Jordan Derivations on Triangular Matrix Rings

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Abstract: Guided by the research line introduced by Martindale III in [5] on the study of the additivity of maps, this article aims establish conditions on triangular matrix rings in order that an map φ satisfying

$$\varphi(ab + ba) = \varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a)$$

for all a, b in a triangular matrix ring becomes additive.

Key words: Additivity, Jordan derivation, triangular matrix ring, nest algebras.

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1. Introduction

Let $\mathfrak{R}$ be an associative ring. An additive mapping φ from ring $\mathfrak{R}$ into itself is called a *Jordan derivation* if for any $a \in \mathfrak{R}$

$$\varphi(a^2) = \varphi(a)a + a\varphi(a).$$

The study on the question of when a particular application between two rings is additive has become an area of great interest in the theory of associative rings. One of the first results ever recorded was given by Martindale III which in his condition requires that the ring possess idempotents, see [5]. Daif in [2] considered the same problem for multiplicative derivations. In more recent works Yu Wang in [6], [7] and Bruno L. M. Ferreira in [4] discusses the additivity of certain maps. Although these authors use a standard argument to prove their results the math involved is sufficiently substantial to this line of research. This motivated us to ask the question: When is a Jordan derivation additive in a triangular matrix ring? In this paper, we give a full answer for this question. At the end of the article we present an application of our results in nest algebras.

2. Definition and Notation

We will use the following definition given by Yu Wang (see, [7]), becoming our results more general than if we use the definition of triangular algebra given by Wai-Shun Cheung (see, [1])

DEFINITION 2.1. Let $\mathfrak{R}_1, \mathfrak{R}_2$ be rings ($\mathfrak{R}_1, \mathfrak{R}_2$ need not have an identity element) and $\mathfrak{M}$ ($\mathfrak{R}_1, \mathfrak{R}_2$)-bimodule. We consider

(a) $\mathfrak{M}$ is faithful as a left $\mathfrak{R}_1$-module and faithful as a right $\mathfrak{R}_2$-module,

(b) if $m \in \mathfrak{M}$ is such that $\mathfrak{R}_1 m \mathfrak{R}_2 = 0$ then $m = 0$.

Let

$$\mathfrak{T} = \left\{ \begin{pmatrix} r_1 & m \\ & r_2 \end{pmatrix} : r_1 \in \mathfrak{R}_1, r_2 \in \mathfrak{R}_2 \text{ and } m \in \mathfrak{M} \right\}$$

be the set of all 2×2 matrices. Observe that, with the obvious matrix operations of addition and multiplication, $\mathfrak{T}$ is a ring, it is called a *triangular matrix ring*.

Set

$$\mathfrak{T}_{11} = \left\{ \begin{pmatrix} r_1 & 0 \\ & 0 \end{pmatrix} : r_1 \in \mathfrak{R}_1 \right\}, \mathfrak{T}_{12} = \left\{ \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} : m \in \mathfrak{M} \right\}$$

and

$$\mathfrak{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ & r_2 \end{pmatrix} : r_2 \in \mathfrak{R}_2 \right\}.$$

Then we can write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$. Henceforth the element a_{ij} belongs $\mathfrak{T}_{ij}$ and the corresponding elements are in $\mathfrak{R}_1, \mathfrak{R}_2$ or $\mathfrak{M}$. By a direct calculation $a_{ij} a_{kl} = 0$ if $j \neq k$ where $i, j, k \in \{1, 2\}$.

DEFINITION 2.2. A ring $\mathfrak{R}$ is called $k-$torsion free if $kx = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}, k > 0$.

3. MAIN RESULT

Let's state our main result.

THEOREM 3.1. *Let $\mathfrak{T}$ be a triangular matrix ring satisfying the following conditions:*

(i) *If $r_1 \mathfrak{R}_1 + \mathfrak{R}_1 r_1 = 0$ then $r_1 = 0$,*

(ii) *If $r_2 \mathfrak{R}_2 + \mathfrak{R}_2 r_2 = 0$ then $r_2 = 0$.*

If a mapping $\varphi : \mathfrak{T} \longrightarrow \mathfrak{T}$ satisfies

$$\varphi(ab + ba) = \varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a)$$

for all $a, b \in \mathfrak{T}$, then φ is additive. In addition, if $\mathfrak{T}$ is $2-$torsion free, then φ is a Jordan derivation.

4. JORDAN DERIVATIONS ON TRIANGULAR MATRIX RINGS

In this section we always assume the conditions (i) and (ii) hold and also assume that mapping $\varphi : \mathfrak{T} \longrightarrow \mathfrak{T}$ satisfies

$$\varphi(ab + ba) = \varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a)$$

for all $a, b \in \mathfrak{T}$. Note that clearly $\varphi(0) = 0$.

To prove Theorem 3.1 we need some auxiliary lemmas. Let's begin with

LEMMA 4.1. *For any $r_1 \in \mathfrak{T}_{11}$, $r_2 \in \mathfrak{T}_{22}$, $m \in \mathfrak{T}_{12}$ we have*

(1) $\varphi(r_1 + m) = \varphi(r_1) + \varphi(m)$.

(2) $\varphi(r_2 + m) = \varphi(r_2) + \varphi(m)$.

Proof. We prove only (1) because the proof of (2) is similar. For any $a_2 \in \mathfrak{T}_{22}$, we compute

$$\varphi[(r_1 + m)a_2 + a_2(r_1 + m)]$$
$$= \varphi(r_1 + m)a_2 + (r_1 + m)\varphi(a_2) + \varphi(a_2)(r_1 + m) + a_2\varphi(r_1 + m).$$

On the other hand,

$$\varphi[(r_1 + m)a_2 + a_2(r_1 + m)]$$
$$= \varphi(ma_2) = \varphi(0) + \varphi(ma_2 + 0)$$
$$= \varphi(r_1 a_2 + a_2 r_1) + \varphi(ma_2 + a_2 m)$$
$$= \varphi(r_1)a_2 + r_1\varphi(a_2) + \varphi(a_2)r_1 + a_2\varphi(r_1)$$
$$+ \varphi(m)a_2 + m\varphi(a_2) + \varphi(a_2)m + a_2\varphi(m).$$

Observe that comparing these two equalities we obtain

$$[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]a_2 + a_2[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)] = 0.$$

Thus

$$[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{12}a_2 = 0;$$

and

$$[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{22}a_2 + a_2[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{22} = 0.$$

By Conditions (b) and (ii) of the Definition 2.1 and Theorem 3.1 respectively, we have

$$[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{12} = 0;$$

$$[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{22} = 0.$$

In order to complete the proof, we now show that $[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{11} = 0$. For any $n \in \mathfrak{T}_{12}$, note that

$$\varphi[(r_1 + m)n + n(r_1 + m)] = \varphi(r_1 n) = \varphi(r_1 n + n r_1) + \varphi(mn + nm).$$

It follows that

$$[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]n + n[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)] = 0.$$

Consequently, $[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{11}n = 0$. It follows from Condition (a) of the Definition 2.1 that $[\varphi(r_1 + m) - \varphi(r_1) - \varphi(m)]_{11} = 0$, which completes the proof. ∎

LEMMA 4.2. *For any* $r_1 \in \mathfrak{T}_{11}$, $r_2 \in \mathfrak{T}_{22}$ *and* $m, n \in \mathfrak{T}_{12}$ *we have* $\varphi(r_1 m + nr_2) = \varphi(r_1 m) + \varphi(nr_2)$.

Proof. Note that $r_1 m + n r_2 = (r_1 + n)(m + r_2) + (m + r_2)(r_1 + n)$. Thus by Lemma 4.1 we have,

$$
\begin{aligned}
\varphi(r_1 m + n r_2) &= \varphi((r_1 + n)(m + r_2) + (m + r_2)(r_1 + n)) \\
&= \varphi(r_1 + n)(m + r_2) + (r_1 + n)\varphi(m + r_2) \\
&\quad + \varphi(m + r_2)(r_1 + n) + (m + r_2)\varphi(r_1 + n) \\
&= (\varphi(r_1) + \varphi(n))(m + r_2) + (r_1 + n)(\varphi(m) + \varphi(r_2)) \\
&\quad + (\varphi(m) + \varphi(r_2))(r_1 + n) + (m + r_2)(\varphi(r_1) + \varphi(n)) \\
&= \varphi(r_1 m + m r_1) + \varphi(n r_2 + r_2 n) \\
&= \varphi(r_1 m) + \varphi(n r_2).
\end{aligned}
$$

∎

LEMMA 4.3. *For any $r_2 \in \mathfrak{T}_{22}$ and $m, n \in \mathfrak{T}_{12}$ we have $\varphi(m r_2 + n r_2) = \varphi(m r_2) + \varphi(n r_2)$.*

Proof. For any $r_1 \in \mathfrak{T}_{11}$ and $r_2 \in \mathfrak{T}_{22}$, we calculate $\varphi[r_1((m + n)r_2) + ((m + n)r_2)r_1]$. On one hand,

$$
\begin{aligned}
\varphi[r_1((m + n)r_2) + ((m + n)r_2)r_1] &= \varphi(r_1)((m + n)r_2) \\
&\quad + r_1\varphi((m + n)r_2) + \varphi((m + n)r_2)r_1 + ((m + n)r_2)\varphi(r_1).
\end{aligned}
$$

Now on the other hand, by Lemma 4.2

$$
\begin{aligned}
\varphi[r_1((m + n)r_2) + ((m + n)r_2)r_1] &= \varphi(r_1(m r_2) + (r_1 n)r_2) \\
&= \varphi(r_1 m r_2) + \varphi(r_1 n r_2) = \varphi(r_1(m r_2) + (m r_2)r_1) + \varphi(r_1(n r_2) + (n r_2)r_1) \\
&= \varphi(r_1)(m r_2) + r_1\varphi(m r_2) + \varphi(m r_2)r_1 + (m r_2)\varphi(r_1) + \varphi(r_1)(n r_2) \\
&\quad + r_1\varphi(n r_2)\varphi(n r_2)r_1 + (n r_2)\varphi(r_1).
\end{aligned}
$$

Thus

$$
[\varphi((m + n)r_2) - \varphi(m r_2) - \varphi(n r_2)]r_1 + r_1[\varphi((m + n)r_2) - \varphi(m r_2) - \varphi(n r_2)] = 0
$$

for all $r_1 \in \mathfrak{T}_{11}$. It follows that

$$
r_1[\varphi((m + n)r_2) - \varphi(m r_2) - \varphi(n r_2)]_{12} = 0;
$$

and

$$
[\varphi((m+n)r_2) - \varphi(m r_2) - \varphi(n r_2)]_{11} r_1 + r_1[\varphi((m+n)r_2) - \varphi(m r_2) - \varphi(n r_2)]_{11} = 0.
$$

By Conditions (b) and (i) of the Definition 2.1 and Theorem 3.1 respectively, we have

$$[\varphi((m+n)r_2) - \varphi(mr_2) - \varphi(nr_2)]_{12} = 0;$$
$$[\varphi((m+n)r_2) - \varphi(mr_2) - \varphi(nr_2)]_{11} = 0.$$

Let us show now that $[\varphi((m+n)r_2) - \varphi(mr_2) - \varphi(nr_2)]_{22} = 0$. To this end, for any $p \in \mathfrak{T}_{12}$, we compute

$$\varphi((m+n)r_2)p + ((m+n)r_2)\varphi(p) + \varphi(p)((m+n)r_2) + p\varphi((m+n)r_2)$$
$$= \varphi[((m+n)r_2)p + p((m+n)r_2)] = 0$$
$$= \varphi((mr_2)p + p(mr_2)) + \varphi((nr_2)p + p(nr_2))$$
$$= \varphi(mr_2)p + (mr_2)\varphi(p) + \varphi(p)(mr_2) + p\varphi(mr_2)$$
$$+ \varphi(nr_2)p + (nr_2)\varphi(p) + \varphi(p)(nr_2) + p\varphi(nr_2).$$

This yields that

$$[\varphi((m+n)r_2) - \varphi(mr_2) - \varphi(nr_2)]p + p[\varphi((m+n)r_2) - \varphi(mr_2) - \varphi(nr_2)] = 0.$$

Therefore, we have $p[\varphi((m+n)r_2) - \varphi(mr_2) - \varphi(nr_2)]_{22} = 0$ for all $p \in \mathfrak{T}_{12}$. By Condition (a) of the Definition 2.1 we can infer that $[\varphi((m+n)r_2) - \varphi(mr_2) - \varphi(nr_2)]_{22} = 0$, which completes the proof. ∎

LEMMA 4.4. *For any $m, n \in \mathfrak{T}_{12}$ we have $\varphi(m+n) = \varphi(m) + \varphi(n)$.*

Proof. For any $r_2 \in \mathfrak{T}_{22}$, we calculate $\varphi[(m+n)r_2 + r_2(m+n)]$. On one hand,

$$\varphi[(m+n)r_2 + r_2(m+n)]$$
$$= \varphi(m+n)r_2 + (m+n)\varphi(r_2) + \varphi(r_2)(m+n) + r_2\varphi(m+n).$$

Now on the other hand, by Lemma 4.3

$$\varphi[(m+n)r_2 + r_2(m+n)]$$
$$= \varphi(mr_2 + nr_2)$$
$$= \varphi(mr_2) + \varphi(nr_2)$$
$$= \varphi(mr_2 + r_2m) + \varphi(nr_2 + r_2n)$$
$$= \varphi(m)r_2 + m\varphi(r_2) + \varphi(r_2)m + r_2\varphi(m) + \varphi(n)r_2$$
$$+ n\varphi(r_2) + \varphi(r_2)n + r_2\varphi(n).$$

Thus

$$[\varphi(m+n) - \varphi(m) - \varphi(n)]r_2 + r_2[\varphi(m+n) - \varphi(m) - \varphi(n)] = 0$$

for all $r_2 \in \mathfrak{T}_{22}$. It follows that

$$[\varphi(m+n) - \varphi(m) - \varphi(n)]_{12}r_2 = 0;$$

and

$$[\varphi(m+n) - \varphi(m) - \varphi(n)]_{22}r_2 + r_2[\varphi(m+n) - \varphi(m) - \varphi(n)]_{22} = 0.$$

By Conditions (b) and (ii) of the Definition 2.1 and Theorem 3.1 respectively, we have

$$[\varphi(m+n) - \varphi(m) - \varphi(n)]_{12} = 0;$$

$$[\varphi(m+n) - \varphi(m) - \varphi(n)]_{22} = 0.$$

Let us show now that $[\varphi(m+n) - \varphi(m) - \varphi(n)]_{11} = 0$. To this end, for any $p \in \mathfrak{T}_{12}$, we compute

$$\begin{aligned}
\varphi(m+n)p + (m+n)\varphi(p) &+ \varphi(p)(m+n) + p\varphi(m+n) \\
&= \varphi[(m+n)p + p(m+n)] = 0 \\
&= \varphi(mp+pm) + \varphi(np+pn) \\
= \varphi(m)p + m\varphi(p) &+ \varphi(p)m + p\varphi(m) \\
&+ \varphi(n)p + n\varphi(p) + \varphi(p)n + p\varphi(n)
\end{aligned}$$

This yields that

$$[\varphi(m+n) - \varphi(m) - \varphi(n)]p + p[\varphi(m+n) - \varphi(m) - \varphi(n)] = 0.$$

Therefore, we have $[\varphi(m+n) - \varphi(m) - \varphi(n)]_{11}p = 0$ for all $p \in \mathfrak{T}_{12}$. By Condition (a) of the Definition 2.1 we can infer that $[\varphi(m+n) - \varphi(m) - \varphi(n)]_{11} = 0$, which completes the proof. ∎

LEMMA 4.5. *For any* $r_1, s_1 \in \mathfrak{T}_{11}$, $r_2, s_2 \in \mathfrak{T}_{22}$ *we have*

(1) $\varphi(r_1 + s_1) = \varphi(r_1) + \varphi(s_1)$.

(2) $\varphi(r_2 + s_2) = \varphi(r_2) + \varphi(s_2)$.

Proof. We only prove (1). For any $t_2 \in \mathfrak{T}_{22}$, we have

$$\varphi(r_1 + s_1)t_2 + (r_1 + s_1)\varphi(t_2) + \varphi(t_2)(r_1 + s_1) + t_2\varphi(r_1 + s_1)$$
$$= \varphi[(r_1 + s_1)t_2 + t_2(r_1 + s_1)] = 0$$
$$= \varphi(r_1 t_2 + t_2 r_1) + \varphi(s_1 t_2 + t_2 s_1)$$
$$= \varphi(r_1)t_2 + r_1\varphi(t_2) + \varphi(t_2)r_1 + t_2\varphi(r_1)$$
$$+ \varphi(s_1)t_2 + s_1\varphi(t_2) + \varphi(t_2)s_1 + t_2\varphi(s_1).$$

This gives us

$$[\varphi(r_1 + s_1) - \varphi(r_1) - \varphi(s_1)]t_2 + t_2[\varphi(r_1 + s_1) - \varphi(r_1) - \varphi(s_1)] = 0;$$

which implies that

$$[\varphi(r_1 + s_1) - \varphi(r_1) - \varphi(s_1)]_{12} = 0;$$

$$[\varphi(r_1 + s_1) - \varphi(r_1) - \varphi(s_1)]_{22} = 0.$$

Similarly, by considering $(r_1 + s_1)m + m(r_1 + s_1)$ and using Lemma 4.4 one can deduce that $[\varphi(r_1 + s_1) - \varphi(r_1) - \varphi(s_1)]_{11} = 0.$ ∎

LEMMA 4.6. *For any $r_1 \in \mathfrak{T}_{11}$, $m \in \mathfrak{T}_{12}$, $r_2 \in \mathfrak{T}_{22}$ we have $\varphi(r_1 + m + r_2) = \varphi(r_1) + \varphi(m) + \varphi(r_2)$.*

Proof. For any $a_1 \in \mathfrak{T}_{11}$, by Lemma 4.1, we have

$$\varphi(r_1 + m + r_2)a_1 + (r_1 + m + r_2)\varphi(a_1)$$
$$+ \varphi(a_1)(r_1 + m + r_2) + a_1\varphi(r_1 + m + r_2)$$
$$= \varphi[(r_1 + m + r_2)a_1 + a_1(r_1 + m + r_2)]$$
$$= \varphi(r_1 a_1 + a_1 r_1 + a_1 m)$$
$$= \varphi(r_1 a_1 + a_1 r_1) + \varphi(a_1 m)$$
$$= \varphi(r_1 a_1 + a_1 r_1) + \varphi(a_1 m + m a_1) + \varphi(r_2 a_1 + a_1 r_2).$$

This gives us

$$[\varphi(r_1 + m + r_2) - \varphi(r_1) - \varphi(m) - \varphi(r_2)]a_1$$
$$+ a_1[\varphi(r_1 + m + r_2) - \varphi(r_1) - \varphi(m) - \varphi(r_2)] = 0.$$

We can infer by Condition (i) of the Theorem 3.1 and Condition (b) of Definition 2.1 respectively that

$$[\varphi(r_1 + m + r_2) - \varphi(r_1) - \varphi(m) - \varphi(r_2)]_{11} = 0;$$
$$[\varphi(r_1 + m + r_2) - \varphi(r_1) - \varphi(m) - \varphi(r_2)]_{12} = 0;$$

Similarly, one can get

$$[\varphi(r_1 + m + r_2) - \varphi(r_1) - \varphi(m) - \varphi(r_2)]_{22} = 0;$$

which completes the proof. ∎

Now we are ready to prove our main result.

Proof of the THEOREM 3.1. For any $a, b \in \mathfrak{T}$, we write $a = a_{11} + a_{12} + a_{22}$ and $b = b_{11} + b_{12} + b_{22}$. Applying Lemmas 4.4 - 4.6, we have

$$
\begin{aligned}
\varphi(a &+ b) \\
&= \varphi(a_{11} + a_{12} + a_{22} + b_{11} + b_{12} + b_{22}) \\
&= \varphi[(a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{22} + b_{22})] \\
&= \varphi(a_{11} + b_{11}) + \varphi(a_{12} + b_{12}) + \varphi(a_{22} + b_{22}) \\
&= \varphi(a_{11}) + \varphi(b_{11}) + \varphi(a_{12}) + \varphi(b_{12}) + \varphi(a_{22}) + \varphi(b_{22}) \\
&= \varphi(a_{11} + a_{12} + a_{22}) + \varphi(b_{11} + b_{12} + b_{22}) \\
&= \varphi(a) + \varphi(b),
\end{aligned}
$$

that is, φ is additive.

In addition, if $\mathfrak{T}$ is $2-$torsion free, then for any $a \in \mathfrak{T}$, we have

$$2\varphi(a^2) = \varphi(2a^2) = \varphi(aa + aa) = 2[\varphi(a)a + a\varphi(a)].$$

Therefore, φ is a Jordan derivation.

5. APPLICATION IN NEST ALGEBRAS

A *nest* $\mathcal{N}$ is a totally ordered set of closed subspaces of a Hilbert space $\mathcal{H}$ such that $\{0\}, \mathcal{H} \in \mathcal{N}$, and $\mathcal{N}$ is closed under the taking of arbitrary intersections and closed linear spans of its elements. The *nest algebra* associated to $\mathcal{N}$ is the set $\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathcal{H}) \ : \ TN \subseteq N \text{ for all } N \in \mathcal{N}\}$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators over a complex Hilbert space $\mathcal{H}$.

We recall the standard result ([1], Proposition 16) that say we can view $\mathcal{T}(\mathcal{N})$ as triangular algebra $\begin{pmatrix} A & M \\ & B \end{pmatrix}$ where A, B are themselves nest algebras.

PROPOSITION 5.1. *If $N \in \mathcal{N} \setminus \{0, H\}$ and E is the orthonormal projection onto N, then $E\mathcal{N}E$ and $(1-E)\mathcal{N}(1-E)$ are nest, $\mathcal{T}(E\mathcal{N}E) = E\mathcal{T}(\mathcal{N})E$ and $\mathcal{T}((1-E)\mathcal{N}(1-E)) = (1-E)\mathcal{T}(\mathcal{N})(1-E)$. Furthermore*

$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{T}(E\mathcal{N}E) & E\mathcal{T}(\mathcal{N})(1-E) \\ & \mathcal{T}((1-E)\mathcal{N}(1-E)) \end{pmatrix}.$$

We refer the reader to [3] for the general theory of nest algebras.

COROLLARY 5.2. *Let P_n be an increasing sequence of finite dimensional subspaces such that their union is dense in $\mathcal{H}$. Consider $\mathcal{P}=\{\{0\}, P_n, n \geq 1, \mathcal{H}\}$ a nest and $\mathcal{T}(\mathcal{P})$ the set consists of all operators which have a block upper triangular matrix with respect to $\mathcal{P}$. If a mapping $\varphi : \mathcal{T}(\mathcal{P}) \longrightarrow \mathcal{T}(\mathcal{P})$ satisfies*

$$\varphi(fg + gf) = \varphi(f)g + f\varphi(g) + \varphi(g)f + g\varphi(f)$$

for all $f, g \in \mathcal{T}(\mathcal{P})$, then φ is additive.

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