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Spectral Rank of Maximal Finite-Rank Elements in Banach Jordan Algebras

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Abstract: We give a new proof to a spectral characterisation of the spectral rank established by Aupetit by replacing his deep analytic arguments by the new characterisation of the connected component of the group of invertible elements obtained by O. Loos.

Key words: Banach Jordan algebra, spectral rank, maximal finite-rank element.

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1. Preliminaries

Let A be a semisimple complex unital Banach Jordan algebra and $\Omega(A)$ its set of invertible elements. For $x \in A$ we denote $\mathrm{Sp}(x) = \{\lambda : \lambda 1 - x \notin \Omega(A)\}$ and $\rho_A(x) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(x)\}$ the spectrum and spectral radius of x.

For each nonnegative integer m, let

$$\mathcal{F}_m = \{a \in A : \sharp(\mathrm{Sp}\,\mathrm{U}_x\, a \setminus \{0\}) \leq m \text{ for all } x \in A\},$$

where the symbol $\sharp K$ denotes the number of distinct elements in a set $K \in \mathbb{C}$. Following [1], we define the rank of an element a of A as the smallest integer m such that $a \in \mathcal{F}_m$, if it exists; otherwise the rank is infinite. In other words,

$$\mathrm{rank}(a) = \{\sup \sharp(\mathrm{Sp}\,\mathrm{U}_x\, a \setminus \{0\}), x \in A\}.$$

If $a \in A$ is a finite-rank element, then

$$\mathcal{E}(a) = \{x \in A : \sharp(\mathrm{Sp}\,\mathrm{U}_x\, a \setminus \{0\}) = \mathrm{rank}(a)\}$$

is a dense open subset of A [2, Theorem 2.1].

It is shown in [1] that the socle, denoted Soc A, of a semisimple Banach Jordan algebra A coincides with the collection $\cup_{m=0}^{\infty} \mathcal{F}_m$ of finite-rank elements.

We first recall a very important theorem obtained by O. Loos [2], saying that the connected component of $\Omega(A)$ is arcwise connected as in the case of Banach algebras.

THEOREM 1. (O. Loos [2]) *Let A be a real or complex Banach Jordan algebra with unit element 1. Then*

$$\Omega_1 = \{\mathrm{U}_{(\exp x_1)} \cdots \mathrm{U}_{(\exp x_n)}(1) : x_i \in A, n \geq 1\}$$

is the connected component of 1 *of the set* Ω *of invertible elements of* A*.*

With the help of this Theorem 1 we are able now to eliminate the deep and difficult analytic arguments used by Aupetit to prove the next theorem.

2. The rank in Banach Jordan algebras

THEOREM 2. ([1], Theorem 3.1) *Let A be a Banach Jordan algebra with identity. Suppose that $a \in A$ and that $m \geq 0$ is an integer. The following properties are equivalent:*

(1) $\sharp(\mathrm{Sp}(\mathrm{U}_x\, a) \setminus \{0\}) \leq m$ *for every* $x \in A$,

(2) $\{t \in \mathbb{C} : 0 \in \mathrm{Sp}(y + ta)) \leq m\}$ *for every* y *invertible in* A,

(3) $\bigcap_{t \in F} \mathrm{Sp}(y + ta) \subset \mathrm{Sp}\, y$ *for every* $y \in A$ *and every subset* F *of* $\mathbb{C}$ *having* $m + 1$ *non-zero elements.*

Proof. (1) $\Rightarrow$ (2) First suppose that $0 \notin \sigma(y)$. By the Holomorphic Functional Calculus Theorem applied to y and a branch of $\sqrt{z}$, there exists an invertible x such that $y = x^2$. Since

$$y + ta = x^2 + ta = \mathrm{U}_x(1 + t\mathrm{U}_{x^{-1}} a)$$

we get $y + ta$ is non-invertible if and only if $-\frac{1}{t} \in \mathrm{Sp}(\mathrm{U}_{x^{-1}} a)$. By Hypothesis (1) this set $\mathrm{Sp}(\mathrm{U}_x^{-1} a)$ contains at most m non-zero points. Thus (2) is proved in this situation. Now $\{y \in A : 0 \notin \sigma(y)\}$ is an open subset of Ω, by upper semicontinuity of the spectrum. Let $y \in \Omega_1$, then by O. Loos's Theorem 1:

$$y = \mathrm{U}_{\exp(x_1)} \cdots \mathrm{U}_{\exp(x_n)} 1$$

so

$$y + ta = \mathrm{U}_{\exp(x_1)} \cdots \mathrm{U}_{\exp(x_n)} \left[1 + t \cdot \mathrm{U}_{\exp(-x_n)} \cdots \mathrm{U}_{\exp(-x_1)} a\right].$$

Then

$$0 \in \mathrm{Sp}(y + ta) \Longleftrightarrow -\frac{1}{t} \in \mathrm{Sp}(\mathrm{U}_{\exp(-x_n)} \cdots \mathrm{U}_{\exp(-x_1)} a).$$

The set $\mathrm{Sp}(\mathrm{U}_{\exp(-x_n)} \cdots \mathrm{U}_{\exp(-x_1)} a)$ contains at most m points by (1) because $\exp(-x_i)$ is invertible in A.

(2) $\Rightarrow$ (3) If $\lambda \in \varepsilon(y)$ then $\lambda - y \in \Omega_1$, so $\sharp\{t : \lambda \in \mathrm{Sp}(y + ta)\} \leq m$, hence $\lambda \notin \cap_{t \in F} \mathrm{Sp}(y + ta)$.

(3) $\Rightarrow$ (1) Same as in [1]. ∎

References

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