log n} \quad \mathbb{P}\text{-almost surely.}$$

Due to the fact that $\nu(n)/n \to 1/\mathbb{E}[\tau_2 - \tau_1] \in (0,\infty)$ $\mathbb{P}$-a.s. by the strong law of large numbers, we get

$$\begin{aligned}\limsup_{n\to\infty} \left|\frac{X_n - n\overline{v}}{n^{1/\alpha}}\right|^{1/\log\log n} &= \limsup_{n\to\infty} \left|\overline{v}\left(\frac{\nu(n)}{n}\right)^{1/\alpha}\right|^{1/\log\log n} \left|\frac{\nu(n)\mathbb{E}[\tau_2 - \tau_1] - n}{\nu(n)^{1/\alpha}}\right|^{1/\log\log n} \\ &= \limsup_{n\to\infty} \left|\frac{\nu(n)\mathbb{E}[\tau_2 - \tau_1] - n}{\nu(n)^{1/\alpha}}\right|^{1/\log\log n} \quad \mathbb{P}\text{-almost surely.}\end{aligned}$$

For $n \in \mathbb{N}$, set $S_n := \sum_{k=1}^{n} \tau_{k+1} - \tau_k$. It follows from the definition of $k(n)$ and $\nu(n) = k(n)+1$ that $S_{k(n)} \le n \le S_{\nu(n)}$. Hence, the usual sandwich estimate with the renewal counting process and the first passage time gives

$$\frac{\nu(n)\mathbb{E}[\tau_2-\tau_1] - S_{\nu(n)}}{\nu(n)^{1/\alpha}} \le \frac{\nu(n)\mathbb{E}[\tau_2-\tau_1] - n}{\nu(n)^{1/\alpha}} \le \frac{\nu(n)\mathbb{E}[\tau_2-\tau_1] - S_{k(n)}}{\nu(n)^{1/\alpha}}. \tag{2.6.7}$$

Note that the right-hand side of (2.6.7) can be written as

$$\frac{\nu(n)\mathbb{E}[\tau_2-\tau_1] - S_{k(n)}}{\nu(n)^{1/\alpha}} = \frac{k(n)\mathbb{E}[\tau_2-\tau_1] - S_{k(n)}}{k(n)^{1/\alpha}}\left(\frac{k(n)}{\nu(n)}\right)^{1/\alpha} + \frac{\mathbb{E}[\tau_2-\tau_1]}{\nu(n)^{1/\alpha}},$$

and that due to the fact that $\nu(n) = k(n)+1 \to \infty$ $\mathbb{P}$-a.s., we have $k(n)/\nu(n) \to 1$ and $\mathbb{E}[\tau_2-\tau_1]/\nu(n)^{1/\alpha} \to 0$ $\mathbb{P}$-a.s.
From Lemma 2.6.4, it follows that

$$\limsup_{n\to\infty} \left|\frac{S_n - n\mathbb{E}[\tau_2-\tau_1]}{n^{1/\alpha}}\right|^{1/\log\log n} = e^{\beta} \quad \mathbb{P}\text{-almost surely}$$

where $\beta = \inf\big\{b \in \mathbb{R} : \mathbb{E}\big[\frac{(\tau_2-\tau_1)^\alpha}{(\log(e\vee(\tau_2-\tau_1)))^{b\alpha}}\big] < \infty\big\}$. Our fine estimates for the tails of the regeneration times from Proposition 2.3.5 imply $\beta = 1/\alpha$.
As $\tau_{k+1} - \tau_k \ge 1$ for all $k \in \mathbb{N}$, it follows that the sets $\{S_n - n\mathbb{E}[\tau_2-\tau_1] : n \in \mathbb{N}\}$ and $\{S_{\nu(n)} - \nu(n)\mathbb{E}[\tau_2-\tau_1] : n \in \mathbb{N}\}$ coincide. Therefor,

$$\begin{aligned}\limsup_{n\to\infty} \left|\frac{S_{\nu(n)} - \nu(n)\mathbb{E}[\tau_2-\tau_1]}{\nu(n)^{1/\alpha}}\right|^{1/\log\log\nu(n)} &= \limsup_{n\to\infty} \left|\frac{S_n - n\mathbb{E}[\tau_2-\tau_1]}{n^{1/\alpha}}\right|^{1/\log\log n} \\ &= e^{1/\alpha} \quad \mathbb{P}\text{-almost surely.}\end{aligned} \tag{2.6.8}$$

Since $\nu(n) = k(n)+1$, the same holds true when $\nu(n)$ is replaced by $k(n)$.
Writing $1/\log\log n$ as

$$\frac{1}{\log\log n} = \frac{1}{\log\log\nu(n)} \frac{\log\Big(\log n + \log\frac{\nu(n)}{n}\Big)}{\log\log n},$$

we see that the second factor on the right-hand side of this equation converges to 1 $\mathbb{P}$-a.s. as $n \to \infty$. This remains true when $\nu(n)$ is replaced by $k(n)$. Thus, the limit in (2.6.8) carries over to the upper and lower bound in (2.6.7), completing the proof. ☐

PROOF OF THEOREM 2.6.5(B). Suppose that $\alpha \in (0,1)$. Along the same lines as in the proof of Theorem 2.6.3 (c), we write

$$\frac{\rho_{k(n)}}{k(n)}\frac{k(n)}{n^\alpha} = \frac{\rho_{k(n)}}{n^\alpha} \le \frac{X_n}{n^\alpha} \le \frac{\rho_{\nu(n)}}{n^\alpha} = \frac{\rho_{\nu(n)}}{\nu(n)}\frac{\nu(n)}{n^\alpha}. \tag{2.6.9}$$

We perform the following estimates only for the upper bound, as the lower bound can be treated along exactly the same lines. Since $\nu(n) \to \infty$ $\mathbb{P}$-a.s. as $n \to \infty$, Lemma 2.3.4 and the law of large numbers imply

$$\frac{\rho_{\nu(n)}}{\nu(n)} = \frac{1}{\nu(n)}\sum_{k=1}^{\nu(n)}(\rho_k - \rho_{k-1}) \to \mathbb{E}[\rho_2-\rho_1] \quad \mathbb{P}\text{-almost surely.}$$

Thus, we can neglect the factor $\rho_{\nu(n)}/\nu(n)$ in the $|\cdot|^{1/\log\log n}$-scaling,

$$\liminf_{n\to\infty}\left|\frac{\rho_{\nu(n)}}{n^\alpha}\right|^{1/\log\log n} = \liminf_{n\to\infty}\left|\frac{\nu(n)}{n^\alpha}\right|^{1/\log\log n} \quad \mathbb{P}\text{-almost surely.} \tag{2.6.10}$$

We investigate the reciprocal of the remaining factor. More precisely, we estimate $n/\nu(n)^{1/\alpha}$. Let $S_n := \sum_{k=1}^n \tau_{k+1} - \tau_k$, $k \in \mathbb{N}$. It follows from the definition of $k(n)$ and $\nu(n)$ that

$$\frac{S_{\nu(n)}}{\nu(n)^{1/\alpha}} \leq \frac{n}{\nu(n)^{1/\alpha}} \leq \frac{S_{k(n)}}{(k(n)+1)^{1/\alpha}}. \tag{2.6.11}$$

Now, Lemma 2.6.4 gives

$$\limsup_{n\to\infty} \left| \frac{S_n}{n^{1/\alpha}} \right|^{1/\log\log n} = e^{\beta} \quad \mathbb{P}\text{-almost surely}$$

where $\beta = \inf\left\{b \in \mathbb{R} : \mathbb{E}\left[\frac{(\tau_2-\tau_1)^\alpha}{(\log(e\vee(\tau_2-\tau_1)))^{b\alpha}}\right] < \infty\right\}$. Our estimates for the tails of the regeneration times from Proposition 2.3.5 imply $\beta = 1/\alpha$.

As $\tau_{k+1} - \tau_k \geq 1$ for all $k \in \mathbb{N}$, the sets $\{S_n : n \in \mathbb{N}\}$, $\{S_{\nu(n)} : n \in \mathbb{N}\}$ and $\{S_{k(n)} : n \in \mathbb{N}\}$ coincide up to a single point. Therefor, the limit transfers to the upper and lower bounds in (2.6.11). More precisely,

$$\limsup_{n\to\infty} \left| \frac{S_{k(n)}}{k(n)^{1/\alpha}} \right|^{1/\log\log k(n)} = \limsup_{n\to\infty} \left| \frac{S_{\nu(n)}}{\nu(n)^{1/\alpha}} \right|^{1/\log\log \nu(n)} = e^{1/\alpha} \quad \mathbb{P}\text{-almost surely.} \tag{2.6.12}$$

Due to the fact that $k(n)/(k(n)+1) \to 1$ $\mathbb{P}$-a.s., the limit remains unchanged when the denominator in the first term of (2.6.12) is replaced by $k(n)+1$. Writing $1/\log\log n$ as

$$\frac{1}{\log\log n} = \frac{1}{\log\log \nu(n)} \frac{\log\left(\log n + \log \frac{\nu(n)}{n}\right)}{\log\log n},$$

we see that the second factor on the right-hand side of this equation converges to 1 $\mathbb{P}$-a.s. as $n \to \infty$. The same holds true when $\nu(n)$ is replaced by $k(n)$. Therefor, with (2.6.10), (2.6.11) and (2.6.12) we infer

$$\begin{aligned} \liminf_{n\to\infty} \left| \frac{\rho_{\nu(n)}}{n^\alpha} \right|^{1/\log\log n} &= \liminf_{n\to\infty} \left| \left(\frac{n}{\nu(n)^{1/\alpha}} \right)^{-\alpha} \right|^{1/\log\log n} = \left(\limsup_{n\to\infty} \left| \frac{n}{\nu(n)^{1/\alpha}} \right|^{1/\log\log n} \right)^{-\alpha} \\ &= e^{-1} \quad \mathbb{P}\text{-almost surely.} \end{aligned}$$

Since the analogue estimate for the lower bound in (2.6.9) leads to the same limit, this concludes the proof. □

APPENDIX A

Uniform integrability of renewal counting processes

In our proof of Theorem 2.6.3, we use that the suitably renormalised renewal counting process of a delayed renewal process is uniformly integrable. The following result is (more than) sufficient for our purposes.

PROPOSITION A.1.1. *Let $\xi_2, \xi_3, \dots$ be a sequence of i.i.d. random variables independent of ξ_1 such that $\mathrm{P}(\xi_k > 0) = 1$ for $k \in \mathbb{N}$, where P denotes the underlying probability measure. Suppose there are constants $d > 0$ and $\alpha \in (1, 2]$ such that $\mathrm{P}(\xi_2 > t) \le dt^{-\alpha}$ for all $t \ge 1$. Then, with $\mu := \mathrm{E}[\xi_2]$, $S_n := \sum_{k=1}^n \xi_k$, $\nu(t) := \inf\{n \in \mathbb{N} : S_n > t\}$ and $a(t) := t^{1/\alpha}$ if $\alpha \in (1, 2)$ and $a(t) := \sqrt{t \log t}$ if $\alpha = 2$, it holds that*

$$\left(\exp\Big(\theta \frac{\nu(t) - t/\mu}{a(t)}\Big)\right)_{t \ge 2} \quad \text{is uniformly integrable for every } \theta > 0 \tag{A.1.1}$$

and

$$\left(\Big(\frac{\nu(t) - t/\mu}{a(t)}\Big)_-^p\right)_{t \ge 2} \quad \text{is uniformly integrable for every } p \in (1, \alpha) \tag{A.1.2}$$

for which there exists an $r > p$ with $\mathbb{E}[\xi_1^r] < \infty$.

The statements (A.1.1) and (A.1.2) have been shown in [**29**] in the case where the ξ_k, $k \in \mathbb{N}$ are i.i.d. and ξ_1 is in the domain of attraction of an α-stable law. Unfortunately, we have not been able to apply a coupling argument in order to deduce uniform integrability here from the main results in the cited source. However, the proofs given in [**29**] apply. We will provide a sketch of these proofs with the necessary changes needed here.

SKETCH OF THE PROOF OF PROPOSITION A.1.1. Let $\theta > 0$, and denote by ψ and φ the Laplace transforms of ξ_1 and ξ_2, respectively, i.e., $\psi(\lambda) = \mathrm{E}[\exp(-\lambda\xi_1)]$ and $\varphi(\lambda) := \mathrm{E}[\exp(-\lambda\xi_2)]$ for $\lambda \ge 0$. Arguing as in (2.2) of [**29**], we infer

$$\mathrm{E}\Big[\exp\Big(\theta \frac{\nu(t) - t/\mu}{a(t)}\Big)\Big] \le 1 + \frac{\psi(\lambda)}{\varphi(\lambda)} \big(e^{\lambda\mu}\varphi(\lambda)\big)^{\frac{t}{\mu}} \int_0^\infty e^x \varphi(\lambda)^{\frac{x a(t)}{\theta} - 1} \,\mathrm{d}x$$

where the difference to (2.2) in [**29**] is a factor $\psi(\lambda)/\varphi(\lambda)$, which appears here since we allow the first step to have a different law than the other steps. Equation (2.7) in [**20**, XIII.2] and Proposition 2.3.5 give

$$\varphi(\lambda) = 1 - \mu\lambda + \lambda \int_0^\infty (1 - e^{-\lambda x}) \mathrm{P}(\xi_2 > x) \,\mathrm{d}x \le 1 - \mu\lambda + \int_0^\infty (1 - e^{\lambda x})(1 \wedge dx^{-\alpha}) \,\mathrm{d}x.$$

The third summand on the right hand side is the second-order term of the Laplace transform of a random variable with tail probability $1 \wedge dx^{-\alpha}$ for $x > 0$. From [**13**, Theorem 8.1.6], we thus infer that it is $\mathcal{O}(\lambda^\alpha)$ as $\lambda \to 0$ if $\alpha \in (1, 2)$ and $\mathcal{O}(\lambda^2 |\log\lambda|)$ if $\alpha = 2$. Choosing $\lambda^* := \lambda/a(t)$, this gives

$$e^{\lambda^*\mu}\varphi(\lambda^*) \le \Big(1 + \frac{\mu\lambda}{a(t)} + \mathcal{O}\big(t^{-\frac{2}{\alpha}}\big)\Big)\Big(1 - \frac{\mu\lambda}{a(t)} + \frac{\lambda}{a(t)} \int_0^\infty \big(1 - e^{-\frac{\lambda x}{a(t)}}\big)(1 \wedge dx^{-\alpha}) \,\mathrm{d}x\Big) = 1 + \mathcal{O}(t^{-1}),$$

thus

$$\sup_{t\geq 2}\left(e^{\lambda^*\mu}\varphi(\lambda^*)\right)^{t/\mu} < \infty.$$

Further, the proof of (2.3) in [**29**] applies and gives

$$\sup_{t\geq t_0}\int_0^\infty e^x\varphi(\lambda^*)^{\frac{xa(t)}{\theta}-1}\,\mathrm{d}x < \infty$$

for t_0 and λ sufficiently large. Uniform integrability of $(\exp(\theta\frac{\nu(t)-t/\mu}{a(t)}))_{t\geq 2}$ now follows from the Vallée-Poussin criterion.

Turning to the second assertion, pick $1 < p < \alpha$ and $r \in (p, \alpha)$ such that $\mathbb{E}[\xi_1^r] < \infty$. Following the proof of (2.5) in [**29**] with mild adaptions, we obtain

$$\mathrm{E}\big[(\nu\big(\mathbb{E}[S_n]\big) - n)_-^r\big] \leq r + \mathrm{const}\cdot\mathrm{E}[|S_n - \mathbb{E}[S_n]|^r] = \mathcal{O}\big(a(n)^r\big)$$

as $n\to\infty$. Here, the last step follows from

$$\mathrm{E}[|S_n - n\mu|^r] \leq 2^{r-1}\big(\mathrm{E}[|S_1-\mu|^r] + \mathrm{E}[|S_n - S_1 - (n-1)\mu|^r]\big).$$

By assumption, $\mathrm{E}[S_1^r] = \mathrm{E}[\xi_1^r] < \infty$. Further, positive and negative part of $\xi_2 - \mu$ can be stochastically dominated by a nonnegative random variable with tails of order $x^{-\alpha}$. Hence it follows from [**28**, Lemma 5.2.2] that

$$\mathrm{E}[|S_n - S_1 - (n-1)\mu|^r] = \mathcal{O}(a(n)^r) \quad \text{as } n\to\infty.$$

The rest of the proof is as in [**29**]. □

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