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Characterizations of Complete Linear Weingarten Spacelike Submanifolds in a Locally Symmetric Semi-Riemannian Manifold

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Abstract: In this paper, we deal with n-dimensional complete spacelike submanifolds M^n with flat normal bundle and parallel normalized mean curvature vector immersed in an $(n+p)$-dimensional locally symmetric semi-Riemannian manifold L_p^{n+p} of index p obeying some standard curvature conditions which are naturally satisfied when the ambient space is a semi-Riemannian space form. In this setting, we establish sufficient conditions to guarantee that, in fact, $p=1$ and M^n is isometric to an isoparametric hypersurface of L_1^{n+1} having two distinct principal curvatures, one of which is simple.

Key words: Locally symmetric semi-Riemannian manifold, complete linear Weingarten spacelike submanifolds, isoparametric submanifolds.

AMS *Subject Class.* (2010): 53C42, 53A10, 53C20, 53C50.

1. Introduction

Let L_p^{n+p} be an $(n+p)$-dimensional semi-Riemannian space, that is, a semi-Riemannian manifold of index p. An n-dimensional submanifold M^n immersed in L_p^{n+p} is said to be spacelike if the metric on M^n induced from that of L_p^{n+p} is positive definite. Spacelike submanifolds with parallel normalized mean curvature vector field (that is, the mean curvature function is positive and that the corresponding normalized mean curvature vector field is parallel as a section of the normal bundle) immersed in semi-Riemannian manifolds have been deeply studied for several authors (see, for example, [2, 3, 15, 19]). More recently, in [12] the second, third and fourth authors showed that complete linear Weingarten spacelike submanifolds must be isometric to certain hyperbolic cylinders of a semi-Riemannian space form $\mathbb{Q}_p^{n+p}(c)$ of constant sectional curvature c, under suitable constraints on the values of the mean curvature and of the norm of the traceless part of the second fundamental form. We recall that a spacelike submanifold is said to be linear Weingarten when its mean and normalized scalar curvature functions are linearly related.

Now, let L_p^{n+p} be a locally symmetric semi-Riemannian space, that is, the curvature tensor $\bar{R}$ of L_p^{n+p} is parallel in the sense that $\overline{\nabla}\bar{R} = 0$, where $\overline{\nabla}$ denotes the Levi-Civita connection of L_p^{n+p}. In 1984, Nishikawa [16] introduced an important class of locally symmetric Lorentz spaces satisfying certain curvature constraints. In this setting, he extended the classical results of Calabi [4] and Cheng-Yau [6] showing that the only complete maximal spacelike hypersurface immersed in such a locally symmetric space having nonnegative sectional curvature are the totally geodesic ones. This seminal Nishikawa's paper induced the appearing of several works approaching the problem of characterizing complete spacelike hypersurfaces immersed in such a locally symmetric space (see, for instance, [1, 10, 11, 13, 14]).

Our purpose in this paper is establish characterization results concerning complete linear Weingarten submanifolds immersed in a locally symmetric manifold obeying certain curvature conditions which extend those ones due to Nishikawa [16]. For this, we need to work with a Cheng-Yau modified operator L and we establish a generalized maximum principle. Afterwards, under suitable constrains, we apply our Omori-Yau maximum principle to prove that such a submanifold must be isometric to an isoparametric hypersurface with two distinct principal curvatures, one of them being simple. Our purpose in this work is to extend the results of [10] for the case that the ambient space is a locally symmetric semi-Riemannian manifold L_p^{n+p} obeying certain geometric constraints. For this, in Section 3 we develop a suitable Simons type formula concerning spacelike submanifolds immersed in L_p^{n+p} and having certain positive curvature function. Afterwards, in Section 4 we prove an extension of the generalized maximum principle of Omori [17] to a Cheng Yau modified operator L (see Lemma 3). Moreover, we use our Simons type formula to obtain an appropriated lower estimate to the operator L acting on the mean curvature function of a linear Weingarten spacelike submanifold (cf. Proposition 1) and, next, we establish our characterization theorems (see Theorems 1 and 2).

2. Preliminaries

Let M^n be a spacelike submanifold immersed in a locally symmetric semi-Riemannian space L_p^{n+p}. In this context, we choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in L_p^{n+p}, with dual coframes $\omega_1, \ldots, \omega_{n+p}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n. We will use the following convention of indices

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

In this setting, the semi-Riemannian metric of L_p^{n+p} is given by

$$ds^2 = \sum_A \epsilon_A \, \omega_A^2,$$

where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$, $1 \leq i \leq n$, $n+1 \leq \alpha \leq n+p$. Denoting by $\{\omega_{AB}\}$ the connection forms of L_p^{n+p}, we have that the structure equations of L_p^{n+p} are given by:

$$d\omega_A = -\sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D \overline{R}_{ABCD} \, \omega_C \wedge \omega_D, \tag{2.2}$$

where, $\overline{R}_{ABCD}$, $\overline{R}_{CD}$ and $\overline{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Lorentz space L_p^{n+p}. In this setting, we have

$$\overline{R}_{CD} = \sum_B \varepsilon_B \overline{R}_{CBDB}, \quad \overline{R} = \sum_A \varepsilon_A \overline{R}_{AA}. \tag{2.3}$$

Moreover, the components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor L_p^{n+p} are defined by

$$\begin{aligned} \sum_E \varepsilon_E \overline{R}_{ABCD;E} \omega_E \; &= \; d\overline{R}_{ABCD} - \sum_E \varepsilon_E \big(\overline{R}_{EBCD} \omega_{EA} + \overline{R}_{AECD} \omega_{EB} \\ &\qquad + \overline{R}_{ABED} \omega_{EC} + \overline{R}_{ABCE} \omega_{ED} \big). \end{aligned}$$

Next, we restrict all the tensors to M^n. First of all,

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p.$$

Consequently, the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since

$$-\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0,$$

from Cartan's Lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.4}$$

This gives the second fundamental form of M^n, $B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j e_\alpha$, and its square length from second fundamental form is $S = |B|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$. Furthermore, we define the mean curvature vector field $\mathbf{H}$ and the mean curvature function H of M^n respectively by

$$\mathbf{H} = \frac{1}{n} \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha \quad \text{and} \quad H = |\mathbf{H}| = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2}.$$

The structure equations of M^n are given by

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of the curvature tensor of M^n. Using the previous structure equations, we obtain Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} - \sum_\beta \left(h_{ik}^\beta h_{jl}^\beta - h_{il}^\beta h_{jk}^\beta \right). \tag{2.5}$$

and

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijij} - n^2 H^2 + S. \tag{2.6}$$

We also state the structure equations of the normal bundle of M^n

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0,$$

$$d\omega_{\alpha\beta} = -\sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\omega_k \wedge \omega_l.$$

We suppose that M^n has flat normal bundle, that is, $R^{\perp} = 0$ (equivalently $R_{\alpha\beta jk} = 0$), then $\overline{R}_{\alpha\beta jk}$ satisfy Ricci equation

$$\overline{R}_{\alpha\beta ij} = \sum_{k} \left(h^{\alpha}_{ik}h^{\beta}_{kj} - h^{\alpha}_{kj}h^{\beta}_{ik}\right). \tag{2.7}$$

The components h^{α}_{ijk} of the covariant derivative ∇B satisfy

$$\sum_{k} h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_{ij} - \sum_{k} h^{\alpha}_{ik}\omega_{kj} - \sum_{k} h^{\alpha}_{jk}\omega_{ki} - \sum_{\beta} h^{\beta}_{ij}\omega_{\beta\alpha}. \tag{2.8}$$

In this setting, from (2.4) and (2.8) we get Codazzi equation

$$\overline{R}_{\alpha ijk} = h^{\alpha}_{ijk} - h^{\alpha}_{ikj}. \tag{2.9}$$

The first and the second covariant derivatives of h^{α}_{ij} are denoted by h^{α}_{ijk} and h^{α}_{ijkl}, respectively, which satisfy

$$\sum_{l} h^{\alpha}_{ijkl}\omega_l = dh^{\alpha}_{ijk} - \sum_{l} h^{\alpha}_{ljk}\omega_{li} - \sum_{l} h^{\alpha}_{ilk}\omega_{lj} - \sum_{l} h^{\alpha}_{ijl}\omega_{lk} - \sum_{\beta} h^{\beta}_{ijk}\omega_{\beta\alpha}.$$

Thus, taking the exterior derivative in (2.8), we obtain the following Ricci identity

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = -\sum_{m} h^{\alpha}_{im}R_{mjkl} - \sum_{m} h^{\alpha}_{mj}R_{mikl}. \tag{2.10}$$

Restricting the covariant derivative $\overline{R}_{ABCD;E}$ of $\overline{R}_{ABCD}$ on M^n, then $\overline{R}_{\alpha ijk;l}$ is given by

$$\begin{aligned}\overline{R}_{\alpha ijkl} = \overline{R}_{\alpha ijk;l} &+ \sum_{\beta} \overline{R}_{\alpha\beta jk}h^{\beta}_{il} + \sum_{\beta} \overline{R}_{\alpha i\beta k}h^{\beta}_{jl} + \sum_{\beta} \overline{R}_{\alpha ij\beta}h^{\beta}_{kl}\\ &+ \sum_{m,k} \overline{R}_{mijk}h^{\alpha}_{lm}.\end{aligned} \tag{2.11}$$

where $\overline{R}_{\alpha ijkl}$ denotes the covariant derivative of $\overline{R}_{\alpha ijk}$ as a tensor on M^n.

3. Locally symmetric spaces and some auxiliary results

Proceeding with the context of the previous section, along this work we will assume that there exist constants c_1, c_2 and c_3 such that the sectional curvature $\overline{K}$ and the curvature tensor $\overline{R}$ of the ambient space L_p^{n+p} satisfies the following constraints:

$$\overline{K}(u,\eta) = \frac{c_1}{n}, \tag{3.1}$$

for any $u \in TM$ and $\eta \in TM^\perp$; when $p > 1$, suppose that

$$\langle \overline{R}(\xi, u)\eta, u\rangle = 0, \tag{3.2}$$

for $u \in TM$ and $\xi, \eta \in TM^\perp$, with $\langle \xi, \eta\rangle = 0$. Suppose also

$$\overline{K}(u,v) \geq c_2, \tag{3.3}$$

for any $u, v \in TM$; and

$$\overline{K}(\eta,\xi) = \frac{c_3}{p}, \tag{3.4}$$

for any $\eta, \xi \in TM^\perp$.

The curvature conditions (3.1) and (3.3), are natural extensions for higher codimension of conditions assumed by Nishikawa [16] in context of hypersurfaces. Obviously, when the ambient manifold L_p^{n+p} has constant sectional curvature c, then it satisfies conditions (3.1), (3.2), (3.3) and (3.4). On the other hand, the next example gives us a situation where the curvature conditions (3.1), (3.2), (3.3) and (3.4) are satisfied but the ambient space is not a space form.

Example 1. Let $L_p^{n+p} = \mathbb{R}_p^{n_1+p} \times \mathbb{N}_\kappa^{n_2}$ be a semi-Riemannian manifold, where $\mathbb{R}_p^{n_1+p}$ stands for the (n_1+p)-dimensional semi-Euclidean space of index p and $\mathbb{N}_\kappa^{n_2}$ is a n_2-dimensional Riemannian manifold of constant sectional curvature κ. We consider the spacelike submanifold $M^n = \Gamma^{n_1} \times \mathbb{N}_\kappa^{n_2}$ of L_p^{n+p}, where Γ^{n_1} is a spacelike submanifold of $\mathbb{R}_p^{n_1+p}$.

Taking into account that the normal bundle of $\Gamma^{n_1} \hookrightarrow \mathbb{R}_p^{n_1+p}$ is equipped with p linearly independent timelike vector fields $\xi^1, \xi^2, \ldots, \xi^p$, it is not difficult to verify that the sectional curvature $\overline{K}$ of L_p^{n+p} satisfies

$$\begin{aligned} \overline{K}\,(\xi_i, X) &= \left\langle R_{\mathbb{R}_p^{n_1+p}}(\xi^i, X_1)\xi^i, X_1\right\rangle_{\mathbb{R}_p^{n_1+p}} \\ &\quad + \left\langle R_{N_\kappa^{n_2}}(0, X_2)0, X_2\right\rangle_{N_\kappa^{n_2}} = 0, \end{aligned} \tag{3.5}$$

for each $i \in \{1, \ldots, p\}$, where $R_{\mathbb{R}_p^{n_1+p}}$ and $R_{N_\kappa^{n_2}}$ denote the curvature tensors of $\mathbb{R}_p^{n_1+p}$ and $\mathbb{N}_\kappa^{n_2}$, respectively, $\xi_i = (\xi^i, 0) \in T^\perp M$ and $X = (X_1, X_2) \in TM$ with $\langle \xi_i, \xi_i \rangle = \langle X, X \rangle = 1$.

On the other hand, by a direct computation we obtain

$$\overline{K}(X,Y) = \big\langle R_{\mathbb{R}_p^{n_1+p}}(X_1,Y_1)X_1,Y_1\big\rangle_{\mathbb{R}_p^{n_1+p}} + \big\langle R_{N_\kappa^{n_2}}(X_2,Y_2)X_2,Y_2\big\rangle_{N_\kappa^{n_2}} \quad (3.6)$$

for every $X = (X_1, X_2), Y = (Y_1, Y_2) \in TM$ such that $\langle X, Y \rangle = 0, \langle X, X \rangle = \langle Y, Y \rangle = 1$.

Consequently, from (3.6) we get

$$\overline{K}(X,Y) = \kappa\big(|X_2|^2|Y_2|^2 - \langle X_2, Y_2\rangle^2\big) \geq \min\{\kappa, 0\}. \quad (3.7)$$

Moreover, we have that

$$\overline{K}(\xi_i, \xi_j) = 0, \quad \text{for all} \quad i,j \in \{1,\ldots,p\} \quad (3.8)$$

and

$$\big\langle \overline{R}(\xi_i, X)\xi_j, X\big\rangle = 0, \quad \text{for all} \quad i,j \in \{1,\ldots,p\}. \quad (3.9)$$

We observe from (3.5), (3.7), (3.8) and (3.9) that the curvature constraints (3.1), (3.2), (3.3) and (3.4) are satisfied with $c_1 = c_3 = 0$ and $c_2 \leq \min\{\kappa, 0\}$.

Denote by $\overline{R}_{CD}$ the components of the Ricci tensor of L_p^{n+p}, then the scalar curvature $\overline{R}$ of L_p^{n+p} is given by

$$\overline{R} = \sum_A \varepsilon_A \overline{R}_{AA} = \sum_{i,j} \overline{R}_{ijij} - 2\sum_{i,\alpha} \overline{R}_{i\alpha i\alpha} + \sum_{\alpha,\beta} \overline{R}_{\alpha\beta\alpha\beta}.$$

If L_p^{n+p} satisfies conditions (3.1) and (3.4), then

$$\overline{R} = \sum_{i,j} \overline{R}_{ijij} - 2pc_1 + (p-1)c_3. \quad (3.10)$$

But, it is well known that the scalar curvature of a locally symmetric Lorentz space is constant. Consequently, $\sum_{i,j} \overline{R}_{ijij}$ is a constant naturally attached to a locally symmetric Lorentz space satisfying conditions (3.1) and (3.4). For sake of simplicity, in the course of this work we will denote the constant $\frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$ by $\overline{\mathcal{R}}$. In order to establish our main results, we devote this section to present some auxiliary lemmas. Using the ideas of the Proposition 2.2 of [19] we have

Lemma 1. *Let M^n be a linear Weingarten spacelike submanifold immersed in locally symmetric space L_p^{n+p} satisfying conditions* (3.1) *and* (3.4)*, such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \geq 0. \tag{3.11}$$

Then,

$$|\nabla B|^2 \geq n^2 |\nabla H|^2. \tag{3.12}$$

Moreover, if the equality holds in (3.12) *on M^n, then H is constant on M^n.*

Proof. Since we are supposing that $R = aH + b$ and L_p^{n+p} satisfies the conditions (3.1) and (3.4) then from equation (2.6) we get

$$2\sum_{i,j,\alpha} h_{ij}^\alpha h_{ijk}^\alpha = \big(2n^2 H + n(n-1)a\big) H_k, \tag{3.13}$$

where H_k stands for the k-th component of ∇H.

Thus,

$$4\sum_k \left(\sum_{i,j,\alpha} h_{ij}^\alpha h_{ijk}^\alpha\right)^2 = \big(2n^2 H + n(n-1)a\big)^2 |\nabla H|^2.$$

Consequently, using Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} 4S|\nabla B|^2 &= 4\sum_{i,j,\alpha} \left(h_{ij}^\alpha\right)^2 \sum_{i,j,k,\alpha} \left(h_{ijk}^\alpha\right)^2 \geq 4\sum_k \left(\sum_{i,j,\alpha} h_{ij}^\alpha h_{ijk}^\alpha\right)^2 \\ &= \big(2n^2 H + n(n-1)a\big)^2 |\nabla H|^2. \end{aligned} \tag{3.14}$$

On the other hand, since $R = aH + b$, from equation (2.6) we easily see that

$$\big(2n^2 H + n(n-1)a\big)^2 = n^2(n-1)\left[(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right)\right] + 4n^2 S. \tag{3.15}$$

Thus, from (3.14) and (3.15) we have

$$4S|\nabla B|^2 \geq n^2(n-1)\left[(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right)\right] + 4n^2 S|\nabla H|^2, \tag{3.16}$$

and taking account that since $(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \geq 0$, from (3.16) we obtain

$$S|\nabla B|^2 \geq S n^2 |\nabla H|^2.$$

Therefore, either $S = 0$ and $|\nabla B|^2 = n^2|\nabla H|^2 = 0$ or $|\nabla B|^2 \geq n^2|\nabla H|^2$.

Now suppose that $|\nabla B|^2 = n^2|\nabla H|^2$. If $(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) > 0$ then from (3.16) we have that H is constant. If $(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) = 0$, then from (3.15)

$$\left(2n^2H + n(n-1)a\right)^2 - 4n^2S = 0. \tag{3.17}$$

This together with (3.13) forces that

$$S_k^2 = 4n^2SH_k^2, \quad k = 1, \ldots, n, \tag{3.18}$$

where S_k stands for the k-th component of ∇S.

Since the equality in (3.14) holds, there exists a real function c_k on M^n such that

$$h_{ijk}^{n+1} = c_k h_{ij}^{n+1}; \quad h_{ijk}^{\alpha} = c_k h_{ij}^{\alpha}, \quad \alpha > n+1; \quad i,j,k = 1, \ldots, n. \tag{3.19}$$

Taking the sum on both sides of equation (3.19) with respect to $i = j$, we get

$$H_k = c_k H; \quad H_k^{\alpha} = 0, \quad \alpha > n+1; \quad k = 1, \ldots, n. \tag{3.20}$$

From second equation in (3.20) we can see that e_{n+1} is parallel. It follows from (3.19) that

$$S_k = 2\sum_{i,j,k,\alpha} h_{ij}^{\alpha} h_{ijk}^{\alpha} = 2c_k S, \quad k = 1, \ldots, n. \tag{3.21}$$

Multiplying both sides of equation (3.21) by H and using (3.20) we have

$$HS_k = 2H_k S, \quad k = 1, \ldots, n. \tag{3.22}$$

It follows from (3.18) and (3.22) that

$$H_k^2 S = H_k^2 n^2 H^2, \quad k = 1, \ldots, n. \tag{3.23}$$

Hence we have

$$|\nabla H|^2\left(S - n^2H^2\right) = 0. \tag{3.24}$$

We suppose that H is not constant on M^n. In this case, $|\nabla H|$ is not vanishing identically on M^n. Denote $M_0 = \{x \in M; |\nabla H| > 0\}$ and $T = S - n^2H^2$. It follows form (3.24) that M_0 is open in M and $T = 0$ over M_0. From the continuity of T, we have that $T = 0$ on the closure $cl(M_0)$ of M_0. If $M - cl(M_0) \neq \emptyset$, then H is constant in $M - cl(M_0)$. It follows that S is constant and hence T is constant in $M - cl(M_0)$. From the continuity of T, we have that $T = 0$ and hence $S = n^2H^2$ on M^n. It follows that H is constant on M^n, which contradicts the assumption. Hence we complete the proof. ∎

In our next result, we will deal with submanifolds M^n of L_p^{n+p} having parallel normalized mean curvature vector field, which means that the mean curvature function H is positive and that the corresponding normalized mean curvature vector field $\frac{\mathbf{H}}{H}$ is parallel as a section of the normal bundle. Extending the ideas of [9] we obtain the following Simons type formula for locally symmetric spaces.

LEMMA 2. *Let M^n be an n-dimensional ($n \geq 2$) submanifold with flat normal bundle and parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space L_p^{n+p}. Then, we have*

$$\begin{aligned}\frac{1}{2}\Delta S = |\nabla B|^2 &+ 2\Bigg(\sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha \overline{R}_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik}\Bigg)\\ &+ \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha i\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha k\beta i}\\ &+ \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta \overline{R}_{\alpha k\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{kk}^\beta \overline{R}_{\alpha i\beta j} \qquad (3.25)\\ &+ n\sum_{i,j} h_{ij}^{n+1} H_{ij} - nH \sum_{i,j,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1}\\ &+ \sum_{\alpha,\beta}\left[tr\left(h^\alpha h^\beta\right)\right]^2 + \frac{3}{2}\sum_{\alpha,\beta} N\left(h^\alpha h^\beta - h^\beta h^\alpha\right),\end{aligned}$$

where $N(A) = \mathrm{tr}(AA^t)$, for all matrix $A = (a_{ij})$.

Proof. Note that

$$\frac{1}{2}\Delta S = \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha + \sum_{i,j,k,\alpha}\left(h_{ijk}^\alpha\right)^2.$$

Using the definition of $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$ and the fact that $|\nabla B|^2 = \sum_{i,j,k}(h_{ijk}^\alpha)^2$ we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} h_{ij}^\alpha h_{ijkk}^\alpha + |\nabla B|^2.$$

Using the Codazzi equation (2.9) and the fact that $h_{ij}^\alpha = h_{ji}^\alpha$ we get

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} h_{ij}^\alpha \overline{R}_{\alpha ijkk} + \sum_{i,j,k,\alpha} h_{ij}^\alpha h_{kijk}^\alpha + |\nabla B|^2.$$

From (2.10) we obtain

$$\frac{1}{2}\Delta S = |\nabla B|^2 + \sum_{i,j,k,\alpha} h^\alpha_{ij}\overline{R}_{\alpha ijkk} + \sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{kikj} + \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{km}R_{mijk} + \\ + \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}.$$

Thence,

$$\frac{1}{2}\Delta S = |\nabla B|^2 + \sum_{i,j,k,\alpha} h^\alpha_{ij}\overline{R}_{\alpha ijkk} + \sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{kkij} + \sum_{i,j,k,\alpha} h^\alpha_{ij}\overline{R}_{\alpha kikj} + \\ + \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{km}R_{mijk} + \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}.$$

Using the Gauss equation (2.5) we get

$$\sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{km}R_{mijk} = \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{km}\overline{R}_{mijk} - \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\alpha_{km}h^\beta_{mj}h^\beta_{ik} + \\ + \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\alpha_{km}h^\beta_{mk}h^\beta_{ij}$$

and

$$\sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} = \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{mi}\overline{R}_{mkjk} - n\sum_{i,j,m,\alpha,\beta} h^\alpha_{ij}h^\alpha_{mi}h^\beta_{mj}H^\beta + \\ + \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj}.$$

Since we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $e_{n+1} = \frac{\mathbf{H}}{H}$, we have that

$$H^{n+1} = \frac{1}{n}tr(h^{n+1}) = H \quad \text{and} \quad H^\alpha = \frac{1}{n}tr(h^\alpha) = 0, \quad for \quad \alpha \geq n+2.$$

Thus, we get

$$\sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} = \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{mi}\overline{R}_{mkjk} - n\sum_{i,j,m,\alpha} h^\alpha_{ij}h^\alpha_{mi}h^{n+1}_{mj}H + \\ + \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj}.$$

Therefore,

$$\begin{aligned}\sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} &+ \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk}\\ &= \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} \overline{R}_{mijk} - \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{km}^{\alpha} h_{mj}^{\beta} h_{ik}^{\beta}\\ &\quad + \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{km}^{\alpha} h_{mk}^{\beta} h_{ij}^{\beta} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{mi}^{\alpha} \overline{R}_{mkjk}\\ &\quad - n \sum_{i,j,m,\alpha} h_{ij}^{\alpha} h_{mi}^{\alpha} h_{mj}^{n+1} H + \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{mi}^{\alpha} h_{mk}^{\beta} h_{kj}^{\beta}.\end{aligned}$$

From (2.11) we have

$$\begin{aligned}\sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha ijkk} &= \sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha ijk;k} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} \overline{R}_{\alpha\beta jk} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{jk}^{\beta} \overline{R}_{\alpha i\beta k}\\ &\quad + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kk}^{\beta} \overline{R}_{\alpha ij\beta} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} \overline{R}_{mijk}.\end{aligned}$$

Using the Ricci equation (2.7) , we conclude that

$$\sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} \overline{R}_{\alpha\beta jk} = \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} h_{jm}^{\alpha} h_{mk}^{\beta} - \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} h_{mk}^{\alpha} h_{jm}^{\beta}.$$

Thence,

$$\begin{aligned}\sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha ijkk} &= \sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha ijk;k} + \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} h_{jm}^{\alpha} h_{mk}^{\beta}\\ &\quad - \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} h_{mk}^{\alpha} h_{jm}^{\beta} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{jk}^{\beta} \overline{R}_{\alpha i\beta k}\\ &\quad + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kk}^{\beta} \overline{R}_{\alpha ij\beta} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} \overline{R}_{mijk}.\end{aligned}$$

On other hand

$$\begin{aligned}\sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha kikj} &= \sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha kik;j} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} \overline{R}_{\alpha\beta ik} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ij}^{\beta} \overline{R}_{\alpha k\beta k}\\ &\quad + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} \overline{R}_{\alpha ki\beta} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{jm}^{\alpha} \overline{R}_{mkik}.\end{aligned}$$

Thence,

$$\sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} \overline{R}_{\alpha\beta ik} = \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} h_{im}^{\alpha} h_{mk}^{\beta} - \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} h_{mk}^{\alpha} h_{im}^{\beta}.$$

Therefore,

$$\begin{aligned}\sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha kikj} = \sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha kik;j} &+ \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} h_{im}^{\alpha} h_{mk}^{\beta} \\ &- \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} h_{mk}^{\alpha} h_{im}^{\beta} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ij}^{\beta} \overline{R}_{\alpha k\beta k} \\ &+ \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} \overline{R}_{\alpha ki\beta} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{jm}^{\alpha} \overline{R}_{mkik}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i,j,k,\alpha} h_{ij}^{\alpha} (\overline{R}_{\alpha ijkk} + \overline{R}_{\alpha kikj}) = &\sum_{i,j,k,\alpha} h_{ij}^{\alpha} (\overline{R}_{\alpha ijk;k} + \overline{R}_{\alpha kik;j}) \\ &+ \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} h_{jm}^{\alpha} h_{mk}^{\beta} - \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\beta} h_{mk}^{\alpha} h_{jm}^{\beta} \\ &+ \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{jk}^{\beta} \overline{R}_{\alpha i\beta k} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kk}^{\beta} \overline{R}_{\alpha ij\beta} \\ &+ \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} \overline{R}_{mijk} + \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} h_{im}^{\alpha} h_{mk}^{\beta} \\ &- \sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} h_{mk}^{\alpha} h_{im}^{\beta} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ij}^{\beta} \overline{R}_{\alpha k\beta k} \\ &+ \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kj}^{\beta} \overline{R}_{\alpha ki\beta} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{jm}^{\alpha} \overline{R}_{mkik}.\end{aligned}$$

Since L_p^{n+p} is locally symmetric, we have that

$$\sum_{i,j,k,\alpha} h_{ij}^{\alpha} (\overline{R}_{\alpha ijk;k} + \overline{R}_{\alpha kik;j}) = 0.$$

Thus,

$$\begin{aligned}\sum_{i,j,k,\alpha} h^\alpha_{ij}(\overline{R}_{\alpha ijkk} + \overline{R}_{\alpha kikj}) = &\sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{ik}h^\alpha_{jm}h^\beta_{mk} - \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{ik}h^\alpha_{mk}h^\beta_{jm}\\ &+ \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{jk}\overline{R}_{\alpha i\beta k} + \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{kk}\overline{R}_{\alpha ij\beta}\\ &+ \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{km}\overline{R}_{mijk} + \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{kj}h^\alpha_{im}h^\beta_{mk}\\ &- \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{kj}h^\alpha_{mk}h^\beta_{im} + \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}\overline{R}_{\alpha k\beta k}\\ &+ \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{kj}\overline{R}_{\alpha ki\beta} + \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{jm}\overline{R}_{mkik}.\end{aligned}$$

Now, observe that

$$\sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{kkij} = n\sum_{i,j,\alpha} h^\alpha_{ij}H^\alpha_{ij}.$$

Using the fact that $H_{kj} = H^{n+1}_{kl}$ and $H^\alpha_{kj} = 0$, for $\alpha > n+1$ we have

$$\sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{kkij} = n\sum_{i,j} h^{n+1}_{ij}H_{ij}.$$

Finally, we conclude that

$$\begin{aligned}\frac{1}{2}\Delta S = |\nabla B|^2 &+ \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{ik}h^\alpha_{jm}h^\beta_{mk} - \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{ik}h^\alpha_{mk}h^\beta_{jm}\\ &+ \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{jk}\overline{R}_{\alpha i\beta k} + nH\sum_{i,j,k,\alpha} h^\alpha_{ij}\overline{R}_{\alpha ijn+1}\\ &+ \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{km}\overline{R}_{mijk} + \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{kj}h^\alpha_{im}h^\beta_{mk}\\ &- \sum_{i,j,k,m,\alpha,\beta} h^\alpha_{ij}h^\beta_{kj}h^\alpha_{mk}h^\beta_{im} + \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}\overline{R}_{\alpha k\beta k} +\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha}h_{kj}^{\beta}\overline{R}_{\alpha ki\beta}+\sum_{i,j,k,m,\alpha} h_{ij}^{\alpha}h_{jm}^{\alpha}\overline{R}_{mkik}+n\sum_{i,j} h_{ij}^{n+1}H_{ij}\\
&+\sum_{i,j,k,m,\alpha} h_{ij}^{\alpha}h_{km}^{\alpha}\overline{R}_{mijk}-\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{km}^{\alpha}h_{mj}^{\beta}h_{ik}^{\beta}\\
&+\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}+\sum_{i,j,k,m,\alpha} h_{ij}^{\alpha}h_{mi}^{\alpha}\overline{R}_{mkjk}\\
&-nH\sum_{i,j,m,\alpha} h_{ij}^{\alpha}h_{mi}^{\alpha}h_{mj}^{n+1}+\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{mi}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}.
\end{aligned}
\tag{3.26}
$$

Note that

$$
\begin{aligned}
&\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{ik}^{\beta}h_{jm}^{\alpha}h_{mk}^{\beta}-\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{ik}^{\beta}h_{mk}^{\alpha}h_{jm}^{\beta}\\
&\qquad=\sum_{\alpha,\beta} tr\big(h^{\alpha}h^{\beta}h^{\beta}h^{\alpha}\big)-\sum_{\alpha,\beta} tr\big(h^{\alpha}h^{\beta}\big)^{2},
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}-\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{km}^{\alpha}h_{mj}^{\beta}h_{ik}^{\beta}\\
&\qquad=\sum_{\alpha,\beta} \big[tr\big(h^{\alpha}h^{\beta}\big)\big]^{2}-\sum_{\alpha,\beta} tr\big(h^{\alpha}h^{\beta}\big)^{2},
\end{aligned}
\tag{3.28}
$$

$$
\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{kj}^{\beta}h_{im}^{\alpha}h_{mk}^{\beta}=\sum_{\alpha,\beta} tr\big(h^{\alpha}h^{\beta}h^{\beta}h^{\alpha}\big)
\tag{3.29}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{mi}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}-\sum_{i,j,k,m,\alpha,\beta} h_{ij}^{\alpha}h_{kj}^{\beta}h_{mk}^{\alpha}h_{im}^{\beta}\\
&\qquad=\frac{1}{2}\sum_{\alpha,\beta} N\big(h^{\alpha}h^{\beta}-h^{\beta}h^{\alpha}\big).
\end{aligned}
\tag{3.30}
$$

Therefore, inserting (3.27), (3.28), (3.29) and (3.30) into (3.26) we complete the proof. ∎

In order to study linear Weingarten submanifolds, we will consider, for each $a \in \mathbb{R}$, an appropriated Cheng-Yau's modified operator, which is given by

$$
L=\Box+\frac{n-1}{2}a\Delta,
\tag{3.31}
$$

where, according to [7], the square operator is defined by

$$\Box f = \sum_{i,j} \left(nH\delta_{ij} - nh^{n+1}\right) f_{ij}, \tag{3.32}$$

for each $f \in C^\infty(M)$, and the normal vector field e_{n+1} is taken in the direction of the mean curvature vector field, that is, $e_{n+1} = \frac{\mathbf{H}}{H}$.

The next lemma guarantees us the existence of an Omori-type sequence related to the operator L.

Lemma 3. *Let M^n be a complete linear Weingarten spacelike in a locally symmetric semi-Riemannian space $L_p^{n+p}(c)$ satisfying conditions* (3.1), (3.3) *and* (3.4), *such that $R = aH + b$, with $a \geq 0$ and $(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \geq 0$. If H is bounded on M^n, then there is a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^n$ such that*

$$\lim_k nH(q_k) = \sup_M nH, \quad \lim_k |\nabla nH(q_k)| = 0 \quad \textit{and} \quad \limsup_k L(nH(q_k)) \leq 0.$$

Proof. Let us choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$. From (3.31) we have that

$$L(nH) = n\sum_i \left(nH + \frac{n-1}{2}a - \lambda_i^{n+1}\right) H_{ii}.$$

Thus, for all $i = 1, \ldots, n$ and since that L_p^{n+p} satisfies the conditions (3.1) and (3.4) then from (2.6) and with straightforward computation we get

$$\begin{aligned}
(\lambda_i^{n+1})^2 \leq S &= n^2H^2 + n(n-1)\left(aH + b - \overline{\mathcal{R}}\right)\\
&= \left(nH + \frac{n-1}{2}a\right)^2 - \frac{n-1}{4}\left[(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right)\right]\\
&\leq \left(nH + \frac{n-1}{2}a\right)^2,
\end{aligned}$$

where we have used our assumption that $(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \geq 0$ to obtain the last inequality. Consequently, for all $i = 1, \ldots, n$, we have

$$|\lambda_i^{n+1}| \leq \left|nH + \frac{n-1}{2}a\right|. \tag{3.33}$$

Thus, from (2.6) we obtain

$$\begin{aligned} R_{ijij} &= \overline{R}_{ijij} - \sum_\alpha h^\alpha_{ii} h^\alpha_{jj} + \sum_\alpha (h^\alpha_{ij})^2 \\ &\geq \overline{R}_{ijij} - \sum_\alpha h^\alpha_{ii} h^\alpha_{jj}. \end{aligned}$$

Since

$$S \leq \left(nH + \frac{n-1}{2} a \right)^2,$$

we get that

$$(h^\alpha_{ij})^2 \leq \left(nH + \frac{n-1}{2} a \right)^2,$$

for every α, i, j and, hence, from (3.33) we have

$$h^\alpha_{ii} h^\alpha_{jj} \leq |h^\alpha_{ii}||h^\alpha_{jj}| \leq \left(nH + \frac{n-1}{2} a \right)^2.$$

Therefore, since we are supposing that H is bounded on M^n and L^{n+p}_p satisfies the condition (3.3), this is, $\overline{R}_{ijij} \geq c_2$, it follows that the sectional curvatures of M^n are bounded from below. Thus, we may apply the well known generalized maximum principle of Omori [17] to the function nH, obtaining a sequence of points $\{q_k\}_{k\in\mathbb{N}}$ in M^n such that

$$\begin{aligned} &\lim_k nH(q_k) = \sup nH, \quad \lim_k |\nabla nH(q_k)| = 0, \quad \text{and} \\ &\limsup_k \sum_i nH_{ii}(q_k) \leq 0. \end{aligned} \tag{3.34}$$

Since $\sup_M H > 0$, taking subsequences if necessary, we can arrive to a sequence $\{q_k)_{k\in\mathbb{N}}$ in M^n which satisfies (3.34) and such that $H(q_k) \geq 0$. Hence, since $a \geq 0$, we have

$$\begin{aligned} 0 &\leq nH(q_k) + \frac{n-1}{2} a - |\lambda^{n+1}_i(q_k)| \leq nH(q_k) + \frac{n-1}{2} a - \lambda^{n+1}_i(q_k) \\ &\leq nH(q_k) + \frac{n-1}{2} a + |\lambda^{n+1}_i(q_k)| \leq 2nH(q_k) + (n-1)a. \end{aligned}$$

This previous estimate shows that function $nH(q_k) + \frac{n-1}{2}a - \lambda_i^{n+1}(q_k)$ is non-negative and bounded on M^n, for all $k \in \mathbb{N}$. Therefore, taking into account (3.34), we obtain

$$\limsup_k (L(nH)(q_k)) \leq n \sum_i \limsup_k \left[\left(nH + \frac{n-1}{2}a - \lambda_i^{n+1} \right) (q_k) H_{ii}(q_k) \right] \leq 0. \blacksquare$$

We close this section with the following algebraic lemma, whose proof can be found in [18].

Lemma 4. *Let $A, B : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $tr(A) = tr(B) = 0$. Then*

$$\left| tr(A^2 B) \right| \leq \frac{n-2}{\sqrt{n(n-1)}} N(A) \sqrt{N(B)},$$

where $N(A) = \mathrm{tr}(AA^t)$, for all matrix $A = (a_{ij})$. Moreover, the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B and corresponding eigenvalues y_i of A satisfy

$$|x_i| = \sqrt{\frac{N(B)}{n(n-1)}}, \; x_i y_i \geq 0 \quad \text{and} \quad y_i = \sqrt{\frac{N(A)}{n(n-1)}} \; \left(\text{resp.} -\sqrt{\frac{N(A)}{n(n-1)}} \right).$$

4. Main results

As before, the normal vector field e_{n+1} is taken in the direction of the mean curvature vector field, that is, $e_{n+1} = \frac{\mathbf{H}}{H}$. In this setting, we will consider the following symmetric tensor

$$\Phi = \sum_{i,j,\alpha} \Phi_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha},$$

where $\Phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij}$ and $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha}$, $n+2 \leq \alpha \leq n+p$. Let $|\Phi|^2 = \sum_{i,j,\alpha} (\Phi_{ij}^{\alpha})^2$ be the square of the length of Φ.

Remark 1. Since the normalized mean curvature vector of M^n is parallel, we have $\omega_{n+1\alpha} = 0$, for $\alpha > n+1$. Thus, from of the structure equations of the normal bundle of M^n, it follows that $\overline{R}_{n+1\beta ij} = 0$, for all α, i, j. Hence, from Ricci equation, we have that $h^{n+1}h^{\alpha} - h^{\alpha}h^{n+1} = 0$, for all α. This implies that the matrix h^{n+1} commutes with all the matrix h^α. Thus, being $\Phi^\alpha = (\Phi^\alpha_{ij})$, we have that $\Phi^\alpha = h^\alpha - H^\alpha$ and, hence $\Phi^{n+1} = h^{n+1} - H^{n+1}$ and $\Phi^\alpha = h^\alpha$, for $\alpha > n+1$. These form, Φ^{n+1} commutes with all the matrix Φ^α. Since the matrix Φ^α is traceless and symmetric, once the matrix h^α are symmetric, we can use Lemma 4 for the matrix Φ^α and Φ^{n+1} in order to obtain

$$\left|tr\big((\Phi^\alpha)^2\Phi^{n+1}\big)\right| \le \frac{n-2}{\sqrt{n(n-1)}} N(\Phi^\alpha)\sqrt{N(\Phi^{n+1})}. \tag{4.1}$$

Summing (4.1) in α, we have

$$\sum_\alpha \left|tr\big((\Phi^\alpha)^2\Phi^{n+1}\big)\right| \le \frac{n-2}{\sqrt{n(n-1)}} \sum_\alpha N(\Phi^\alpha)\sqrt{N(\Phi^{n+1})}.$$

In order to prove our characterization results, it will be essential the following lower boundedness for the Laplacian operator acting on the square of the length of the second fundamental form. If L^{n+p}_p is a space form then from [8] follows that $R^\perp = 0$ if and only if there exists an orthogonal basis for TM that diagonalizes simultaneously all $B_\xi, \xi \in TM^\perp$.

Proposition 1. *Let M^n be a linear Weingarten spacelike submanifold in a semi-Riemannian locally symmetric space L^{n+p}_p satisfying conditions* (3.1), (3.2), (3.3) *and* (3.4), *with parallel normalized mean curvature vector field and flat normal bundle. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all $B_\xi, \xi \in TM^\perp$. If M^n is such that $R = aH + b$, with $(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \ge 0$ and $c = \frac{c_1}{n} + 2c_2$, then*

$$L(nH) \ge |\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(H^2 - c)\right).$$

Proof. Let us consider $\{e_1, \dots, e_n\}$ a local orthonormal frame on M^n such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$, for all $\alpha \in \{n+1, \dots, n+p\}$. From (3.25), we get

$$\begin{aligned} &2\left(\sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{km}\overline{R}_{mijk} + \sum_{i,j,k,m,\alpha} h^\alpha_{ij}h^\alpha_{jm}\overline{R}_{mkik}\right) \\ &\qquad = 2\sum_{i,k,\alpha}\big((\lambda^\alpha_i)^2\overline{R}_{ikik} + \lambda^\alpha_i\lambda^\alpha_k\overline{R}_{kiik}\big) = \sum_{i,k,\alpha}\overline{R}_{ikik}(\lambda^\alpha_i - \lambda^\alpha_k)^2. \end{aligned}$$

Since that L_p^{n+p} satisfies the condition (3.3) we have

$$\begin{aligned}
2\Bigg(\sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} \overline{R}_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{jm}^{\alpha} \overline{R}_{mkik}\Bigg) \\
\geq c_2 \sum_{i,k,\alpha} \left(\lambda_i^{\alpha} - \lambda_k^{\alpha}\right)^2 = 2nc_2|\Phi|^2.
\end{aligned} \tag{4.2}$$

Now, for each α, consider h^{α} the symmetric matrix (h_{ij}^{α}), and

$$S_{\alpha\beta} = \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta}.$$

Then the $(p \times p)$-matrix $(S_{\alpha\beta})$ is symmetric and we can see that is diagonalizable for a choose of $e_{n+1}, \ldots, e_{n+p}$. Thence,

$$S_{\alpha} = S_{\alpha\alpha} = \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\alpha},$$

and we have that

$$S = \sum_{\alpha} S_{\alpha}.$$

Since that L_p^{n+p} satisfies the condition (3.2) we obtain

$$\begin{aligned}
\sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{jk}^{\beta} \overline{R}_{\alpha i\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{jk}^{\beta} \overline{R}_{\alpha k\beta i} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ij}^{\beta} \overline{R}_{\alpha k\beta k} \\
- \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kk}^{\beta} \overline{R}_{\alpha i\beta j} = \sum_{i,k,\alpha} (\lambda_i^{\alpha})^2 \overline{R}_{\alpha k\alpha k} - nH^2 c_1.
\end{aligned}$$

Since that L_p^{n+p} satisfies the condition (3.1) we conclude that

$$\begin{aligned}
\sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{jk}^{\beta} \overline{R}_{\alpha i\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{jk}^{\beta} \overline{R}_{\alpha k\beta i} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ij}^{\beta} \overline{R}_{\alpha k\beta k} \\
- \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kk}^{\beta} \overline{R}_{\alpha i\beta j} = c_1|\Phi|^2.
\end{aligned} \tag{4.3}$$

Finally note that

$$\sum_{\alpha,\beta} N\left(h^{\alpha} h^{\beta} - h^{\beta} h^{\alpha}\right) \geq 0. \tag{4.4}$$

Therefore, from (3.25) and using (4.2), (4.3) and (4.4) we conclude that

$$\begin{aligned}\frac{1}{2}\Delta S \geq |\nabla B|^2 + cn|\Phi|^2 + n\sum_{i,j} h_{ij}^{n+1}H_{ij}\\ - nH\sum_{i,j,m,\alpha} h_{ij}^{\alpha}h_{mi}^{\alpha}h_{mj}^{n+1} + \sum_{\alpha,\beta}\big[tr\big(h^{\alpha}h^{\beta}\big)\big]^2.\end{aligned} \tag{4.5}$$

From (3.31) we have

$$\begin{aligned}L(nH) &= \Box(nH) + \frac{n-1}{2}a\Delta(nH)\\ &= \sum_{i,j}\big(nH\delta_{ij} - h_{ij}^{n+1}\big)(nH)_{ij} + \frac{n-1}{2}a\Delta(nH)\\ &= n^2H\sum_i H_{ii} - n\sum_{i,j} h_{ij}^{n+1}H_{ij} + \frac{n-1}{2}a\Delta(nH)\\ &= n^2H\Delta H - n\sum_{i,j} h_{ij}^{n+1}H_{ij} + \frac{n-1}{2}a\Delta(nH).\end{aligned}$$

Note that

$$\Delta H^2 = 2H\Delta H + 2|\nabla H|^2.$$

Thus,

$$L(nH) = \frac{1}{2}\Delta\big(n^2H^2\big) - n^2|\nabla H|^2 - n\sum_{i,j} h_{ij}^{n+1}H_{ij} + \frac{n-1}{2}a\Delta(nH).$$

Since that $R = aH + b$ and L_p^{n+p} satisfies the conditions (3.1) and (3.3) we have that $\mathcal{R}$ is constant then from (2.6) we get

$$\frac{1}{2}n(n-1)\Delta(aH) + \frac{1}{2}\Delta\big(n^2H^2\big) = \frac{1}{2}\Delta S.$$

Therefore, using the inequality (4.5) and Lemma 1 we conclude that

$$\begin{aligned}L(nH) &= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - n\sum_{i,j} h_{ij}^{n+1}H_{ij}\\ &\geq |\nabla B|^2 - n^2|\nabla H|^2 + cn|\Phi|^2\\ &\quad - nH\sum_{i,j,m,\alpha} h_{ij}^{\alpha}h_{mi}^{\alpha}h_{mj}^{n+1} + \sum_{\alpha,\beta}\big[tr\big(h^{\alpha}h^{\beta}\big)\big]^2\\ &\geq cn|\Phi|^2 - nH\sum_{i,j,m,\alpha} h_{ij}^{\alpha}h_{mi}^{\alpha}h_{mj}^{n+1} + \sum_{\alpha,\beta}\big[tr\big(h^{\alpha}h^{\beta}\big)\big]^2.\end{aligned}$$

On the other hand, with a straightforward computation we guarantee that

$$\begin{aligned} &-nH\sum_{\alpha} tr\left[h^{n+1}\big(h^{\alpha}\big)^2\right]+\sum_{\alpha,\beta}\left[tr\big(h^{\alpha}h^{\beta}\big)\right]^2 \\ &\quad = -nH\sum_{\alpha} tr\left[\Phi^{n+1}\big(\Phi^{\alpha}\big)^2\right]-nH^2|\Phi|^2+\sum_{\alpha,\beta}\left[tr\big(\Phi^{\alpha}\Phi^{\beta}\big)\right]^2 \\ &\quad \geq \frac{-n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3-nH^2|\Phi|^2+\frac{|\Phi|^4}{p}. \end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned} L(nH) &\geq cn|\Phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3-nH^2|\Phi|^2+\frac{|\Phi|^4}{p} \\ &= |\Phi|^2 P_{H,p,c}(|\Phi|), \end{aligned} \tag{4.7}$$

where

$$P_{H,p,c}(x)=\frac{x^2}{p}-\frac{n(n-2)}{\sqrt{n(n-1)}}Hx-n\big(H^2-c\big).$$

∎

When $c>0$, if $H^2\geq\frac{4(n-1)c}{Q(p)}$, where $Q(p)=(n-2)^2p+4(n-1)$, then the polynomial $P_{H,p,c}$ defined by

$$P_{H,p,c}(x)=\frac{x^2}{p}-\frac{n(n-2)}{\sqrt{n(n-1)}}Hx-n(H^2-c)$$

has (at least) a positive real root given by

$$C(n,p,H)=\frac{\sqrt{n}}{2\sqrt{n-1}}\left(p(n-2)H+\sqrt{pQ(p)H^2-4p(n-1)c}\right).$$

On the other hand, in the case that $c\leq 0$, the same occurs without any restriction on the values of the mean curvature function H. Now, we are in position to present our first theorem.

THEOREM 1. *Let M^n be a complete linear Weingarten spacelike submanifold in locally symmetric L_p^{n+p} satisfying conditions* (3.1), (3.2), (3.3) *and* (3.4), *with parallel normalized mean curvature vector and flat normal bundle, such that $R=aH+b$ with $a\geq 0$ and $(n-1)a^2+4n\left(\overline{\mathcal{R}}-b\right)\geq 0$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously* *all $B_\xi, \xi \in TM^\perp$. When $c > 0$, assume in addition that $H^2 \geq \frac{4(n-1)c}{Q(p)}$. If H is bounded on M^n and $|\Phi| \geq C(n, p, \sup H)$, then $p = 1$ and M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

Proof. Since we are assuming that $a \geq 0$ and that inequality (3.11) holds, we can apply Lemma 3 to the function nH in order to obtain a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^n$ such that

$$\lim_k nH(q_k) = \sup_M nH, \quad \text{and} \quad \limsup_k L(nH)(q_k) \leq 0. \tag{4.8}$$

Thus, from (4.7) and (4.8) we have

$$0 \geq \limsup_k L(nH)(q_k) \geq \sup_M |\Phi|^2 P_{\sup H,p,c}\left(\sup_M |\Phi|\right). \tag{4.9}$$

On the other hand, our hypothesis imposed on $|\Phi|$ guarantees us that $\sup_M |\Phi| > 0$. Therefore, from (4.9) we conclude that

$$P_{\sup H,p,c}\left(\sup_M |\Phi|\right) \leq 0. \tag{4.10}$$

Suppose, initially, the case $c > 0$. From our restrictions on H and $|\Phi|$, we have that $P_{H,p,c}(|\Phi|) \geq 0$, with $P_{H,p,c}(|\Phi|) = 0$ if, and only if, $|\Phi| = C(n, p, H)$. Consequently, from (4.10) we get

$$\sup_M |\Phi| = C(n, p, \sup H).$$

Taking into account once more our restriction on $|\Phi|$, we have that $|\Phi|$ is constant on M^n. Thus, since M^n is a linear Weingarten submanifold, from (3.11) we have that H is also constant on M^n. Hence, from (4.7) we obtain

$$0 = L(nH) \geq |\Phi|^2 P_{H,p,c}(|\Phi|) \geq 0.$$

Since $|\Phi| > 0$, we must have $P_{H,p,c}(|\Phi|) = 0$. Thus, all inequalities obtained along the proof of Proposition 1 are, in fact, equalities. In particular, from inequality (4.6) we conclude that

$$tr(\Phi^{n+1}) = |\Phi|^2.$$

So, from (2.6) we get

$$tr(\Phi^{n+1})^2 = |\Phi|^2 = S - nH^2. \tag{4.11}$$

On the other hand, we also have that

$$tr\big(\Phi^{n+1}\big)^2 = S - \sum_{\alpha>n+1}\sum_{i,j}\big(h_{ij}^{\alpha}\big)^2 - nH^2. \tag{4.12}$$

Thus, from (4.11) and (4.12) we conclude that $\sum_{\alpha>n+1}\sum_{i,j}(h_{ij}^{\alpha})^2 = 0$. But, from inequality (4.6) we also have that

$$|\Phi|^4 = p\sum_{\alpha}\big[N\big(\Phi^{\alpha}\big)\big]^2 = pN\big(\Phi^{n+1}\big)^2 = p|\Phi|^4. \tag{4.13}$$

Hence, since $|\Phi| > 0$, we must have that $p = 1$. In this setting, from (3.12) and (4.13) we get

$$\sum_{i,j,k}\big(h_{ijk}^{n+1}\big)^2 = |\nabla B|^2 = n^2|\nabla H|^2 = 0,$$

that is, $h_{ijk}^{n+1} = 0$ for all i, j. Hence, we obtain that M^n is an isoparametric hypersurface of L_p^{n+p}.

When $c \leq 0$, we proceed as before until reach equation (4.10) and, from $|\Phi| \geq C(n, p, \sup H)$, we have that $P_{H,p,c}(|\Phi|) \geq 0$. At this point, we can reason as in the previous case to obtain that H is constant, $p = 1$ and, consequently, we also conclude that M^n is an isoparametric hypersurface of L_p^{n+p}. Hence, since the equality occurs in (4.1), we have that also occurs the equality in Lemma 4. Consequently, M^n has at most two distinct constant principal curvatures. ∎

In particular, when the immersed submanifold has constant scalar curvature, from Theorem 1 we obtain the following

Corollary 1. *Let M^n be a complete spacelike submanifold in locally symmetric semi-Riemannian L_p^{n+p} satisfying conditions* (3.1), (3.2), (3.3) *and* (3.4), *with parallel normalized mean curvature vector, flat normal bundle and constant normalized scalar curvature R satisfying $R \leq c$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all $B_{\xi}, \xi \in TM^{\perp}$. When $c > 0$, assume in addition that $H^2 \geq \frac{4(n-1)c}{Q(p)}$. If H is bounded on M^n and $|\Phi| \geq C(n, p, \sup H)$, then $p = 1$ and M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

In order to establish our next theorem, we will need of the following lemma obtained by Caminha, which can be regarded as an extension of Hopf's maximum principle for complete Riemannian manifolds (cf. Proposition 2.1 of [5]). In what follows, let $\mathcal{L}^1(M)$ denote the space of Lebesgue integrable functions on M^n.

LEMMA 5. *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold M^n, such that $div_M X$ does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then $div_M X = 0$.*

We close our paper stating and proving our second characterization theorem.

THEOREM 2. *Let M^n be a complete linear Weingarten spacelike submanifold in locally symmetric Einstein semi-Riemannian L_p^{n+p} satisfying conditions (3.1), (3.2), (3.3) and (3.4), with parallel normalized mean curvature vector, flat normal bundle such that $R = aH + b$, with $(n-1)a^2 + 4n(\overline{\mathcal{R}} - b) \geq 0$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all $B_\xi, \xi \in TM^\perp$. When $c > 0$, assume in addition that $H^2 \geq \frac{4(n-1)c}{Q(p)}$. If H is bounded on M^n, $|\Phi| \geq C(n, p, H)$ and $|\nabla H| \in \mathcal{L}^1(M)$, then $p = 1$ and M^n is a isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

Proof. Since the ambient space L_p^{n+p} is supposed to be Einstein, reasoning as in the first part of the proof of Theorem 1.1 in [10], from (3.31) and (3.32) it is not difficult to verify that

$$L(nH) = div_M(P(\nabla H)), \tag{4.14}$$

where

$$P = \left(n^2 H + \frac{n(n-1)}{2} a\right) I - nh^{n+1}. \tag{4.15}$$

On the other hand, since $R = aH + b$ and H is bounded on M^n, from equation (2.6) we have that B is bounded on M^n. Consequently, from (4.15) we conclude that the operator P is bounded, that is, there exists C_1 such that $|P| \leq C_1$. Since we are also assuming that $|\nabla H| \in \mathcal{L}^1(M)$, we obtain that

$$|P(\nabla H)| \leq |P||\nabla H| \leq C_1 |\nabla H| \in \mathcal{L}^1(M). \tag{4.16}$$

So, from Lemma 5 and (4.14) we obtain that $L(nH) = 0$ on M^n. Thus,

$$0 = L(nH) \geq |\Phi|^2 P_{H,p,c}(|\Phi|) \geq 0 \tag{4.17}$$

and, consequently, we have that all inequalities are, in fact, equalities. In particular, from (3.11) we obtain

$$|\nabla B|^2 = n^2 |\nabla H|^2. \tag{4.18}$$

Hence, Lemma 1 guarantees that H is constant. At this point, we can proceed as in the last part of the proof of Theorem 1 to conclude our result. ∎

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