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Vieta's formulae for regular polynomials of a quaternionic variable

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Abstract: Vieta's classical formulae explicitly determine the coefficients of a polynomial $p \in \mathbb{F}[x]$ in terms of the roots of p, where $\mathbb{F}$ is any commutative ring. In this paper, Vieta's formulae are obtained for slice-regular polynomials over the noncommutative algebra of quaternions, by an argument which essentially relies on induction, without invoking quasideterminants or noncommutative symmetric functions.

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Vieta's well-known formulae (named after Francois Viéte, a French mathematician of the sixteenth century, often referred to by his latinised name Franciscus Vieta) relate the coefficients of a polynomial and its roots and have many applications in algebra. In symbols, if $p(x) = x^n a_n + x^{n-1} a_{n-1} + \cdots + x a_1 + a_0$ is a polynomial of degree n whose coefficients are real or complex numbers (hence $a_n \neq 0$), then, by the Fundamental Theorem of Algebra, p has n complex roots, say $x_1, x_2, \dots, x_n$ (which are not necessarily distinct). Vieta's formulae state that a_{n-k} is related to the roots of p in the following way

$$(-1)^k a_{n-k}/a_n = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

where the right-hand sides are the *elementary symmetric functions* of the roots of p. We observe that, given $x_1, x_2, \dots, x_n$ (not necessarily distinct), Vieta's formulae provide a family of polynomials of degree n whose roots are precisely $x_1, x_2, \dots, x_n$; actually, if $b_0, b_1, \dots, b_n$ are the coefficients of such a polynomial, so are $\lambda b_0, \lambda b_1, \dots, \lambda b_n$ with $\lambda \neq 0$. In particular, since $b_n \neq 0$, we can consider in this family the monic polynomial whose coefficients are $b_0 b_n^{-1}, b_1 b_n^{-1}, \dots, 1$, or, in other words, we can take $[b_0 b_n^{-1} : b_1 b_n^{-1} : \dots : 1]$ as homogeneous coordinates for the coefficients of the family of polynomials we are interested in.

For a version of Vieta's formulae for polynomials with coefficients from a noncommutative ring (or from a skew field) and for an introduction to noncommutative symmetric functions see [5; 8; 24]. This approach requires the introduction of quasideterminants, see e.g. [1; 6; 4; 24], and the (abstract) algebra of symmetric functions together with the plactic action of Lascoux and Schützenberger, now known to be a particular case of Kashiwara's action of Weyl groups on crystal graphs (see also [19]).

1 Recent results on regular polynomials of a quaternionic variable

Let $\mathbb{H}$ denote the skew field of real quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$ where the x_l are real, and i, j, k, are imaginary units (i.e. their square equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. We denote by $\mathbb{S}$ the 2-dimensional sphere of imaginary units of $\mathbb{H}$, i.e. $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$. Every nonreal quaternion q can be written in a unique way as $q = x + yI$, with $I \in \mathbb{S}$ and $x, y \in \mathbb{R}$, $y > 0$. We refer to $x = \mathrm{Re}(q)$ as the real part of q and to $y = \mathrm{Im}(q)$ as the imaginary part of q.

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In [12; 13] a new theory of regularity for functions of a quaternionic variable has been introduced, inspired by an idea of Cullen [2]. *Regular* polynomials in the sense of Gentili and Struppa are polynomials of the form

$$p(q) = \sum_{k=0}^{n} q^k a_k \quad \text{with } a_k \in \mathbb{H} \text{ for } k = 0, \dots, n.$$

In general, it can be proven that a function f of quaternionic variable q is (*Cullen*) *regular* or *slice regular* in a ball B centered at 0 if and only if f admits a (converging) power series expansion

$$f(q) = \sum_n q^n a_n$$

in B with $a_n \in \mathbb{H}$ for any n. Therefore it is very natural to expect that Cullen regular functions have many properties in common with holomorphic functions of a complex variable. In particular, it is easy to prove that every (Cullen) regular function $f(q) = \sum_n q^n a_n$ is C^∞, with (Cullen) derivative f' still (Cullen) regular, namely $f'(q) = \sum_{n\geq 1} q^n n a_{n-1}$.

Below we simply say polynomials when referring to (Cullen) regular polynomials. The papers [10; 14; 15; 25; 26] deepened our understanding of the structure of such polynomials but in general it requires a certain effort to extend some notions from the complex (holomorphic) case to the quaternionic case. To begin with, we observe that the product of two regular polynomials (functions) is not regular in general. For example, even the simple product $(q-\alpha)(q-\beta) = q^2 - \alpha q - q\beta + \alpha\beta$ is not regular when α is not real. Thus, as for polynomials over skew-fields, one defines a different product $*$ which guarantees that the product of regular functions is regular. For polynomials, this product is defined as follows.

Definition 1.1. Let $p(q) = \sum_{i=0}^{n} q^i a_i$ and $s(q) = \sum_{j=0}^{m} q^j b_j$ be two polynomials. We define *the regular product of p and s* as the polynomial $p * s(q) = \sum_{k=0}^{mn} q^k c_k$, where $c_k = \sum_{l=0}^{k} a_l b_{k-l}$ for all k.

Remark 1.2. This definition, see e.g. [18], has the effect that multiplication of polynomials is performed as if the coefficients were chosen in a commutative field; as a consequence, the resulting polynomial is still a regular polynomial with all the coefficients on the right. In [11] the regular product is extended to regular functions and a Leibniz-rule for the regular product of regular functions is proven true.

In general the Fundamental Theorem of Algebra fails to be valid for quaternionic polynomials, as shown in the following example:

Example 1.3. For any $n \in \mathbb{N}$ and for any quaternion q, the polynomial $aq^n - q^n a + 1$ (with coefficient the quaternion a) has real part identically equal to 1.

However, for regular polynomials this is not the case, since in [15] one can find a "universal" proof of the Fundamental Theorem of Algebra for regular polynomials over Hamilton and Cayley numbers. To find explicit roots of quaternionic algebraic equations remains a difficult problem in general; see [20; 21]. We begin by analyzing three simple examples which, however, contain all the features which distinguish the theory of polynomials in $\mathbb{H}$ from the standard theory of complex polynomials.

Remark 1.4. Consider the polynomial $p_1(q) = (q-\alpha) * (q-\beta) = q^2 - q(\alpha+\beta) + \alpha\beta$, where α and β are nonreal quaternions with $|\operatorname{Im}(\alpha)| \neq |\operatorname{Im}(\beta)|$. It is immediate to verify, by direct substitution, that α is a solution of $p_1(q) = 0$, while β is not a root of the polynomial. In fact, one can prove (see Theorem 1.7 below) that p_1 has a second root given by $(\overline{\beta} - \alpha)^{-1}\beta(\overline{\beta} - \alpha)$. Thus, as one would expect, the polynomial has two roots (and in fact only two roots), though they are not what one would expect from a first look at the polynomial (this is a consequence of the fact that the valuation is not a homomorphism of rings).

Remark 1.5. Consider the polynomial $p_2(q) = (q-\alpha) * (q-\overline{\alpha}) = q^2 - q(2\operatorname{Re}(\alpha)) + |\alpha|^2$. In this case α is called a spherical root, see [10; 13], and it is easy to verify that every point on the 2-sphere $S_\alpha = \operatorname{Re}(\alpha) + \operatorname{Im}(\alpha)\mathbb{S}$ is a root for p_2. More precisely we say that α is a generator of the spherical root S_α.

Remark 1.6. Let $p_3(q) = (q-\alpha) * (q-\beta) = q^2 - q(\alpha+\beta) + \alpha\beta$, where α and β are nonreal quaternions with $\beta \in S_\alpha$ and $\beta \neq \overline{\alpha}$. In this case, as shown in [10], the only root of the polynomial p_3 is α.

These three examples exhibit a behavior that is very different from the one we are used to in the complex case. First we observed that, as already clarified in [13], some polynomials of a quaternionic variable admit spherical zeroes, i.e. entire 2-spheres of the form $x + y\mathbb{S}$ for some real values x, y. Secondly, even when the polynomial is factored as a $*$ product of monomials, we cannot guarantee that each monomial contributes a zero. Indeed, in the case of p_1, when both monomials contribute a zero, the contribution of the second monomial depends explicitly on the first monomial. This is a direct consequence of Theorem 3.3 in [10], which we repeat here for the sake of completeness (but see also [18] for the same statement in the case of polynomials).

Theorem 1.7 (Zeros of a regular product). *Let f, g be given quaternionic power series with radii greater than R and let $p \in B(0,R)$. Then $f * g(p) = 0$ if and only if $f(p) = 0$ or $f(p) \neq 0$ and $g(f(p)^{-1}pf(p)) = 0$.*

Remark 1.8. We observe here that $f(p)^{-1}pf(p)$ has the same real part as p but a different imaginary part, even though they have the same module. In short, we usually say that p and $f(p)^{-1}pf(p)$ "lie on the same sphere"; for a detailed investigation on this phenomenon, also known as *camshaft effect*, see [17].

Furthermore, see again [10], the following result holds true.

Theorem 1.9. *Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R and let $\alpha \in B(0,R)$. Then $f(\alpha) = 0$ if and only if there exists a quaternionic power series g with radius of convergence R such that*

$$f(q) = (q - \alpha) * g(q). \tag{1}$$

Remark 1.10. For a similar result in the case of noncommutative polynomials see also [22].

Now we come to the peculiarity described in Example 1.6. Here the polynomial p_3 has degree two, hence one would expect either two solutions, or one solution with "multiplicity" two. As pointed out in [23], to define a good notion of multiplicity for zeros of quaternionic polynomials is rather complicated and has required some efforts, but it was finally successfully established in [14] after obtaining this important result:

Theorem 1.11. *Let p be a regular polynomial of degree m. Then there exist $r, m_1, \dots, m_r \in \mathbb{N}$ and $w_1, \dots, w_p \in \mathbb{H}$, generators of the spherical roots of p, such that*

$$P(q) = (q^2 - 2q\,\mathrm{Re}(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2q\,\mathrm{Re}(w_r) + |w_r|^2)^{m_r} Q(q), \tag{2}$$

where $\mathrm{Re}(w_i)$ denotes the real part of w_i and Q is a regular polynomial with coefficients in $\mathbb{H}$ having only non-spherical zeroes. Moreover, if $n = m - 2(m_1 + \cdots + m_r)$, then there exists a constant $c \in \mathbb{H}$ such that

$$Q(q) = \left[\boxed{\prod_{i=1}^{t}}_{*} \boxed{\prod_{j=1}^{n_i}}_{*} (q - \alpha_{ij}) \right] c, \tag{3}$$

where $\boxed{\prod}_*$ is the analog of $\prod$ with respect to the $*$-product, $n_1, \dots, n_t$ are integers with $n_1 + \cdots + n_t = n$, and the quaternions $\alpha_{ij} \in S_i$ with $i = 1, \dots, t$ and $j = 1, \dots, n_i$ belong to t distinct 2-spheres $S_1 = x_1 + y_1\mathbb{S}, \dots, S_t = x_t + y_t\mathbb{S}$.

From the results in [10; 14; 25; 26], we recall the following.

Definition 1.12. Let $p : U \to \mathbb{H}$ be a regular polynomial. If $x + Iy$ is a spherical zero of p, its *spherical* multiplicity is defined as two times the largest integer m for which it is possible to write $p(q) = (q^2 - 2qx + (x^2 + y^2))^m s(q)$ with $s : U \to \mathbb{H}$ a regular polynomial. Furthermore, we say that a zero $\alpha_1 \in \mathbb{H} \setminus \mathbb{R}$ of p has *isolated* multiplicity k if s can be written as

$$s(q) = (q - \alpha_1) * (q - \alpha_2) * \cdots * (q - \alpha_k) * h(q)$$

with all α_j on the sphere S_{α_1} and such that $\alpha_j \neq \overline{\alpha}_{j+1}$ for $j = 1, \dots k-1$ and $h : U \to \mathbb{H}$ is a regular polynomial that does not vanish at any point of the sphere S_{α_1}. Finally, if $x \in \mathbb{R}$ is a zero of p, we say that it has *isolated* multiplicity n if we can write $s(q) = (q - x)^n h(q)$ with $h : U \to \mathbb{H}$ some regular polynomial which does not vanish at x.

2 Vieta's formulae for regular polynomials over the quaternions

In [3] a version of Vieta's formulae in a noncommutative skew-field is obtained without invoking quasideterminants or noncommutative symmetric functions, essentially by using induction. We follow this approach as far as it applies to our case, namely Vieta's formulae for slice-regular quaternionic polynomials.

Proposition 2.1. *The coefficients of monic slice-regular polynomial p can be inductively expressed in terms of (the real and imaginary parts of) the roots of p.*

First part of the proof of Proposition 2.1. First we obtain the coefficients of a monic slice-regular polynomial p_n which has n distinct simple isolated roots $\alpha_1, \dots, \alpha_n$, i.e. each α_j is a nonspherical root of multiplicity 1 of p_n. For $n = 2$ we define

$$p_2(q) := (q - \alpha_1) * (q - \widetilde{\alpha_2}) \qquad \text{where} \quad \widetilde{\alpha_2} := (\alpha_2 - \alpha_1)^{-1}\alpha_2(\alpha_2 - \alpha_1).$$

In other words, $p_2(q) := q^2 - q(\alpha_1 + \tilde{\alpha}_2) + \alpha_1\tilde{\alpha}_2$ is a monic regular polynomial of degree 2, and by the result of the previous section one can easily check that α_1 and α_2 are its only roots. (The trivial case $p_1(q) = q - \alpha_1$ has some importance for the next considerations; actually, we shall see that it makes sense to consider also the case $p_0 = 1$.) Note that

$$\widetilde{\alpha_2} := (p_1(\alpha_2))^{-1}\alpha_2(p_1(\alpha_2)).$$

Therefore we define $\widetilde{\alpha_1} := \alpha_1$, and by induction on k with $k > 1$ we introduce

$$p_k(q) := p_{k-1}(q) * (q - \widetilde{\alpha_k}) \qquad \text{where} \quad \widetilde{\alpha_k} := (p_{k-1}(\alpha_k))^{-1}\alpha_k p_{k-1}(\alpha_k))$$

(recall that $p_{k-1}(\alpha_k) \neq 0$; one can consider the previous definition also for $k = 1$ if $p_0 = 1$). It turns out that $p_k(q)$ is a monic polynomial of degree k which vanishes at $\alpha_1, \dots \alpha_k$. Conversely if one considers

$$\tilde{p}_k(q) := p_{k-1}(q) * (q - \alpha)$$

then it follows that $\tilde{p}_k(q) = qp_{k-1}(q) - p_{k-1}(q)\alpha$. In particular, from the request $\tilde{p}_k(\alpha_k) = 0$, one obtains equivalently $\alpha_k p_{k-1}(\alpha_k) - p_{k-1}(\alpha_k)\alpha = 0$, and finally concludes that

$$\alpha = (p_{k-1}(\alpha_k))^{-1}\alpha_k(p_{k-1}(\alpha_k)) = \widetilde{\alpha_k}.$$

So the coefficients $a_0, a_1, \dots, a_{n-1}$ of the (monic) polynomial

$$p_n(q) = (q - \tilde{\alpha}_1) * (q - \tilde{\alpha}_2) * \cdots * (q - \tilde{\alpha}_n) = q^n + \sum_{k=0}^{n-1} q^k a_k$$

are uniquely determined in terms of the "shifted" roots $\tilde{\alpha}_j$ and so of $\alpha_1, \alpha_2, \dots, \alpha_n$. □

With a different approach, the same result has been obtained already in [14]. In order to complete our task we need to prove that the coefficients a_j are independent of the ordering of $\alpha_1, \alpha_2, \dots, \alpha_n$, but the adapted argument given in [3] (which presumably should consist of the inductive application of the correct formula $a(a-b)^{-1}b = b(a-b)^{-1}a$, for $a \neq b$, even though the author gives a direct proof only for the case $n = 3$) cannot be applied for our case; we provide a different approach. We recall that the entire theory of slice functions over quaternions can be reinterpreted by considering the induced functions of $\mathbb{H}$-stem functions on an open set $D \subset \mathbb{C}$; see [16], where the construction is more generally carried out for any finite-dimensional alternative real algebra A with unit. In particular, if F and G are two holomorphic $\mathbb{H}$-stem functions, it turns out that the induced functions $\mathcal{I}(F)$ and $\mathcal{I}(G)$ are slice-regular functions and, moreover,

$$\mathcal{I}(F \cdot G) = \mathcal{I}(F) * \mathcal{I}(G).$$

Since any factor $f_j(q) = (q - \tilde{\alpha}_j)$ of P_n is a slice-regular function obtained by a corresponding holomorphic $\mathbb{H}$-stem function $F_j(z)$, we can also write $\mathcal{I}(F_1 \cdot F_2 \cdots F_n)$; in particular, the polynomial $f := F_1 \cdot F_2 \cdots F_n$ is monic and has coefficients in a division algebra (not necessarily commutative) whereas the formal variable of f commutes with the coefficients of f. As proven in [8; 9], under these assumptions the coefficients of f do not depend on the ordering of $\alpha_1, \dots, \alpha_n$, therefore also the coefficients of $p := \mathfrak{I}(f)$ are independent of the ordering of $\alpha_1, \dots, \alpha_n$.

Remark 2.2. We point out that this is in accordance with the results proved in [14], where it is shown that the regular factorization of a monic regular polynomial in terms of linear factors is not unique. Indeed, in [14] it is shown that if a, b lie on different spheres, then

$$(q-a) * (q-b) = (q-a') * (q-b')$$

if and only if $a' = (b-a)^{-1}b(b-a)$ and $b' = (b-a)^{-1}a(b-a)$.

We make a short remark on nonsimple roots. From the results proved in [14] it is clear that any analog of a Vieta formulae requires more efforts for quaternionic regular polynomials with nonsimple roots; indeed the polynomial

$$p(q) = (q-\alpha_1) * (q-\alpha_2) * \cdots * (q-\alpha_n),$$

where each α_j belongs to the same sphere of α_1 and $\alpha_{j+1} \neq \overline{\alpha_j}$ for $j = 1, \dots, n-1$, has a unique root, namely α_1.

Viceversa, if a monic slice-regular polynomial

$$P(q) = q^n + q^{n-1}a_{n-1} + \cdots + qa_1 + a_0$$

such that $P(\alpha_1) = 0$ is given, there is a quick test to check if $\alpha_1 = x + I_1 y$ has multiplicity n. In fact, if $(a_{n-1} - nx)^2$ is not real, then α_1 cannot have multiplicity n, but if $(a_{n-1} - nx)^2$ is real, then α_1 may have multiplicity n. This depends on the fact that $\alpha_1 = x + I_1 y$ is a root of multiplicity n for p *if and only if*

$$p^s(q) = [(q-x)^2 + y^2]^n,$$

where p^s is the *symmetrized* of p, or $p^s = p * p^c = p^c * p$ with $p^c(q) = q^n + q^{n-1}\overline{a_{n-1}} + \cdots + q\overline{a_1} + \overline{a_0}$. It turns out that

$$p^s(q) = \sum_{k=0}^{n} \left[\sum_{s=0}^{2k} q^s (-x)^{2k-s} \binom{2k}{s} \right] \binom{n}{k} y^{2(n-k)},$$

therefore the coefficient c_m of the monomial of degree m of p^s with $0 \le m \le 2n$ is given by

$$c_m = \sum_{k=0 \text{ or } 2k \ge m}^{n} (-x)^{2k-m} \binom{2k}{m} \binom{n}{k} y^{2(n-k)}. \tag{4}$$

Condition (4) may be checked (possibly with the help of a computer).

Furthermore, the factorization of p is unique, since for any two distinct quaternions α, β belonging to the same sphere it turns out that $(\beta - \alpha)^{-1}\beta(\beta - \alpha) = \overline{\alpha}$; see [14].

Finally, if the polynomial p has multiple roots, we can try to obtain a certain generalized version of Vieta's formulae from the previous results.

Second part of the proof of Proposition 2.1. If α_1 is the only root of the polynomial p, then the polynomial

$$p(q) * (q - [p(\overline{\alpha_1})]^{-1}\overline{\alpha_1}p(\overline{\alpha_1}))$$

has two roots, namely α_1 and $\overline{\alpha_1}$. From the uniqueness of the factorization of p we then conclude that

$$\overline{\alpha_m} = [p(\overline{\alpha_1})]^{-1}\overline{\alpha_1}p(\overline{\alpha_1}) \qquad \text{or} \qquad \alpha_n = \overline{[p(\overline{\alpha_1})]^{-1}\overline{\alpha_1}p(\overline{\alpha_1})} = \overline{[p(\overline{\alpha_1})]}^{-1}\alpha_1\overline{p(\overline{\alpha_1})}.$$

Therefore if α_1 is a multiple root of p of multiplicity n and if we know the value $p(\overline{\alpha_1}) \neq 0$ we obtain α_n. If we apply the same procedure to the polynomial

$$p_1(q) := p(q) * \frac{(q - [p(\overline{\alpha_1})]^{-1}\overline{\alpha_1}p(\overline{\alpha_1}))}{q^2 - 2q\operatorname{Re}(\alpha_1) + |\alpha_1|^2} = (q-\alpha_1) * (q-\alpha_2) * \cdots * (q-\alpha_{n-1})$$

we obtain α_{n-1} from

$$\alpha_{n-1} = \overline{[p_1(\overline{\alpha_1})]^{-1}\overline{\alpha_1}p_1(\overline{\alpha_1})}$$

and so eventually all α_j for $j = 2, \dots, n$. Then, as in the case of nonmultiple roots, from the α_j's (and more specifically the shifted $\tilde{\alpha}_j$'s) we are able to reconstruct the coefficients a_k of the polynomial p. □

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