Cold Atoms in Non-Abelian Gauge Potentials: From the Hofstadter "Moth" to Lattice Gauge Theory

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(Received 3 March 2005; published 28 June 2005)

We demonstrate how to create artificial external non-Abelian gauge potentials acting on cold atoms in optical lattices. The method employs atoms with k internal states, and laser assisted state sensitive tunneling, described by unitary $k \times k$ matrices. The single-particle dynamics in the case of intense $U(2)$ vector potentials lead to a generalized Hofstadter butterfly spectrum which shows a complex mothlike structure. We discuss the possibility to realize non-Abelian interferometry (Aharonov-Bohm effect) and to study many-body dynamics of ultracold matter in external lattice gauge fields.

DOI: 10.1103/PhysRevLett.95.010403 PACS numbers: 03.75.Lm, 03.75.Mn, 03.75.Ss, 11.15.Ha

One of the most significant trends in the physics of ultracold gases nowadays concerns, without any doubts, strongly correlated systems. Apart from systems at the transition from fermionic pair to Bose-Einstein condensation (BCS-BEC crossover), low dimensional systems, and atomic lattice gases, perhaps the most fascinating possibilities are offered by atomic systems subject to artificial magnetic fields, such as gases in rotating traps. When the rotation frequency approaches the trap frequency, such systems are expected to exhibit the fractional quantum Hall effect and behave as Laughlin liquids [1]. Several experiments with gases in rotating traps are currently underway [2,3]. Very recently, the creation of artificial magnetic fields in terms of electromagnetically induced transparency (EIT) has been proposed in Ref. [4].

Alternatively, magnetic field effects can be realized in a lattice by introducing appropriate phase factors for tunneling amplitudes. If multiplied around an elementary plaquette, these lead to phases proportional to the magnetic flux penetrating the plaquette. Jaksch and Zoller, and subsequently other groups, have recently proposed methods to realize such "artificial" magnetic field effects in lattice gases. These employ atoms with multiple internal states, laser assisted tunneling, lattice tilting (acceleration), and other experimentally accessible techniques [5–7]. The physical features in such artificial magnetic fields are extremely rich. For single atoms, the spectrum exhibits the fractal Hofstadter "butterfly" structure [8]. In weakly interacting or weakly disordered systems, the modifications of the butterfly due to interactions and disorder, respectively, may be studied [5]. Finally, in the limit of strong magnetic fields, Laughlin-like states are expected to appear [7]. In this Letter, we show that the use of atoms with more internal states, the application of laser assisted nonuniform and state dependent tunneling (cf. [9]), and coherent transfer between internal states allow for a generalization of the above methods and creation of "artificial external magnetic fields" corresponding to non-Abelian $U(k)$ or $SU(k)$ gauge fields. In this case, tunneling amplitudes are replaced by unitary matrices whose product around a plaquette is nontrivial and its mean trace (Wilson loop) is not equal to k [10]. To our knowledge, the physics of matter in designed non-Abelian gauge fields has not been studied at all in the context of atomic, molecular, and optical physics. Although single particles in $SU(k)$ monopole fields have been studied in high energy physics [11], other field configurations have not attracted such interest. Specific examples of external gauge fields have also been discussed with respect to NMR, nuclear, and molecular physics [12]. However, all of these proposals either do not deal with interacting many particle systems in such gauge fields or do not consider lattice gauge fields of ultrahigh strength.

Having introduced the designed external non-Abelian gauge fields in optical lattices, we study their influence on single-particle dynamics. We generalize the Hofstadter butterfly to the non-Abelian case and show that in case of intense $U(2)$ gauge fields, the spectrum (as a function of two "magnetic fluxes") exhibits a complex "moth" structure of holes. We discuss the possibility to observe the $U(2)$ Aharonov-Bohm effect, to realize non-Abelian interferometry, and to simulate lattice gauge dynamics.

We start from the physics of a single particle in a two-dimensional square lattice of spacing a in the presence of an Abelian magnetic field $\vec{B}$. When the lattice potential is sufficiently strong, the single-particle nonrelativistic Hamiltonian in the tight binding approximation is given by the so-called Peierl's substitution [8], and reads

$$H = -2V_0 \sum_{j=x,y} \cos\left[\frac{a}{\hbar}\left(p_j - i\frac{e}{c}A_j\right)\right], \tag{1}$$

where V_0 is the optical potential strength, $\vec{p}$ is the momentum operator, and $\vec{A}$ is the magnetic vector potential. Given $[p_j, A_j] = 0$, the single-particle wave equation reads

$$e^{-i\Delta A_x}\psi(x+a,y)+e^{i\Delta A_x}\psi(x-a,y) + e^{-i\Delta A_y}\psi(x,y+a)+e^{i\Delta A_y}\psi(x,y-a)=\varepsilon\psi(x,y), \tag{2}$$

where $\Delta = ea/\hbar c$ and $\varepsilon = -E/V_0$. The choice of $\vec{A}$ determines the magnetic field $\vec{B}$, and thus the behavior of the system. For $\vec{B} = B\vec{e}_z$, one may choose $\vec{A} = (0, Bx, 0)$; thus, solely tunnelings in the y direction acquire phases. This makes the problem effectively one dimensional and (2) transforms into Harper's equation [13]:

$$g(m+1)+g(m-1)=[\varepsilon-2\cos(2\pi m\alpha-\nu)]g(m) \tag{3}$$

by using the ansatz $\psi(ma, na) = e^{i\nu n}g(m)$, where $x = ma$ and $y = na$. The eigenvalue problem for rational values of the magnetic flux $\alpha = \frac{ea^2B}{hc}$ per elementary plaquette becomes periodic. This results in a band spectrum whose bands form the famous Hofstadter butterfly [8]. Note that the regime of this spectrum requires finite values of α, i.e., magnetic fields $B \sim 1/a^2$, which in the continuum limit $a \to 0$ become ultraintense.

A scheme to realize Abelian fields $\vec{B}$ using an atomic gas in a 3D optical lattice was proposed in Ref. [5]. In this scheme there is no tunneling in the z direction, so that, effectively, one deals with an array of 2D lattice gases; we restrict ourselves to one copy. The atoms occupy two internal hyperfine states $|g\rangle$ and $|e\rangle$, and the optical potential traps them in the states $|g\rangle$ and $|e\rangle$ in every second column, i.e., for the y coordinate equal to $\ldots, n-1, n+1, \ldots (\ldots, n, n+2, \ldots)$. The resulting 2D lattice has thus the spacing $\lambda/2$ ($\lambda/4$) in the x (y) direction. The tunnel rates in the x direction are due to kinetic energy; they are spatially homogeneous and assumed to be equal for both hyperfine states. The lattice is tilted in the y direction, which introduces an energy shift Δ between neighboring columns. Tilting can be achieved by accelerating the lattice or by placing it in a static electric field. By doing this, standard tunneling rates due to kinetic energy are suppressed in the y direction. Instead, tunneling is driven by two different lasers resonant with Raman transitions $|g\rangle \to |e\rangle$, for $n \to n \pm 1$. The lasers must be different because the offset energy for both transitions is different and equals $\pm\Delta$. Their detunings are chosen in such a way that the effect of tilting is canceled in the rotating frame of reference. The lasers generate running waves in the $\pm x$ direction, so that the corresponding tunneling rates acquire local phases $\exp(\pm iqx) = \exp(\pm i\alpha m)$, where we define $\alpha = qa$.

In order to realize artificial non-Abelian fields in a similar scheme as in Ref. [5], one may use atoms with degenerate Zeeman sublevels in the hyperfine ground state manifolds $\{|g_i\rangle, |e_i\rangle\}$ with $i = 1, \ldots, k$, whose degeneracy is lifted in external magnetic fields. These states may be thought of as "colors" of the gauge fields. Promising fermionic candidates with these properties are heavy alkali atoms, for instance, ^{40}K atoms in states $F = 9/2$, $M_F = 9/2, 7/2, \ldots,$ and $F = 7/2$, $M_F = -7/2, -5/2, \ldots$; in particular, they allow for "spin" dependent hopping [9].

Having identified the colors, one modifies the scheme of Ref. [5]: For a given link $|g_i\rangle$ to $|e_i\rangle$, laser assisted tunneling rates along the y axis should be described by a nontrivial unitary matrix $U_y(x)$ being a member of the "color" group [$U(k)$, $SU(k)$, $GL(k)$, etc.]. For unitary groups, $U_y(x)$ can be represented as $\exp(i\tilde{\alpha}A_y(x))$, with $\tilde{\alpha}$ real, and $A_y(x)$ a Hermitian matrix from the gauge algebra, e.g., $\mathfrak{u}(k)$ or $\mathfrak{su}(k)$. The transitions from $|g_i\rangle$ to $|e_i\rangle$ correspond to different frequencies for $n \to n \pm 1$ due to the offset of the lattice sites. In general, depending on the Zeeman splittings, they can also differ for each i. They are driven by different running wave lasers with Rabi frequencies $\Omega_{\pm i}$, and attain different factors $\exp(\pm iq_i x) = \exp(\pm i\alpha_i m)$.

In order to create gauge potentials that cannot simply be reduced to two independent Abelian components, tunneling in the x direction must also be laser assisted and should allow for coherent transfer between internal Zeeman states. This can be achieved using a nonresonant Raman transition via excited states trapped between the sites (m, n) and $(m + 1, n)$. We assume that the same unitary matrix U_x describes tunneling for both hyperfine state manifolds, although more general situations are feasible and basically interesting. To assure a genuine non-Abelian character of the fields, it is necessary that $[U_x, U_y(x)] \neq 0$. We stress that all elements of our scheme, as shown in Fig. 1, are experimentally accessible. Nevertheless, consistent gauge group realizations demand tunneling matrix amplitudes to be controlled in an appropriate way.

The scheme of Fig. 1 allows one to generalize the Hamiltonian (1) to the case of non-Abelian vector potentials. In fact, we replace the components of $\vec{A}$ by the corresponding matrices from the group algebra. In particular, the illustrated setup generates artificial gauge potentials of the form $\vec{A} = (A_x, A_y(m), 0)$, with

$$\vec{A}=\frac{\hbar c}{ea}\left(\begin{pmatrix} -\frac{\pi}{2} & \frac{\pi}{2}e^{i\phi} \\ \frac{\pi}{2}e^{-i\phi} & -\frac{\pi}{2}\end{pmatrix}, \begin{pmatrix} 2\pi m\alpha_1 & 0 \\ 0 & 2\pi m\alpha_2\end{pmatrix}, 0\right). \tag{4}$$

More precisely, the scheme associates a unitary tunneling operator with every link in analogy with standard lattice gauge theory prescriptions [10]: $U(m-1, n \to m, n) \equiv U_x$, $U(m, n \to m-1, n) \equiv U_x^\dagger$, $U(m, n \to m, n+1) \equiv U_y(m)$, $U(m, n \to m, n-1) \equiv U_y^\dagger(m)$, where $U_x = \exp(-ieaA_x/c\hbar)$ and $U_y = \exp(-ieaA_y(m)/c\hbar)$. The only difference is that $\vec{A}$ acquires an overall factor of $\hbar c/ea$; in effect, though it does not behave well in the continuous limit $a \to 0$, the "magnetic flux" per plaquette, $\alpha_{1,2}$ remains finite. Thus, we are in the same limit of ultraintense fields $\sim 1/a^2$ as in the "classic" Hofstadter case of Abelian magnetic fields. The ansatz $\psi(ma, na) = e^{i\nu n}g(m)$ leads to a generalized Harper wave equation

$$\begin{pmatrix} g(m+1) \\ g(m)\end{pmatrix} = B(m)\begin{pmatrix} g(m) \\ g(m-1)\end{pmatrix} \tag{5}$$

with

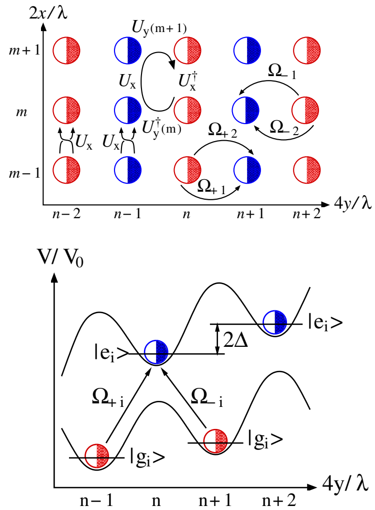

FIG. 1 (color online). *Optical lattice setup for U(2) gauge fields:* Red and blue open semicircles (closed semicircles) denote atoms in states $|g_1\rangle$ and $|g_2\rangle$, respectively ($|e_1\rangle$ and $|e_2\rangle$). Top: Hopping in the x direction is laser assisted and allows for unitary exchange of colors; it is described by the same unitary hopping matrix U_x for both $|g_i\rangle$ and $|e_i\rangle$ states. Hopping along the y direction is also laser assisted and attains ''color dependent'' phase factors. Bottom: Trapping potential in the y direction. Adjacent sites are set off by an energy Δ due to the lattice acceleration, or a static inhomogeneous electric field. The lasers $\Omega_{\pm i}$ are resonant for transitions $|g_i\rangle \to |e_i\rangle$ for $n \to n \pm 1$. Because of the spatial dependence of $\Omega_{\pm i}$ (running waves in the $\pm x$ direction), the atoms hopping around the plaquette get the unitary transformation $\hat{U} = U_y^\dagger(m) U_x U_y(m+1) U_x^\dagger$, where $U_y(m) = \exp(2\pi i m \mathrm{diag}[\alpha_1, \alpha_2])$, as indicated in the upper figure.

$$B(m) = \begin{pmatrix} \begin{pmatrix} 0 & e^{i\phi}\varepsilon_m(\alpha_2, \nu) \\ e^{-i\phi}\varepsilon_m(\alpha_1, \nu) & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

where $\varepsilon_m(\alpha, \nu) = \varepsilon - 2\cos(2\pi m\alpha - \nu)$ is the Harper energy term. Both $\psi(m, n)$ and $g(m)$ are now two-component objects. In the particular case of Eq. (4), when two successive transfer matrices $B(m)$ are multiplied by each other, Eq. (5) decomposes into a pair of equivalent 2D equations. Nevertheless, to obtain the spectrum, one has to rely on numerical methods. For rational $\alpha_i = p_i/q_i$ with p_i, q_i integers without a common divisor, the problem is Q periodic (where Q equals the least common multiple of q_1 and q_2). The allowed energies are those for which the product of Q successive matrices $B(m)$ has solely eigenvalues of modulus 1. The spectrum shows a band structure and is bounded by two hyperplanes (depicted in gray in Fig. 2). It exhibits a very complex formation of holes of finite measure and various sizes, which we name the Hofstadter ''moth.'' Although a rigorous proof cannot be provided, the moth reminds one of a fractal structure. Obviously, this fractal structure will be very sensitive to any sort of perturbation (finite size, trapping, etc.) on very small scales. But, since the holes are 3D objects with finite volume, the spectrum will be more robust on a larger scale to perturbations than the Hofstadter ''butterfly.''

To measure the spectral structure, one could load a dilute Bose condensate into the lattice and look at the evolution of the particle density, as suggested in Ref. [5]. Alternatively, one could load an ultracold polarized Fermi gas and measure its Fermi energy as a function of the number of particles.

It is interesting to consider yet another effect that becomes particularly spectacular in the limit of ultraintense gauge fields and that can be measured in the proposed system: a non-Abelian Aharonov-Bohm effect, to be considered as an example of non-Abelian interference. In order to realize it, one should prepare, for instance, a weakly interacting Bose condensate in a definite internal state $|\psi_0\rangle$ around a location P_1. Then, the BEC (or parts of it) should be split by Raman scattering, and piecewise dragged (using, e.g., laser tweezers) to a meeting point P_2 on two distinct paths. These correspond to the unitary transporters U_1 and $U_2^\dagger$, respectively. A measurement of the density of atoms at P_2 will reveal a non-Abelian interferometer signal; i.e., it will detect the interference term $n_{\rm int} \propto \langle\Psi_0|U_2 U_1|\psi_0\rangle$. Choosing, e.g., the rectangular loop, consisting of L_y steps in the y direction, L_x steps in the x direction, L_y steps in the $-y$ direction, and finally L_x steps in the $-x$ direction, we obtain for the gauge potential of Eq. (4): $n_{\rm int} \propto \langle\Psi_0|\exp[\pm 2\pi i\hat{\alpha}_A L_y L_x]|\psi_0\rangle$, if L_x is even, and $n_{\rm int} \propto \langle\Psi_0|\exp[\pm 2\pi i\hat{\alpha}_B L_y L_x]|\psi_0\rangle$, if L_x is odd, where $\hat{\alpha}_A = \mathrm{diag}(\alpha_1, \alpha_2)$ and $\hat{\alpha}_B = \mathrm{diag}(\alpha_2, \alpha_1)$. The signal is thus extremely sensitive to L_x. If one attempted to measure the phase shifts by introducing obstacles on the y arms of this interferometer, the result would strongly depend on the x coordinate of the obstacles, and the location of P_1, P_2, a non-Abelian manifestation of the external gauge potential.

Obviously, the properties of the considered system in the limit of ultraintense fields are quite complex. Though, to get a better intuition concerning the scope of this scheme, it is useful to consider also the ''continuum'' limit $a \to 0$ with $V_0 \to a^2/m$. Then, the Hamiltonian becomes

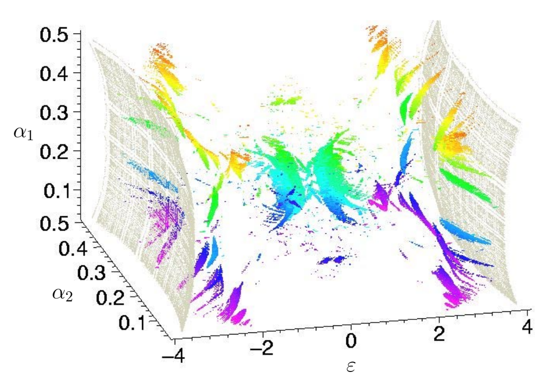

FIG. 2 (color online). The Hofstadter moth spectrum. Forbidden eigenenergies ε are plotted versus $\alpha_i = p_i/q_i, \in [0, = 0.5]$ $(i = 1, 2)$, where $q_i \leq 41$ and $\alpha_1 \neq \alpha_2$.

$\mathcal{H} = (\vec{p} - e/c\vec{A})^2/2m$. A natural question to ask is what kinds of gauge fields of "normal," i.e., a-independent strength, can be realized. In other words, what are the possible artificial gauge potentials that can be created? In general, phase factors resulting from running wave vectors can be introduced for all tunneling matrices: $\vec{A}(\vec{r}) = \frac{c}{el}(M_1 + [M_2(\frac{x}{l}) + M_3(\frac{y}{l})], N_1 + [N_2(\frac{x}{l}) + N_3(\frac{x}{l})], 0)$, where M_i, N_i are arbitrary (in general noncommuting), dimensionless, and a-independent matrices from, e.g., $\mathfrak{u}(2)$, and l is the characteristic length on which $\vec{A}$ varies. Furthermore, local disorder may be introduced in a controlled way that allows for fluctuations of the matrices M_i. In particular, disorder can be annealed and can mimic thermal fluctuations of significant amplitude. One can also realize more complicated, e.g., piecewise linear, spatial dependencies and include arbitrary local temporal components of $\vec{A}$.

Finally, let us discuss in which sense the proposed 2D scheme might be useful to study lattice gauge theories (LGT) in $(2 + 1)$D. There, one uses the framework of Euclidean field theory, and methods of statistical physics, such as Monte Carlo methods, to sum over all configurations of gauge and matter fields. Thus, gauge fields are dynamical variables in LGT, whereas they are obviously not in the proposed scheme. Moreover, our scheme is realized in real rather than in imaginary time. Nevertheless, the big advantage of our proposal is that given a gauge field configuration, the dynamics of matter fields in real time are given for free. By generating various configurations of gauge fields, we may try to "mimic" the Monte Carlo sampling of LGT. Averaging over both annealed disorder and quantum fluctuations should approximate the statistical average in LGT. However, this inevitably requires that generated configurations represent the characteristic or statistically relevant ones.

Although the gauge fields accessible in the proposed scheme are limited, at least some of them share characteristics of LGT phases, e.g., an area law fulfilled by Wilson loops in the confinement sector. In fact, the $SU(2)$ gauge potential of Eq. (4) with fluctuating, but anticorrelated fluxes $\alpha_1 = -\alpha_2$ yields an area law for Wilson loops in the xy plane, provided the probability distribution of the fluxes is Lorentzian. However, it would be desirable to create configurations that exhibit other characteristics of the confinement phase such as appropriate distributions of center vortices, Abelian magnetic monopoles, instantons, merons, calorons, etc. [14].

We expect that this program will lead to many fascinating results. We stress that Yang-Mills theories in $(2 + 1)$D are in the center of interest in high energy physics, as they describe the high temperature behavior of 4D models [15,16]. Recently, there has been progress in understanding these theories [15,17] in the pure gauge sector: the gauge-invariant degrees of freedom and the Hamiltonian have been identified. The ground state wave function is approximately a Gaussian function of the currents, and the string tension is known exactly. Our lattice models may shed new light on these recent discoveries. Another promising method of creating non-Abelian gauge potentials based on EIT has been proposed in Ref. [18].

We acknowledge fruitful discussions with B. Damski, H.-W. Diehl, O. Lechtenfeld, M. Fleischhauer, H. Frahm, F. Haake, C. Sämann, and H. Schulz. This work was supported by the Deutsche Forschungsgemeinschaft (SPP1116, SFB407, 436POL), Alexander von Humboldt Foundation, the ESF PESC QUDEDIS, and the Spanish MEC Grants No. HA2002-0036 and No. BFM 2002-02588.

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