

## Signature reversal invariance

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### Abstract

We consider the signature reversing transformation of the metric tensor  $g_{\mu\nu} \rightarrow -g_{\mu\nu}$  induced by the chiral transformation of the curved space gamma matrices  $\gamma_\mu \rightarrow \gamma \gamma_\mu$  in spacetimes with signature  $(S, T)$ , which also induces a  $(-1)^T$  spacetime orientation reversal. We conclude: (1) It is a symmetry only for chiral theories with  $S - T = 4k$ , with  $k$  integer. (2) Yang-Mills theories require dimensions  $D = 4k$  with  $T$  even for which even rank antisymmetric tensor field strengths and mass terms are also allowed. For example,  $D = 10$  super Yang-Mills is ruled out. (3) Gravitational theories require dimensions  $D = 4k + 2$  with  $T$  odd, for which the symmetry is preserved by coupling to odd rank field strengths. In  $D = 10$ , for example, it is a symmetry of N=1 and Type IIB supergravity but not Type IIA. A cosmological term and also mass terms are forbidden but non-minimal  $R\phi^2$  coupling is permitted. (4) Spontaneous compactification from  $D = 4k + 2$  leads to interesting but different symmetries in lower dimensions such as  $D = 4$ , so Yang-Mills terms, Kaluza-Klein masses and a cosmological constant may then appear. As a well-known example, IIB permits  $AdS_5 \times S^5$ .

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# 1 Signature reversal

## 1.1 Bicoastal theories

An interesting parlour game is to write down the equations of motion of various theories and then have your friends guess what spacetime signature you had in mind. Was it East coast conventions  $(-+++)$  or West coast  $(+---)$ ? For example, a free massive scalar field obeying

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi = 0 \quad (1)$$

is unambiguously West coast, since the East coast kinetic term has the opposite sign. The same is true for a Dirac spinor. If the gamma matrices obey

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1} , \quad (2)$$

then on the West coast

$$(-i\gamma^\mu\partial_\mu + m)\psi = 0 , \quad (3)$$

while there is no factor of  $i$  on the East. Had the fields been massless, on the other hand, there would have been no way to tell.

Even in the massless case, coupling to electromagnetism gives the game away:

$$\partial^\mu F_{\mu\nu} = -e\bar{\psi}\gamma^\mu\psi \quad (4)$$

$$-i\gamma^\mu(\partial_\mu - ieA_\mu)\psi = 0 \quad (5)$$

is once again West coast since the right hand side of the Maxwell equation would acquire a factor of  $i$  on the East.

On the West Coast the empty space Einstein's equations with a cosmological constant are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}\Lambda g_{\mu\nu} = 0 , \quad (6)$$

but the sign of the cosmological term is opposite on the East. Had there been no cosmological constant, on the other hand, there would be no way to tell. In that case, moreover you could have coupled gravity to a massless scalar field and even allowed a non-minimal coupling

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + \xi R)\phi = 0 \quad (7)$$

and your friends would still be unable to guess the signature. We shall refer to this class of theory as “bicoastal”.

A natural question to ask, therefore, is whether this can be promoted to a symmetry principle. We shall consider theories in spacetime signature  $(S, T)$  that are invariant under reversing the sign of the metric tensor, which may be regarded as a reversal of signature  $(S, T) \rightarrow (T, S)$ . Since we prefer to transform fields rather than constants, we work with the curved space metric, henceforth denoted  $G_{MN}(x)$ , rather than the Minkowski metric  $\eta_{MN}$ , and consider the transformation

$$G_{MN}(x) \rightarrow -G_{MN}(x) . \quad (8)$$

Similarly, we shall work with the curved space gamma matrices

$$\left\{ \Gamma_M(x), \Gamma_N(x) \right\} = 2G_{MN}(x) \mathbf{1} \quad (9)$$

rather than the flat space ones.

In the presence of fermions, it is useful to implement this reversal by the transformation

$$\Gamma_M \rightarrow \Gamma \Gamma_M, \quad (10)$$

where  $\Gamma$  is the normalised chirality operator

$$\Gamma \equiv \frac{1}{\sqrt{G}} \frac{1}{D!} \varepsilon^{M_1 \dots M_D} \Gamma_{M_1 \dots M_D} \quad (11)$$

and where

$$\sqrt{G} \equiv \sqrt{(-1)^T \det G_{MN}(x)}. \quad (12)$$

Let us denote the Clifford algebra by  $\text{Cliff}(S, T)$ . A change of signature may change the Clifford algebra generated by the gamma matrices. There are, however, special dimensions where the Clifford algebra is isomorphic in opposite signatures. This is necessary so that the fermion representations do not jump, and that the fermion interactions in the action remain real. As explained in appendix A, in order that the Clifford algebra remain the same under signature reversal

$$\text{Cliff}(S, T) = \text{Cliff}(T, S) \quad (13)$$

we require

$$S - T = 4k' \quad (14)$$

for some integer  $k'$ . See tables 2, 3 and 4. This rules out odd  $D$ , and one can express the dimension of the spacetime in terms of an arbitrary integer  $k$  and the number of time-like directions  $T$  as

$$D = 4k' + 2T. \quad (15)$$

By redefining  $k$  we have therefore two prototypical admissible dimensionalities:

- The Minkowskian type with an odd number of time-like directions  $D = 4k + 2$ ;
- The Euclidean type with an even number of time-like directions  $D = 4k$ .

In both cases,

$$\Gamma^2 = +1. \quad (16)$$

Since  $\Gamma$  anticommutes with  $\Gamma_M$  the operation (10) will reverse the sign of the metric as desired. We also note that under (10)

$$\Gamma \rightarrow (-1)^T \Gamma \quad (17)$$

As explained in section 3.2 this means that for dimensions  $D = 4k$  the sign of the volume form  $\text{Vol}(M) \equiv \sqrt{G} d^D x$  remains the same while for  $D = 4k + 2$  it changes sign corresponding to reversal of orientation. The link between the orientation and the signature arises from the Clifford algebra. In the signatures  $S - T = 4k'$  where Clifford algebra remains isomorphic under change of signature, we have  $D/2 \equiv T \pmod{2}$ , and the naïve transformation

$$\sqrt{G} d^D x \rightarrow (-1)^T \sqrt{G} d^D x \quad (18)$$

and the transformation induced by (10)

$$\frac{1}{D!} \varepsilon^{M_1 \dots M_D} \Gamma_{M_1 \dots M_D} \rightarrow (-1)^T \frac{1}{D!} \varepsilon^{M_1 \dots M_D} \Gamma_{M_1 \dots M_D} \quad (19)$$

have an identical effect. Note that the density  $\sqrt{G} > 0$  is always positive, and does therefore not change sign.

The change in sign of the volume element then raises an important question. Do we demand only that the equations of motion be invariant, which would allow the action to change by an overall sign, or do we insist on the stronger requirement that the action be invariant? Let us first consider the case of pure gravity.

## 1.2 Pure gravity

The D-dimensional gravitational action functional is

$$S_E = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{G} R. \quad (20)$$

Under (8) the volume element transforms as

$$\sqrt{G} d^D x \rightarrow (-1)^{D/2} \sqrt{G} d^D x \quad (21)$$

while the curvature scalar flips sign for all  $D$

$$R \rightarrow -R. \quad (22)$$

The requirement of invariance selects out the dimensions

$$D = 4k + 2, \quad k = 0, 1, 2, 3 \dots \quad (23)$$

and forbids a bulk cosmological constant  $\Lambda$  in the action

$$S_\Lambda = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{G} \Lambda. \quad (24)$$

As we have seen, the weaker requirement, that the Einstein equations be invariant, rules out a cosmological constant in any dimension at the classical level but at the quantum level there will be  $L$ -loop counterterms of the form

$$S_c \sim \frac{1}{2\kappa_D^2} \int d^D x \sqrt{G} \kappa_D^{2L} R^{\frac{(D-2)L+2}{2}} \quad (25)$$

where  $R^n$  is symbolic for a scalar contribution of  $n$  Riemann tensors each of dimension 2. This again requires  $D = 4k + 2$  for invariance. So we shall take the view that the action, and not just the equations of motion should be invariant.

Of course, one could argue that this is giving the game away since the requirement that the Hamiltonian be positive then forces a specific sign for scalar and gravity kinetic terms which will flip in the opposite signature. So, if one prefers, one could adopt the view that a bicoastal gravitational theory is one for which the field equations, including their quantum corrections, are insensitive to signature flip. This would lead to the same  $D = 4k + 2$  requirement and, if fermions are involved, the same  $S - T = 4k$  condition with  $T$  odd. This point of view may be especially compelling for theories with self-dual field strengths such as Type IIB supergravity in  $D = 10$  and chiral supergravity in  $D = 6$  which have no Lorentz-invariant action principle.

### 1.3 Comparison with other authors

Motivated primarily by the desire to explain the (approximate) vanishing of the cosmological constant, several authors, Erdem [1, 2], Quiros [3], 't Hooft and Nobbenhuis [4, 5], Kaplan and Sundrum [6], have recently considered the transformation

$$x^M \rightarrow i x^M \quad (26)$$

with the understanding that  $G_{MN}(ix) = G_{MN}(x)$ . This also induces a signature reversal

$$ds^2 = G_{MN} dx^M dx^N \rightarrow -ds^2. \quad (27)$$

Signature reversal and the cosmological constant problem was also discussed in signature (3,3) by Bonelli and Boyarsky in [7]. For earlier work on flipping the sign of the metric and its relation to Clifford algebras, see [8, 9].

For clarity, therefore, we note the following differences and similarities of our approach:

- 1) *There is more to our chiral transformation (10) than just signature reversal (27).* There is also the change of orientation (17) which depends on whether  $T$  is even or odd. So we are excluding some theories that would be allowed by (26).
- 2) *We work only in  $D = 4k' + 2T$  dimensions.* 't Hooft and Nobbenhuis [4, 5] work in four dimensions and note that the requirement that the Einstein equations be invariant under (26) rules out a cosmological constant. As noted by Erdem [1, 2], the stronger requirement of invariance of the action under (26) again selects out the dimensions

$$D = 4k + 2, \quad k = 0, 1, 2, 3, \dots \quad (28)$$

In this respect we agree with Erdem [1, 2] because, as noted above, quantum corrections require this stronger condition. We differ from Erdem in the

transformation rules for matter fields, however. We also differ from Quiros [3] who compensates for the flip of sign of the four-dimensional action by a flip in the sign of Newton’s constant.

- 3) *We consider only positive energies.* As we have just seen, by working in  $D = 4k' + 2T$  dimensions, we avoid transformations that change the sign of the action and hence avoid all discussion of ghosts and other interpretational problems that negative energies involve. This differs from ‘t Hooft and Nobbenhuis [4, 5], Kaplan and Sundrum [6], and the earlier work of Linde [10].
- 4) *Real coordinates transform only into real coordinates.* By reversing the signature using a transformation on the metric (8) and the curved space gamma matrices (10) we avoid the introduction of complex coordinates or Wick rotations. (However, since in the appropriate signature the complex transformation (26) on the coordinate  $x^M$  reproduces formally the same transformation rules for the volume element (21) and the curvature (22) as the chiral transformation (10) on the curved space gamma matrices, it is sometimes useful in providing a check on our results. That metric reversal (8) yields the same transformation rules as (26) was first stated by Erdem [2], but without the Clifford algebra result (17) it is difficult to justify the orientation reversal (21) in  $D = 4k + 2$  that changes the sign of the volume element.)
- 5) *Real fields transform only into real fields.* We avoid transformations which take real fields into imaginary ones. This is important and restricts our choice of field parameterizations. For example, the metric and curved space gamma matrices are “good” variables in this respect, while the vielbein  $e_M^A$  for which

$$G_{MN} = e_M^A e_N^B \eta_{AB} \quad (29)$$

is a “bad” variable, since there is in general no real transformation on the vielbeins that induces our basic transformation (10). See appendix B. For this reason we avoid vielbeins and work directly with the curved-space Dirac matrices. As shown in section 3, it is in fact possible to describe the coupling of fermions to gravity using just the gammas, without ever having to introduce vielbeins (an interesting observation in its own right).

Similarly, the scalar field  $H(x)$  whose vev is the string coupling constant

$$\langle H(x) \rangle = g_s \quad (30)$$

is a good variable whose sign we can flip [11], while the usual  $\Phi$  parameterization

$$H = e^\Phi \quad (31)$$

is bad from this point of view.

- 6) *Constants do not transform.* We do not consider transforming constants such as the Minkowski metric  $\eta_{MN}$  or flat space gamma matrices. Nor do we transform Newton’s constant [3] or particle masses [1].

In making the above comparisons, we do not wish to imply that these alternative approaches are without merit. We merely wish to note that, in some respects, our approach is more conservative.

## 2 Inclusion of boson matter fields

### 2.1 Antisymmetric tensor fields

Extension of this symmetry to include matter in  $D = 4k + 2$  requires that the kinetic terms should transform in the same way as the curvature scalar, with a reversal of sign. This allows antisymmetric tensor field strengths of odd rank but not even

$$S_F \equiv \int d^D x \sqrt{G} G^{M_1 N_1} G^{M_2 N_2} \dots G^{M_n N_n} F_{M_1 M_2 \dots M_n} F_{N_1 N_2 \dots N_n} \quad (32)$$

and also rules out all mass terms, although a non-minimal  $R\phi^2$  scalar coupling is allowed. It is at this stage that the two versions of signature reversal (8) and (26) may diverge since one is free to assign different transformation rules to the matter fields. To keep them the same would require covariant vectors to transform with the opposite sign to contravariant under (26).

In  $D = 4k$ , on the other hand, signature reversal allows antisymmetric tensor field strengths with even rank, as well as mass terms.

### 2.2 Pure Yang-Mills

The kinetic term in pure Yang-Mills

$$S_{YM} = \frac{1}{4g_D^2} \int d^D x \sqrt{G} \text{Tr} |F_2|^2 \quad (33)$$

contains two contractions with the background metric, and is therefore invariant under reversal of signature. The same is true of Chern-Simons terms as well. Invariance then requires that the volume form should not change sign under signature reversal. Consequently Yang-Mills theory is invariant only in dimensions  $D = 4k$ . If the theory is coupled to fermions, and we require  $S - T = 4k'$ , this leads to  $D = 4k' + 2T$  so that there would have to be an even number of time-like dimensions. This is the case in the Euclidean four-dimensional spacetime, for instance.

Quantum corrections that involve only the field strength  $F$  are invariant under signature reversal. Corrections that involve powers of the d'Alembertian  $\square$  need not be, however.  $L$ -loop corrections that may on dimensional grounds arise in perturbation theory are, schematically, of the form

$$S_c \sim \frac{1}{g_D^2} \int d^D x \sqrt{G} \text{Tr} F_2 \left( g_D^2 \square^{(D-4)/2} \right)^L F_2 . \quad (34)$$



These terms are invariant for all  $L$  under signature reversal only when  $D = 4k$ . This is consistent with the fact that Yang-Mills theories are signature reversal invariant in signatures of Euclidean type.

There are several caveats, however:

- 1) Although forbidden in  $D = 4k + 2$ , Maxwell and Yang-Mills terms can arise after compactification to lower dimensions. See, for example, section 5.
- 2) The absence in  $D = 4k + 2$  applies in pure Yang-Mills theory only. In the Yang-Mills sector of Type I supergravity the kinetic term is multiplied by a dilaton factor  $H(x)$  that may also change sign to compensate for the change in sign of the volume element [11].
- 3) Yang-Mills interaction may also appear on branes, for example D3, where the rules of signature reversal are different from those in the bulk [11].

## 2.3 Self-duality

A real  $n$ -form can be (anti-)self-dual in  $D = S + T$  dimensions provided that the Hodge star operation  $\star$ , which obeys

$$\star^2 = (-1)^{n(D-n)+T} \quad (35)$$

when operated on  $n$ -forms, is nilpotent. As the dimension  $D = 2n$  is even, this happens for even  $n$  in signatures of Euclidean type ( $T$  even), and for odd  $n$  in signatures of Minkowskian type ( $T$  odd). Self-dual fields arise therefore in gravity-like theories in Minkowskian signature and in Yang-Mills like theories in Euclidean space.

Under signature reversal the Hodge star operated on an  $n$ -form  $F_n$  transforms as

$$\star F_n \longrightarrow (-1)^{n+T} \star F_n . \quad (36)$$

This is because the Hodge star carries information of the orientation of the space-time, and the volume form picks up the sign  $(-1)^T$  under signature reversal; further, the Hodge star involves  $D - n$  lowered indices, and hence  $D - n$  occurrences of the metric tensor. An (anti-)self-duality equation is therefore left form invariant under signature reversal precisely when it is defined.

## 3 Inclusion of fermion matter fields

### 3.1 Issues of signature

Fermions and their interactions are sensitive to the choice of signature in several ways. As explained above, in this paper we restrict to such signatures  $S - T = 4k'$  and field theories which happen to be independent of whether the actual signature is  $(S, T)$  or  $(T, S)$ . To appreciate the importance of making this restriction, it is instructive to see what would happen otherwise:

First of all, fermions belong to representations of the Clifford algebra. If the Clifford algebra changes, the minimal fermion representations change, as discussed at length in appendix A. An example of this is the three dimensional Minkowski signature  $(2, 1)$ , where the Clifford algebra  $\mathbb{R}(2) \oplus \mathbb{R}(2)$  consists of pairs of real  $2 \times 2$  matrices in the mostly plus, or East coast, signature and of  $2 \times 2$  complex matrices  $\mathbb{C}(2)$  in the West coast signature. (A similar statement applies in eleven dimensions as well, as both have  $S - T = 4k' + 1$ , and not  $4k'$ .)

Changing signature when  $S - T \neq 4k'$  changes the reality properties of fermion interactions. In many cases a change of signature amounts indeed to multiplying gamma matrices with the imaginary unit: in such cases, if fermion interactions involve odd numbers of gamma matrices (*e.g.* the kinetic term), this makes the action imaginary and the theory non-unitary. In  $S - T = 4k'$  the Lagrangian is the same in both signatures.

These observations mentioned above are local in nature. The final difference concerns defining fermions globally. Though we shall not discuss global issues in the present paper, we recall it briefly here: namely, if the background spacetime is non-orientable, the global structure of fermion fields may change when the signature is changed [8, 9]. In particular, the topological obstruction to defining a global fermion field then typically changes. As with the differences mentioned above, this does not happen when  $S - T = 4k'$ .

## 3.2 Clifford volume

In section 1.1 we argued that the transformation properties of the volume element should be consistent with the representation of the volume form in terms of elements of the Clifford algebra. We shall here show in more detail in what sense the chirality operator  $\Gamma$  and the volume element  $\sqrt{G}d^Dx$  can be identified.

Arbitrary linear combinations of products of gamma-matrices generate the full Clifford algebra  $\text{Cliff}(TM)$ . At a given point  $x \in M$  the Clifford algebra is a vector space with the basis

$$\mathbf{1}, \Gamma_M, \Gamma_{MN}, \dots, \Gamma_{M_1 \dots M_{D-1}}, \Gamma, \quad (37)$$

where  $\Gamma$  was defined in (11). In even dimensions, only  $\mathbf{1}$  has a non-zero trace. A typical element  $\Sigma$  at degree  $n$  of the Clifford algebra  $\text{Cliff}(TM)$  can then be expanded locally as

$$\Sigma = \frac{1}{n!} \Sigma_{M_1 \dots M_n} \Gamma^{M_1 \dots M_n}. \quad (38)$$

As a vector bundle on  $M$ , the Clifford algebra is the same as the space of differential forms

$$\text{Cliff}(TM) \simeq \Omega^*(M). \quad (39)$$

The multiplicative structure is different: Clifford product ‘ $\cdot$ ’ in the former, wedge product ‘ $\wedge$ ’ in the latter.

As we have the metric  $G_{MN}$  at our disposal, we may define the Hodge dual of a differential form as well as of an element of the Clifford algebra

$$\star \Sigma = \frac{(-1)^T}{(D-n)!n! \sqrt{G}} \varepsilon^{M_1 \dots M_n N_{n+1} \dots N_D} \Sigma_{M_1 \dots M_n} \Gamma^{N_{n+1} \dots N_D} . \quad (40)$$

The definition of  $\sqrt{G}$  was given in (12); the orientation implied in  $d^D x$  is

$$dx^{M_1} \wedge \dots \wedge dx^{M_D} = \epsilon^{M_1 \dots M_D} d^D x . \quad (41)$$

This determines the standard volume element

$$\text{Vol}(M) = \star 1 \equiv \sqrt{G} d^D x \quad (42)$$

$$= (-1)^T \Gamma \quad (43)$$

that can be expressed as a differential form as well as an element of the Clifford algebra.

### 3.3 Kinetic terms

Defining kinetic terms and interaction potentials for fermions requires one piece of additional structure, a spin (or pin) invariant inner product

$$\overline{\psi} \chi \equiv \psi^\dagger C \chi , \quad (44)$$

where  $C$  is a constant matrix. As we are working in  $S-T=4k'$  a multiple of four, the dagger  $^\dagger$  stands for transpose for even  $k'$  and for quaternionic conjugation when  $k'$  is odd.<sup>3</sup> The inner product must be invariant in order that the Lagrangean not break Lorentz symmetry, and in order that the Leibnitz rule for a spin connection hold.

There are generally two such invariant inner products: given one  $C$  the other candidate is proportional to  $\Gamma C$ . A convenient way to distinguish these inner products [12] is to keep track of the Hermitean conjugate of a rank  $n$  Clifford matrix  $\Sigma$

$$(\Sigma \psi)^\dagger \check{C} \chi = (-1)^{\frac{1}{2}n(n-1)} \psi^\dagger \check{C} \Sigma \chi \quad (45)$$

$$(\Sigma \psi)^\dagger \hat{C} \chi = (-1)^{\frac{1}{2}n(n+1)} \psi^\dagger \hat{C} \Sigma \chi . \quad (46)$$

We must choose such an inner product that the fermion kinetic terms

$$\overline{\lambda} \Gamma^M D_M \lambda \quad \text{and} \quad \overline{\psi}_M \Gamma^{MNP} D_N \psi_P \quad (47)$$

for the dilatino and the gravitino are nontrivial. Keeping in mind that fermion fields are Grassmann odd, this leads to a symmetry requirement for the inner product

- If  $C = \check{C}$  the inner product must be symmetric  $\check{C}^T = \check{C}$ ;
- If  $C = \hat{C}$  the inner product must be anti-symmetric  $\hat{C}^T = -\hat{C}$ .

This is not always possible. We shall here discuss the implementation of this separately in  $D=4k$  and  $D=4k+2$  dimensions. The results have been summarised in table 1.

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<sup>3</sup>In terms of imaginary Pauli matrices, this is of course just Hermitean conjugation.

$\mathcal{P}$	$n = 0 \quad n = 1 \quad n = 2 \quad n = 3$				$n = 2k + 1$	
					$k$ even	$k$ odd
$D = 4k + 2$	0	$\mathcal{P}_-$	<b>1</b>	$\mathcal{P}_+$	ASD	SD
$D = 4k$	<b>1</b>	$\mathcal{P}_+$	0	$\mathcal{P}_-$	<b>1</b>	0

Table 1: Projections  $\mathcal{P}$  appearing in invariant Yukawa couplings  $\bar{\psi} F_n \mathcal{P} \chi$ , where (A)SD refers to the projection to the (anti-)self-dual component, and where in the middle column  $n$  is the rank of the tensor  $F_n$  modulo four.

### 3.3.1 Minkowskian type $D = 4k + 2$

In dimensions  $D = 4k + 2$  we have an odd number of time-like directions. The typical example is the Minkowski signature: it turns out that in that signature both of the inner products  $\check{C}$  and  $\hat{C}$  have the symmetry property we required in section 3.3. To be concrete, these Minkowski signatures are  $(S, T) = (1, 1), (5, 1)$  and  $(9, 1)$  in dimensions less or equal to 14.

Apart from the Minkowski case, the middle-dimensional signature with  $S = T$  is interesting. The symmetry property for inner products  $\check{C}$  and  $\hat{C}$  is satisfied then only if the number of time-like directions is  $T = 1$  modulo 4. This means that  $(S, T) = (1, 1)$  and  $(5, 5)$  have a nontrivial kinetic term, but  $(S, T) = (3, 3)$  and  $(7, 3)$  do not. This does not prohibit us from writing down equations of motion for them, though.

These turn out to be all the admissible signatures with  $D = 4k + 2 \leq 12$ . We are now ready to discuss the behaviour of kinetic terms under change of signature. In the above signatures, the invariant fermion kinetic terms are

$$\int d^D x \sqrt{G} \left( \overline{\mathcal{P}_- \lambda} \Gamma^M D_M \lambda \right) \quad (48)$$

$$\int d^D x \sqrt{G} \left( \overline{\mathcal{P}_+ \psi_M} \Gamma^{MNP} D_N \psi_P \right). \quad (49)$$

The gravitino and the dilatino must therefore be Weyl fermions of opposite chirality. This is indeed the case in Type IIB and Type I supergravities in  $D = 10$ , as well as in chiral supergravity in  $D = 6$ . Type IIA, for instance, does not have this property.

### 3.3.2 Euclidean type $D = 4k$

When  $D = 4k$  we have an even number of time-like directions. The prime example of this is the Euclidean signature with  $T = 0$ . Then only  $\check{C}$  produces non-trivial kinetic terms for fermions, and the admissible signatures are  $(S, T) = (4, 0), (8, 0)$  and  $(12, 0)$ .

In the middle dimensions we have signatures  $(S, T) = (2, 2)$  and  $(6, 6)$  with  $\hat{C}$ , and  $(S, T) = (4, 4)$  with  $\check{C}$ . The other admissible signatures in  $D = 4k \leq 12$  that have not been mentioned above are  $(S, T) = (6, 2)$ ,  $(10, 2)$  and  $(8, 4)$ .

The invariant fermionic kinetic terms are the same as in (48) and (49). This is because the volume element  $\sqrt{G} d^D x$  picks up the opposite sign in the change of signature to compensate for the fact that the chirality operator  $\Gamma$  picks up the opposite sign under conjugation.

### 3.4 Yukawa coupling to tensors

The isomorphism (39) between differential forms and the Clifford algebra enables one to write actions for form fields in terms of the corresponding Clifford matrices. This relies on the fact that (37) is an orthogonal basis. For the rank- $n$  field  $F_n$  we can therefore write the action

$$\int \frac{1}{g^2} F_n \wedge \star F_n + \theta F_n \wedge F_n \sim \int \sqrt{G} d^D x \operatorname{Tr} \left( \frac{1}{g^2} \mathbf{1} + \theta \Gamma \right) \cdot \mathcal{F}_n \cdot \mathcal{F}_n, \quad (50)$$

where  $g^2$  and  $\theta$  are arbitrary couplings, and ‘ $\cdot$ ’ stands for Clifford multiplication. To emphasise the rôle played by the gamma matrices, we write

$$F_n \equiv \frac{1}{n!} F_{M_1 \dots M_n} dx^{M_1} \wedge \dots \wedge dx^{M_n} \quad (51)$$

$$\mathcal{F}_n \equiv \frac{1}{n!} F_{M_1 \dots M_n} \Gamma^{M_1 \dots M_n}. \quad (52)$$

The topological term proportional to  $\theta$  vanishes consistently on both sides when  $D \neq 2n$  or  $n$  is odd.

This allows one to revisit the symmetry properties of the form field action under signature reversal. For this we note first that on even tensors  $F_n$  with rank  $n$  a change of signature induces a change of sign

$$\mathcal{F}_n \longrightarrow (-1)^{n/2} \mathcal{F}_n. \quad (53)$$

This means that for even  $n$  the invariance properties of the action follow purely from the properties of  $\Gamma$  and  $\sqrt{G} d^D x$ : the kinetic term is invariant only when  $\sqrt{G} d^D x$  is, whereas the topological term is always invariant.

If the rank  $n$  is odd and the dimension  $D$  even, we get

$$\mathcal{F}_n \longrightarrow -\star \mathcal{F}_n. \quad (54)$$

Using the standard results

$$\Gamma \mathcal{F}_n = (-1)^{\frac{1}{2}n(n+1)+nD} (\star \mathcal{F}_n) \quad (55)$$

$$\star^2 \mathcal{F}_n = (-1)^{n(D-n)+T} \mathcal{F}_n \quad (56)$$

one can show that the action transforms to

$$\int \sqrt{G} d^D x \operatorname{Tr} \left( (-1)^{T+n} \frac{1}{g^2} + \theta \Gamma \right) \cdot \mathcal{F}_n \cdot \mathcal{F}_n. \quad (57)$$

This reproduces the result of section 2 that followed simply from counting the number of times one had to use the metric to raise indices in the kinetic term: we find that odd-rank theories when  $T$  is odd, such as in Minkowski signature, and even rank theories when  $T$  is even.

Yukawa couplings are of the form

$$\int d^D x \sqrt{G} F_{M_1 \dots M_n} \bar{\psi} \Gamma^{M_1 \dots M_n} \chi. \quad (58)$$

One of the reasons why we had to restrict to dimensions  $S - T = 4k'$  was that these interactions should remain unitary under change of signature, and this can be guaranteed only where the Clifford algebra is isomorphic in opposite signatures. One can show that the Yukawa couplings transform to

$$(-1)^{\frac{1}{2}n(n-1)+T} \int d^D x \sqrt{G} F_{M_1 \dots M_n} \bar{\psi} \Gamma^{M_1 \dots M_n} (\Gamma)^n \chi. \quad (59)$$

The factor  $(-1)^T$  comes from (17). Note that

$$\Gamma^M \longrightarrow -\Gamma^M \quad (60)$$

with a contravariant index.

### 3.4.1 Minkowskian type $D = 4k + 2$

Yukawa type interactions are not admissible for  $n = 0 \pmod{4}$  when the number of time-like directions is odd, as is the case here. This excludes mass terms, for instance. The other even-rank interactions with  $n = 2 \pmod{4}$  are automatically invariant in this dimensionality.

Changing signature in tensors that have odd rank introduces a chirality operator in them as we saw already in discussing kinetic terms in section 3.3. Restricting to the part of the interaction term invariant under signature reversal leads therefore to chirality projections. When the rank of the tensor is  $n = 1 \pmod{4}$  even terms are of the form

$$\int d^D x \sqrt{G} F_{M_1 \dots M_n} \bar{\psi} \Gamma^{M_1 \dots M_n} \mathcal{P}_- \chi. \quad (61)$$

Note that, as  $n$  is odd, only the negative chirality component of  $\psi$  couples here to the tensor field. For  $n = 3 \pmod{4}$  the chirality projection is opposite

$$\int d^D x \sqrt{G} F_{M_1 \dots M_n} \bar{\psi} \Gamma^{M_1 \dots M_n} \mathcal{P}_+ \chi, \quad (62)$$

and only the positive chirality components of  $\psi$  and  $\chi$  are concerned.

If the rank of the tensor is precisely half of the dimension of the spacetime  $n = 2k + 1$ , the presence of these projection operators means that only the self-dual, or the anti-self-dual, part of the tensor field couples to fermions. Which one it should be, depends on whether  $k$  is even or odd. If  $k$  should be odd, such as is the

case for instance in the six-dimensional Minkowski signature, it is the anti-self-dual field

$$\mathcal{F}_n = -\star \mathcal{F}_n \quad (63)$$

$$= (-1)^k \Gamma \cdot \mathcal{F}_n \quad (64)$$

that couples to the (spin-half) fermions, consistent with the case  $n = 3 \bmod 4$  above. (For even  $k$ ,  $n = 1 \bmod 4$  and fermions couple to the self-dual part.) This anti-self-dual field  $F_n$  is indeed odd under the change of signature as a Clifford matrix, so that in  $D = 4k + 2$  the Yukawa coupling is invariant. Note that in this case the kinetic term (50) vanishes, however, and there is no covariant action principle.

### 3.4.2 Euclidean type $D = 4k$

In this dimensionality the number of time-like directions is even and interactions with  $n = 0 \bmod 4$  are symmetric under signature reversal. This means in particular that mass terms are invariant under it. The other even-rank interactions with  $n = 2 \bmod 4$  are not invariant, however. The chirality projection in an invariant interaction term with  $n = 1 \bmod 4$  is

$$\int d^D x \sqrt{G} F_{M_1 \dots M_n} \bar{\psi} \Gamma^{M_1 \dots M_n} \mathcal{P}_+ \chi. \quad (65)$$

In  $n = 3 \bmod 4$  the chirality projection is again opposite

$$\int d^D x \sqrt{G} F_{M_1 \dots M_n} \bar{\psi} \Gamma^{M_1 \dots M_n} \mathcal{P}_- \chi. \quad (66)$$

The fermions appearing in these interactions must have opposite chiralities: for instance in the  $n = 3 \bmod 4$  case  $\chi$  has negative and  $\psi$  has positive chirality. As the invariant kinetic terms for all spin-half fields involve the same negative chirality projection, it follows that these odd-rank Yukawa couplings are trivial as such in  $D = 4k$ . However, when one of the fermions is a Rarita-Schwinger field  $\psi_M$ , the interaction is nontrivial: in  $n = 1 \bmod 4$

$$\int d^D x \sqrt{G} F_{M_0 M_1 \dots M_n} \bar{\psi}^{M_0} \Gamma^{M_1 \dots M_n} \mathcal{P}_- \chi, \quad (67)$$

as  $\Gamma \psi_M = +\psi_M$  and  $\Gamma \chi = -\chi$ . (Now  $\chi$  has negative chirality as we introduced a new occurrence of the metric in  $\psi^M = G^{MN} \psi_N$ .) In  $n = 3 \bmod 4$  the rôles are interchanged, and we have  $\psi$  and  $\chi_M$ .

In  $D = 4k$  the middle-dimensional form field with  $n = 2k$  picks up the sign  $(-1)^k$  under signature reversal. This merely reproduces the result that  $n = 0 \bmod 4$  couplings are invariant, and  $n = 2 \bmod 4$  are not. In particular, where these couplings are consistent in  $D = 8l$  and  $n = 4l$ , both the self-dual and the anti-self-dual part of the tensor couple to fermions.

This is to be contrasted with the fact that only the (anti-)self-dual parts of the middle-dimensional forms in  $D = 4k + 2$  coupled to fermions. No such halving of

degrees of freedom arises in  $D = 4k$ . This is related to the fact that in  $D = 4k + 2$  the fermions in a Yukawa coupling had the same chirality, and one could in fact set  $\psi = \chi$ ; in  $D = 4k$  the two fermions must have opposite chiralities, and will therefore have to be independent, or else the Yukawa coupling is trivial.

### 3.5 Fermions without vielbeins

So far, we have managed to discuss both kinetic and interaction terms for fermions without having to mention vielbeins, but working only with the curved space gamma matrices. Since, as explained in Section 1.3, we wish to avoid the introduction of vielbeins altogether, we show in this section how, in second order formalism, the covariant derivative is related to these curved space gammas.

The spin covariant derivative is defined in terms of how it acts on fermions  $\psi$ :

$$D_M \psi \equiv (\partial_M + \Omega_M) \psi. \quad (68)$$

The gamma matrices  $\Gamma_M$  and the spin-connection  $D_M$  are required to be compatible in the sense that

$$\partial_M \Gamma_N - \Gamma_{MN}^K \Gamma_K + [\Omega_M, \Gamma_N] = 0, \quad (69)$$

where  $\Gamma_{MN}^K$  is the Christoffel symbol. For this equation to have solutions we notice that  $\partial_M \Gamma_N$  should be expressible in terms of a linear combination of the original gamma matrices, for instance  $\partial_M \Gamma_N = a_{MN}^K \Gamma_K$ , which indeed follows from the Clifford algebra (9). One can then verify that the vielbein-independent expression

$$\Omega_M = \frac{1}{4} \left( \Gamma_N \partial_M \Gamma^N + \partial_M \log \sqrt{G} \mathbf{1} + \Gamma^{KL} \partial_{[K} G_{L]M} \right) \quad (70)$$

reproduces the usual torsionless relation between spin connections and vielbeins when expressed in a vielbein basis.

Now that we have the curved space gamma matrices  $\Gamma_M$  and the corresponding compatible spin connection  $\Omega_M$  at our disposal, it is always possible to write a Lagrangian involving couplings to spinors in a form that does not involve explicit use of vielbeins, as long as all the tensor fields in the theory are written in terms of world indices rather than tangent space.

As a concrete example, let us consider  $D = 10, N = 1$  supergravity in Einstein frame

$$\begin{aligned} S_{N=1} = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left[ R + \bar{\psi}_M \Gamma^{MNK} D_N \psi_K + \frac{3}{2} H^{-2} H_{MNK}^2 + \bar{\lambda} \Gamma^M D_M \lambda \right. \\ & + \frac{1}{2} \left( \partial_M \ln H^{-2} \right)^2 - \frac{1}{\sqrt{2}} \bar{\psi}_M \left( \Gamma^N \partial_N \ln H^{-2} \right) \Gamma^M \lambda \\ & \left. + \frac{\sqrt{2}}{8} H^{-1} H_{MNK} \left( \bar{\psi}_L \Gamma^{MNKLR} \psi_R - 6 \bar{\psi}^M \Gamma^N \psi^K + \sqrt{2} \bar{\psi}_L \Gamma^{MNK} \Gamma^L \lambda \right) \right]. \end{aligned} \quad (71)$$

We have identified  $\phi^{3/4} \equiv H$  in the notation of [13].



The supersymmetry transformation rule for the graviton can be expressed as

$$\delta G_{MN} = \bar{\epsilon} \Gamma_{(M} \Psi_{N)} . \quad (72)$$

The curved gamma matrices transform under supersymmetry as well:

$$\delta \Gamma_M = \left( \frac{1}{2} \bar{\epsilon} \Gamma^N \Psi_M \right) \Gamma_N . \quad (73)$$

Note that the rule is of the same form as  $\partial_M \Gamma_N = a_{MN}^K \Gamma_K$  and that it does not involve explicit vielbeins. The rest of the supersymmetry transformation rules are [13]

$$\delta H = -\frac{1}{2\sqrt{2}} \bar{\eta} \lambda H \quad (74)$$

$$\delta C_{MN} = \frac{1}{2\sqrt{2}} H \left( \bar{\eta} \Gamma_M \psi_N - \bar{\eta} \Gamma_N \psi_M - \frac{1}{\sqrt{2}} \bar{\eta} \Gamma_{MN} \lambda \right) \quad (75)$$

$$\delta \lambda = -\frac{1}{\sqrt{2}} \left( \Gamma^N \partial_N \ln H \right) \eta + \frac{1}{8} H^{-1} \Gamma^{MNK} \eta \hat{H}_{MNK} \quad (76)$$

$$\begin{aligned} \delta \psi_M = & \hat{D}_M \eta + \frac{\sqrt{2}}{32} H^{-1} \left( \Gamma_M^{NKL} - 9 \delta_M^N \Gamma^{KL} \right) \eta \hat{H}_{NKL} \\ & - \frac{1}{512} \left( \Gamma_M^{NKL} - 5 \delta_M^N \Gamma^{KL} \right) \eta \bar{\lambda} \Gamma_{NKL} \lambda + \frac{\sqrt{2}}{96} \left[ \left( \bar{\psi}_M \Gamma_{NK} \lambda \right) \Gamma^{NK} \eta \right. \\ & \left. + \left( \bar{\lambda} \Gamma_{NK} \eta \right) \Gamma^{NK} \psi_M + 2 \left( \bar{\psi}_M \lambda \right) \eta - 2 \left( \bar{\lambda} \eta \right) \psi_M + 4 \left( \bar{\psi}_M \Gamma_N \eta \right) \Gamma^N \lambda \right] . \end{aligned} \quad (77)$$

Since supersymmetry is sensitive to the number of degrees of freedom this indicates that  $\Gamma_M(x)$  correctly propagates the same number of physical degrees of freedom as the metric  $G_{MN}(x)$ . Although we do not attempt it here, it would be interesting to generalize this approach to superspace.

## 4 Example: D=10 supergravities

### 4.1 N=1 supergravity

The bosonic part of the N=1 supergravity action is given by [14]

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} H^{-2} \left( R + 4H^{-2} (\partial H)^2 - \frac{1}{12} |H_3|^2 \right) \quad (78)$$

$$(79)$$

where

$$H_3 = dB_2 . \quad (80)$$

This is invariant under signature flip since the change in sign of the volume element is compensated by a change in sign of the Einstein term and the scalar and 2-form

kinetic terms. It remains invariant when the fermions are included according to the rules of sections 3.3, 3.4 and 3.5. The supersymmetry transformation rules given in 3.5 are also invariant.

Let us now compare this with heterotic supergravity whose action is given by the inclusion of a Yang-Mills term [14]

$$S_{het} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} H^{-2} \left( R + 4H^{-2}(\partial H)^2 - \frac{1}{12}|\tilde{H}_3|^2 - \frac{\kappa_{10}^2}{g_{10}^2} \text{Tr} |F_2|^2 \right) \quad (81)$$

where

$$\tilde{H}_3 = dB_2 - \frac{\kappa_{10}^2}{g_{10}^2} \omega_3. \quad (82)$$

Since the Yang-Mills term does not change sign, the action is no longer invariant.

## 4.2 Type IIB supergravity versus Type IIA

The bosonic part of the Type IIB supergravity action is given by [14]

$$S_{IIB} = S_{NS} + S_R + S_{CS} \quad (83)$$

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} H^{-2} \left( R + 4H^{-2}(\partial H)^2 - \frac{1}{12}|H_3|^2 \right) \quad (84)$$

$$S_R = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{G} \left( |F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right) \quad (85)$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3, \quad (86)$$

where

$$F_{p+1} = dC_p \quad (87)$$

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3 \quad (88)$$

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \quad (89)$$

and where we must impose the extra self-duality constraint

$$\star \tilde{F}_5 = \tilde{F}_5. \quad (90)$$

Since both  $S_{NS}$  and  $S_R$  contain field strengths only of odd rank, it is invariant under signature reversal.

To exhibit the  $SL(2, R)$  symmetry, it is more convenient to change to Einstein frame and define

$$G^E_{MN} = H^{-1/2} G_{MN} \quad (91)$$

$$\tau = C_0 + iH^{-1} \quad (92)$$

$$\mathcal{M}_{ij} = \frac{1}{\text{Im } \tau} \begin{pmatrix} |\tau|^2 & -\text{Re } \tau \\ -\text{Re } \tau & 1 \end{pmatrix} \quad (93)$$

$$F_3^i = \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}. \quad (94)$$

Then

$$\begin{aligned} S_{IIB} = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left( R^E - \frac{\partial \bar{\tau} \partial \tau}{2(\text{Im } \tau)^2} - \frac{1}{2} \mathcal{M}_{ij} F_3^i \cdot F_3^j - \frac{1}{4} |\tilde{F}_5|^2 \right) \\ & - \frac{1}{8\kappa_{10}^2} \epsilon_{ij} \int C_4 \wedge F_3^j \wedge F_3^i, \end{aligned} \quad (95)$$

In Einstein frame the fermion interactions are included via the complex supercovariant quantities [15]

$$\hat{F}_M = F_M - \kappa^2 \overline{\psi}_M^* \lambda \quad (96)$$

$$\hat{\Omega}_{MNP} = \Omega_{MNP} - \frac{3}{2} \kappa^2 \left( \overline{\psi}_{[M} \Gamma_N (i\psi_{P]} - \overline{\psi}_{[M}^* \Gamma_N (i\psi_{P]}^*) \right) \quad (97)$$

$$\hat{F}_{MNP} = F_{MNP} - 3\kappa \overline{\psi}_{[M} \Gamma_{NP]} \lambda + 6\kappa \overline{(i\psi_{[M}^*)} \Gamma_N \psi_{P]} \quad (98)$$

$$\hat{F}_{MNPQR} = F_{MNPQR} - 5\kappa \overline{\psi}_{[M} \Gamma_{NPQ} \psi_{R]} - \frac{1}{16} \kappa \bar{\lambda} \Gamma_{MNPQR} \lambda. \quad (99)$$

From sections 3.3 and 3.4, we see that the form of these supercovariant quantities as well as the supersymmetry transformations themselves are left invariant under change of signature provided  $F_{MNP}$  is odd under signature reversal, and both  $\psi_M$  and  $\lambda$  are invariant. The fact that the 3-forms are odd under signature reversal is a consequence of covariance under T-duality, and of supersymmetry. With these conventions, we conclude that Type IIB supergravity is invariant under change of signature.

We contrast this metric reversal invariance with Type IIA supergravity, whose action is given by [14]

$$S_{IIA} = S_{NS} + S_R + S_{CS} \quad (100)$$

where

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} H^{-2} \left( R + 4H^{-2} (\partial H)^2 - \frac{1}{12} |H_3|^2 \right) \quad (101)$$

$$S_R = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{G} \left( |F_2|^2 + |\tilde{F}_4|^2 \right) \quad (102)$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4, \quad (103)$$

and where

$$\tilde{F}_4 = dC_3 - C_1 \wedge H_3 . \quad (104)$$

Since  $S_R$  contains RR field strengths of even rank, it is not invariant under signature reversal.

## 5 Spontaneous compactification

### 5.1 Example of dimensional reduction from $D = 6$ to $D = 4$

Since signature reversal for gravity is allowed in  $D = 6$  but not in  $D = 4$ , it is of interest to examine the effects of dimensional reduction. To see this, we look at  $N = 1$  supergravity. The equations of motion of both the gravity multiplet  $(G_{MN}, \mathcal{P}_+ \psi_M, B_{MN}^+)$  and the tensor multiplet  $(B_{MN}^-, \mathcal{P}_- \chi, \varphi)$  are separately signature reversal invariant. As a simple example we consider the two combined.

Denoting the  $D = 6$  spacetime indices by  $(M, N = 0, \dots, 5)$ , the bosonic part of the action takes the form

$$S_{N=1} = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{G} H^{-2} \left[ R_G + 4H^{-2} G^{MN} \partial_M H \partial_N H - \frac{1}{12} G^{MQ} G^{NR} G^{PS} H_{MNP} H_{QRS} \right] . \quad (105)$$

Note that there is no self-duality condition on  $H_{MNP}$ . The metric  $G_{MN}$  is related to the canonical Einstein metric  $G^E_{MN}$  by

$$G_{MN} = H G^E_{MN} . \quad (106)$$

The combination of the six-dimensional  $N = 1$  supergravity and tensor multiplets reduce to give the  $D = 4$ ,  $N = 2$  graviton multiplet with helicities  $(\pm 2, 2(\pm 3/2), \pm 1)$  and three vector multiplets with helicities  $(\pm 1, 2(\pm 1/2), 2(0))$ . In order to make this explicit, we use a standard decomposition of the six-dimensional metric

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu^m A_\nu^n G_{mn} & A_\mu^m G_{mn} \\ A_\nu^n G_{mn} & G_{mn} \end{pmatrix} , \quad (107)$$

where the spacetime indices are  $\mu, \nu = 0, 1, 2, 3$  and the internal indices are  $m, n = 1, 2$ . The remaining two vectors arise from the reduced  $B$  field

$$B_{MN} = \begin{pmatrix} B_{\mu\nu} + \frac{1}{2}(A_\mu^m B_{m\nu} + B_{\mu n} A_\nu^n) & B_{\mu n} + A_\mu^m B_{mn} \\ B_{m\nu} + B_{mn} A_\nu^n & B_{mn} \end{pmatrix} . \quad (108)$$

Four of the six resulting scalars are moduli of the 2-torus. We parametrize the internal metric and 2-form as

$$G_{mn} = CB^{-1} \begin{pmatrix} C^{-2} + c^2 & -c \\ -c & 1 \end{pmatrix}, \quad (109)$$

and

$$B_{mn} = b \epsilon_{mn}. \quad (110)$$

The four-dimensional metric, given by  $g_{\mu\nu}$ , is related to the four-dimensional canonical Einstein metric,  $g_{\mu\nu}^E$ , by

$$g_{\mu\nu} = A g_{\mu\nu}^E, \quad (111)$$

where  $A$  is the four-dimensional shifted dilaton:

$$A^{-1} = \int dx^5 dx^6 H^{-2} \sqrt{\det G_{mn}} = H^{-2} B^{-1}. \quad (112)$$

Thus the remaining two scalars are the dilaton  $A$  and axion  $a$ , where the axion field  $a$  is defined by

$$\epsilon^{\mu\nu\rho\sigma} \partial_\sigma a = \sqrt{g} A^{-1} g^{\mu\sigma} g^{\nu\lambda} g^{\rho\tau} H_{\sigma\lambda\tau}, \quad (113)$$

and where

$$\begin{aligned} H_{\sigma\lambda\tau} &= 3 \left( \partial_{[\sigma} B_{\lambda\tau]} + \frac{1}{2} A_{[\sigma}^m F_{\lambda\tau]m} + \frac{1}{2} B_{m[\sigma} F_{\lambda\tau]}^m \right) \\ F_{\lambda\tau}^m &= \partial_\lambda A_\tau^m - \partial_\tau A_\lambda^m \\ F_{\lambda\tau m} &= \partial_\lambda B_{m\tau} - \partial_\tau B_{m\lambda}. \end{aligned} \quad (114)$$

We may now combine the above six scalars into the complex axion/dilaton field  $S$ , the complex Kähler form field  $T$  and the complex structure field  $U$  according to

$$\begin{aligned} S &= S_1 + iS_2 = a + iA^{-1} \\ T &= T_1 + iT_2 = b + iB^{-1} \\ U &= U_1 + iU_2 = c + iC^{-1}. \end{aligned} \quad (115)$$

Define the matrices  $\mathcal{M}_S$ ,  $\mathcal{M}_T$  and  $\mathcal{M}_U$  via

$$\mathcal{M}_S = \frac{1}{S_2} \begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix}, \quad (116)$$

with similar expressions for  $\mathcal{M}_T$  and  $\mathcal{M}_U$ . We also define the four  $U(1)$  gauge fields  $A^a$  by

$$A^1_\mu = B_{4\mu}, \quad A^2_\mu = B_{5\mu}, \quad A^3_\mu = A^5_\mu, \quad A^4_\mu = -A^4_\mu. \quad (117)$$

and the 3-form

$$H_{\mu\nu\rho} = 3 \left( \partial_{[\mu} B_{\nu\rho]} - \frac{1}{2} A_{[\mu}^T (\epsilon_T \otimes \epsilon_U) F_{\nu\rho]} \right). \quad (118)$$

The action (105) now becomes that of the  $N = 2$   $STU$  model [16]

$$\begin{aligned}
S_{N=2} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g} A^{-1} & \left[ R_g + A^{-2} g^{\mu\nu} \partial_\mu A \partial_\nu A - \frac{1}{12} g^{\mu\lambda} g^{\nu\tau} g^{\rho\sigma} H_{\mu\nu\rho} H_{\lambda\tau\sigma} \right. \\
& + \frac{1}{4} \text{Tr} (\partial \mathcal{M}_T^{-1} \partial \mathcal{M}_T) + \frac{1}{4} \text{Tr} (\partial \mathcal{M}_U^{-1} \partial \mathcal{M}_U) \\
& \left. - \frac{1}{4} F_{S\mu\nu}^T (\mathcal{M}_T \otimes \mathcal{M}_U) F_S^{\mu\nu} \right]. \quad (119)
\end{aligned}$$

The action is invariant under  $\text{SL}(2, \mathbb{R})_T \times \text{SL}(2, \mathbb{R})_U$  and the equations of motion have an additional  $\text{SL}(2, \mathbb{R})_S$ .

It is now of interest to see how the original reversal of the  $D = 6$  metric and 2-form

$$G_{MN} \rightarrow -G_{MN} \quad B_{MN} \rightarrow -B_{MN} \quad (120)$$

manifests itself in  $D = 4$ . The transformation property of the  $B_{MN}$ -field is forced on us by supersymmetry. From (107), (109) and (116) we find

$$g_{\mu\nu} \longrightarrow -g_{\mu\nu} \quad (121)$$

$$A_\mu^{1,2} \longrightarrow -A_\mu^{1,2} \quad (122)$$

$$A_\mu^{3,4} \longrightarrow A_\mu^{3,4} \quad (123)$$

$$S \longrightarrow -S \quad (124)$$

$$T \longrightarrow -T \quad (125)$$

$$U \longrightarrow U \quad (126)$$

Thus, remarkably, the noninvariance under metric reversal is compensated by reversing the sign<sup>4</sup> of the shifted dilaton  $A$ . In other words, the  $D = 4$  Einstein metric (111) does not change sign. Similarly, the non-invariance of the Yang-Mills term is compensated by reversing the sign of  $T$ .

In the quantum theory  $\text{SL}(2, \mathbb{R})$  is restricted to  $\text{SL}(2, \mathbb{Z})$ . It may seem unusual to find an action of a discrete group  $\text{SL}(2, \mathbb{Z})$  on the complex parameters  $S$  and  $T$  which take values not restricted to the upper half of the complex plane. Interestingly enough, such a situation was also recently encountered in [17]. The context was somewhat different but also involved orientation reversal.

## 5.2 Kaluza-Klein mass terms

Although mass terms are forbidden in the  $D = 4k + 2$  gravitational theories, they may nevertheless appear in lower dimensions à la Kaluza-Klein. To see this consider a massless scalar whose field equation is signature reversal invariant:

$$G^{MN} \nabla_M \nabla_N \phi = 0. \quad (127)$$

---

<sup>4</sup>This is reminiscent of flipping the sign of Newton's constant [3] but here we are transforming a field, not a constant.

Now consider a product manifold  $M \times K$  and decompose the coordinates as  $x^M = (x^\mu, y^m)$ . Then

$$(G^{\mu\nu}\nabla_\mu\nabla_\nu + G^{mn}\nabla_m\nabla_n)\phi(x, y) = 0 . \quad (128)$$

Fourier expanding  $\phi(x, y)$  in terms of harmonics on the internal manifold  $K$ , we see that the mass matrix will transform the same way as the kinetic term under signature invariance, thus preserving the symmetry.

### 5.3 The cosmological constant

Although a cosmological term  $\int \Lambda \sqrt{G} d^D x$  is forbidden in  $D = 4k + 2$ , a cosmological constant may nevertheless arise from the vev of an antisymmetric tensor field strength [18, 19]

$$\left\langle \sqrt{g} g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \dots g^{\mu_n \nu_n} F_{\mu_1 \mu_2 \dots \mu_n} \right\rangle \sim \epsilon^{\nu_1 \nu_2 \dots \nu_n} . \quad (129)$$

For example, by setting

$$F_5 \sim \epsilon_5 + \star \epsilon_5 \quad (130)$$

we could obtain  $AdS^5 \times S^5$  from a Freund-Rubin [20] compactification of Type IIB in  $D = 10$ .

## 6 Conclusions

Signature reversal invariance favours two kinds of theory: Yang-Mills theories of the Euclidean type in  $D = 4k$ , and gravitational theories of the Minkowskian type in  $D = 4k + 2$ , although Yang-Mills interactions may appear in the latter after spontaneous compactification.

In many ways Type IIB supergravity is the archetypal bicoastal theory, making use of all the ingredients:  $S - T = 4k'$ ,  $T$  odd,  $D = 4k + 2$ , no mass or cosmological terms, chiral, with gravitinos and dilatinos of opposite chirality, involving field strengths only of odd rank which are self-dual in the generalized sense of section 3.4. In fact it also works in the midwest and points in between since it is invariant under metric reversal in all its various signatures: (9,1), (7,3), (5,5), (3,7) and (1,9). If Type IIB supergravity did not already exist, signature reversal invariance would have forced us to invent it. However, by virtue of its  $AdS_5 \times S^5$  vacuum, it also clearly illustrates that this symmetry does not rule out a lower-dimensional cosmological constant. The situation with the Type IIB string is equally interesting, with some unexpected consequences. Signature reversal in string theory, together with reversal of the string coupling field,  $H(x)$  is the subject of an accompanying paper [11].

Chiral supergravity in  $D = 6$  (for example the (2,0) theory obtained by compactifying Type IIB on K3) is equally archetypically bicoastal, for all the same reasons, including self-duality. The latter property means that such theories do not possess a Lorentz invariant action principle. It could be argued that they occupy

a privileged position in that with no action, there is truly no way for your friends to guess what signature you had in mind when writing down the field equations.

Although the flipping of the sign of the metric tensor may leave the equations of motion invariant, a choice has to be made when choosing the boundary conditions. A metric vacuum expectation value

$$\langle G_{MN}(x) \rangle = \eta_{MN} \quad (131)$$

breaks the reversal symmetry spontaneously. Similarly the dilaton field expectation value

$$\langle H(x) \rangle = g_s \quad (132)$$

breaks spontaneously the sign reversal  $H \rightarrow -H$  to be discussed in the accompanying paper [11]. In this context, it may be worth reviving speculations about a phase of quantum gravity or string theory in which these expectation values vanish and the symmetries are restored, notwithstanding the noninvertibility [21].

## A Fermion representations and Clifford algebra

Consider a metric  $G_{MN}$  with  $S$  positive and  $T$  negative eigenvalues. In this appendix we shall now discuss the structure of the Clifford algebra  $\text{Cliff}(\mathbb{R}^{S,T})$ .

The structure of the full Clifford algebra depends on the signature of the metric; in general  $\text{Cliff}(\mathbb{R}^{S,T})$  differs from  $\text{Cliff}(\mathbb{R}^{T,S})$ . Only if the difference  $S - T$  is a multiple of four does it happen that the two algebras in opposite signatures are isomorphic. Then, denoting  $D = S + T$ ,

$$\text{Cliff}(\mathbb{R}^{S,T}) = \begin{cases} \text{Mat}_n(\mathbb{R}), n = 2^{\frac{1}{2}D} & \text{for } S - T = 8k' \\ \text{Mat}_n(\mathbb{H}), n = 2^{\frac{1}{2}D-1} & \text{for } S - T = 8k' + 4 \end{cases} . \quad (133)$$

In particular, the four-dimensional Minkowski signature gives rise to un-isomorphic Clifford algebras in the two signatures. Examples where the Clifford algebras are the same irrespective the signature are the  $D = 4k + 2$  dimensional Minkowski spaces where self-dual tensor theories have odd field strengths, and the symmetric cases  $S = T$  with an equal number of space and time directions.

In even dimensions the chirality operator anticommutes with the generators of the Clifford algebra. We can decompose the full algebra into two algebras span by even, resp. odd, products of the generators

$$\text{Cliff}(\mathbb{R}^{S,T}) = \text{Cliff}^{\text{even}}(\mathbb{R}^{S,T}) \oplus \text{Cliff}^{\text{odd}}(\mathbb{R}^{S,T}) . \quad (134)$$

Note that the even part is isomorphic to that in the opposite signature

$$\text{Cliff}^{\text{even}}(\mathbb{R}^{S,T}) \simeq \text{Cliff}^{\text{even}}(\mathbb{R}^{T,S}) \quad (135)$$



$T - S \pmod 8$	$N$	$\text{Cliff}(\mathbb{R}^{S,T}) \simeq$
0,6	$2^{\frac{1}{2}D}$	$\text{Mat}_N(\mathbb{R})$
2,4	$2^{\frac{1}{2}(D-2)}$	$\text{Mat}_N(\mathbb{H})$
1,5	$2^{\frac{1}{2}(D-1)}$	$\text{Mat}_N(\mathbb{C})$
3	$2^{\frac{1}{2}(D-3)}$	$\text{Mat}_N(\mathbb{H}) \oplus \text{Mat}_N(\mathbb{H})$
7	$2^{\frac{1}{2}(D-1)}$	$\text{Mat}_N(\mathbb{R}) \oplus \text{Mat}_N(\mathbb{R})$

Table 2: The isomorphism classes of Clifford algebras,  $D = S + T$ .

in any dimensionality. The various symmetry groups are subgroups of these algebras

$$\text{Spin}(S, T) \subset \text{Cliff}^{\text{even}}(\mathbb{R}^{S,T}) \quad (136)$$

$$\text{Pin}(S, T) \subset \text{Cliff}(\mathbb{R}^{S,T}) . \quad (137)$$

It follows from the isomorphism of the algebras (135) that

$$\text{Spin}(S, T) \simeq \text{Spin}(T, S) . \quad (138)$$

In the special dimensions where  $S - T = 0 \pmod 4$  we have also

$$\text{Pin}(S, T) \simeq \text{Pin}(T, S) , \quad (139)$$

which is not true in general.

	Euclidean		Minkowskian		Complexified
$D$	Space-like $(D, 0)$	Time-like $(0, D)$	Space-like $(D - 1, 1)$	Time-like $(1, D - 1)$	Complexified $D$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{R}(2)$	$\mathbb{H}$	$\mathbb{R}(2)$	$\mathbb{R}(2)$	$\mathbb{C}(2)$
3	$\mathbb{C}(2)$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{R}(4)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$
5	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$
6	$\mathbb{H}(4)$	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
7	$\mathbb{C}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$
9	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{C}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{C}(16) \oplus \mathbb{C}(16)$
10	$\mathbb{R}(32)$	$\mathbb{H}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$
11	$\mathbb{C}(32)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{C}(32) \oplus \mathbb{C}(32)$
12	$\mathbb{H}(32)$	$\mathbb{H}(32)$	$\mathbb{R}(64)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$

Table 3: Clifford algebras for Minkowskian and Euclidean signatures, where  $\mathbb{R}(n) \equiv \text{Mat}_n(\mathbb{R})$  etc.

		Space-like dimensions											
		0	1	2	3	4	5	6	7	8	9	10	11
Time-like dimensions	11	$\mathbb{H}(16) \oplus \mathbb{H}(16)$											
	10	$\mathbb{H}(16)$	$\mathbb{C}(32)$										
	9	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$									
	8	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$								
	7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$							
	6	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$						
	5	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$					
	4	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$				
	3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$			
	2	$\mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$		
	1	$\mathbb{C}$	$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	
	0	0	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$

Table 4: Isomorphism classes of Clifford algebras  $D < 12$  in arbitrary signatures, where  $\mathbb{R}(n) \equiv \text{Mat}_n(\mathbb{R})$  etc. Boxed entries are symmetric in signature reversal.

Fermions belong to representations of the Clifford algebra. Let us concentrate in  $S - T = 4k'$ , so that the Clifford algebra can be thought of as an algebra of real (or quaternionic) matrices of a given dimensionality. There the representations of the fermions that do not satisfy a chirality condition are referred to as Majorana (resp. quaternionic Majorana) fermions. In Mathematics Literature these representations are usually referred to as *pinors*  $P$ .

In these dimensionalities  $S - T = 4k'$ , the chirality operator is nilpotent  $\Gamma^2 = \mathbf{1}$ , and the pinor representations  $P$  split to positive and negative eigenspaces  $S_{\pm}$  with respect to it  $P = S_+ \oplus S_-$ . These (quaternionic) Majorana-Weyl fermions are in Mathematics Literature usually referred to as *spinors*. The even part of the Clifford algebra acts diagonally on  $S_+ \oplus S_-$ , whereas the odd part does not. Note that a similar split does not occur in other even dimensionalities, as there the two chiralities are typically related by complex conjugation.

## B Real structure

In most literature the gamma matrices  $\Gamma_M(x) \equiv e_M^A(x)\Gamma_A$  are defined in terms of a set of constant reference matrices  $\Gamma_A$  that satisfy  $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$ . The defining equation for vielbeins  $e_M^A$  is

$$e_M^A e_N^B \eta_{AB} = G_{MN} . \quad (140)$$

One may ask whether one could account for the change of signature  $G_{MN} \longrightarrow -G_{MN}$  by a redefinition of vielbeins

$$\bar{e}_M^A \equiv e_M^B J_B^A . \quad (141)$$

This requires

$$J_C^A J_D^B \eta_{AB} = -\eta_{CD} \quad (142)$$

$$J_A^C J_C^B = +\delta_A^B . \quad (143)$$

When indices are lowered with  $\eta_{AB}$ , this means that  $J_{AB}$  is antisymmetric, and  $\det J = \pm 1$ .

One can show that neither in Euclidean, nor in  $D > 2$  Minkowski signature is there a non-trivial solution to these equations. The problem really is the involution (143). To see this, let us write

$$(J_{AB}) = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} \quad (144)$$

$$(J_{AB}\eta^{BC}) = \begin{pmatrix} -A & B \\ B^T & C \end{pmatrix} , \quad (145)$$

where  $A$  and  $C$  are antisymmetric  $T \times T$  resp.  $S \times S$  matrices. The square of this is

$$\mathbf{1}_T \oplus \mathbf{1}_S = J\eta^{-1}J\eta^{-1} \quad (146)$$

$$= \begin{pmatrix} A^2 + BB^T & -AB + BC \\ -B^TA + CB^T & B^TB + C^2 \end{pmatrix}. \quad (147)$$

Let us consider the different cases for  $T = 0, 1$  and  $T > 1$  respectively.

- In Euclidean signature  $T = 0$ , so that  $A = B = 0$  and

$$\mathbf{1}_S = C^2 = -C^TC \leq 0. \quad (148)$$

This implies

$$S = -\text{tr } C^TC \leq 0; \quad (149)$$

this is a contradiction, and it follows that there are no matrices  $C$  that satisfy the constraints imposed above.

- In Minkowskian signature  $T = 1$  and  $A$  is an antisymmetric real number, hence zero  $A = 0$ . We have

$$1 \oplus \mathbf{1}_S = \begin{pmatrix} BB^T & BC \\ CB^T & B^TB + C^2 \end{pmatrix} \quad (150)$$

$$. \quad (151)$$

It follows

$$1 = BB^T \quad (152)$$

$$S = \text{tr}(B^TB + C^2) = BB^T - \text{tr } C^TC \quad (153)$$

so that

$$S - 1 = -\text{tr } C^TC \leq 0. \quad (154)$$

This means that  $S \leq 1$ , and there is a real structure available in a Minkowski space only for  $(S, T) = (1, 1)$ .

- This argument does not generalise to higher  $T > 1$  as then the equation

$$\mathbf{1}_T = -A^TA + BB^T \quad (155)$$

has more than one solution.

More generally, however, in  $\mathbb{R}^{p,p}$  we can choose

$$(\eta_{AB}) = \sigma_3 \otimes \mathbf{1}_p \quad (156)$$

$$(J_A^B) = \sigma_1 \otimes \mathbf{1}_p \quad (157)$$

$$(J_{AB}) = i\sigma_2 \otimes \mathbf{1}_p, \quad (158)$$

which solves the two constraints.

## References

- [1] R. Erdem, “A symmetry for vanishing cosmological constant in an extra dimensional toy model,” *Phys. Lett. B* **621**, 11 (2005) [arXiv:hep-th/0410063].
- [2] R. Erdem, “A symmetry for vanishing cosmological constant: Another realization,” [arXiv:gr-qc/0603080].
- [3] I. Quiros, “Symmetry relating gravity with antigravity: A possible resolution of the cosmological constant problem?” [arXiv:gr-qc/0411064].
- [4] S. Nobbenhuis, “Categorizing different approaches to the cosmological constant problem,” [arXiv:gr-qc/0411093].
- [5] G. ’t Hooft and S. Nobbenhuis, “Invariance under complex transformations, and its relevance to the cosmological constant problem,” [arXiv:gr-qc/0602076].
- [6] D. E. Kaplan and R. Sundrum, “A symmetry for the cosmological constant,” [arXiv:hep-th/0505265].
- [7] G. Bonelli and A.M. Boyarsky, “Six dimensional topological gravity and the cosmological constant problem,” *Phys. Lett. B* **490** (2000) 147 [arXiv:hep-th/0004058].
- [8] S. Carlip and C. DeWitt-Morette, “Where the sign of the metric makes a difference,” *Phys. Rev. Lett.* **60** (1988) 1599.
- [9] M. Berg, C. DeWitt-Morette, S. Gwo and E. Kramer, “The Pin groups in physics: C, P, and T,” *Rev. Math. Phys.* **13**, 953 (2001) [arXiv:math-ph/0012006].
- [10] A.D. Linde, “The universe multiplication and the cosmological constant problem,” *Phys. Lett. B* **200**, 272 (1988).
- [11] M.J. Duff and J. Kalkkinen, “Signature and coupling reversal in string theory,” [arXiv:hep-th/0605274].
- [12] F. Reese Harvey, “Spinors and calibrations,” (Academic Press , 1990)
- [13] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, “Ten-Dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields,” *Nucl. Phys. B* **195** (1982) 97.
- [14] J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond.”
- [15] J.H. Schwarz, “Covariant field equations of chiral  $N = 2$   $D = 10$  supergravity,” *Nucl. Phys. B* **226**, 269 (1983).
- [16] M.J. Duff, J.T. Liu and J. Rahmfeld, “Four-dimensional string/string/string triality,” *Nucl. Phys. B* **459** (1996) 125 [arXiv:hep-th/9508094].
- [17] A. Kapustin and E. Witten, “Electric-magnetic duality and the geometric Langlands program,” [arXiv:hep-th/0604151].
- [18] M.J. Duff and P. van Nieuwenhuizen, “Quantum inequivalence of different field representations,” *Phys. Lett. B* **94**, 179 (1980).

- [19] A. Aurilia, H. Nicolai and P.K. Townsend, “Hidden constants: The theta parameter of QCD and the cosmological constant of  $N = 8$  supergravity,” Nucl. Phys. B **176**, 509 (1980).
- [20] P.G.O. Freund and M.A. Rubin, “Dynamics of dimensional reduction,” Phys. Lett. B **97**, 233 (1980).
- [21] S. Deser, “Why is the metric invertible?” [arXiv:gr-qc/0603125].