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Real Analytic Fréchet Algebras Containing Algebras of Holomorphic Functions

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Abstract: A class of Fréchet algebras of real analytic functions is constructed, with a weaker condition than complex analyticity. These algebras consist of functions on Stein spaces; a related construction on CR manifolds of a certain type is also given.

Key words: Fréchet algebras, CR extension.

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A fundamental, well-known property of the algebra of holomorphic functions on a domain is that it is closed under the topology of uniform convergence on compacta. Considering the holomorphic functions as a class of real analytic functions, we see that this class of real analytic functions is closed in the given topology. More generally, solutions of elliptic PDEs are real analytic, or smooth, given suitable conditions on the coefficients; furthermore, uniformly convergent sequences of solutions to the PDE are again solutions, hence real analytic or smooth [5].

All of this is old theory, familiar to analysts. In this paper, we demonstrate an entirely new way of creating algebras of functions, closed under the topology of uniform convergence on compacta, strictly containing the holomorphic functions (on the given domain), such that all functions in this algebra are real analytic. The functions obtained thereby satisfy no elliptic PDE; the real analyticity stems from moment conditions, and is proved by the technique of complexification and CR wedge extension. We construct examples on $\mathbf{C^n}$ for any n, on some Stein spaces, and even related algebras on CR hypersurfaces, such that there is a real analytic extension on one side which is holomorphic in some directions, but not fully holomophic. There are infinite dimensional examples, but we do not explore this in detail in this paper. The techniques of proof are those used by several authors to prove holomorphicity of functions on domains, or holomorphic extensions from boundaries, via moment conditions.

The paper has a short review of the 1-dimensional extension property and the relevant CR wedge extension theorems, after which the main theorems are stated and proved.

1. The 1-dimensional extension property and the technique of CR wedge extension

The *1-dimensional extension property* applied to a family of curves is one method of determining if a function is holomorphic, or in the case of boundary values, of determining whether a function has a holomorphic extension to a domain. An early version of this was a paper by Stout [10]. He proved that a continuous function f on the boundary of a C^1 domain $D \subseteq \mathbf{C^n}$ has a holomorphic extension to D if for every affine complex line L such that $L \cap D \neq \emptyset$, $f|_{bD\cap L}$ has a continuous holomorphic extension to $D \cap L$. Many theorems improved on this result by limiting the family of lines, including results in papers by Dinh and the author [3] [8]. A different but related case of the 1-dimensional extension property was examined in papers by Tumanov, Globevnik, Agranovsky, as well as a preprint by the author [1] [6] [11]. These papers were considered with the so-called "strip problem", its variants and generalization. In the strip problem one asks whether a function on a horizontal strip in the plane, having the property that it extends holomorphically from translates of a curve (whose height equals the height of the strip) must be holomorphic. In the papers of Globevnik and Tumanov, the method of CR wedge extension was used.

All of the above-mentioned papers were concerned with holomorphic extension, or whether a function is holomorphic. In this paper, we analyze how *real* analyticity can be a consequence of the 1-dimensional extension phenomenon, even when holomorphicity is not obtained. For a discussion of the wedge extension theorems which we need, see [2], [12] [13].

We make the convention that a real analytic function of may be written as $h(z)$ or $h(z, \overline{z})$. The complexified function, where one sets $w = \overline{z}$, is also written $h(z, w)$. We call a function $f(z, \overline{z})$ *entire real analytic* if it is the restriction of an entire holomorphic function $f(z, w)$ to the set $\{(w, z) : w = \overline{z}\}$.

All examples are based on the following definition.

Definition 1. Let $X \subseteq \mathbf{C}$ be a discrete set. The algebra $\mathcal{A}_X$ consists of all continuous functions f on $\mathbf{C}$ such that if $p \in X$ and $r > 0$, $f|_{\{z:|z-p|=r\}}$ has a holomorphic extension to $\{z : |z - p| < r\}$.

If $X = \{a_1, a_2, \dots\}$, we may write $\mathcal{A}_{a_1 a_2 \dots}$.

Our construction is based on a property of $\mathcal{A}_{a_1 a_2 a_3}$. Following the method of Globevnik and Tumanov, we complexify the circles and the discs they bound. Set $V_{i,r} = \{(z,w) : 0 < |z - a_i| < r, w = \frac{r^2}{z-a} + \overline{a}\}$. Denote by F_i the function on $M_i = \cup_r V_{i,r}$ defined by $F_i(z,w) = F_i(z)$, the holomorphic extension of f. Let $M = \cup_i M_i$. This is a union of three Levi-flat hypersurfaces meeting in the totally real edge $w = \overline{z}$; let F be the function which is equal to F_i on each M_i. M_i is contained in the quadric $\rho_i(z,w) = Im\,((w - \overline{a}_i)(z - a_i)) = 0$. Let us first consider the extension from the union of two M_i's. Let us take $a_0 = 0$ and $a_1 = 1$ for convenience of exposition. The change of coordinates $\tau = zw$ and $\zeta = zw - z - w$ is 1-1 in a neighborhood of any point $(z, \overline{z})$ with $z \neq \mathbf{R}$ and sends $\rho_0 = 0$ to $Im(\tau) = 0$ and $\rho_1 = 0$ to $Im(\zeta) = 0$. In the case of two *linear* CR manifolds N_1 and N_2 in $\mathbf{C^2}$ intersecting transversally in a totally real edge (see e.g. [13]), a continuous CR function F extends locally onto a full neighborhood on the side of $N_1 \cup N_2$ near a point in the edge, if N_1 and N_2 intersect in an angle less than π; the extension is on the side of the angle less than π, of course. For M_0 and M_1, the extension criteria applies at any point $(z, \overline{z})$ with $z \neq \mathbf{R}$; in general, when the point z is not on the line determined by the two points in X.

The fiber of M_i over z, denoted by $(M_i)_z$, is the line $w = \overline{a}_i + \frac{r^2}{z-a_i}$, $|z - a_i| \le r < \infty$, which is a ray from $\overline{z}$ to ∞ with slope $\overline{z - a_i}$. Suppose that for some z_0, all three angles between adjacent rays are $< \pi$. Let α be one of these angles in $\mathbf{C}_{z_0}$. Write $\alpha(z)$ for the angle in $\mathbf{C}_z$ between the rays of the same pair of CR manifolds, for nearby z. As noted in the last paragraph, the function F extends to an open set U with $(z, \overline{z}) \in U$, which fills out one side of the union of two of the M_i's, the side whose intersection with the vertical fiber $\mathbf{C}_z$ is the angle $\alpha(z)$ between the two rays. If z is contained in the interior of the triangle determined by a_1, a_2 and a_3 and if that triangle is non-degenerate, then the angles between adjacent rays will be less than π and we can conclude that the function F extends holomorphically to a neighborhood of $(z, \overline{z})$ in $\mathbf{C^2}$.

The above criterion immediately gives our first theorem.

THEOREM 1. *Suppose that a_1, a_2 and a_3 are not collinear. Then any continuous function $f(z) \in \mathcal{A}_{a_1,a_2,a_3}$ is real analytic inside the triangle with vertices a_1, a_2 and a_3.*

A couple of remarks. First, if the circles are replaced by any real analytic curves whose complexifications are sufficiently close to those of the circles, near $\{w = \overline{z}\}$, then there will be real analytic extension into some piecewise-smoothly bounded region close to a triangle.

By modifying this construction, we can create an algebra, closed under the topology of uniform convergence on compact sets, such that any function in this algebra is real analytic on all of $\mathbf{C}$.

THEOREM 2. *Let $X \subseteq \mathbf{C}$ be a discrete set with the following property: for some $z \in \mathbf{C}$, there exist a set of triangles T_n, containing z in the interior, with vertices in X, such that in T_n are uniformly bounded below and such that for all $r > 0$, $\exists N$ with the ball $B(z, r)$ contained in the interior of T_n for all $n > N$. Then if $f \in \mathcal{A}_X$, f is entire real analytic.*

Proof. Since the triangles T_n eventually fill up the plane, the claim that f is real analytic is proved. To show that f is the restriction of an entire function, we use scaling of the argument from Theorem 1. A function f which is in $\mathcal{A}_{a_1,a_2,a_3}$ extends to a holomorphic function on $\mathbf{C^2}$ on the ball $B(z, r)$ for some r if z is in the triangle with vertices at a_1, a_2, a_3. The value of r is bounded below, depending on the distance to the boundary of the triangle and the smallest angle. This follows from the continuity of the wedges depending on the CR manifolds under consideration. As the vertices of the triangle tend of ∞, then by scaling, the value of r also goes to ∞. This proves that f is entire real analytic. To prove that there are non-holomorphic functions in $\mathcal{A}_X$, we note that the function $|z|^2(z - a)$ has a holomorphic extension from every circle $|z - a| = r$. An example of a non-holomorphic function is $f(z) = |z|^2 g(z)$, where g is an entire function vanishing at all points of X. ∎

Since the functions in $\mathcal{A}_X$ extend to functions on $\mathbf{C^2}$, let us also consider the algebra $\mathcal{A}_X^*$ consisting of entire holomorphic functions $f(z, w)$ such that $f(z, \overline{z}) \in \mathcal{A}_X$. The proof of Theorem 2 contains the following fact:

LEMMA 1. *There exists a constant $C > 1$ and a sequence $R_n \to \infty$ such that*

$$\sup_{|z| \le R_n, |w| \le R_n} |f(z, w)| \le \sup_{|z| \le CR_n} f(z, \overline{z})$$

for $f \in \mathcal{A}_X^*$. The constant C is independent of f.

The proof of existence of non-holomorphic functions depends on the curves all being circles. The author believes that given two generic concentric curve families, a function which possesses holomorphic extensions from every curve in both families must be holomorphic. This is related to the strip problem, as well as theorems of Agranovsky. However, those theorems only apply to one continuous family of curves. A further remark is that the author has shown in [7] that in the setting of the 1-dimensional extension property, CR wedge extension theorems hold if a function is in L^1_{loc}.

2. Examples in higher dimensions

A natural attempt to generalize Theorem 2 is to use a construction with concentric spheres. Let S^1_r and S^2_r be two families of concentric spheres in $\mathbf{C^2}$ with centers at z_1 and z_2. Assume that C^1 function $f(z)$ has holomorphic extensions from all of the spheres. Let S_1 and S_2 be the two spheres, one from each family, which contain a point z. Then the sum of the complex tangent spaces is the whole tangent space at z: $T^{\mathbf{C}}_z S_1 + T^{\mathbf{C}}_z S_2 = T^{\mathbf{C}}_z \mathbf{C^2}$. Since f satisfies the Cauchy-Riemann equations in the tangential directions of each sphere, it satisfies those equations in all directions, thus is holomorphic.

Extension from spheres gives too much information. Instead, assume that f extends from concentric circles contained in parallel complex planes. Given a point $p = (a, b)$, we say that $f \in \mathcal{A}_p$ if for every w, $f(z, w)$ extends holomorphically from each circle $|z - a| = r$; also for fixed z, f extends holomorphically from all circles $|w - b| = r$. Given centers p_1 and p_2, a non-holomorphic function in $\mathcal{A}_{p_1 p_2}$ is $(z - a_1)|z - a_2|^2$ or $(w - a_2)|w - a_1|^2$. The directions of the z and w axes can be changed, so that we add the requirement of extension from circles $|\alpha z + \beta w| = r$ in the planes $-\overline{\beta} z + \overline{\alpha} w = k$, where $|\alpha|^2 + |\beta|^2 = 1$. In this case, $(\overline{\alpha} w - \overline{\beta} z)(z - a_1)(w - b_1)|\alpha z + \beta w|^2$ is a non-holomorphic function with extension from the concentric circles in all four directions: z, w, $\alpha z + \beta w$, $\overline{\alpha} w - \overline{\beta} z$.

Following the same procedure as in Theorem 2, we find a set of centers X in the z-plane and a set Y in the w-plane, both satisfying the conditions for creating entire extensions of functions in $\mathcal{A}_X$ and $\mathcal{A}_Y$. Denote by $\mathcal{A}_{X,Y}$ the algebra determined by the circles in the horizontal planes with centers in X, and circles in the vertical planes with centers in Y. Thus if $f \in \mathcal{A}_{X,Y}$, then for fixed z, $f(z, w)$ is the restriction of an entire function of z, w and $\overline{w}$ to the complexification of the w-plane, and for each fixed w, $f(z, w)$ is the restriction of an entire function of z, $\overline{z}$, w to the complexification of the z-plane. In other words, f is separately real analytic, in a strong sense. In general, separate real analyticity is not enough to guarantee real analyticity, even when the extensions on slices are entire. In this case, however, we have the following lemma.

LEMMA 2. *Let $f(z,w) \in \mathcal{A}_{X,Y}$ Then f is entire real analytic on $\mathbf{C^2}$.*

Proof. The w and $\overline{w}$ derivatives of $f(z,w)$ depend on the uniform norm on each z-slice of $\mathbf{C^2}$; hence, for any compact set $K \subseteq \mathbf{C}$ and any large enough R, there is a constant C_R such that

$$\left|\frac{\partial^{k+l}(f(z,0))}{\partial w^k \partial \overline{w}^l}\right| \leq \frac{C_R}{R^{k+l}}$$

for $z \in K$. Applying this to a circle $|u-a| = r$ with $a \in X$, we conclude that for a fixed z with $|z-a| < r$,

$$\int_{|u-a|=r} \frac{f(u,w)}{u-z}\frac{du}{2\pi i} \tag{1}$$

is entire real analytic in w, by differentiating with respect to w and $\overline{w}$ under the integral and evaluating at 0, then letting $R \to \infty$. Let M_1 and M_2 be two of the CR manifolds coming from the complexification of the circle families. M_1 can be written as $(Im(\rho_1) = 0) \cap (Im(\rho_2 > 0))$ and M_2 as $(Im(\rho_1) > 0) \cap (Im(\rho_2 = 0)$ for some ρ_i's which are holomorphic quadratic in z and τ. Also, let ρ_i^* be the function on $\mathbf{C^2}(z,\tau,w,\zeta)$ defined by $\rho_i^*(z,\tau,w,\zeta) = \rho_i(z,\tau)$; define M_i^* similarly. For each fixed w, we have a CR function in z (since the leaves are always graphs, we can leave out τ independent from other considerations). Because the integral in (1) is entire real analytic in w, and using Hartogs' separate analyticity theorems on the leaves of M_i^*, we conclude that F can be extended as a CR function (holomorphic in z, w and ζ on leaves) on each M_i^*. The same considerations used in the proof of Theorem 1 show that the extension from three wedges create to full neighborhood of any point $(z_0, \overline{z}_0)$ in the (z,τ) coordinates can be be taken to be holomorphic in a full neighborhood of $(z_0, \overline{z}_0, w_0, \overline{w}_0)$ in the (z,τ,w,ζ) coordinates for any (z_0, w_0). ∎

This construction can be lifted to closed analytic subvarieties of $\mathbf{C^n}$, in some cases. Let M be Stein space of pure dimension k. Suppose there is an embedding $\iota : M \to \mathbf{C^n}$. Let $V = \iota(M)$.

THEOREM 3. *Let $V \subseteq \mathbf{C^n}$ be a k-dimensional complex subvariety. Suppose that the projection π of V onto some $\mathbf{C^k} \subseteq \mathbf{C^n}$ is a proper, finite mapping of multiplicity l. Let $W \subseteq \mathbf{C^k}$ be the set of points such that $\#(\pi^{-1}(z) \cap V) < l$.*

Then there exist subalgebras of $C(V)$, strictly larger than $\mathcal{O}(V)$, closed under the topology of uniform convergence on compacta, such that every function in one of these sub-algebras is real analytic on $V \setminus (\pi^{-1}(W) \cap V)$.

Proof. Without loss of generality, we may assume that the projection from the theorem is onto the first k coordinates. The hypotheses of the theorem imply that $\pi : V \to \mathbf{C^k}$ is l to 1, for some integer l. Write $z = (z', z'')$, $z' = (z_1, \ldots z_k)$ and $z'' = (z_{k+1}, \ldots z_n)$. Let $X = X_1 \times X_2 \cdots \times X_k$ be a set such that each X_i satisfies the hypotheses of Theorem 2. Then by Theorem 3, the algebra $\mathcal{A}_{\mathcal{X}}$ consists of entire real analytic functions on $\mathbf{C^k}$. Membership in $\mathcal{A}_X$ consists of a function satisfying various moment conditions on circles in affine coordinate slices. Denote by π the projection onto the first k coordinates. We define an algebra on V as follows.

Definition 2. $\mathcal{A}_{V,\pi,X}$ is the set of all $f \in C(V)$ such that for every circle C in $\mathbf{C^k}$ associated to X, $\pi_{\#}(f)|C$ has a holomorphic extension to the affine disc which C bounds. $\pi_{\#}(f)$ is the pushforward of f—i.e., the value of $\pi_{\#}(f)$ is the sum of the values of f in the fiber.

From this definition it is clear that this algebra is closed under uniform convergence on compacta. Also, for any $f \in V$, $\pi_{\#}(f)$ is entire real analytic.

Given a point $p \in \mathbf{C^k} \setminus W$, pick a ball $B \subseteq \mathbf{C^k}$ centered at p, and disjoint from W. For any $h \in C(V)$ and for $z \in B$ let $[h](z)$ denote the 1 by l row vector whose entries are the values of h at the l points of V lying above z (arranged in some fixed order). Then on a possibly smaller ball B_1 we can find functions $g_1, g_2 \ldots g_l \in \mathcal{O}(V \cap (\pi^{-1}(B_1))$ such that the matrix A whose rows are the $[g_i]$'s is invertible. Write $A([f]^T) = w$. The entries of w are real analytic on B_1, and from $[f]^T = A^{-1}w$ we see that the entries of $[f]$ are real analytic. This proves that f is real analytic away from the branch locus of π. ∎

3. Order of growth, zero varieties and vanishing ideals

Let X be any discrete set in the plane satisfying the hypotheses of Theorem 2. The moment conditions of $\mathcal{A}_X$ impose growth restrictions from below on non-holomorphic functions in $\mathcal{A}_X$. This is based on another lemma.

Lemma 3. *Let $f \in \mathcal{A}_0$ be entire real analytic. Then $\frac{\partial f}{\partial \bar{z}} = zg(z)$, where g is entire real analytic.*

Proof. Examination of Fourier coefficients on the circles centered at 0 implies that

$$f(z,\overline{z}) = \Sigma_{0\leq j\leq i} a_{ij} z^i \overline{z}^j.$$

This gives $\frac{\partial f}{\partial \overline{z}}(z) = \Sigma_{0\leq j<i} b_{ij} z^i \overline{z}^j$, which implies the lemma. ∎

For a function $f \in \mathcal{A}_X$ which is not holomorphic, with associated entire function $f(z,w)$, this shows that $\frac{\partial f}{\partial w}(z,w)$ has order of growth on all horizontal slices which is bounded below, because of the zeroes on the set X (excluding the possible discrete set of lines where the functions is identically zero). We can quantify this notion if the set X satisfies an additional hypothesis.

THEOREM 4. *Suppose that the set X from Theorem 2 satisfies the condition that for some $\alpha > 1$, $T_n \subseteq \{z : \alpha^n < |z| < \alpha^{n+1}\}$. Let ρ be the minimum order of an entire holomorphic function $h(z)$ which vanishes on the set X. Let ρ_1 be the order of growth of an $f \in \mathcal{A}_X$ which is not a holomorphic function. Then $\rho \leq \rho_1$.*

Proof. The hypotheses of the theorem imply that there exist $C > 1$, $s > 0$ such that

$$\sup_{|z|\leq s\alpha^n, |w|\leq s\alpha^n} |f(z,w)| \leq \sup_{|z|\leq Cs\alpha^n} |f(z,\overline{z})|$$

for $n = 1, 2 \ldots$. Set $R_n = s\alpha^n$ Thus, if $|f(z)| \leq e^{(R_n)^k}$ on $|z| \leq R_n$ for every $n \geq n_0$, then

$$|f(z,w| \leq e^{C^k \alpha^k r^k} \tag{2}$$

on $|z| \leq r, |w| \leq r$ for any $r > R_{n_0}$. Write $f(z,w) = \Sigma_{j=0}^{\infty} a_j(w) z^j$. From (2) and from [9] p. 5, we get that for $|w| \leq 1$,

$$|a_j(w)| < \left(\frac{eC^k \alpha^k k}{j}\right)^{\frac{j}{k}} \tag{3}$$

for $j > j_0$, with j_0 independent of w. By Cauchy's estimates this gives that

$$|a_j'(0)| < \left(\frac{eC^k \alpha^k k}{j}\right)^{\frac{j}{k}}.$$

According to [9] p. 5, this implies that

$$\sup_{|w|\leq r} \left|\frac{\partial f}{\partial w}(z,0)\right| < e^{(C^k\alpha^k+\epsilon)r^k}$$

for any $\epsilon > 0$, for all sufficiently large r. Since $\frac{\partial f}{\partial w}(z,0)$ is an entire holomorphic function vanishing on X, this implies the theorem. If $\frac{\partial f}{\partial w}(z,0) \equiv 0$, then vary 0 slightly so get a non-zero function, then apply the argument. ∎

The proof of Lemma 3. can be improved to describe the generators of $\mathcal{A}_X$. The author is indebted to Z. Slodkowski for making this observation.

Proposition 1. *For a set X satisfying the conditions of Theorem 2., let $h(z)$ be an entire function which vanishes precisely on X, with all zeros of order 1. Then $\mathcal{A}_X$ is generated by z and $\overline{z}h(z)$.*

Proof. We use Theorem 2. only for the power series; the structure X plays no role in the proof.

Let $f \in \mathcal{A}_X$. From the proof of Lemma 3., we have $\frac{\partial f}{\partial \overline{z}}(z) = \Sigma_{0\leq j<i} b_{ij} z^i \overline{z}^j$. From this we see that $\frac{\partial f}{\partial \overline{z}}(z) = zg(z)$, where $g \in \mathcal{A}_X$. We also have $\frac{\partial f}{\partial \overline{z}}(z) = G(z)h(z)$, where G is entire real analytic. Suppose for convenience that $0 \in X$. Let $d_0 = dist(0, (X \setminus \{0\})$. On $|z| < d_0$, we have $g(z) = G(z)\phi(z)$ where ϕ is a holomorphic function which is non-vanishing on $|z| < d_0$. This shows that G satisfies the moment conditions to be in A_0 on circles of radius $< d_0$. Since these moment conditions are real analytic, we conclude that $G \in \mathcal{A}_X$, after repeating the argument at the other points of X.

Now write $f(z,w) = \Sigma_{n=0}^{\infty} a_n(z) w^n$. We have

$$\frac{\partial f}{\partial w} = a_1(z) + \Sigma_{n=1}^{\infty}(n+1)a_{n+1}(z)w^n =$$

$$h(z)G(z,w) = h(z)(g_1(z) + 2g_2(z)w\dots),$$

which gives us $a_1(z) = h(z)g_1(z)$. Thus we can write $f(z) = a_0(z) + h(z)g_1(z)\overline{z} + \dots$.

The rest of the terms are obtained in the same manner, noting that $\frac{\partial^n f}{\partial \overline{z}^n} = z^n k(z)$, where $k \in \mathcal{A}_0$. ∎

Given the set X, one may ask what are the zero varieties of $\mathcal{A}_X$. Consider an entire holomorphic function $g(z)$ which vanishes at all points of X. Let $p(z,\overline{z})$ be a real-valued polynomial of degree n in $\overline{z}$. Then $g(z)p(z,\overline{z}) \in \mathcal{A}_X$ and vanishes on $X \cup K$, where K is the zero variety for p. Thus, the zero varieties of $\mathcal{A}_X$ need not be discrete. We can also construct real analytic varieties which cannot be contained in non-trivial zero variety of a function in $\mathcal{A}_X$. Consider a graph $y = \phi(x)$, where ϕ is entire real analytic and vanishes to order 2 at $z = 0$ (we assume here that $0 \in X$). Changing coordinates to z and

$\overline{z}$, we can re-write the defining function as $\phi_1(z,\overline{z}) = z - \overline{z} + \Sigma_{n=2}^{\infty} c_n (z+\overline{z})^n$. Noting that the $V = \{\phi_1 = 0\}$ is irreducible, we infer that if $f \in \mathcal{A}_X$ vanishes on V, then $f(z,\overline{z}) = \phi_1(z,\overline{z})\psi(z,\overline{z})$. Using the equation from Lemma 3 on the power series for f, we can rule out V being a zero variety for $\mathcal{A}_X$, with one extra condition. In fact, we show that V cannot be the zero variety for an entire real analytic function in $\mathcal{A}_0$.

Proposition 2. *Suppose that $\alpha(z) = z + \Sigma_{n=2}^{\infty} c_n z^n$ has infinitely many zeroes, and is not the zero function. If*

$$\Sigma_{0\le j\le i} a_{ij} z^i \overline{z}^j = (z - \overline{z} + \Sigma_{n=2}^{\infty} c_n (z+\overline{z})^n)\,(\Sigma_{i,j\ge 0} b_{ij} z^i \overline{z}^j),$$

then $a_{ij} = 0$ for all i, j.

Proof. We prove that $a_{ij} = 0$ and $b_{ij} = 0$ for all j, by induction on i. For $i = 0$, we collect the anti-holomorphic and constant terms on both sides, getting the equation

$$a_{00} = (\Sigma_{j=0}^{\infty} b_{0j} \overline{z}^j)(\overline{z} + \Sigma_{n=2}^{\infty} c_n \overline{z}^n).$$

The left side is constant, while the right side is anti-holomorphic, vanishing at 0; therefore $a_{00} = 0$; since α is not zero, this means that $b_{0j} = 0$ for all j.

Assuming that $a_{ij} = 0$, $b_{ij} = 0$ for all j and for $i < k$, we collect the terms with degree k in z, obtaining the equation

$$\Sigma_{j=0}^{k} a_{kj} z^k \overline{z}^j = (\Sigma_{j=0}^{\infty} b_{kj} z^k \overline{z}^j)(\overline{z} + \Sigma_{n=2}^{\infty} c_n \overline{z}^n).$$

Factoring out z^k we get that $\alpha(\overline{z})$ times an anti-holomorphic function is equal to a polynomial in $\overline{z}$, contradicting the assumption about the zeroes of α. Therefore all the a_{kj}'s and b_{kj}'s are 0, as claimed. ∎

4. Algebras on CR manifolds

An algebra $\mathcal{A}_X$ on $\mathbf{C^k}$ can be lifted to a CR manifold of hypersurface type embedded in $\mathbf{C^{k+1}}$. We cover the case $k = 1$ for illustration. Let $M \subseteq \mathbf{C^2}$ be a connected hypersurface such that $\pi : M \to \mathbf{C}$ is a submersion, where π is the projection onto the first coordinate (the coordinates are (z,w)), and such that for every z, M_z, the fiber of M over z, is a simple closed curve. If f is a continuous CR function on M, then for any holomorphic function $g(z,w)$, $h(z) = \int_{M_z} fg\,dw$ is holomorphic. Let X be a discrete set satisfying the hypotheses of Theorem 2. Define $\mathcal{A}_{\mathcal{M},\pi,\mathcal{X}}$ to be the set of $f \in C(M)$ such that $\int_{M_z} f dw \in \mathcal{A}_X$. Immediately we see that $CR(M) \subseteq \mathcal{A}_{M,\pi,X}$. By Theorem 2, we have that $\int_{M_z} f dw$ is real analytic for $f \in \mathcal{A}_{M,\pi,X}$.

We can strengthen the result by restricting the algebra. Let $\mathcal{O} = \mathcal{O}(\mathbf{C^2})|M$. Let $\pi^*(\mathcal{A}_X) = \{f \in C(M) : \exists g \in \mathcal{A}_X, f(z,w) = g(z)\}$. Let $\mathcal{B} = \overline{\mathcal{O} \otimes (\pi^*(\mathcal{A}_X))}$, with the closure taken in the topology of uniform convergence on compacta. Let Ω be the domain in C^2 whose fibers over the z-axis are bounded by the M_z's.

Theorem 5. *Suppose $\{w = 0\} \subseteq \Omega$. Let $f \in \mathcal{B}$. Then*

1. *For each z, $f(z,w)$ extends holomorphically from M_z to Ω_z.*
2. *Let $f(z,w)$ denote the extension of f which is holomorphic on vertical slices. Then f is real analytic on Ω.*

Proof. If $f \in \mathcal{B}$, then for every z, $f|_{M_z}$ extends holomorphically to the interior of M_z. We call the extension F as well. L. Suppose that $\{w = 0\} \subseteq \Omega$. Then for all non-negative integers n, $\int_{M_z} \frac{f dw}{w^n} \in \mathcal{A}_{M,\pi,X}$. Given a point z_0, choose r such that $\{(z,w) : |z - z_0| \leq r, |w| < r\} \subseteq \Omega$. Set $h_n(z) = \int_{M_z} \frac{f dw}{w^n}$, $n = 1, 2, \ldots$. Assume without loss of generality that $0 \in X$. Pick R_1 such that $|z_0| + r < R_1$. Then, by Lemma 1, there exist R_2 such that for any $g \in \mathcal{A}_X$, $\sup_{|z| \leq R_1, |u| \leq R_1} |g(z,u)| \leq \sup_{|z| \leq R_2} |g(z,\overline{z})|$. Let $\delta = \min_{(z,w) \in M, |z| = R_2} |w|$, Then
$$|h_n(z)| \leq \frac{2\pi R_2 (\sup_{w \in M_z, |z| = R_2} |f(z,w)|)}{\delta^n},$$
which implies that $h_n(z,u)$ has the same bound in $(|z - z_0| < r) \times (|u - \overline{z}_0) < r)$. Hence the power series expansion in terms of powers of w can be extended real analytically in the z and w variables. let $r_1 = \min(\delta, r)$. Then we can conclude that f is real analytic in on $(|z - z_0| < r) \times (|w| < r_1)$. We now apply a theorem of Fefferman and Narasimhan [4]

Theorem 6. *Suppose $f(x,w)$ be a function on $M \times U$, where M is a real analytic manifold and U is a complex manifold. Suppose that for for some open $V \subseteq U$, f is real analytic on $M \times U$. Suppose also that for each fixed x, f is holomorphic on V. Then f is real analytic on $M \times U$.*

For a fixed z_0, apply this theorem with $M = \{|z - z_0| < \delta\}$ and U any open subset of Ω_z containing 0, with $M \times U \subseteq \Omega$ to conclude that f is real analytic on Ω. ∎

We now construct an example of a Hartogs domain such that $L^\infty(M) \cap \mathcal{CR}(M) \subset \mathcal{B} \cap L^\infty M$ is a proper inclusion. Fix an algebra $\mathcal{A}_X$ on the z-plane. Let $\Omega = \{(z, w) : |w| < e^{-\phi(z)}\}$, where ϕ is plurisubharmonic, chosen so that for some non-holomorphic $f \in \mathcal{A}_X$, $g(z) = f(z)e^{-\phi(z)} \in L^\infty(\mathbf{C})$; set $M = \partial\Omega$. Set $\mathcal{C} = \overline{(L^\infty(M) \cap \mathcal{CR}(M)) \otimes (L^\infty(M) \cap \mathcal{A}_{M,\pi,X})}$; where this time the closure is taken in L^∞. By construction $\mathcal{C}$ contains functions which are not CR; for example, $wf(z)$, where $f \in \mathcal{A}_X$ is non-holomorphic with $f(z)e^{-\phi(z)}$ bounded. The real analytic regularity follows from two things. First, for an L^∞ CR function f, $\int_{M_z} f dw$ is holomorphic, by the definition of weak CR functions. Second, the author showed in [7] that in the context of the 1-dimensional extension property on 1-dimensional families of curves, the CR functions constructed on Levi-flat CR manifolds intersecting in a totally real edge can be approximated locally in L^p for $p \geq 1$. This implies that the wedge extension theorems used in Theorems 1 and 2 apply in this setting.

Obviously, within a vertical slice, non-tangential limits exist for any $f \in \mathcal{C}$. We do not know if limits exist along paths which are not contained in a vertical slice.

5. Further questions

Let $X_i \subseteq \mathbf{C}_{\mathbf{z_i}}$ be discrete sets which make the argument of Theorem 2 go through, and let $\mathcal{A}_n = \mathcal{A}_{X_1, X_2, \ldots X_n}$ be the algebra on $\mathbf{C^n}$. We have injective homomorphisms $\phi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ given by $\phi \circ f(z_1, z_2, \ldots z_{n+1}) = f(z_1, z_2 \ldots z_n)$. Let $\mathcal{A}_\infty$ be the direct limit of this sequence of algebras. These examples may have interesting properties, depending on the sets X_i; we have not begun to examine this question.

Questions remain about the algebra $\mathcal{A}_X$ where $X \subseteq \mathbf{R}$. We are unable to construct functions in $\mathcal{A}_{01}$ which are not real analytic. If X is an infinite discrete set, especially if there are points tending to both infinities, we wonder if there are additional smoothness properties of $\mathcal{A}_X$ of any kind. The results of Section 3 indicate that the ideal structure of $\mathcal{A}_X$ may be quite intricate, even in the 1-dimensional case. The examples for CR manifolds and Stein spaces might allow even further generalizations.

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