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Condensation and superfluidity of dilute Bose gases with finite-range interaction

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Abstract

We investigate an ultracold and dilute Bose gas by taking into account a finite-range two-body interaction. The coupling constants of the resulting Lagrangian density are related to measurable scattering parameters by following the effective-field-theory approach. A perturbative scheme is then developed up to the Gaussian level, where both quantum and thermal fluctuations are crucially affected by finite-range corrections. In particular, the relation between spontaneous symmetry breaking and the onset of superfluidity is emphasized by recovering the renowned Landau's equation for the superfluid density in terms of the condensate one.

1. Introduction

Since the seminal investigation on liquid helium by Kamerlingh Onnes, the research on low-temperature physics has been focused on the understanding and engineering of superfluid states of matter [1]. Nowadays, superfluids and superconducting materials are at the core of a new technological revolution centered around the development of quantum devices [2–4].

From a theoretical point of view, an enormous effort has been devoted to finally provide a microscopic theory accounting for the transition between a normal state and a superfluid one, where dissipationless flow occurs. This research oscillated between the necessity of statistical mechanics to identify general mechanisms leading to supertransport and the interests of condensed matter theorists and material scientists on peculiar setups with unique properties. Concerning metallic superconductors, the theoretical investigation reached one of its peaks with the BCS theory and its consequent refinements [5, 6].

Moving to the superfluid side, liquid helium remains for decades the only efficient platform to probe the markers of superfluid behavior, like the absence of viscous forces and the vorticity quantization [7, 8]. The formulation of a microscopic theory for the superfluid phase of liquid helium proves to be an exceptionally demanding task. Up to now, a reliable picture can be achieved via *ab initio* numerical simulations such as path-integral Monte Carlo algorithms [9]. Indeed, liquid helium has to be classified as a strongly-correlated system, due to the experimental values at play for density and interaction strength. In this situation, even identifying a *smallness* parameter is non trivial, preventing the effective implementation of a perturbative expansion [10].

The two-fluid phenomenological approach by Landau [11, 12] has been much more fruitful, enabling the derivation of a self-consistent hydrodynamic theory over few crucial assumptions. Remarkably, the original formulation of Landau did not rely upon an atomic point of view and, moreover, did not invoke any symmetry breaking related to the modern characterization of phase transitions [13].

Since the pioneering guess of London in 1938 [14, 15], it has been a common approach to interpret the superfluid transition in terms of Bose–Einstein condensation, where a macroscopic fraction of helium atoms can be described by a macroscopic wavefunction. Unfortunately, analytical result can be obtained only in the weakly-interacting limit, which does not hold up to the experimental values for helium. Applied to helium, the

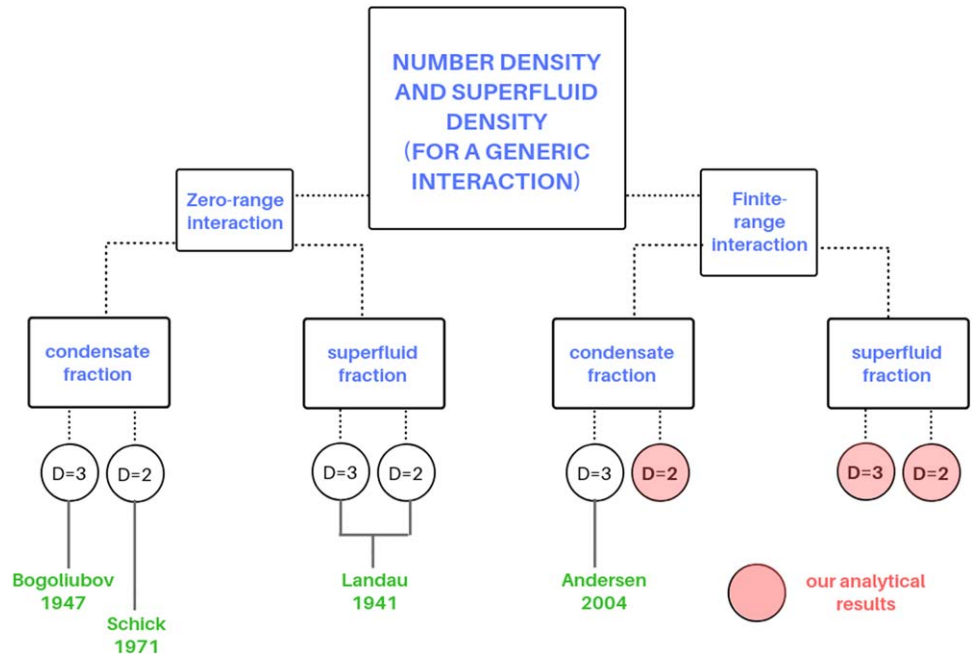

Figure 1. Flowchart diagram showing some the key theoretical developments related to our analysis of three- and two-dimensional Bose gases with a finite-range interaction potential.

resulting picture is only qualitative since, for instance, it does not even manage to capture the peculiar rotonic minimum of the excitation spectrum.

In 1995, the experimental realization of a Bose–Einstein condensates [13] changed the scenario in a crucial way: for the first time, the predictions of the Bogoliubov theory [16] have been checked in actual weakly-interacting quantum gases. At the same time, within a field-theory approach, it is possible to recover the Landau main results moving from a microscopic Lagrangian for ultracold atoms.

The several successful theoretical studies based on the Bogoliubov framework move from the crucial assumption that the *true* atom–atom interaction can be replaced by a contact (i.e. zero-range) pseudopotential whose strength is given by the *s*-wave scattering length a_s [17, 18]. The resulting thermodynamics is universal since there is no dependence on the potential shape, with only a_s playing a relevant role. The same point can be made for transport quantities as the superfluid fraction. Despite the many achievements of this strategy, current experiments deal with higher density setups, reduced dimensionalities and more complex interactions [19, 20]. Thus, it is pressing to extend the usual two-body zero-range framework in order to capture more realistic and interesting experimental regimes. Within a functional integration formalism, atoms are represented by a bosonic field whose dynamics is governed by a microscopic interacting Lagrangian density. The coupling constants of the finite-range theory can be determined in terms of the *s*-wave scattering parameters, namely a_s and the corresponding effective range r_e. In [21–24], the finite-range thermodynamics is derived up to the Gaussian level for a three-dimensional uniform Bose gas, while the non-trivial case of two spatial dimensions is addressed in [25, 26]. In figure 1 we report a visual summary of the major analytical approaches to modeling bosonic quantum gases.

A similar analysis concerning the superfluid properties of a finite-range theory is still missing and it is the main subject of this work. By adopting a functional integration point of view as in [23, 25], we are going to show that both condensate and superfluid depletion are modified by the finite-range character of the two-body interaction. Moreover, they are not independent from each other but the familiar Landau equation for the superfluid density can be formulated in terms of the condensate one.

The paper is organized as follows: in the next section we discuss the Landau two-fluid model from a field-theory perspective moving from a microscopic Langrangian density. The key point consists in properly accounting the different response of normal and superfluid components to a Galilean boost [27]. Technically, this corresponds to performing a phase twist on the order parameter [28, 29], which has to be considered throughout the whole perturbative expansion. By following this scheme, we will derive a Landau-like equation relating superfluid and condensate density. Our calculation are carried on in a generic dimension D, simplifying the application of dimensional regularization to heal the UV-divergent zero-point energy. We then specify to the cases $D = 3$ and $D = 2$.

2. Two-fluid model of a superfluid

The thermodynamic properties of a physical system can be described calculating the grand canonical partition function $\mathcal{Z}$, which is related to the grand potential Ω by

$$\Omega = -\frac{1}{\beta}\ln(\mathcal{Z}). \tag{1}$$

We consider a uniform D-dimensional Bose gas of identical cold atoms described by the complex scalar field $\psi(\vec{r}, \tau)$. We calculate the grand canonical partition function $\mathcal{Z}$ as the functional integral

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi]\ \mathrm{e}^{-\frac{S[\bar{\psi},\psi]}{\hbar}}, \tag{2}$$

where

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} \mathrm{d}\tau \int_{L^D} \mathrm{d}^D r\ \mathcal{L}(\bar{\psi}, \psi) \tag{3}$$

is the Euclidean action and $\beta = 1/(k_B T)$, with k_B the Boltzmann constant and T the absolute temperature. We introduce the non-relativistic Lagrangian

$$\begin{aligned}\mathcal{L} = \bar{\psi}(\vec{r}, \tau)&\left(\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu\right)\psi(\vec{r}, \tau)\\ &+ \frac{1}{2}\int \mathrm{d}^D r'\ |\psi(\vec{r}', \tau)|^2\ V(|\vec{r} - \vec{r}'|)\ |\psi(\vec{r}, \tau)|^2,\end{aligned} \tag{4}$$

where μ is the chemical potential and $V(|\vec{r} - \vec{r}'|)$ is the generic interaction between bosons, assuming it is dependent only on the relative distance $|\vec{r} - \vec{r}'|$.

In the context of the two-fluid model of a superfluid [11, 12] we describe the fluid behavior of the system as composed by a mixture of a normal component and a superfluid component. In particular, we consider the normal part as a fluid current moving with velocity $\vec{v}$ with respect to the laboratory frame of reference. Working with imaginary time, we describe this motion by substituting the time derivative in the Lagrangian (4) with the Lagrangian fluid derivative

$$\partial_\tau \to \partial_\tau - \mathrm{i}\vec{v}\cdot\vec{\nabla}. \tag{5}$$

Moreover, since the superfluid part does not exchange momentum with the normal part, we impose a superfluid current with a phase twist of the bosonic field [28]

$$\psi(\vec{r}, \tau) \to \mathrm{e}^{\mathrm{i}\frac{m\vec{v}_s\cdot\vec{r}}{\hbar}}\psi(\vec{r}, \tau). \tag{6}$$

Substituting these expressions in the Lagrangian (4) we obtain

$$\begin{aligned}\mathcal{L} = \bar{\psi}(\vec{r}, \tau)&\left(\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu_e + (\vec{v} - \vec{v}_s)\cdot(-\mathrm{i}\hbar\vec{\nabla})\right)\psi(\vec{r}, \tau)\\ &+ \frac{1}{2}\int \mathrm{d}^D r'\ |\psi(\vec{r}', \tau)|^2\ V(|\vec{r} - \vec{r}'|)\ |\psi(\vec{r}, \tau)|^2,\end{aligned} \tag{7}$$

where we define the effective chemical potential μ_e as [27]

$$\mu_e = \mu - \frac{1}{2}m\vec{v}_s\cdot(\vec{v}_s - 2\vec{v}). \tag{8}$$

Considering that in the condensed phase the U(1) global symmetry is spontaneously broken we use the bosonic field parametrization

$$\psi(\vec{r}, \tau) = \psi_0 + \eta(\vec{r}, \tau), \tag{9}$$

where $\eta(\vec{r}, \tau)$ is the complex field describing the fluctuation around the uniform field configuration ψ_0, which represents the order parameter of the condensate phase transition. Substituting the parametrization (9) in the action (3) and keeping only quadratic terms in the fluctuation field $\eta(\vec{r}, \tau)$ we obtain the homogeneous system action

$$S_0 = \beta\hbar L^D\left(-\mu_e\psi_0^2 + \frac{1}{2}g_0\psi_0^4\right) \tag{10}$$

and we calculate the Gaussian action in the Fourier space as [30]

$$S_g[\bar{\eta}, \eta] = \frac{\hbar}{2}\sum_k (\bar{\eta}(k)\ \eta(-k))\ \mathcal{M}\begin{pmatrix}\eta(k)\\ \bar{\eta}(-k)\end{pmatrix}, \tag{11}$$

where $k = (\vec{k},\ \omega_n)$ are the $D+1$ wavevectors, $\omega_n = \frac{2\pi n}{\beta\hbar}$ are the bosonic Matsubara frequencies and

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \tag{12}$$

with

$$\begin{aligned} \mathcal{M}_{11} &= -\mathrm{i}\hbar\omega_n + \frac{\hbar^2k^2}{2m} - \mu_e + g_0\psi_0^2 + \psi_0^2\,\tilde{V}(\vec{k}) + \hbar\,(\vec{v}-\vec{v}_s)\cdot\vec{k} \\ \mathcal{M}_{22} &= +\mathrm{i}\hbar\omega_n + \frac{\hbar^2k^2}{2m} - \mu_e + g_0\psi_0^2 + \psi_0^2\,\tilde{V}(\vec{k}) - \hbar\,(\vec{v}-\vec{v}_s)\cdot\vec{k} \\ \mathcal{M}_{12} &= \mathcal{M}_{21} = \psi_0^2\tilde{V}(\vec{k}). \end{aligned} \tag{13}$$

Performing the functional integration of the Gaussian action (11) we obtain the partition function (2) and the grand potential (1). In particular, we find that the grand potential Ω is the sum of two contributions

$$\Omega = \Omega_0 + \Omega_g, \tag{14}$$

where

$$\Omega_0 = L^D\left(-\mu_e\psi_0^2 + \frac{1}{2}g_0\psi_0^4\right) \tag{15}$$

is the order parameter grand potential and

$$\Omega_g = \frac{1}{2\beta}\sum_{\vec{k},\omega_n}\ln[\beta^2(E_{\vec{k}}(\psi_0^2)^2 - (\mathrm{i}\hbar\omega_n - \hbar\,(\vec{v}-\vec{v}_s)\cdot\vec{k}\ )^2)] \tag{16}$$

is the Gaussian contribution to the grand potential, where

$$E_{\vec{k}}(\psi_0^2) = \sqrt{\left(\frac{\hbar^2k^2}{2m} - \mu_e + g_0\psi_0^2 + \psi_0^2\,\tilde{V}(\vec{k})\right)^2 - (\psi_0^2\,\tilde{V}(\vec{k}))^2} \tag{17}$$

is the excitation spectrum of the Bose gas. The Hugenholtz–Pines theorem is guaranteed imposing the saddle point condition $\partial\Omega/\partial\psi_0 = 0$, for which the excitation spectrum becomes gapless and the chemical potential reads

$$\mu_e = g_0\psi_0^2. \tag{18}$$

Moreover, we identify the condensate density as $n_0 = \psi_0^2$. Notice that, within the Gaussian approximation of the action, the excitation spectrum does not contain the anomalous density, which is instead included by adopting the next-next-to-leading Hartree–Fock–Bogoliubov scheme [31–33]. The initial assumption that the real space interaction depends only on the distance between bosons implies that the interaction potential is left unchanged by a reflection of the momenta: $\tilde{V}(\vec{k}) = \tilde{V}(-\vec{k})$. Due to this property we are able to perform the summation over Matsubara frequencies ω_n in the Gaussian grand potential (16) [27, 34], obtaining the grand potential Ω in the form

$$\Omega = \Omega_0 + \Omega_g^{(0)} + \Omega_g^{(T)}, \tag{19}$$

where Ω_0 is given by equation (15)

$$\Omega_g^{(0)} = \frac{1}{2}\sum_{\vec{k}} E_{\vec{k}}(\psi_0^2) \tag{20}$$

is the zero-temperature Gaussian contribution, written as the sum of noninteracting elementary excitations with spectrum $E_{\vec{k}}(\psi_0^2)$, and

$$\Omega_g^{(T)} = \frac{1}{\beta}\sum_{\vec{k}}\ln(1 - \mathrm{e}^{-\beta(E_{\vec{k}}(\psi_0^2)+\hbar\,(\vec{v}-\vec{v}_s)\cdot\vec{k})}) \tag{21}$$

is the finite-temperature Gaussian contribution. The beyond-mean-field Gaussian equation of state with finite-range interaction has been analyzed in previous papers [21, 23, 25]. For the sake of completeness we report the main results in the appendix.

We now calculate the superfluid density n_s of the system with a self-consistent approach, employing the Gaussian grand potential (19). In particular, we will identify n_s from the calculation of the total momentum density $\vec{\mathcal{P}}$ of the fluid, which is obtained as follows

$$\vec{\mathcal{P}} = \frac{1}{L^D}\frac{\partial\Omega(\mu_e,\ \psi_0)}{\partial(-\vec{v})}\bigg|_{\mu_e=g_0\psi_0^2=g_0n_0}, \tag{22}$$

where we take the derivative of the grand potential with respect to the velocity $-\vec{v}$ first and then we substitute the mean field value of the chemical potential $\mu_e = g_0\psi_0^2$. We find that the momentum density $\vec{\mathcal{P}}$ is given by

$$\vec{\mathcal{P}} = \vec{\mathcal{P}}_0 + \vec{\mathcal{P}}_g^{(0)} + \vec{\mathcal{P}}_g^{(T)}, \tag{23}$$

where, since the grand potential (19) is given by the sum of three sum contributions, the three terms of the momentum density $\vec{\mathcal{P}}$ are defined accordingly. In particular, we find

$$\vec{\mathcal{P}}_0 = n_0 m\vec{v}_s. \tag{24}$$

The Gaussian zero-temperature contribution $\vec{\mathcal{P}}_g^{(0)}$ is

$$\vec{\mathcal{P}}_g^{(0)} = f_g^{(0)}(n_0)\ m\vec{v}_s, \tag{25}$$

where we define the zero-temperature number density contribution $f_g^{(0)}(n_0)$ as

$$f_g^{(0)}(n_0) = \frac{1}{2L^D}\sum_{\vec{k}} \frac{1}{E_{\vec{k}}(n_0)}\left(\frac{\hbar^2k^2}{2m} + n_0\tilde{V}(\vec{k})\right). \tag{26}$$

The Gaussian thermal contribution $\vec{\mathcal{P}}_g^{(T)}$ to the momentum density (23) is more involved:

$$\vec{\mathcal{P}}_g^{(T)} = \frac{1}{L^D}\sum_{\vec{k}} \frac{1}{e^{\beta(E_{\vec{k}}(\psi_0^2)+\hbar(\vec{v}-\vec{v}_s)\cdot\vec{k})}-1}\left[\frac{\partial E_{\vec{k}}(\psi_0^2)}{\partial\mu_e}\frac{\partial\mu_e}{\partial(-\vec{v})} - \hbar\vec{k}\right]\Bigg|_{\mu_e=g_0\psi_0^2=g_0n_0}. \tag{27}$$

Assuming that the difference between the velocity $\vec{v}$ of the normal fluid and the velocity $\vec{v}_s$ of the superfluid is small, we can expand the exponential and, taking into account that some terms are zero for symmetry reasons, we obtain

$$\vec{\mathcal{P}}_g^{(T)} = f_g^{(T)}(n_0)\ m\vec{v}_s + n_n(n_0,\ T)\ m(\vec{v}-\vec{v}_s), \tag{28}$$

where $f_g^{(T)}(n_0)$ is the Gaussian thermal density contribution

$$f_g^{(T)}(n_0) = \frac{1}{L^D}\sum_{\vec{k}} \frac{1}{E_{\vec{k}}(n_0)}\left(\frac{\hbar^2k^2}{2m} + n_0\tilde{V}(\vec{k})\right)\frac{1}{e^{\beta E_{\vec{k}}(n_0)}-1} \tag{29}$$

and we have defined the normal density of the fluid as

$$n_n(n_0,\ T) = \frac{\beta\hbar^2}{mDL^D}\sum_{\vec{k}} k^2\ \frac{e^{\beta E_{\vec{k}}(n_0)}}{(e^{\beta E_{\vec{k}}(n_0)}-1)^2} \tag{30}$$

Notice that, in the thermodynamic limit $L^D \to \infty$, the normal fluid density (30) is fully consistent with the familiar Landau result [12]. In conclusion, putting together the contributions (24), (25) and (28) we rewrite the momentum density $\vec{\mathcal{P}}$ as

$$\vec{\mathcal{P}} = [n_0 + f_g^{(0)}(n_0) + f_g^{(T)}(n_0)]m\vec{v}_s + n_n(n_0,\ T)\ m(\vec{v}-\vec{v}_s). \tag{31}$$

In the square bracket we identify the number density $n(n_0, T)$, expressed as a function of the condensate number density n_0 and the temperature T as follows [27]

$$n(n_0,\ T) = -\frac{1}{L^D}\frac{\partial\Omega(\mu_e,\ \psi_0,\ T)}{\partial\mu_e}\Bigg|_{\mu_e=g_0n_0} = n_0 + f_g^{(0)}(n_0) + f_g^{(T)}(n_0). \tag{32}$$

We remark that we take the derivative of the grand potential with respect to the chemical potential first and then we substitute the mean field value of the chemical potential $\mu_e = g_0\psi_0^2$, with the identification for the condensate density $n_0 = \psi_0^2$. This procedure can be justified considering that the same procedure is implemented to calculate the condensate fraction of a noninteracting Bose gas [35]. With this identification, we express the momentum density $\vec{\mathcal{P}}$ as

$$\vec{\mathcal{P}} = n(n_0,\ T)m\vec{v}_s + n_n(n_0,\ T)\ m(\vec{v}-\vec{v}_s). \tag{33}$$

Finally, we identify the superfluid density n_s as

$$n_s = n(n_0,\ T) - n_n(n_0,\ T) \tag{34}$$

which allows us to express the momentum density $\vec{\mathcal{P}}$ as the sum of the momentum density $n_s m\vec{v}_s$ of the superfluid part of the fluid and the momentum density $n_n(n_0,\ T)m\vec{v}$ of the normal part of the fluid, namely

$$\vec{\mathcal{P}} = n_s m\vec{v}_s + n_n(n_0,\ T)m\vec{v}. \tag{35}$$

We emphasize that equation (34) constitutes the main result of this paper, since it highlights the non-trivial relationship between the superfluid density n_s, the condensate density n_0 and the temperature T. This result may be regarded as the explicit formulation, at a Gaussian level, of the Josephson relation [36].

Notice that our number density (32) and the superfluid fraction (34) are equivalent to the result obtained with Beliaev diagrammatic technique in reference [37] if we approximate $n_0 \approx n$ in equations (26), (29) and (30) and we consider the zero-range interaction $\tilde{V}(\vec{k}) = g_0$. In the following section we implement the superfluid density calculation for bosons with finite-range interaction in three- and two- dimensional systems.

3. Superfluid density of bosons with finite-range interaction

In order to obtain explicit formulas for the superfluid density n_s, in this section we shall implement equation (34) in the three- and in the two- dimensional Bose gas, considering the explicit form of the interaction $V(\vec{r})$. The usual approximation to study a weakly-interacting Bose gas of ultracold atoms is constituted by the zero-range interaction $V(\vec{r}) = g_0\, \delta(\vec{r})$, which in the Fourier space gives

$$\tilde{V}(\vec{k}) = g_0. \tag{36}$$

Here we improve this approximation, considering the finite-range effective interaction

$$\tilde{V}(\vec{k}) = g_0 + g_2 k^2 \tag{37}$$

which is obtained adding to the zero-range interaction strength g_0 the first nonzero correction in the gradient expansion of the real interaction potential $V(|\vec{r} - \vec{r}'|)$, namely $g_2 k^2$, where we define

$$g_0 = \int \mathrm{d}^D r\ V(r), \qquad g_2 = -\frac{1}{2}\int \mathrm{d}^D r\ r^2\ V(r) \tag{38}$$

In the three-dimensional case, the values of the couplings g_0 and g_2 are determined with the scattering theory in terms of the s-wave scattering length a_s and the effective range r_e as follows [23]

$$g_0 = \frac{4\pi\hbar^2 a_s}{m}, \qquad g_2 = \frac{2\pi\hbar^2 a_s^2 r_e}{m}. \tag{39}$$

In the two-dimensional case we choose the zero-range interaction coupling g_0 according to [38] and we derive the coupling g_2 from the definition of the characteristic range $R = 2\sqrt{|g_2/g_0|}$ discussed in [25], obtaining

$$g_0 = \frac{4\pi\hbar^2}{m}\frac{1}{|\ln(n a_s^2)|} \qquad g_2 = \frac{\pi\hbar^2}{m}\frac{R^2}{|\ln(n a_s^2)|}. \tag{40}$$

We now explicitly implement the superfluid density n_s calculation for the finite-range effective interaction (37). According to equation (34), n_s is obtained by subtracting the normal density $n_n(n_0, T)$ from the number density $n(n_0, T)$. We first calculate the number density $n(n_0, T)$, which is given as the sum of the three contributions of equation (32). The first contribution is the condensate density n_0. The second contribution is the zero-temperature Gaussian contribution to the normal density, namely

$$f_g^{(0)}(n_0) = \frac{S_D}{2(2\pi)^D}\int_0^{+\infty} \mathrm{d}k\ k^{D-1}\frac{1}{E_k(n_0)}\left(\frac{\hbar^2 k^2}{2m}\tilde{\lambda}(g_2, n_0) + g_0 n_0\right), \tag{41}$$

where $S_D = 2\pi^{D/2}/\Gamma[D/2]$ is the D-dimensional solid angle, in which $\Gamma[x]$ is the Euler gamma function and where we define the excitation spectrum

$$E_k(n_0) = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m}\lambda(g_2, n_0) + 2 n_0 g_0\right)} \tag{42}$$

with

$$\lambda(g_2, n_0) = 1 + \frac{4m}{\hbar^2} g_2 n_0 \qquad \tilde{\lambda}(g_2, n_0) = 1 + \frac{2m}{\hbar^2} g_2 n_0. \tag{43}$$

Notice that the excitation spectrum $E_k(n_0)$ reproduces the familiar Bogoliubov spectrum if the zero-range interaction is restored by putting $g_2 = 0$. Since $f_g^{(0)}(n_0)$ is ultraviolet divergent, we will regularize it using dimensional regularization [39, 40]. In particular, we obtain an adimensional integral using the integration variable $t = \hbar^2 k^2 \lambda(g_2, n_0)/(4 m g_0 n_0)$, then we extend the spatial dimension D to the complex value $\mathcal{D} = D - \varepsilon$. We remark that this additional step is needed because the dimensional regularization procedure is not always able to heal the ultraviolet divergence of the integrals [35]. After the integration, we obtain $f_g^{(0)}(n_0)$ in the form

$$f_g^{(0)}(n_0) = \frac{\kappa^{\varepsilon}\lambda(g_2,\, n_0)^{1/2}}{4\Gamma(\mathcal{D}/2)}\left(\frac{mg_0n_0}{\pi\hbar^2\lambda(g_2,\, n_0)}\right)^{\mathcal{D}/2}\left[2\frac{\tilde{\lambda}(g_2,\, n_0)}{\lambda(g_2,\, n_0)}\cdot B\left(\frac{\mathcal{D}+1}{2}, \frac{-\mathcal{D}}{2}\right) + B\left(\frac{\mathcal{D}-1}{2}, \frac{2-\mathcal{D}}{2}\right)\right], \tag{44}$$

where $B(x, y)$ is the Euler Beta function and κ is an ultraviolet cutoff wavevector introduced for dimensional reasons.

The third term in the number density of equation (32) is the finite-temperature Gaussian contribution $f_g^{(T)}(n_0)$, which, unlike the zero-temperature one, is convergent. However, this integral can be calculated analitically only in the low-temperature regime, where it is useful to introduce the integration variable $x = \beta E_k(n_0)$. Doing so in equation (29) in which the finite-range interaction (37) is substituted, we get

$$f_g^{(T)}(n_0) = \frac{S_D}{(2\pi)^D k_B T}\int_0^{+\infty} \mathrm{d}x\, \frac{\mathrm{d}k(x)}{\mathrm{d}x}\frac{k(x)^{D-1}}{x(e^x - 1)}\left(\frac{\hbar^2k(x)^2}{2m}\lambda(g_2,\, n_0) + n_0g_0\right) \tag{45}$$

with

$$k(x) = \sqrt{\frac{2\, mn_0g_0}{\hbar^2\lambda(g_2,\, n_0)}}\sqrt{-1 + \sqrt{1 + \frac{(k_B T)^2\lambda(g_2,\, n_0)x^2}{n_0^2g_0^2}}}. \tag{46}$$

Before substituting the spatial dimension D to calculate explicitly the number density contributions obtained in the previous equations, let us also calculate the normal density $n_n(n_0,\, T)$, which is given by equation (30) where the finite-range interaction (37) is substituted. In analogy with the finite-temperature density contribution $f_g^{(T)}(n_0)$, the normal density $n_n(n_0,\, T)$ can be calculated analytically only in the low-temperature regime: as before we introduce the integration variable $x = \beta E_k(n_0)$, obtaining

$$n_n(n_0,\, T) = \frac{\beta\hbar^2S_D}{mD(2\pi)^D}\int_0^{+\infty} \mathrm{d}x\, \frac{\mathrm{d}k}{\mathrm{d}x}(x)\, k(x)^{D+1}\, \frac{e^x}{(e^x - 1)^2}, \tag{47}$$

where $k(x)$ is given again by equation (46). We now calculate explicitly the number density $n(n_0, T)$ and the superfluid density $n_s(n_0, T)$ in $D = 3$ and in $D = 2$

3.1. $D = 3$

As a preliminar result for the superfluid density n_s, we employ equation (32) to calculate the density $n(n_0,\, T)$ of bosons with finite-range interaction. In $D = 3$, the zero-temperature Gaussian contribution $f_g^{(0)}(n_0)$ is regularized simply by the means of dimensional regularization, therefore we put $\varepsilon = 0$ into the equation (44). In the limit of small g_2 we get

$$f_g^{(0)}(n_0) = \frac{1}{3\pi^2}\left(\frac{mg_0n_0}{\hbar^2}\right)^{3/2}\left[1 - \frac{12m}{\hbar^2}g_2n_0\right]. \tag{48}$$

This result allows us to calculate the zero-temperature number density $n(n_0, T = 0)$ as the sum of the condensate density n_0 and the zero-temperature density contribution $f_g^{(0)}(n_0)$. In particular, we substitute explicitly the expressions of the couplings g_0 and g_2 as given by equation (39), obtaining $n(n_0, T = 0)$ as a function of the three-dimensional s-wave scattering length a_s and the effective range r_e:

$$n(n_0,\, T = 0) = n_0\left[1 + \frac{8}{3\sqrt{\pi}}(n_0a_s^3)^{1/2} - 64\sqrt{\pi}\frac{r_e}{a_s}(n_0a_s^3)^{3/2}\right] \tag{49}$$

which differs from the result of [22] by a factor 2 in the finite-range correction. In figure 2 we compare our finite-range condensate fraction n_0/n (black dotted–dashed line) obtained from the numerical solution of equation (49) with the zero-range result (blue solid line) obtained putting $r_e = 0$ and the classical result of Bogoliubov (red dashed line) [16].

Let us also calculate the finite-temperature contribution $f_g^{(T)}(n_0)$ in $D = 3$, given by the integration of equation (45) in the low-temperature regime

$$f_g^{(T)}(n_0) = \frac{(k_B T)^2}{12(n_0g_0)^{1/2}}\left(\frac{m}{\hbar^2}\right)^{3/2}\left(1 - \frac{\pi^2(k_B T)^2\lambda(g_2,\, n_0)}{20(n_0g_0)^2}\right) + o(k_B T)^5. \tag{50}$$

The finite-temperature number density $n(n_0, T)$, according to equation (32), is given by the sum of the condensate density n_0 and the Gaussian density contributions of equations (48) and (50)

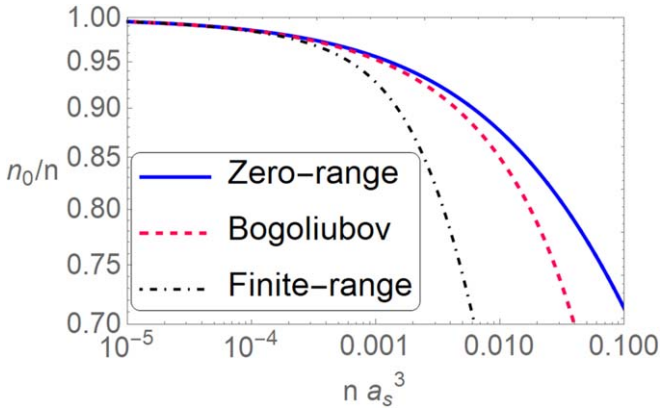

Figure 2. Zero-temperature condensate fraction n_0/n as a function of the gas parameter na_s^3 in $D = 3$. Here we represent the finite-range condensate fraction (black dotted–dashed line) for the effective range value $r_e/a_s = -10$ in comparison with the result for zero-range interaction (blue solid line) and the Bogoliubov's result (red dashed line). Notice that, with a weak dependence on the choice of r_e, the finite-range correction becomes relevant for values of the gas parameter na_s^3 greater than 10^{-3}.

$$n(n_0,\,T) = n_0 + \frac{1}{3\pi^2}\left(\frac{mg_0n_0}{\hbar^2}\right)^{3/2}\left[1 - \frac{12m}{\hbar^2}g_2n_0\right] + \frac{(k_BT)^2}{12(n_0g_0)^{1/2}}\left(\frac{m}{\hbar^2}\right)^{3/2}. \tag{51}$$

We also rewrite the general form of $n(n_0, T)$ in terms of the three-dimensional gas parameter na_s^3 and the effective range r_e, employing the explicit form of the couplings g_0 and g_2 given by equation (39), namely

$$n(n_0,\,T) = n_0 + \frac{8(n_0a_s)^{3/2}}{3\sqrt{\pi}}\frac{1 - 8\pi^2n_0a_s^2r_e}{(1 + 8\pi^2n_0a_s^2r_e)^2} + \frac{(k_BT)^2}{24\sqrt{\pi}\,(n_0a_s)^{1/2}}\left(\frac{m}{\hbar^2}\right)^2. \tag{52}$$

This equation can be used to express the condensate fraction n_0/n explicitly in the very weakly-interacting regime in which $a_s,\ r_e \to 0$, where one can approximate in the second and third subleading terms the condensate density n_0 with the density n, since the phenomenon of quantum depletion is absent in the noninteracting zero-temperature limit and these terms are finer corrections with respect to the first. We obtain

$$\frac{n_0}{n} = 1 - \frac{8}{3\sqrt{\pi}}(na_s^3)^{1/2}\left[1 - 24\pi(na_s^3)\frac{r_e}{a_s} + \frac{\pi^2}{16\zeta(3/2)^{4/3}(na_s^3)^{2/3}}\cdot\left(\frac{T}{T_{\rm BEC}}\right)^2\left(1 - \frac{\pi^2}{80\zeta(3/2)^{4/3}(na_s^3)^{2/3}}\left(\frac{T}{T_{\rm BEC}}\right)^2\left(1 + 8\pi\frac{r_e}{a_s}(na_s^3)\right)\right)\right], \tag{53}$$

where we have rescaled the temperature in terms of $T_{\rm BEC} = (2\pi\hbar^2n^{2/3})/(mk_B\zeta(3/2)^{2/3})$ for noninteracting bosons, and $\zeta(x)$ is the Riemann zeta function. We emphasize that, at $T = 0$ and for a zero-range interaction for which $r_e = 0$, the Bogoliubov result for the condensate quantum depletion is reproduced [16]. Finally, we calculate the normal density n_n of equation (47) substituting $D = 3$ and considering the low-temperature regime, in which we get

$$n_n = \frac{2\pi^2}{45}\left(\frac{m}{\hbar^2}\right)^{3/2}\frac{(k_BT)^4}{(n_0g_0)^{5/2}}\left(1 - \frac{5\pi^2\lambda(g_2,\,n_0)(k_BT)^2}{2(n_0g_0)^2}\right) + o(k_BT)^7. \tag{54}$$

The superfluid density n_s for a system of bosons interacting with the finite-range interaction is obtained substituting equations (49), (50) and (54) in equation (34), namely

$$n_s = n_0 + \frac{1}{3\pi^2}\left(\frac{mg_0n_0}{\hbar^2}\right)^{3/2} + \frac{(k_BT)^2}{12(n_0g_0)^{1/2}}\left(\frac{m}{\hbar^2}\right)^{3/2} - \frac{(8\lambda(n_0) + 3)\pi^2}{360}\left(\frac{m}{\hbar^2}\right)^{3/2}\frac{(k_BT)^4}{(n_0g_0)^{5/2}}. \tag{55}$$

As for the condensate fraction, we obtain an explicit expression of the superfluid fraction n_s/n in the very weakly-interacting limit in which one can approximate the condensate density n_0 with the number density n in the subleading terms, rewriting the previous equation as

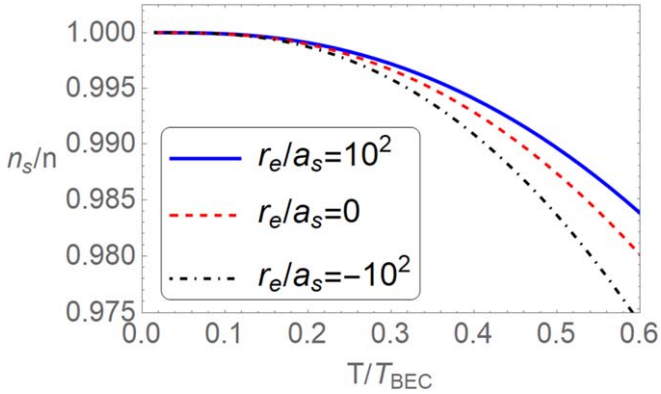

Figure 3. Superfluid fraction n_s/n in $D = 3$ for the gas parameter value $na_s^3 = 10^{-4}$, obtained as the numerical integral of equation (47). The zero-range interaction result is reported as the red dashed line, while the finite-range corrections are reported as the solid blue line (for $r_e/a_s = 10^2$) and the black dotted–dashed line (for $r_e/a_s = 10^{-2}$).

$$\frac{n_s}{n} = 1 - \frac{\pi^{7/4}}{45(na_s^3)^{5/6}}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^4\left[1 - \frac{5\pi^2}{8\,\zeta(3/2)^{4/3}(na_s^3)^{2/3}}\left(\frac{T}{T_{\mathrm{BEC}}}\right)^2 \cdot \left(1 + 8\pi\frac{r_e}{a_s}(na_s^3)\right)\right], \tag{56}$$

where we substitute also the expressions for g_0 and g_2 of equation (39). Notice that, while the superfluid fraction $n_s/n = 1$ at zero temperature, the condensate fraction is $n_0/n < 1$ due to the quantum depletion. Obviously, equation (56) is reliable only in the deep $T/T_{\mathrm{BEC}} \ll 1$ regime but, more generally, one must consider the full solution of equation (47).

In figure 3 we report the superfluid fraction n_s/n as a function of the scaled temperature T/T_{BEC} for three values of the ratio r_e/a_s at fixed gas parameter na_s^3 by numerically solving equation (47). Figure 3 shows that a positive r_e/a_s slightly enhances the superfluid fraction while the opposite occurs for negative values.

3.2. $D = 2$

Here we formally follow the same path we have introduced in the three-dimensional case, obtaining the number density n at zero-temperature and employing it to calculate the superfluid density n_s. In the two-dimensional case we calculate the regularized zero-temperature density contribution $f_g^{(0)}(n_0)$ from equation (44) where $D = 2$ is substituted. Notice that $f_g^{(0)}(n_0)$ is obtained as a sum of the two terms inside the square bracket of (44): while the first term is finite, the divergence of the second term is healed by performing a Taylor expansion around $\epsilon = 0$ and deleting the $o(\epsilon^{-1})$ divergence [41, 42]. The regularized zero-temperature density contribution reads

$$f_g^{(0)}(n_0) = \frac{mg_0n_0}{4\pi\hbar^2}\frac{1}{\lambda(g_2,\, n_0)^{3/2}}\left[\tilde{\lambda}(g_2,\, n_0) + \frac{2m}{\hbar^2}g_2n_0\ln\left(\frac{2\epsilon_0}{g_0n_0}\lambda(g_2,\, n_0)\right)\right], \tag{57}$$

where we identify the ultraviolet energy scale ϵ_0 as

$$\epsilon_0 = \frac{4\pi\hbar^2\kappa^2}{me^{\gamma}}. \tag{58}$$

If the finite-range interaction strength g_2, as we suppose, constitutes a small correction of the zero-range term g_0, one can expand the previous equation for small values of the adimensional parameter $\frac{2m}{\hbar^2}g_2n_0$, thus

$$f_g^{(0)}(n_0) = \frac{1}{4\pi}\frac{mg_0n_0}{\hbar^2} + \frac{1}{2\pi}\left(\frac{m}{\hbar^2}\right)^2 g_0g_2n_0^2\left[\ln\left(\frac{2\epsilon_0}{g_0n_0}\right) - 2\right]. \tag{59}$$

In this limit, the zero-temperature density $n(n_0,\, T = 0)$ reads

$$n(n_0,\, T = 0) = n_0 + \frac{1}{4\pi}\frac{mg_0n_0}{\hbar^2} + \frac{1}{2\pi}\left(\frac{m}{\hbar^2}\right)^2 g_0g_2n_0^2\left[\ln\left(\frac{2\epsilon_0}{g_0n_0}\right) - 2\right]. \tag{60}$$

In analogy with the three-dimensional case, we derive an explicit—but approximated—formula for the zero-temperature condensate fraction n_0/n in $D = 2$ considering the very weakly-interacting regime in which $g_0,\, g_2 \to 0$. As before, in this regime we can approximate $n_0^2 \approx n_0n$ in the third term of equation (60) and

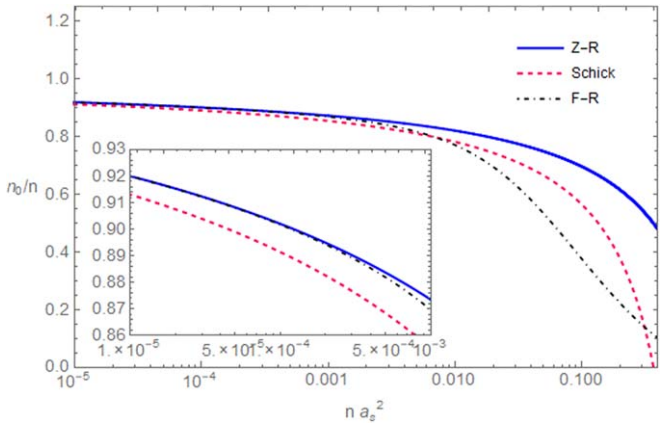

Figure 4. Condensate fraction n_0/n for bosons in $D = 2$ at $T = 0$, reported as a function of the gas parameter na_s^2. The blue solid line is the condensate fraction for bosons with zero-range interaction (Z-R), obtained from the numerical solution of equation (60) with $R = 0$. The black dotted–dashed line is the condensate fraction for bosons with finite-range interaction (FR) given by equation (60) with a characteristic range R value given by $R = 2a_s$ and a ultraviolet cutoff $\kappa = 2\pi/a_s$. The red dashed line is the analytical formula of Schick [43], i.e. Equation (61) with $R = 0$. The inset highlights the small differences between our theoretical scheme and the Schick one in the weakly-interacting regime.

$n_0 \approx n$ inside the logarithm, since it is a subleading term with respect to the first one. Let us come back to ε_0, by choosing the cutoff as $\kappa = 2\pi/a_s$. Then, by taking g_0 and g_2 as in equation (40), we obtain the approximated condensate fraction n_0/n as a function of the two-dimensional gas parameter na_s^2 and the characteristic range R, namely

$$\frac{n_0}{n} = 1 - \frac{1}{|\ln(na_s^2)|} - \frac{2\pi(R/a_s)^2}{|\ln(na_s^2)|^2}\ln\left(\frac{8\pi^2|\ln(na_s^2)|}{na_s^2 e^{\gamma+2}}\right). \tag{61}$$

Notice that the zero-range interaction result by Schick for the condensate fraction n_0/n [43] is easily reproduced by setting $R = 0$. Working outside the very weakly-interacting limit, one can also obtain the zero-temperature condensate fraction from the numerical solution of equation (60). We report it as the black dotted–dashed line in figure 4, in comparison with our zero-range result (blue solid line) and the result by Schick (red dashed line), which is reproduced in the weakly-interacting regime in which $na_s^3 \ll 1$.

Finally, following the three-dimensional case, we may want to calculate also the finite-temperature density contribution $f_g^{(T)}(n_0)$. However, substituting $D = 2$ in equation (45) we find that

$$f_g^{(T)}(n_0) = \frac{mk_BT}{2\pi\hbar^2}\int_0^{+\infty} dx\ \frac{1}{e^x - 1} \tag{62}$$

is infrared divergent, therefore cannot be regularized with dimensional regularization. This result is indeed correct and reflects the absence of Bose–Einstein condensation at finite-temperature in two-dimensional systems [44], as already pointed out in [45].

The two-dimensional normal density $n_n(n_0,\ T = 0)$ of bosons with finite-range interaction is obtained from the integration of equation (47), in which we expand the integrand in the low-temperature limit, namely

$$n_n(n_0,\ T = 0) = \frac{3m}{2\pi\hbar^2}\frac{(k_BT)^3}{(n_0g_0)^2}\left(\zeta(3) - \frac{15\zeta(5)\lambda(g_2,\ n_0)}{(n_0g_0)^2}(k_BT)^2\right) + o(k_BT)^6. \tag{63}$$

Since the thermal contribution to the density $f_g^{(T)}(n_0)$ is divergent in $D = 2$, we cannot express the superfluid density n_s as a function of the condensate density n_0 at a finite temperature T. Therefore we employ the zero-temperature number density of equation (60) to obtain the condensate density in the implicit form $n_0 = n_0(n,\ T = 0)$, namely as a function of the density n. In this way we calculate the superfluid density substituting equation (63) into equation (34)

$$n_s = n - \frac{3m}{2\pi\hbar^2}\frac{(k_BT)^3}{(g_0\ n_0(n,\ T = 0))^2}\left(\zeta(3) - \frac{15\zeta(5)\lambda(g_2,\ n_0(n,\ T = 0))}{(g_0\ n_0(n,\ T = 0))^2}(k_BT)^2\right) \tag{64}$$

which we expect to provide a reliable approximation in the low-temperature regime, remembering that—outside it—our approach would in any case be incorrect due to the Berezinski–Kosterlitz–Thouless transition [46].

Finally, we calculate an approximated expression of the superfluid fraction n_s/n in the very weakly-interacting regime where $g_0,\ g_2 \to 0$, in which we can approximate the condensate density inside equation (64)

as $n_0(n,\ T=0)\approx n$. In the context of this approximation, we also substitute in equation (64) the explicit form of the parameter $\lambda(g_2, n_0)$ of equation (43). Moreover, we remember that the two-dimensional interaction strengths g_0 and g_2 are given by equation (40), obtained in terms of the s-wave scattering length a_s and the characteristic range R of the interatomic potential with the scattering theory. The approximated superfluid fraction n_s/n reads

$$\frac{n_s}{n}=1-\frac{3}{32\pi^3}|\ln(na_s^2)|^2\left(\frac{T}{T^*}\right)^3\left[\zeta(3)-\frac{15\zeta(5)}{16\pi^2}\left(\frac{T}{T^*}\right)^2|\ln(na_s^2)|^2\cdot\left(1+4\pi\frac{R^2}{a_s^2}\frac{na_s^2}{|\ln(na_s^2)|}\right)\right], \tag{65}$$

where we rescale the result in terms of the temperature $T^*=\hbar^2n/(mk_{\mathrm{B}})$ of quantum degeneracy.

4. Conclusions

We have used finite-temperature one-loop functional integration to reproduce the density momentum equation of the two-fluid model. An analytical relationship between the density n and the condensate density n_0 has been obtained at zero-temperature and in the low-temperature limit. This result has been used to express the low-temperature superfluid density n_s as a function of n_0 and T for bosons with finite-range interaction, which can be regarded as an explicit implementation of the Josephson relation. We analyze thoroughly the $D=3$ and $D=2$ case, but our approach could be applied also in $D=1$, where we expect to reproduce the Lieb–Liniger theory except in the strong coupling regime [47].

We expect that our theory is meaningful under the conditions of diluteness of the bosonic gas, for which $na_s^D\ll 1$ and $nr_e^D\ll 1$. Moreover, finite-temperature results must be considered in the limit $k_{\mathrm{B}}T/(g_0n)\ll 1$, for which the mean thermal energy is much lower than the gas healing length. Our finite-range corrections to the condensate fraction n_0/n and to the superfluid fraction n_s/n can be detected in $D=3$ in the regime $a_s/r_e\leqslant 1$ and in $D=2$ for $a_s/R\leqslant 1$ but *not* where they are *much lower* than 1. In that case, the higher order terms which we are neglecting in the gradient expansion of the interaction potential (37) become relevant. Notice that, in $D=3$ and in $D=2$, these may represent different regimes. In fact, while the effective range value r_e can be tuned by the means of a Feshbach resonance, the characteristic range R is fixed, being essentially a geometric property of the real two-body interaction potential between the atoms. Indeed, we expect that R is proportional to the Van der Waals radius of the atoms and it can be numerically computed, following its definition, using a model two-body potential $V(\vec{r})$.

An extension of this work consists in the numerical calculation of the thermal integrals $f_g^{(T)}(n_0)$ and $n_n(n_0,\ T)$ outside the zero-temperature limit in $D=2$, which we have considered for obtaining analytical results. In any case, we expect that our predictions fail to describe the superfluid fraction of the two-dimensional Bose gas at sufficiently high temperature, due to the occurrence of the BKT transition. In this case it is needed a profound rethinking of our approach, including explicitly in the bosonic field parametrization the contribution of vortex configurations of the phase field, which cause the BKT topological phase transion.

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Appendix

In this appendix, we calculate the zero-temperature Gaussian grand potential per unit of volume $\Omega^{(0)}/L^D$, which is given by equations (15) and (20) in which the mean field condition equation (18) is substituted, namely

$$\frac{\Omega^{(0)}}{L^D}=-\frac{\mu_e^2}{2g}+\frac{S_D}{2(2\pi)^D}\int_0^{+\infty}\mathrm{d}k\ k^{D-1}\sqrt{\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m}\lambda(g_2,\ \mu_e/g_0)+2\mu_e\right)}, \tag{66}$$

where λ is defined in equation (43). Performing dimensional regularization of the ultraviolet divergence accordingly to the superfluid density calculation, we obtain the regularized zero-temperature Gaussian grand potential $\Omega_g^{(0)}/L^D$

$$\frac{\Omega^{(0)}}{L^D} = -\frac{\mu_e^2}{2g} - \frac{\mu_e^{\frac{D+2}{2}}}{2\,\pi^{\frac{D+1}{2}}}\left(\frac{m}{\hbar^2}\right)^{D/2}\frac{1}{\lambda(g_2,\,\mu_e/g_0)^{\frac{D+1}{2}}} \times\left[1+\frac{\varepsilon}{2}\ln\left(\frac{\pi\hbar^2\kappa^2\lambda(g_2,\,\mu_e/g_0)}{m\mu_e}\right)+o(\varepsilon^2)\right]\frac{\Gamma[(D-\varepsilon+1)/2]\,\Gamma[(\varepsilon-D-2)/2]}{\Gamma[(D-\varepsilon)/2]}. \tag{67}$$

In the three-dimensional case, it is sufficient to substitute the dimension $D = 3$ and put $\varepsilon = 0$ in equation (67), to get the regularized zero-temperature Gaussian grand potential as

$$\frac{\Omega^{(0)}}{L^3} = -\frac{\mu_e^2}{2g} - \frac{\mu_e^2}{2g}\frac{8}{15\pi^2}\left(\frac{m}{\hbar^2}\right)^{3/2}\frac{\mu_e^{5/2}}{\lambda(g_2,\,\mu_e/g_0)^2} \tag{68}$$

which reproduces a previously known result [23]. In $D = 2$ we expand equation (67) and retain only $o(\varepsilon^0)$ terms, then we identify the energy cutoff ϵ_0 as

$$\epsilon_0 = \frac{4\pi\hbar^2\kappa^2}{m\exp\left(\gamma+1-\frac{4\pi\hbar^2\lambda(g_2,\,\mu_e/g_0)^{3/2}}{mg_0}\right)} \tag{69}$$

to get the zero-temperature Gaussian grand potential

$$\frac{\Omega^{(0)}}{L^2} = -\frac{m\mu_e^2}{8\pi\hbar^2\lambda(g_2,\,\mu_e/g_0)^{3/2}}\left[\ln\left(\frac{\epsilon_0}{\mu_e}\lambda(g_2,\,\mu_e/g_0)\right)+\frac{1}{2}\right]. \tag{70}$$

This equation of state corrects the one obtained in [25]: here $\lambda(g_2, \mu_e/g_0)$ appears also inside the logarithm. Moreover, it is important to stress that, in the case of zero-range interaction, Mora and Castin [48] were able to extend equation (70) by including a beyond-Gaussian term. This next-next-to-leading extension in the finite-range case is highly non trivial and it surely deserves a separate detailed investigation.

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References

[1] Tilley D R and Tilley J 1990 *Superfluidity and Superconductivity* (Boca Raton, FL: CRC Press)
[2] Pezzé L, Smerzi A, Oberthaler M K, Schmied M and Treutlein P 2018 *Rev. Mod. Phys.* **90** 035005
[3] Streltsov A, Adesso G and Plenio M 2017 *Rev. Mod. Phys.* **89** 041003
[4] Chang D E, Douglas J S, Gonzalez-Tudela A, Hung C L and Kimble H J 2018 *Rev. Mod. Phys.* **90** 031002
[5] Bardeen J, Cooper L N and Schrieffer J R 1957 *Phys. Rev.* **106** 1
[6] Larkin A and Varlamov A 2005 *Theory of Fluctuations in Superconductors* (Oxford: Oxford University Press)
[7] Hohenberg P C and Martin P C 1965 *Ann. Phys., NY* **281** 636–705
[8] Anderson P W 1966 *Rev. Mod. Phys.* **38** 2
[9] Ceperley D M 1995 *Rev. Mod. Phys.* **67** 279
[10] Schmitt A 2014 *Introduction to Superfluidity: Field-theoretical Approach and Applications* (Basel: Springer)
[11] Landau L D 1941 *Phys. Rev.* **60** 356
[12] Landau L D and Lifshitz E M 1981 *Statistical Physics—Part 2* (Oxford: Pergamon)
[13] Leggett A J 2006 *Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-matter Systems* (Oxford: Oxford University Press)
[14] London F 1938 *Nature* **141** 643
[15] London F 1938 *Phys. Rev.* **54** 937
[16] Bogoliubov N N 1947 *J. Phys. U.S.S.R* **11** 23
[17] Huang K 1987 *Statistical Mechanics* (New York: Wiley)
[18] Stringari S and Pitaevskii L P 2016 *Bose–Einstein Condensation and Superfluidity* (Oxford: Oxford University Press)
[19] Hadzibabic Z and Dalibard J 2011 *Riv. Nuovo Cimento* **34** 389
[20] Dalibard J, Gerbier F, Juzeliunas G and Öhberg P 2011 *Rev. Mod Phys.* **83** 1523
[21] Braaten E, Hammer H W and Hermans S 2001 *Phys. Rev.* A **63** 063609
[22] Andersen J O 2004 *Rev. Mod. Phys.* **76** 599
[23] Cappellaro A and Salasnich L 2017 *Phys. Rev.* A **95** 033627
[24] Yukalov V I and Yukalova E P 2016 *Laser Phys.* **26** 045501
[25] Salasnich L 2017 *Phys. Rev. Lett.* **118** 130402
[26] Beane S R 2018 *Eur. Phys. J.* D **72** 55
[27] Schakel A M J 2008 *Boulevard of Broken Symmetries* (Singapore: World Scientific)
[28] Fisher M E, Barber M and Jasnow D 1973 *Phys. Rev.* A **8** 1111
[29] Taylor E, Griffin A, Fukushima N and Ohashi Y 2006 *Phys. Rev.* A **74** 063626
[30] Nagaosa N 1999 *Quantum Field Theory in Condensed Matter Physics* (Berlin: Springer)

[31] Boudjemâa A and Benarous M 2011 *Phys. Rev.* A **84** 043633
[32] Boudjemâa A 2012 *Phys. Rev.* A **86** 043608
[33] Boudjemaa A and Guebli N 2017 *J. Phys. A: Math Theor.* **50** 425004
[34] Altland A and Simons B D 2010 *Condensed Matter Field Theory* (Cambridge: Cambridge University Press)
[35] Salasnich L and Toigo F 2016 *Phys. Rep.* **640** 1–29
[36] Josephson B D 1966 *Phys. Lett.* **21** 608
[37] Svistunov B, Babaev E and Prokof'ev N 2015 *Superfluid States of Matter* (Boca Raton, FL: CRC Press)
[38] Astrakharchik G E, Boronat J, Casulleras J, Kurbakov I L and Lozovik Y E 2009 *Phys. Rev.* A **79** 051602
[39] 't Hooft G and Veltman M 1972 *Nucl. Phys.* B **44** 189–213
[40] Leibbrandt G 1975 *Rev. Mod. Phys.* **47** 849
[41] Kleinert H and Schulte-Frohlinde V 2001 *Critical Properties of* ϕ^4 *Theories* (Singapore: World Scientific)
[42] Zeidler E 2009 *Quantum Field Theory II: Quantum Electrodynamics* (Berlin: Springer)
[43] Schick M 1971 *Phys. Rev.* A **3** 1067
[44] Mermin N D and Wagner H 1966 *Phys. Rev. Lett.* **17** 1133
[45] Stoof H T C, Dickerscheid D B M and Gubbels K 2009 *Ultracold Quantum Fields* (Dordrecht: Springer)
[46] Kosterlitz L M and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 1181
[47] Cappellaro A and Salasnich L 2017 *Phys. Rev.* A **96** 063610
[48] Mora C and Castin Y 2009 *Phys. Rev. Lett.* **102** 180404