EXTRACTA MATHEMATICAE Vol. 30, Núm. 2, 235 – 250 (2015)

Virtually $(r; r_1, \dots, r_n; s)$-nuclear multilinear operators

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Presented by Manuel Maestre Received April 7, 2014

Abstract: In this paper, the space of virtually $(r; r_1, \dots, r_n; s)$-nuclear multilinear operators between Banach spaces is introduced, some of its properties are described and its topological dual is characterized as a Banach space of multiple absolutely $(r'; r'_1, \dots, r'_n; s')$-summing multilinear operators.

Key words: Multilinear operators, nuclear operators, summing operators.

AMS *Subject Class.* (2010): 47H60, 47B10.

1. Introduction

The nuclear operators between Banach spaces appeared in [5] when the author studied an infinite dimensional extension of the Malgrange theorem on existence and approximation of solutions for convolution equations (see also [7]). The concept of nuclear multilinear operators was extended and studied in [8]. For other related results we mention [9] and [10]. Matos [9] studied virtually $(r; r_1, \dots, r_n)$-nuclear n-linear operators from $X_1 \times \cdots \times X_n$ into Y, and proved that, if the spaces X_k^*'s $(k = 1, \dots, n)$ have the λ_k-bounded approximation property; then for $r, r_1, \dots, r_n \in [1, +\infty[$ the topological dual of the space of these operators, endowed with a natural linear topology, is isomorphic isometrically to the space of all absolutely $(r', r'_1, \dots, r'_n)$-summing operators from $X_1^* \times \cdots \times X_n^*$ into Y^* with $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{r_k} + \frac{1}{r'_k} = 1$; for r, r_k and $s \in [1, +\infty]$, $k = 1, \dots, n$.

In [3] Cerna established the definition of $(r; r_1, \dots, r_n; s)$-nuclear multilinear operators, which are the natural generalization of the concept of (r, p, s)-nuclear linear operator introduced by Laprest́e [6] (see also [11]).

[1]The authors acknowledge with thanks the support of the MESRS (Algeria) under project CNEPRU B05620120016.

Motivated by these ideas and developments, in this paper we introduce and study the virtually $(r;r_1,\ldots,r_n;s)$-nuclear n-linear operators and we will prove a relation between the topological dual of virtually $(r;r_1,\ldots,r_n;s)$-nuclear n-linear operators and the multiple $(r';r_1',\ldots,r_n';s')$-summing operators [2]. As a consequence we get the same result between the topological dual of the space of $(r;r_1,\ldots,r_n;s)$-nuclear n-linear operators from $X_1\times\cdots\times X_n$ into Y [4] and to the space of all absolutely $(r',r_1',\ldots,r_n',s')$-summing operators from $X_1^*\times\cdots\times X_n^*$ into Y^* [1], for r,r_k and $s\in[1,+\infty]$, $k=1,\ldots,n$.

The definitions and notations used in this paper are, in general, standard. Let $n\in\mathbb{N}$. As usual, an element j from $\mathbb{N}^n$ will be represented by $(j_1,\ldots,j_n)$ with $j_k\in\mathbb{N}$ and $k=1,\ldots,n$. We also consider the finite families $(y_j)_{j\in\mathbb{N}_m^n}$ of elements of a Banach space with $\mathbb{N}_m=\{1,\ldots,m\}$. If $n=1$, we omit $\mathbb{N}^n$ in the preceding notations. Let $X_1,\ldots,X_n;Y$ be Banach spaces over $\mathbb{K}$ (either $\mathbb{C}$ or $\mathbb{R}$). The space of all continuous n-linear operators $T:X_1\times\cdots\times X_n \rightarrow Y$ will be denoted by $\mathcal{L}(X_1,\ldots,X_n;Y)$. It becomes a Banach space with the natural norm

$$\|T\|=\sup_{\|x^k\|\leq 1,\ k=1,\ldots,n}\left\|T\left(x^1,\ldots,x^n\right)\right\|.$$

We recall that a n-linear mappings $T\in\mathcal{L}(X_1,\ldots,X_n;Y)$ is said to be of finite type if it has a finite representation of the form

$$T=\sum_{i=1}^{m}\lambda_i\varphi_i^1\times\cdots\times\varphi_i^n b_i,$$

where $\lambda_i\in\mathbb{K}$, $\varphi_i^k\in X_k^*$, $k=1,\ldots,n$, $b_i\in Y$, $i=1,\ldots,m$. We denote by $\mathcal{L}_f(X_1,\ldots,X_n;Y)$ the vector subspace of $\mathcal{L}(X_1,\ldots,X_n;Y)$ of all n-linear mappings of finite type.

If $r\in\,]0,+\infty[$, we denote by $l_r(Y;\mathbb{N}^n)$ or $(l_r(\mathbb{N}^n);$ if $Y=\mathbb{K})$, the vector space of all families $(y_j)_{j\in\mathbb{N}^n}$ of elements of Y such that

$$\left\|(y_j)_{j\in\mathbb{N}^n}\right\|_r=\left(\sum_{j\in\mathbb{N}^n}\|y_j\|_Y^r\right)^{\frac{1}{r}}<\infty.$$

We observe that $\|\cdot\|_r$ is a norm (r-norm, if $r<1$) on $l_r(Y;\mathbb{N}^n)$ and defines a complete metrizable linear topology on it. We denote by $l_\infty(Y;\mathbb{N}^n)$ (or $l_\infty(\mathbb{N}^n)$, if $Y=\mathbb{K}$) the Banach space of all bounded families $(y_j)_{j\in\mathbb{N}^n}$ of elements of Y, with the norm

$$\left\|(y_j)_{j\in\mathbb{N}^n}\right\|_\infty = \sup_{j\in\mathbb{N}^n}\|y_j\|.$$

The Banach subspace of all families $(y_j)_{j\in\mathbb{N}^n}$ such that

$$\lim_{j_k\to+\infty,\ k=1,\ldots,n}\|y_j\| = 0$$

is denoted by $c_0(Y;\mathbb{N}^n)$ (or $c_0(\mathbb{N}^n)$, if $Y=\mathbb{K}$).

If $0<s\leq\infty$, we will write $l_r^w(Y;\mathbb{N}^n)$ (or $l_r^w(\mathbb{N}^n)$, if $Y=\mathbb{K}$) for the vector space of all families $(y_j)_{j\in\mathbb{N}^n}$ of elements of Y such that

$$\left\|(y_j)_{j\in\mathbb{N}^n}\right\|_{w,s} := \sup_{\|\psi\|_{Y^*}\leq 1}\left(\sum_{j\in\mathbb{N}^n}|\psi(y_j)|^s\right)^{\frac{1}{s}} = \sup_{\|\psi\|_{Y^*}\leq 1}\left\|(\psi(y_j))_{j\in\mathbb{N}^n}\right\|_s < \infty,$$

where Y^* denotes the topological dual of Y.

It is well-known that for $1\leq s<\infty$ and $(\varphi_j)_{j\in\mathbb{N}^n}\in l_s^w(Y^*;\mathbb{N}_m^n)$, we have

$$\left\|(\varphi_j)_{j\in\mathbb{N}^n}\right\|_{w,s} = \sup_{\phi\in B_{Y^{**}}}\left(\sum_{j\in\mathbb{N}^n}|\phi(\varphi_j)|^s\right)^{\frac{1}{s}} = \sup_{y\in B_Y}\left\|(\varphi_j(y))_{j\in\mathbb{N}^n}\right\|_s.$$

Let $0<r,\ 1<p,\ s\leq\infty$ such that

$$\frac{1}{t} = \frac{1}{r}+\frac{1}{p}+\frac{1}{s},\ \text{with}\ \ t\in\,]0,1].$$

An operator $T\in\mathcal{L}(X;Y)$ is $(r;p;s)$-nuclear (see, e.g., [6, 11]) if it has a representation of the form

$$T = \sum_{i=1}^{\infty}\lambda_i x_i\otimes y_i, \tag{1}$$

with $(\lambda_i)_i\in l_r$, if $r<\infty$ (or $(\lambda_i)_i\in c_0$, if $r=+\infty$), $(x_i)_i\in l_p^w(X^*)$ and $(y_i)_i\in l_s^w(Y)$. The vector space of all such operators is denoted by $\mathcal{N}_{(r;p;s)}(X;Y)$ and it is a complete metrizable topological vector space under the t-norm

$$\mu_{(r;p;s)}(T) = \inf\left\{\|(\lambda_i)_i\|_r\,\|(x_i)_i\|_{w,p}\,\|(y_i)_i\|_{w,s}\right\}$$

where the infimum is taken over all representations of T as in (1).

The definition of the virtually $(r; r_1, \ldots, r_n)$-nuclear operators below was first given in [9].

We consider $r \in \left]0, +\infty\right]$, $r_k \in [1, +\infty]$, such that $r \leq r_k$, $k = 1, \ldots, n$ and

$$1 \leq \frac{1}{t_n} = \frac{1}{r} + \frac{1}{r_1'} + \cdots + \frac{1}{r_n'}.$$

Definition 1.1. An operator $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is said to be virtually $(r; r_1, \ldots, r_n)$-nuclear if there is a representation of the form

$$T = \sum_{j \in \mathbb{N}^n} \lambda_j \phi_{j_1}^1 \times \cdots \times \phi_{j_n}^n b_j \tag{2}$$

with $(\lambda_j)_{j \in \mathbb{N}^n} \in l_r(\mathbb{N}^n)$, if $r < \infty$ (or $(\lambda_j)_{j \in \mathbb{N}^n} \in c_0(\mathbb{N}^n)$, if $r = +\infty$), $\left(\phi_i^k\right)_{i=1}^{\infty} \in l_{r_k'}^w(X_k^*)$, for $k = 1, \ldots, n$ and $(b_j)_{j \in \mathbb{N}^n} \in l_\infty(Y; \mathbb{N}^n)$.

The vector space of these operators is denoted by $\mathcal{L}_{VN}^{(r; r_1, \ldots, r_n)}(X_1, \ldots, X_n; Y)$ and we consider on it the t_n-norm

$$\|T\|_{VN,(r; r_1, \ldots, r_n)} = \inf \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_r \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_\infty \prod_{k=1}^n \left\| \left(\phi_i^k\right)_{i=1}^\infty \right\|_{w, r_k'},$$

where the infimum is taken over all representations of T as in (2).

The notion of absolutely $(r; r_1, \ldots, r_n; s)$-summing multilinear operators was introduced by the first author in [1].

Definition 1.2. For $0 < r,\ r_1, \ldots, r_n < \infty$ and $0 < s \leq \infty$ with $\frac{1}{r} \leq \frac{1}{r_1} + \cdots + \frac{1}{r_n} + \frac{1}{s}$, an n-linear operator $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is $(r; r_1, \ldots, r_n; s)$-summing if there is a constant $C > 0$ such that for any $x_1^k, \ldots, x_m^k \in X_k$, $(1 \leq k \leq n)$, and any $\varphi_1, \ldots, \varphi_m \in Y^*$, we have

$$\left(\sum_{i=1}^m \left| \varphi_i \left(T \left(x_i^1, \ldots, x_i^n \right) \right) \right|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^n \left\| \left(x_i^k \right)_{i=1}^m \right\|_{w, r_k} \left\| (\varphi_i)_{i=1}^m \right\|_{w,s}.$$

We denote the vector space of these operators by $\mathcal{L}_{as,(r; r_1, \ldots, r_n; s)}(X_1, \ldots, X_n; Y)$ and the smallest C satisfying the above inequality by $\pi_{(r; r_1, \ldots, r_n; s)}^n(T)$ which defines a norm (r-norm if $r < 1$) on $\mathcal{L}_{as,(r; r_1, \ldots, r_n; s)}(X_1, \ldots, X_n; Y)$.

The following multilinear generalization of $(r; r_1, \ldots, r_n; s)$-summing operators was recently introduced by Bernardino et al. in [2].

DEFINITION 1.3. Let $n\in\mathbb{N}$, r, s, $r_1,\dots,r_n\geq 1$ and $X_1,\dots,X_n,Y$ be Banach spaces. A continuous multilinear operator $T:X_1\times\cdots\times X_n\longrightarrow Y$ is multiple $(r;r_1,\dots,r_n;s)$-summing if there is a $C>0$ such that

$$\left(\sum_{j\in\mathbb{N}_m^n}\left|\varphi_j\left(T(x_{j_1}^1,\dots,x_{j_n}^n)\right)\right|^r\right)^{\frac{1}{r}}\leq C\left\|(\varphi_j)_{j\in\mathbb{N}_m^n}\right\|_{w,s}\prod_{k=1}^n\left\|\left(x_i^k\right)_{i=1}^m\right\|_{w,r_k}$$

where $\frac{1}{r}\leq\frac{1}{r_1}+\cdots+\frac{1}{r_n}+\frac{1}{s}$, $x_1^k,\dots,x_m^k\in X_k$, $k=1,\dots,n$ and $(\varphi_j)_{j\in\mathbb{N}_m^n}\in l_s^w\left(Y^*;\mathbb{N}_m^n\right)$.

We denote by $\mathcal{L}_{mas}^{(r;r_1,\dots,r_n;s)}\left(X_1,\dots,X_n;Y\right)$ the vector space of these operators. The smallest C satisfying the above inequality defines a norm (r-norm if $r<1$) on $\mathcal{L}_{mas}^{(r;r_1,\dots,r_n;s)}\left(X_1,\dots,X_n;Y\right)$; it is denoted by $\|T\|_{mas(r;r_1,\dots,r_n;s)}$.

Remark 1.4. By choosing $(s=\infty)$ in Definition 1.3, we obtain the definition of fully (or multiple) $(r;r_1,\dots,r_n)$-summing n-linear operators presented in [9].

We also need the definition of the $(r;r_1,\dots,r_n;s)$-nuclear n-linear operators. The ideal of $(r;r_1,\dots,r_n;s)$-nuclear operators was introduced by Cerna [3] (see also [4]).

DEFINITION 1.5. For $0<r\leq\infty$, $1\leq s$, $r_1,\dots,r_n\leq\infty$, such that $1\leq\frac{1}{r}+\frac{1}{r_1'}+\cdots+\frac{1}{r_n'}+\frac{1}{s'}$, $T\in\mathcal{L}\left(X_1,\dots,X_n;Y\right)$ is called $(r;r_1,\dots,r_n;s)$-nuclear if it has the form

$$T=\sum_{i=1}^{+\infty}\lambda_i\phi_i^1\times\cdots\times\phi_i^n b_i,\tag{3}$$

with $(\lambda_i)_{i\in\mathbb{N}}\in l_r\left(\mathbb{N}\right)$, if $r<\infty$ (or $(\lambda_i)_{i\in\mathbb{N}}\in c_0\left(\mathbb{N}\right)$, if $r=+\infty$), $\left(\phi_i^k\right)_{i\in\mathbb{N}}\in l_{r_k'}^w\left(X_k^*\right)$ for $k=1,\dots,n$ and $(b_i)_{i\in\mathbb{N}}\in l_{s'}^w\left(Y\right)$. The set of $(r;r_1,\dots,r_n;s)$-nuclear operators satisfying the definition is a vector space and is denoted by $\mathcal{N}_{(r;r_1,\dots,r_n;s)}\left(X_1,\dots,X_n;Y\right)$. Considering that

$$N_{(r;r_1,\dots,r_n;s)}\left(T\right)=\inf\left\|(\lambda_i)_{i\in\mathbb{N}}\right\|_r\left\|(b_i)_{i\in\mathbb{N}}\right\|_{w,s'}\prod_{k=1}^n\left\|\left(\phi_i^k\right)_{i\in\mathbb{N}}\right\|_{w,r_k'},$$

where the infimum is taken over all possible representations of T described in (3), we obtain a t-norm with

$$\frac{1}{t}=\frac{1}{r}+\frac{1}{r_1'}+\cdots+\frac{1}{r_n'}+\frac{1}{s'}.$$

2. Virtually $(r;r_1,\dots,r_n;s)$-nuclear n-linear operators

We consider $r \in]0,+\infty]$, $s, r_k \in [1,+\infty]$, $k = 1,\dots,n$, such that $1 \le \frac{1}{t_n} = \frac{1}{r} + \frac{1}{r'_1} + \cdots + \frac{1}{r'_n} + \frac{1}{s'}$.

Definition 2.1. An operator $T \in \mathcal{L}(X_1,\dots,X_n;Y)$ is said to be virtually $(r;r_1,\dots,r_n;s)$-nuclear if there are $(\lambda_j)_{j\in\mathbb{N}^n} \in l_r(\mathbb{N}^n)$, if $r < \infty$ (or $(\lambda_j)_{j\in\mathbb{N}^n} \in c_0(\mathbb{N}^n)$, if $r = +\infty$), $\left(\phi_i^k\right)_{i=1}^{\infty} \in l^w_{r'_k}(X_k^*)$, for $k = 1,\dots,n$ and $(b_j)_{j\in\mathbb{N}^n} \in l^w_{s'}(Y;\mathbb{N}^n)$ such that

$$T = \sum_{j\in\mathbb{N}^n} \lambda_j \phi^1_{j_1} \times \cdots \times \phi^n_{j_n} b_j. \tag{4}$$

We denote the vector space of all such operators by $\mathcal{L}_{VN}^{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y)$, with the t_n-norm

$$\|T\|_{VN,(r;r_1,\dots,r_n;s)} = \inf \left\|(\lambda_j)_{j\in\mathbb{N}^n}\right\|_r \left\|(b_j)_{j\in\mathbb{N}^n}\right\|_{w,s'} \prod_{k=1}^{n} \left\|\left(\phi_i^k\right)_{i=1}^{\infty}\right\|_{w,r'_k},$$

where the infimum is taken over all representations of T as in (4). This t_n-normed space is a complete metrizable topological vector space.

Remarks 2.2. (a) By choosing $s' = \infty$ in Definition 2.1, we obtain virtually $(r;r_1,\dots,r_n)$-nuclear n-linear operators presented in Definition 1.1.
(b) We have $\mathcal{N}_{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y) \subset \mathcal{L}_{VN}^{(r;r_1,\dots,r_m;s)}(X_1,\dots,X_n;Y)$ and

$$\|T\| \le \|T\|_{VN,(r;r_1,\dots,r_n;s)} \le N_{(r;r_1,\dots,r_n;s)}(T),$$

for every T is in $\mathcal{N}_{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y)$.

By definition every T in $\mathcal{L}_f(X_1,\dots,X_n;Y)$ has a finite representation

$$T = \sum_{j\in\mathbb{N}^n_m} \lambda_j \phi^1_{j_1} \times \cdots \times \phi^n_{j_n} b_j. \tag{5}$$

It is clear that we have a t_n-norm on $\mathcal{L}_f(X_1,\dots,X_n;Y)$ defined by

$$\|T\|_{VN_f,(r;r_1,\dots,r_n;s)} = \inf \left\|(\lambda_j)_{j\in\mathbb{N}^n_m}\right\|_r \left\|(b_j)_{j\in\mathbb{N}^n_m}\right\|_{w,s'} \prod_{k=1}^{n} \left\|\left(\phi_i^k\right)_{i=1}^{m}\right\|_{w,r'_k},$$

where the infimum is taken over all finite representations of T as in (5).

The next result collects some elementary facts about virtually $(r;r_1,\dots,r_n;s)$-nuclear n-linear operators.

PROPOSITION 2.3. (i) *The vector space* $\mathcal{L}_f\left(X_1,\dots,X_n;Y\right)$ *of the continuous* n*-linear operators of finite type is dense in* $\mathcal{L}_{VN}^{(r;r_1,\dots,r_n;s)}\left(X_1,\dots,X_n;Y\right)$.

(ii) *Ideal property: If* $E_1,\dots,E_n$, *and* Y_0 *are Banach spaces and* $T\in\mathcal{L}\left(X_1,\dots,X_n;Y\right)$, $S_k\in\mathcal{L}\left(E_k,X_k\right)$, $k=1,\dots,n$, *and* $R\in\mathcal{L}\left(Y,Y_0\right)$ *with* T *virtually* $(r;r_1,\dots,r_n;s)$*-nuclear, then* $R\circ T\circ\left(S_1,\dots,S_n\right)$ *is virtually* $(r;r_1,\dots,r_n;s)$*-nuclear and*

$$\left\|R\circ T\circ\left(S_1,\dots,S_n\right)\right\|_{VN,(r;r_1,\dots,r_n;s)}\leq\|R\|\,\|T\|_{VN,(r;r_1,\dots,r_n;s)}\prod_{k=1}^{n}\|S_k\|.$$

(iii) $T\in\mathcal{L}\left(X_1,\dots,X_n;Y\right)$ *is virtually* $(r;r_1,\dots,r_n;s)$*-nuclear if and only there are bounded linear operators* $A_k\in\mathcal{L}(X_k;l_{r_k'})$, $k=1,\dots,n$, $B\in\mathcal{L}\left(l_1\left(\mathbb{N}^n\right);Y\right)$ *and* $\left(\lambda_j\right)_{j\in\mathbb{N}^n}\in l_r\left(\mathbb{N}^n\right)$, *if* $r<\infty$ *(or* $\left(\lambda_j\right)_{j\in\mathbb{N}^n}\in c_0\left(\mathbb{N}^n\right)$, *if* $r=+\infty$*), such that*

$$T=B\circ\mathcal{D}_{\left(\lambda_j\right)_{j\in\mathbb{N}^n}}\circ\left(A_1,\dots,A_n\right),$$

where $\mathcal{D}_{\left(\lambda_j\right)_{j\in\mathbb{N}^n}}:l_{r_1'}\times\cdots\times l_{r_n'}\longrightarrow l_1\left(\mathbb{N}^n\right)$ *defined by* $\mathcal{D}_{\left(\lambda_j\right)_{j\in\mathbb{N}^n}}\big((\xi^1_{j_1})_{j_1=1}^\infty,\dots,(\xi^n_{j_n})_{j_n=1}^\infty\big)=(\lambda_j\xi^1_{j_1}\cdots\xi^n_{j_n})_{j\in\mathbb{N}^n}$ *for* $(\xi^1_{j_1})_{j_1=1}^\infty\in l_{r_1'}$, *is a virtually* $(r;r_1,\dots,r_n;s)$*-nuclear with*

$$\left\|\mathcal{D}_{\left(\lambda_j\right)_{j\in\mathbb{N}^n}}\right\|_{VN,(r;r_1,\dots,r_n;s)}=\left\|\left(\lambda_j\right)_{j\in\mathbb{N}^n}\right\|_r.$$

In this case

$$\|T\|_{VN,(r;r_1,\dots,r_n;s)}=\inf\|B\|\left\|\left(\lambda_j\right)_{j\in\mathbb{N}^n}\right\|_r\prod_{k=1}^{n}\|A_k\|,$$

where the infimum is taken over all such factorizations.

3. DUALITY

The natural question is to find out when we have

$$\|T\|_{VN,(r;r_1,\dots,r_n;s)}=\|T\|_{VN_f,(r;r_1,\dots,r_n;s)},$$

for each $T\in\mathcal{L}_f\left(X_1,\dots,X_n;Y\right)$.

Of course we have

$$\|T\|_{VN,(r;r_1,\ldots,r_n;s)} \leq \|T\|_{VN_f,(r;r_1,\ldots,r_n;s)}\,.$$

Below we will see that the reverse implication holds to be true for some certain Banach spaces X_k's $(k = 1,\ldots,n)$. We start with finite dimensional spaces X_k's. The following theorem can be proved as in [9, Proposition 4.6].

Theorem 3.1. *If the spaces* X_k $(k = 1,\ldots,n)$ *are finite dimensional vector spaces, then*

$$\|T\|_{VN_f,(r;r_1,\ldots,r_n;s)} \leq \|T\|_{VN,(r;r_1,\ldots,r_n;s)}\,,$$

for every $T \in \mathcal{L}_f\left(X_1,\ldots,X_n;Y\right)$.

As in [9, Proposition 4.8], we get the following, which extends Theorem 3.1 to infinite dimensional Banach spaces with the λ-bounded approximation property (λ-BAP, for short).

Proposition 3.2. *If the spaces* X_k^**'s* $(k = 1,\ldots,n)$ *have the* λ_k*-BAP, then*

$$\|T\|_{VN,(r;r_1,\ldots,r_n;s)} \geq \|T\|_{VN_f,(r;r_1,\ldots,r_n;s)}\,,$$

for all $T \in \mathcal{L}_f\left(X_1,\ldots,X_n;Y\right)$.

Proof. We consider $T_k \in \mathcal{L}\big(X_k;\mathcal{L}\left(X_1,\ldots,X_{k-1},X_{k+1},\ldots,X_n;Y\right)\big)$, defined by

$$T_k\big(x^k\big)\big(x^1,\ldots,x^{k-1},x^{k+1},\ldots,x^n\big) = T\big(x^1,\ldots,x^{k-1},x^k,x^{k+1},\ldots,x^n\big),$$

for $x^k \in X_k,\ k = 1,\ldots,n$.

Since X_k^* has the λ_k-bounded approximation property for some $\lambda_k > 0$, given $\epsilon > 0$, we can find $S_k \in \mathcal{L}_f\left(D_k,X_k\right)$, such that $T_k = T_k \circ S_k$ and $\|S_k\| \leq (1+\epsilon)\,\lambda_k$. Hence, for all $x^k \in X_k$, for $k = 1,\ldots,n$, we have

$$T\big(x^1,\ldots,x^{k-1},S_k\big(x^k\big),x^{k+1},\ldots,x^n\big) = T\big(x^1,\ldots,x^{k-1},x^k,x^{k+1},\ldots,x^n\big).$$

Now, we can write

$$T\left(x^1,\ldots,x^n\right) = T \circ \left(S_1,\ldots,S_n\right)\left(x^1,\ldots,x^n\right), \quad \forall x^k \in X_k,\ k = 1,\ldots,n.$$

If J_k denotes the natural injection from $S_k\left(D_k\right)$ into X_k, we can write $S_k = J_k \circ \widetilde{S}_k$ $\big(\widetilde{S}_k \in \mathcal{L}_f\left(D_k,S_k\left(D_k\right)\right)\big)$, with $\big\|\widetilde{S}_k\big\| = \|S_k\|$. Therefore we can say that $T\circ(J_1,\ldots,J_n)\in\mathcal{L}_f\left(\left(S_1\left(D_1\right),\ldots,S_n\left(D_n\right)\right);Y\right)$. By Theorem 3.1 and Proposition 2.3 (ii) we have

$$\begin{aligned}\|T\|_{VN_f,(r;r_1,\ldots,r_n;s)} &= \|T\circ(S_1,\ldots,S_n)\|_{VN_f,(r;r_1,\ldots,r_n;s)}\\ &\le \|T\|_{VN,(r;r_1,\ldots,r_n;s)}\prod_{k=1}^{n}\|S_k\|\\ &\le \|T\|_{VN,(r;r_1,\ldots,r_n;s)}(1+\epsilon)^n\prod_{k=1}^{n}\lambda_k.\end{aligned}$$

This implies that

$$\|T\|_{VN_f,(r;r_1,\ldots,r_n;s)}\le\left(\prod_{k=1}^{n}\lambda_k\right)\|T\|_{VN,(r;r_1,\ldots,r_n;s)}.$$

For each $\epsilon>0$, we choose a representation

$$T=\sum_{j\in\mathbb{N}^n}\sigma_j\phi^1_{j_1}\times\cdots\times\phi^n_{j_n}y_j$$

such that

$$\left\|(\sigma_j)_{j\in\mathbb{N}^n}\right\|_r\left\|(y_j)_{j\in\mathbb{N}^n}\right\|_{w,s'}\prod_{k=1}^{n}\left\|\left(\phi^k_i\right)_{i=1}^{\infty}\right\|_{w,r'_k}\le(1+\epsilon)\|T\|_{VN,(r;r_1,\ldots,r_n;s)}.$$

We can find $m\in\mathbb{N}$ such that

$$\left(\prod_{k=1}^{n}\lambda_k\right)\left\|\sum_{j\in\mathbb{N}^n/\mathbb{N}^n_m}\sigma_j\phi^1_{j_1}\times\cdots\times\phi^n_{j_n}y_j\right\|_{VN_f,(r;r_1,\ldots,r_n;s)}\le\epsilon\|T\|_{VN,(r;r_1,\ldots,r_n;s)}.$$

We use the triangular inequality for t_n-norms in order to write

$$\begin{aligned}\left(\|T\|_{VN_f,(r;r_1,\ldots,r_n;s)}\right)^{t_n}\le&\left(\left\|\sum_{j\in\mathbb{N}^n_m}\sigma_j\phi^1_{j_1}\times\cdots\times\phi^n_{j_n}y_j\right\|_{VN_f,(r;r_1,\ldots,r_n;s)}\right)^{t_n}\\ &+\left(\left\|\sum_{j\in\mathbb{N}^n/\mathbb{N}^n_m}\sigma_j\phi^1_{j_1}\times\cdots\times\phi^n_{j_n}y_j\right\|_{VN_f,(r;r_1,\ldots,r_n;s)}\right)^{t_n}\end{aligned}$$

$$\leq (1+\epsilon)^{t_n} \left(\|T\|_{VN,(r;r_1,\dots,r_n;s)}\right)^{t_n}$$
$$+ \left(\prod_{k=1}^{n} \lambda_k\right)^{t_n} \left(\left\| \sum_{j\in\mathbb{N}^n/\mathbb{N}^n_m} \sigma_j \phi^1_{j_1} \times \cdots \times \phi^n_{j_n} y_j \right\|_{VN,(r;r_1,\dots,r_n;s)}\right)^{t_n}$$
$$\leq \left[(1+\epsilon)^{t_n} + \epsilon^{t_n}\right] \left(\|T\|_{VN,(r;r_1,\dots,r_n;s)}\right)^{t_n}.$$

Since $\epsilon > 0$ is arbitrary we have

$$\|T\|_{VN_f,(r;r_1,\dots,r_n;s)} \leq \|T\|_{VN,(r;r_1,\dots,r_n;s)},$$

and this proves the theorem. ∎

For Banach spaces with λ-bounded approximation property, Proposition 3.2 can be seen as a generalization of a result obtained by B. Cerna [4, Lemma 2.1].

Now, we also give another generalization of [4, Lemma 2.1].

Proposition 3.3. *Let $T : X_1 \times \cdots \times X_n \longrightarrow L_s(\Omega, \mu)$ be defined by*

$$T\left(x^1, \dots, x^n\right) = \sum_{j\in\mathbb{N}^n_m} \lambda_j \phi^1_{j_1}\left(x^1\right) \cdots \phi^n_{j_n}\left(x^n\right) b_j,$$

where $\frac{1}{s} = \frac{1}{r'_1} + \cdots + \frac{1}{r'_n}$. Then, $\|T\|_{VN_f,(\infty;r_1,\dots,r_n;s)} = \|T\|_{VN,(\infty;r_1,\dots,r_n;s)} = \|T\|$.

Proof. It is clear that for $\frac{1}{s} = \frac{1}{r'_1} + \cdots + \frac{1}{r'_n}$, we have

$$\|T\| \leq \|T\|_{VN,(\infty;r_1,\dots,r_n;s)} \leq \|T\|_{VN_f,(\infty;r_1,\dots,r_n;s)}.$$

Moreover,

$$\|T\| \left\|x^1\right\| \cdots \|x^n\| \geq \left(\int_\Omega \left| \sum_{j\in\mathbb{N}^n_m} \lambda_j \phi^1_{j_1}\left(x^1\right) \cdots \phi^n_{j_n}\left(x^n\right) b_j(t) \right|^s d\mu(t)\right)^{1/s} \tag{6}$$

Since $\phi^i_{j_i}$ is surjective there exists $\xi_i \in X_i$ such that $\phi^i_{j_i}(\xi_i) = M_i/2^{j_i/r'_i}$, where

$$M_i = \sup_{\|x^i\|_{X_i}\leq 1} \left(\sum_{j_i=1}^{m} \left|\left\langle \phi^i_{j_i}, x^i\right\rangle\right|^{r'_i}\right)^{1/r'_i},$$

We will show that $\|\xi_i\| \leq 1$ and $M_i < +\infty$ for $i = 1, \dots, n$. From the definition of M_i for a fixed i and for $\epsilon > 0$ we have

$$M_i \|\xi_i\| < (1+\epsilon) \left(\sum_{j_i=1}^{m} M_i^{r'_i} / 2^{j_i} \right)^{1/r'_i},$$

which implies that

$$\|\xi_i\| < (1+\epsilon), \text{ for all } \epsilon > 0.$$

So, considering $\|\xi_i\| < 1$ in equation (6) we have

$$\|T\| \geq \left(\int_{\Omega} \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j M_1 / 2^{j_1/r'_1} \cdots M_n / 2^{j_n/r'_n} b_j(t) \right|^s d\mu(t) \right)^{1/s},$$

if $k = \max\{j_1, \dots, j_n\}$ we get

$$\|T\| \geq \left(\int_{\Omega} \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j \frac{b_j(t)}{2^{k/s}} \right|^s d\mu(t) \right)^{1/s} \prod_{i=1}^{n} M_i. \tag{7}$$

Let $z(t) = \sum_{j \in \mathbb{N}_m^n} \lambda_j \frac{b_j(t)}{2^{k/s}}$, then for all $s \geq 1$ we have

$$|\langle \varphi, z \rangle| = \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j \left\langle \varphi, \frac{b_j}{2^{k/s}} \right\rangle \right| \leq \|\varphi\| \|z\|. \tag{8}$$

By renumbering multi-finite indices $j \in \mathbb{N}_m^n$, we can rewrite this finite sum as

$$z(t) = \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}}.$$

In addition, let $M = span_{k \in \{1, \dots, f(m,n)\} - k_0} \left\{ \frac{b_k}{2^{k/s}} \right\}$ where k_0 is a fixed number belongs to $\{1, \dots, f(m,n)\}$, and $f(m,n) \in \mathbb{N}$. Moreover, as a consequence of the Hahn-Banach theorem there exists φ such that $\|\varphi\| = \frac{1}{d}$, $\langle \varphi, x \rangle = 0$ for all $x \in M$ and $\left\langle \varphi, \frac{b_{k_0}}{2^{k_0/s}} \right\rangle = 1$, where $d = \inf_{x \in M} \left\| x - \frac{b_{k_0}}{2^{k_0/s}} \right\|$ and further one can choose λ_{k_0} such that

$$|\lambda_{k_0}| = \max_{k=1,\dots,f(m,n)} |\lambda_k| = \left\| (\lambda_j)_{j\in\mathbb{N}^n} \right\|_\infty ; \text{ where } j = (j_1,\dots,j_n) .$$

Taking into account these last relations in equation (8) we can get,

$$\|z\| \geq |\lambda_{k_0}|\, d. \tag{9}$$

Since $x = \sum_{k=1,\, k\neq k_0}^{f(m,n)} \frac{-b_k}{2^{k/s}} \in M$, then for a given $\epsilon > 0$, we have

$$(1+\epsilon)\, d > \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\| .$$

Therefore, from (9) we get

$$(1+\epsilon)\, \|z\| > \left\| (\lambda_j)_{j\in\mathbb{N}^n} \right\|_\infty \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\| . \tag{10}$$

We know that

$$\left\| (b_j)_{j\in\mathbb{N}^n} \right\|_{w,s'} = \sup_{\|\psi\|_s \leq 1} \left(\sum_{j\in\mathbb{N}_m^n} |\psi(b_j)|^{s'} \right)^{1/s'} = \sup_{a\in B_{l_s^{f(m,n)}}} \left\| \sum_{k=1}^{f(m,n)} a_k b_k \right\| ,$$

and since $a_k = \frac{1}{2^{k/s}}$ for $k = 1,\dots,f(m,n)$, given $\widetilde{\epsilon} > 0$ we have

$$(1+\widetilde{\epsilon}) \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\| \geq \left\| (b_j)_{j\in\mathbb{N}^n} \right\|_{w,s'} .$$

From the last relation and the equation (10) we obtain

$$(1+\epsilon)(1+\widetilde{\epsilon})\, \|z\| > \left\| (\lambda_j)_{j\in\mathbb{N}^n} \right\|_\infty \left\| (b_j)_{j\in\mathbb{N}^n} \right\|_{w,s'} \text{ for all } \epsilon \text{ and } \widetilde{\epsilon} > 0. \tag{11}$$

Therefore, from the relations (7) and (11) we get

$$\begin{aligned} \|T\| &\geq \left\| (\lambda_j)_{j\in\mathbb{N}^n} \right\|_\infty \left\| (b_j)_{j\in\mathbb{N}^n} \right\|_{w,s'} \prod_{i=1}^{n} M_i \\ &\geq \|T\|_{VN_{f,(\infty;r_1,\dots,r_n;s)}} . \end{aligned}$$

∎

We will prove a new link between the topological dual of virtually $(r;r_1,\ldots,r_n;s)$-nuclear n-linear operators and multiple $(r';r'_1,\ldots,r'_n;s')$-summing operators. The proof of the next theorem is similar to the proof of Theorem 7.3.1 in [10]. We included the detailed proof here for completeness.

THEOREM 3.4. *If the spaces X_k^*'s $(k=1,\ldots,n)$ have the λ_k- BAP, then the topological dual of $\mathcal{L}_{VN}^{(r;r_1,\ldots,r_n;s)}(X_1,\ldots,X_n;Y)$ is isomorphic isometrically to $\mathcal{L}_{mas}^{(r';r'_1,\ldots,r'_n;s')}(X_1^*,\ldots,X_n^*;Y^*)$, for $r,r_k\in[1,+\infty[$, $k=1,\ldots,n$ through the mapping $\mathcal{B}$ define by*

$$\mathcal{B}(\Psi)\left(\phi^1,\ldots,\phi^n\right)(b)=\Psi\left(\phi^1\times\cdots\times\phi^n b\right),$$

for all $b\in Y$, $\phi^k\in X_k^*,\ k=1,\ldots,n$ and $\Psi\in\left(\mathcal{L}_{VN}^{(r;r_1,\ldots,r_n;s)}(X_1,\ldots,X_n;Y)\right)^*$.

Proof. It is easy to see that the correspondence

$$\Psi\in\left(\mathcal{L}_{VN}^{(r;r_1,\ldots,r_n;s)}(X_1,\ldots,X_n;Y)\right)^*\longrightarrow\mathcal{B}(\Psi)\in\mathcal{L}_{mas}^{(r';r'_1,\ldots,r'_n;s')}(X_1^*,\ldots,X_n^*;Y^*)$$

defined by

$$\mathcal{B}(\Psi)\left(\phi^1,\ldots,\phi^n\right)(b)=\Psi\left(\phi^1\times\cdots\times\phi^n b\right),\ \phi^k\in X_k^*,\ k=1,\ldots,n \text{ and } b\in Y,$$

is linear and injective. To show the surjectivity let $T\in\mathcal{L}_{mas}^{(r';r'_1,\ldots,r'_n;s')}(X_1^*,\ldots,X_n^*;Y^*)$ and consider the linear functional Ψ_T on the space $\left(\mathcal{L}_f(X_1,\ldots,X_n;Y),\|\cdot\|_{VN_{f,(r;r_1,\ldots,r_n;s)}}\right)$ given by

$$\Psi_T(S)=\sum_{j\in\mathbb{N}_m^n}\lambda_j T\left(\varphi_{j_1}^1,\ldots,\varphi_{j_n}^n\right)(b_j)$$

for every $S\in\mathcal{L}_f(X_1,\ldots,X_n;Y)$ with a finite representation of the form

$$S=\sum_{j\in\mathbb{N}_m^n}\lambda_j\varphi_{j_1}^1\times\cdots\times\varphi_{j_n}^n b_j.$$

Hence, by Hölder's inequality and Definition 1.3 it follows that

$$\begin{aligned}|\Psi_T(S)|&\le\left\|(\lambda_j)_{j\in\mathbb{N}_m^n}\right\|_r\left\|\left(T\left(\varphi_{j_1}^1,\ldots,\varphi_{j_n}^n\right)(b_j)\right)_{j\in\mathbb{N}_m^n}\right\|_{r'}\\&\le\|T\|_{mas(r';r'_1,\ldots,r'_n;s')}\left\|(\lambda_j)_{j\in\mathbb{N}_m^n}\right\|_r\prod_{k=1}^n\left\|\left(\varphi_i^k\right)_{i=1}^m\right\|_{w,r'_k}\left\|(b_j)_{j\in\mathbb{N}_m^n}\right\|_{w,s'}.\end{aligned}$$

This shows that

$$|\Psi_T(S)| \leq \|T\|_{mas(r';r'_1,\dots,r'_n;s')} \|S\|_{VN_f,(r;r_1,\dots,r_n;s)},$$

for all $S \in \mathcal{L}_f(X_1,\dots,X_n;Y)$.

Since on $\mathcal{L}_f(X_1,\dots,X_n;Y)$, under our hypothesis for $X_1,\dots,X_n$, we have

$$\|\cdot\|_{VN_f,(r;r_1,\dots,r_n;s)} = \|\cdot\|_{VN,(r;r_1,\dots,r_n;s)},$$

we conclude that Ψ_T is continuous on $\mathcal{L}_f(X_1,\dots,X_n;Y)$ for $\|\cdot\|_{VN,(r;r_1,\dots,r_n;s)}$ and

$$\|\Psi_T\| \leq \|T\|_{mas(r';r'_1,\dots,r'_n;s')}.$$

By Proposition 2.3 (i), $\mathcal{L}_f(X_1,\dots,X_n;Y)$ is dense in $\mathcal{L}_{VN}^{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y)$. Hence we can extend Ψ_T to a continuous functional $\widetilde{\Psi}_T$ on $\mathcal{L}_{VN}^{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y)$ in a unique way, with

$$\left\|\widetilde{\Psi}_T\right\| \leq \|T\|_{mas(r';r'_1,\dots,r'_n;s')}.$$

Finally we note that $\mathcal{B}(\widetilde{\Psi}_T) = T$.

To show the reverse inequality let $\Psi \in \left(\mathcal{L}_{VN}^{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y)\right)^*$ and consider the corresponding n-linear mapping $\mathcal{B}(\Psi) \in \mathcal{L}(X_1^*,\dots,X_n^*;Y^*)$, defined by $\mathcal{B}(\Psi)\left(\phi^1,\dots,\phi^n\right)(b) = \Psi\left(\phi^1 \times \cdots \times \phi^n b\right)$, for $\phi^k \in X_k^*$, $k = 1,\dots,n$ and $b \in Y$. Let us consider $n \in \mathbb{N}$ and $\varphi_{j_k}^k \in X_k^*$, for $k = 1,\dots,n$, and $(b_j)_{j\in\mathbb{N}_m^n} \in l_{s'}^w(Y;\mathbb{N}_m^n)$. There is $(\lambda_j)_{j\in\mathbb{N}_m^n} \in l_r(\mathbb{N}_m^n)$ such that $\left\|(\lambda_j)_{j\in\mathbb{N}_m^n}\right\|_r = 1$ and

$$\left\|\left(\mathcal{B}(\Psi)\left(\varphi_{j_1}^1,\dots,\varphi_{j_n}^n\right)(b_j)\right)_{j\in\mathbb{N}_m^n}\right\|_{r'} = \sum_{j\in\mathbb{N}_m^n} \lambda_j \left|\mathcal{B}(\Psi)\left(\varphi_{j_1}^1,\dots,\varphi_{j_n}^n\right)(b_j)\right|$$

Now we can choose α_j, $|\alpha_j| = 1$, $j \in \mathbb{N}_m^n$ such that

$$\begin{aligned}\sum_{j\in\mathbb{N}_m^n} \lambda_j \left|\mathcal{B}(\Psi)\left(\varphi_{j_1}^1,\dots,\varphi_{j_n}^n\right)(b_j)\right| &= \sum_{j\in\mathbb{N}_m^n} \lambda_j\alpha_j \mathcal{B}(\Psi)\left(\varphi_{j_1}^1,\dots,\varphi_{j_n}^n\right)(b_j) \\ &= \Psi\left(\sum_{j\in\mathbb{N}_m^n} \lambda_j\alpha_j\varphi_{j_1}^1 \times \cdots \times \varphi_{j_n}^n b_j\right) = (*).\end{aligned}$$

By the continuity of Ψ and the Hölder's inequality we have

$$\begin{aligned}(*) &\leq \|\Psi\| \left\| \left(\lambda_{j_k}\alpha_{j_k}\right)_{j_k\in\mathbb{N}_m} \right\|_r \prod_{k=1}^{n} \left\| \left(\varphi^k_{j_k}\right)_{j\in\mathbb{N}^n_m} \right\|_{w,r'_k} \left\| \left(b_j\right)_{j\in\mathbb{N}^n_m} \right\|_{w,s'} \\ &= \|\Psi\| \prod_{k=1}^{n} \left\| \left(\varphi^k_{j_k}\right)_{j\in\mathbb{N}^n_m} \right\|_{w,r'_k} \left\| \left(b_j\right)_{j\in\mathbb{N}^n_m} \right\|_{w,s'} .\end{aligned}$$

This shows that $\mathcal{B}(\Psi) \in \mathcal{L}_{mas}^{(r';r'_1,\dots,r'_n;s')}(X_1^*,\dots,X_n^*;Y^*)$ and

$$\|\mathcal{B}(\Psi)\|_{mas(r';r'_1,\dots,r'_n;s')} \leq \|\Psi\| .$$

∎

If we replace $\mathbb{N}^n$ by $\mathbb{N}$ and s' by ∞ in Theorem 3.4, we obtain the following known cases.

COROLLARY 3.5. *If the spaces X_k^*'s $(k=1,\dots,n)$ have the λ_k- bounded approximation property, then*

(i) *The topological dual of $\mathcal{N}_{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y)$ is isometrically isomorphic to $\mathcal{L}_{as,(r';r'_1,\dots,r'_n;s')}(X_1^*,\dots,X_n^*;Y^*)$, for r, r_k and $s\in[1,+\infty]$, $k=1,\dots,n$ through the mapping $\mathcal{B}(\Psi)$ given as follows:*

$$\mathcal{B}(\Psi)\left(\phi^1,\dots,\phi^n\right)(b) := \Psi\left(\phi^1\times\cdots\times\phi^n b\right),$$

where Ψ is in the topological dual of $\mathcal{N}_{(r;r_1,\dots,r_n;s)}(X_1,\dots,X_n;Y)$, $\phi^k\in X_k^*$, $k=1,\dots,n$ and $b\in Y$.

(ii) *The topological dual of $\mathcal{L}_{VN}^{(r;r_1,\dots,r_n)}(X_1,\dots,X_n;Y)$ is isometrically isomorphic to $\mathcal{L}_{mas}^{(r';r'_1,\dots,r'_n)}(X_1^*,\dots,X_n^*;Y^*)$.*

ACKNOWLEDGEMENTS

The authors wish to thank Professor Erhan Çalışkan for his many useful suggestions concerning this paper. The author would also like to thank the referee for his useful comments and suggestions.

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