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Real Analytic Version of Lévy's Theorem

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Abstract: We obtain real analytic version of the classical theorem of Lévy on absolutely convergent power series. Whence, as a consequence, its harmonic version.

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1. Introduction

Let A be a complex Banach algebra with the involution $x \longmapsto x^*$ and unit e. The spectrum of an element x of A will be denoted by Spx. An element h of A is called hermitian if $h^* = h$. The set of all Hermitian elements of A will be denoted by $H(A)$. We say that the Banach algebra A is Hermitian if the spectrum of every element of $H(A)$ is real ([9]). For scalars λ, we often write simply λ for the element λe of A. Let $p \in]1, +\infty[$. We say that ω is a weight on $\mathbb{Z}$ if $\omega : \mathbb{Z} \longrightarrow [1, +\infty[$, is a map satisfying

$$c(\omega) = \sum_{n\in\mathbb{Z}} \omega(n)^{\frac{1}{1-p}} < +\infty. \tag{1}$$

We consider the following weighted space:

$$\mathcal{A}^p(\omega) = \{ f : \mathbb{R} \longrightarrow \mathbb{C} : f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}, \ \ a_n \in l^p(\mathbb{Z}, \omega) \}.$$

Endowed with the norm $\|.\|_{p,\omega}$ defined by:

$$\|f\|_{p,\omega} = \Big(\sum_{n\in\mathbb{Z}} |a_n|^p \omega(n) \Big)^{\frac{1}{p}}, \text{ for every } f \in \mathcal{A}^p(\omega),$$

the space $\mathcal{A}^p(\omega)$ becomes a Banach space. Moreover, if there exists a constant $\gamma = \gamma(\omega) > 0$ such that

$$\omega^{\frac{1}{1-p}} * \omega^{\frac{1}{1-p}} \leq \gamma \omega^{\frac{1}{1-p}} \tag{2}$$

then $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ is closed under pointwise multiplication and it is a commutative semi-simple Banach algebra with unity element $\widehat{e}$ given by $\widehat{e}(t) = 1$ $(t \in \mathbb{R})$ ([4]). For the weight function ω on $\mathbb{Z}$ satisfying (2) and

$$\omega(n+m) \leq \omega(n)\omega(m), \text{ for every } n, m \in \mathbb{Z}, \tag{3}$$

it is also shown in ([4]), that the character space of $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ can be identified with the closed annulus:

$$\Gamma_\omega(\rho_1, \rho_2) = \{\xi \in \mathbb{C} : \rho_1(\omega) \leq |\xi| \leq \rho_2(\omega)\},$$

in such a way that each character has the form $f \longmapsto \sum_{n\in\mathbb{Z}} a_n \xi^n$ for some $\xi \in \Gamma_\omega(\rho_1, \rho_2)$, where $f = \sum_{n\in\mathbb{Z}} a_n u^n \in \mathcal{A}^p(\omega)$ with $u(t) = e^{it}$, for every $t \in \mathbb{R}$. For ρ_1 and ρ_2, they are given by:

$$\rho_1 = e^{-\sigma_2} \text{ and } \rho_2 = e^{-\sigma_1}$$

where

$$\sigma_1 = \sup\left\{\frac{-1}{np}\ln(\omega(n)),\ n \geq 1\right\} \text{ and } \sigma_2 = \inf\left\{\frac{1}{np}\ln(\omega(-n)),\ n \geq 1\right\}.$$

The real analytic functional calculus is defined and studied in [1]. To make the paper self-contained, we recall the fundamental properties of this calculus. Let U be an open subset of $\mathbb{R}^2$ and $F : U \longrightarrow \mathbb{C}$ be real analytic function. Then there exists an open subset V, of $\mathbb{C}^2$, and an holomorphic function $\widetilde{F} : V \longrightarrow \mathbb{C}$ such that

$$V \cap \mathbb{R}^2 = U \text{ and } \widetilde{F}_{|U} = F.$$

For the construction of V, we have $V = \bigcup_{x\in U} \Omega_x$, where Ω_x is an open of $\mathbb{C}^2$ centered at x. We denote by $\Lambda_0(U)$ the set of all open subset V described us above and we consider, in $\Lambda_0(U)$, the order given in the following way:

$$V \preceq W \Longleftrightarrow W \subset V.$$

For $V \in \Lambda_0(U)$, we denote by $\mathcal{O}(V)$ the set of holomorphic functions on V. Now we consider the family $(\mathcal{O}(V))_{V\in\Lambda_0(U)}$ of algebras and for every $V, W \in \Lambda_0(U)$ with $V \preceq W$, let

$$\pi_{W,V} : \mathcal{O}(V) \longrightarrow \mathcal{O}(W) : F \longmapsto F_{|W}$$

The family of algebras $(\mathcal{O}(V))_{V\in\Lambda_0(U)}$ with the maps $\pi_{W,V}$ is an inductive system of algebras and it is denoted by $(\mathcal{O}(V), \pi_{W,V})$. Let $\varinjlim (\mathcal{O}(V), \pi_{W,V})$ its inductive limit. We shall denote this simply by $\varinjlim\mathcal{O}(V)$ and we have:

$$\varinjlim\mathcal{O}(V) = \bigcup_{V\in\Lambda_0(U)} \mathcal{O}(V)$$

In the sequel, we denote by$\mathcal{A}(U)$ the algebra of real analytic functions on U. By lemma 2.1.1 of [1], the map

$$\Psi : \mathcal{A}(U) \longrightarrow \varinjlim\mathcal{O}(V) : f \longmapsto \Psi(f)$$

is an isomorphism algebra. Now let A be a commutative and unital Hermitian Banach algebra (with continuous involution) and $a \in A$. Then $a = h+ik$ with $h,\ k \in H(A)$. Put $a' = (h, k)$ and $Sp_A a'$ the joint spectrum of (h, k). We denote by $\Theta_{a'}$ the map that defined the holomorphic functional calculus for a'. One has $Sp_A(h, k) \subset Sp_A h \times Sp_A k \subset \mathbb{R}^2$. By the identification $\mathbb{R}^2 \simeq \mathbb{C}$, via the map $(x, y) \longmapsto x + iy$, we can consider that

$$Sp_A a \simeq Sp_A(h, k)$$

and this motivates the following definition:

Definition 1.1. ([1], définition 2.1.2) Let A be a commutative and unital Hermitian Banach with continuous involution, $a \in A$, U an open subset, of $\mathbb{R}^2$, containing $Sp_A a$ and $f \in \mathcal{A}(U)$. We denote by $f(a)$ the element of A defined by:

$$f(a) = \Theta_{a'}(\Psi(f)) = \Psi(f)(h, k),$$

where $a = h + ik$ and $a' = (h, k)$ with $h, k \in H(A)$.

The fundamental properties of this functional calculus are contained in the following result:

Proposition 1.2. ([1]) 1. *The mapping* $f \longmapsto f(a)$ *is a homomorphism of* $\mathcal{A}(U)$ *into* A *that extends the involutive homomorphism from* $h(U)$ *into* A, *where* $h(U)$ *is the set of all harmonic functions on* U.

2. *"Spectral mapping theorem":*

$$Sp_A f(a) = f(Sp_A a), \text{ for every } f \in \mathcal{A}(U).$$

Let $f(t) = \sum\limits_{n \in \mathbb{Z}} a_n e^{int}$ be a periodic function such that $\sum\limits_{n \in \mathbb{Z}} |a_n| < +\infty$. If F is an holomorphic function defined on an open set containing the image of f, then $F(f)$ can be developed in trigonometric series $F(f)(t) = \sum\limits_{n \in \mathbb{Z}} c_n e^{int}$ such that $\sum\limits_{n \in \mathbb{Z}} |c_n| < +\infty$. This result due to P. Lévy ([7]) generalizes the famous theorem of N. Wiener ([10]) which states that the reciprocal of a nowhere vanishing absolutely convergent trigonometric series is also an absolutely convergent trigonometric series. In this paper, we consider the general case of a weight ω on $\mathbb{Z}$ which satisfies (2), (3) and

$$\lim_{|n| \longrightarrow +\infty} (\omega(|n|))^{\frac{1}{n}} = 1. \tag{4}$$

We then consider $f \in \mathcal{A}^p(\omega)$ and F an analytic function in two real variables on a neighborhood U of Spf. In this case, we obtain a weighted analogues of Lévy's theorem which states that $F(f)$ can be developed in trigonometric series $F(f)(t) = \sum\limits_{n \in \mathbb{Z}} c_n e^{int}$ such that

$$\sum_{n \in \mathbb{Z}} |c_n|^p \, \omega(n) < +\infty.$$

To proceed, we consider the Banach algebra $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ endowed with the involution $f \longmapsto f^\star$ defined by:

$$f^\star(t) = \sum_{n \in \mathbb{Z}} \overline{a_{-n}} e^{int}, \text{ for every } f \in \mathcal{A}^p(\omega).$$

We prove that $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ is Hermitian. In the particular case where F is a harmonic function in a neighborhood of $f(\mathbb{R})$, we prove that the expression of $F(f)$ is also given by the Poisson integral formula ([1]).

2. Real analytic version of Levy's theorem

Now we are ready to generalize Levy's theorem for real analytic functions.

THEOREM 2.1. (REAL ANALYTIC VERSION OF LÉVY'S THEOREM) *Let* $p \in]1,+\infty[$ *and* ω *be a weight on* $\mathbb{Z}$ *satisfying* (2), (3) *and* (4). *Let* $f(t) = \sum\limits_{n\in\mathbb{Z}} a_n e^{int}$ *be a periodic function such*

$$\sum_{n\in\mathbb{Z}} |a_n|^p \omega(n) < +\infty.$$

Let F *be an analytic function in two real variables on an open* U *containing the image of* f, *then the function* $F(f)$ *also can be developed in a trigonometric series* $F(f)(t) = \sum\limits_{n\in\mathbb{Z}} c_n e^{int}$ *such that*

$$\sum_{n\in\mathbb{Z}} |c_n|^p \, \omega(n) < +\infty.$$

Proof. We consider the Banach algebra $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ endowed with the involution $f \longmapsto f^\star$ defined by:

$$f^\star(t) = \sum_{n\in\mathbb{Z}} \overline{a_{-n}} e^{int}, \text{ for every } f \in \mathcal{A}^p(\omega).$$

One can prove that the map $f \longmapsto f^\star$ is an algebra involution on $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$. Moreover, it is continuous for the algebra is semi-simple. By the real analytic functional calculus given by Definition 1.1, the proof will be completed by proving that the last involution is hermitian in $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$. By hypothesis, $\lim\limits_{|n|\longrightarrow+\infty} (\omega(|n|))^{\frac{1}{n}} = 1$. Then the character space $\mathcal{M}(\mathcal{A}^p(\omega))$ of $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ can be identified with $[0, 2\pi]$ in such a way that each character is an evaluation at some $t_0 \in [0, 2\pi]$. This implies that

$$Spf = \{f(t) : t \in [0, 2\pi]\}, \text{ for every } f \in \mathcal{A}^p(\omega).$$

Now, it is clear, that $f(t) = \sum\limits_{n\in\mathbb{Z}} a_n e^{int}$, $t \in \mathbb{R}$, is a hermitian element of $\mathcal{A}^p(\omega)$ if and only, if

$$a_{-n} = \overline{a_n}, \text{ for every } n \in \mathbb{Z}$$

and so $Sp(f) \subset \mathbb{R}$. Whence $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ is Hermitian with continuous involution. This completes the proof. ∎

Remark 2.2. Actually, the reader can prove that the algebra $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ is Hermitian if and only if $\lim\limits_{|n|\longrightarrow+\infty} (\omega(|n|))^{\frac{1}{n}} = 1$. Indeed if the algebra $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$ is Hermitian. Let $f : t \longmapsto \sum\limits_{n\in\mathbb{Z}} a_n e^{int}$ be a hermitian element of $(\mathcal{A}^p(\omega), \|.\|_{p,\omega})$. Then $Sp(f) \subset \mathbb{R}$. Hence

$$\Phi_\zeta(f) = \overline{\Phi_\zeta(f)}, \text{ for every } \zeta \in \Gamma_\omega(\rho_1, \rho_2),$$

where

$$\Phi_\zeta(f) = \sum_{n \in \mathbb{Z}} a_n \zeta^n \quad \text{and} \quad \overline{\Phi_\zeta(f)} = \sum_{n \in \mathbb{Z}} a_n \overline{\zeta^{-n}}, \quad \text{for every } \zeta \in \Gamma_\omega(\rho_1, \rho_2).$$

It follows that

$$|\zeta| = 1, \text{ for every } \zeta \in \Gamma_\omega(\rho_1, \rho_2).$$

This yields $\rho_1 = \rho_2 = 1$, and one obtains that

$$\lim_{|n| \longrightarrow +\infty} (\omega(|n|))^{\frac{1}{n}} = 1.$$

Harmonic functions are particular real analytic functions. In this case, we have the following:

Corollary 2.3. (Harmonic version of Lévy's theorem) *Let $p \in]1, +\infty[$ and ω be a weight on $\mathbb{Z}$ satisfying (2), (3) and (4). Let $f(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}$ be a periodic function such*

$$\sum_{n \in \mathbb{Z}} |a_n|^p \omega(n) < +\infty.$$

Let U be an open subset of $\mathbb{C}$, $z_0 \in U$ such that $\overline{D(z_0, r)} \subset U$ $(r > 0)$ and $f(\mathbb{R}) \subset D(z_0, r)$. If $F \in h(U)$, then

$$F(f) = \frac{1}{2\pi} \int_{|z - z_0| = r} F(z) Re[(z + f - 2z_0)(z - f)^{-1}] \frac{|dz|}{r}$$

can be developed in a trigonometric series $F(f)(t) = \sum_{n \in \mathbb{Z}} c_n e^{int}$ such that

$$\sum_{n \in \mathbb{Z}} |c_n|^p \, \omega(n) < +\infty.$$

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