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A Study on Ricci Solitons in Generalized Complex Space Form

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Abstract: In this paper we obtain the condition for the existence of Ricci solitons in non-flat generalized complex space form by using Eisenhart problem. Also it is proved that if (g, V, λ) is Ricci soliton then V is solenoidal if and only if it is shrinking or steady or expanding depending upon the sign of scalar curvature.

Key words: Kähler manifolds, generalized complex space form, parallel second order covariant tensor field, Einstein space, Ricci soliton.

AMS *Subject Class.* (2010): 53C15, 53C21, 53C35, 53C55, 53C56.

1. Introduction

Ricci flow is an excellent tool in simplifying the structure of the manifolds. It is defined for Riemannain manifolds of any dimension. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the irregularity in the metric. It is given by

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(g(t)),$$

where g is Riemannian metric dependent on time t and $Ric(g(t))$ is Ricci tensor.

Let $\phi_t : M \longrightarrow M$, $t \in R$ be a family of diffeomorphisms and $(\phi_t : t \in R)$ is a one parameter family of abelian group called flow. It generates a vector field X_p given by

$$X_p f = \frac{df(\phi_t(p))}{dt}, \quad f \in C^\infty(M).$$

If Y is a vector field then $L_X Y = \lim_{t\to 0} \frac{\phi_t^* Y - Y}{t}$ is known as Lie derivative of Y with respect to X. Ricci solitons move under the Ricci flow under $\phi_t : M \longrightarrow M$ of the initial metric i.e., they are stationary points of the Ricci flow in space of metrics. If g_0 is a metric on the codomain then $g(t) = \phi_t^* g_0$ is the pullback of g_0, is a metric on the domain. Hence if g_0 is a solution of the Ricci flow on the codomain subject to condition $L_V g_0 + 2Ricg_0 + 2\lambda g_0 = 0$ on the codomain then $g(t)$ is the solution of the Ricci flow on the domain subject to the condition $L_V g + 2Ricg + 2\lambda g = 0$ on the domain by [12] under suitable conditions. Here g_0 and $g(t)$ are metrics which satisfy Ricci flow.

Thus the equation in general

$$L_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

is called Ricci soliton. It is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. Thus Ricci solitons are generalizations of Einstein manifolds and they are also called as quasi Einstein manifolds by theoretical physicists.

In 1923, Eisenhart [6] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor then it is reducible. In 1925, Levy [8] obtained the necessary and sufficient conditions for the existence of such tensors. Since then, many others investigated the Eisenhart problem of finding symmetric and skew-symmetric parallel tensors on various spaces and obtained fruitful results. For instance, by giving a global approach based on the Ricci identity. Sharma [11] firstly investigated Eisenhart problem on non-flat real and complex space forms, in 1989.

Using Eisenhart problem Calin and Crasmareanu [4], Bagewadi and Ingalahalli [7, 1], Debnath and Bhattacharyya [5] have studied the existence of Ricci solitons in f-Kenmotsu manifolds, α-Sasakian, Lorentzian α - Sasakian and Trans-Sasakian manifolds.

In 1989 the author Olszak [9] has worked on existence of generalized complex space form. The authors Parveena and Bagewadi [2, 10] extended the study to some curvature tensors on generalized complex space form. Motivated by these ideas, in this paper, we made an attempt to study Ricci solitons of generalized complex space form by using Eisenhart problem.

2. Preliminaries

A Kähler manifold is an n(even)-dimensional manifold, with a complex structure J and a positive-definite metric g which satisfies the following conditions;

$$J^2(X) = -X, \ \ g(JX, JY) = g(X, Y) \text{ and } (\nabla_X J)(Y) = 0, \tag{2.1}$$

where ∇ means covariant derivative according to the Levi-Civita connection. The formulae [3]

$$\begin{aligned} R(X,Y) &= R(JX,JY), && (2.2)\\ S(X,Y) &= S(JX,JY), && (2.3)\\ S(X,JY) &+ S(JX,Y)=0, && (2.4) \end{aligned}$$

are well known for a Kähler manifold.

DEFINITION 2.1. A Kähler manifold with constant holomorphic sectional curvature c is said to be a complex space form and its curvature tensor is given by

$$R(X,Y)Z=\frac{c}{4}\big[g(Y,Z)X-g(X,Z)Y+g(X,JZ)JY - g(Y,JZ)JX+2g(X,JY)JZ\big].$$

The models now are C^n, CP^n and CH^n, depending on $c=0$, $c>0$ or $c<0$.

DEFINITION 2.2. An almost Hermition manifold M is called a generalized complex space form $M(f_1,f_2)$ if its Riemannian curvature tensor R satisfies,

$$R(X,Y)Z=f_1\{g(Y,Z)X-g(X,Z)Y\}+f_2\{g(X,JZ)JY - g(Y,JZ)JX+2g(X,JY)JZ\}. \quad (2.5)$$

3. PARALLEL SYMMETRIC SECOND ORDER COVARIANT TENSOR AND RICCI SOLITON IN A NON-FLAT GENERALIZED COMPLEX SPACE FORM

Let h be a $(0,2)$-tensor which is parallel with respect to ∇ that is $\nabla h=0$. Applying the Ricci identity [11]

$$\nabla^2 h(X,Y;Z,W)-\nabla^2 h(X,Y;W,Z)=0. \quad (3.1)$$

We obtain the relation [11]:

$$h(R(X,Y)Z,W)+h(Z,R(X,Y)W)=0. \quad (3.2)$$

Using equation (2.5) in (3.2) and putting $X=W=e_i$, $1\le i\le n$ after simplification, we get

$$f_1\{g(Y,Z)(tr.H)-h(Y,Z)\}+f_2\{h(JY,JZ)-g(Y,JZ)(tr.HJ) + 2h(JZ,JY)\}-\{(n-1)f_1-3f_2\}h(Z,Y)=0, \quad (3.3)$$

where H is a $(1,1)$ tensor metrically equivalent to h. Symmetrization and anti-symmetrization of (3.3) yield:

$$\frac{[nf_1 - 3f_2]}{f_1} h(Z,Y) - \frac{3f_2}{f_1} h(JY,JZ) = (tr.H)g(Y,Z), \tag{3.4}$$

$$\frac{[(n-2)f_1 - 3f_2]}{f_2} h(Y,Z) + h(JZ,JY) = g(Y,JZ)(tr.HJ). \tag{3.5}$$

Replacing Y, Z by JY, JZ respectively in (3.4) and adding the resultant equation from (3.4), provide we obtain:

$$h_s(Y,Z) = \beta.(tr.H)g(Y,Z), \tag{3.6}$$

where

$$\beta = \frac{f_1}{nf_1 - 6f_2}.$$

Replacing Y, Z by JY, JZ respectively in (3.5) and adding the resultant equation from (3.5), provide we obtain:

$$h_a(Y,Z) = \frac{f_2}{[(n-2)f_1 - Hf_2]}(tr.HJ)g(Y,JZ). \tag{3.7}$$

By summing up (3.6) and (3.7) we obtain the expression:

$$h = \{\beta.(tr.H)g + \rho(tr.HJ)\Omega\}, \tag{3.8}$$

where

$$\rho = \frac{f_2}{[(n-2)f_1 - Hf_2]}.$$

Hence we can state the following.

THEOREM 3.1. *A second order parallel tensor in a non-flat generalized complex space form is a linear combination (with constant coefficients) of the underlying Kaehlerian metric and Kaehlerian 2-form.*

COROLLARY 3.1. *The only symmetric (anti-symmetric) parallel tensor of type $(0,2)$ in a non-flat generalized complex space form is the Kaehlerian metric (Kaehlerian 2-form) up to a constant multiple.*

COROLLARY 3.2. *A locally Ricci symmetric ($\nabla S = 0$) non-flat generalized complex space form is an Einstein manifold.*

Proof. If $H = S$ in (3.8) then $tr.H = r$ and $tr.HJ = 0$ by virtue of (2.4). Equation (3.8) can be written as

$$S(Y,Z) = \beta r\, g(Y,Z). \tag{3.9}$$

∎

Remark 3.1. The following statements for non-flat generalized complex space form are equivalent.

1. Einstein
2. locally Ricci symmetric
3. Ricci semi-symmetric that is $R \cdot S = 0$ if $f_1 \neq 0$.

Proof. The statements (1) → (2) → (3) are trivial. Now, we prove the statement (3) → (1) is true.
Here $R \cdot S = 0$ means

$$(R(X,Y) \cdot S(U,W)) = 0.$$

Which implies

$$S(R(X,Y)U,W) + S(U,R(X,Y)W) = 0. \tag{3.10}$$

Using equations (2.5) in (3.10) and putting $Y = U = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$ we get after simplification that

$$f_1\{nS(X,W) - rg(X,W)\} = 0. \tag{3.11}$$

If $f_1 \neq 0$, then (3.11) reduced to

$$S(X,W) = \frac{r}{n} g(X,W). \tag{3.12}$$

∎

Therefore, we conclude the following.

LEMMA 3.1. *A Ricci semi-symmetric non-flat generalized complex space form is an Einstein manifold if $f_1 \neq 0$.*

Corollary 3.3. *Suppose that on a non-flat generalized complex space form, the $(0,2)$ type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g, V) yield a Ricci soliton if JV is solenoidal. In particular, if the given non-flat generalized complex space form is Ricci semi-symmetric with $L_V g$ parallel, we have same conclusion.*

Proof. From Theorem (3.1) and corollary (3.2), we have $\lambda = -\beta r$ as seen below:

$$\begin{aligned}(L_V g + 2S)(Y,Z) &= \big[\beta\, tr(L_V g + 2S)g(Y,Z) \\ &\quad + \rho.tr((L_V g + 2S)J)\Omega(Y,Z)\big] \\ &= \big[2\beta(\operatorname{div} V + r)g(Y,Z) + \rho[2(\operatorname{div} JV)\Omega(Y,Z) \\ &\quad + 2(tr.SJ)\Omega(Y,Z)\big],\end{aligned} \tag{3.13}$$

by virtue of (2.4) the above equation becomes

$$(L_V g + 2S)(Y,Z) = \big[2\beta(\operatorname{div} V + r)g(Y,Z) + 2\rho(\operatorname{div} JV)\Omega(Y,Z)\big]. \tag{3.14}$$

By definition (g, V, λ) yields Ricci soliton. If $\operatorname{div} JV = 0$ then $\operatorname{div} V = 0$ becouse $JV = iV$ i.e.,

$$(L_V g + 2S)(Y,Z) = 2\beta r\, g(Y,Z) = -2\lambda g(Y,Z). \tag{3.15}$$

Therefore $\lambda = -\beta r$. ∎

Corollary 3.4. *Let (g, V, λ) be a Ricci soliton in a non-flat generalized complex space form. Then V is solenoidal if and only if it is shrinking or steady or expanding depending upon the sign of scalar curvature.*

Proof. Using equation (3.12) in (1.1) we get

$$(L_V g)(Y,Z) + 2\frac{r}{n}g(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{3.16}$$

Putting $Y = Z = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \le i \le n)$, we get

$$(L_V g)(e_i,e_i) + 2\frac{r}{n}g(e_i,e_i) + 2\lambda g(e_i,e_i) = 0. \tag{3.17}$$

The above equation implies

$$\operatorname{div} V + r + \lambda n = 0. \tag{3.18}$$

If V is solenoidal then $\operatorname{div} V = 0$. Therefore the equation (3.18) can be reduced to

$$\lambda = \frac{-r}{n}. \qquad \blacksquare$$

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