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A generalization of the parallelogram law to higher dimensions

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Abstract

We propose a generalization of the parallelogram identity in any dimension $N \geq 2$, establishing the ratio of the quadratic mean of the diagonals to the quadratic mean of the faces of a parallelotope. The proof makes use of simple properties of the exterior product of vectors.

Keywords: Parallelogram law, parallelotope.

Math. Subj. Class.: 51M04

1 Introduction and statement of the result

The well known parallelogram law states:

> *For any parallelogram, the sum of the squares of the lengths of its two diagonals is equal to the sum of the squares of the lengths of its four sides.*

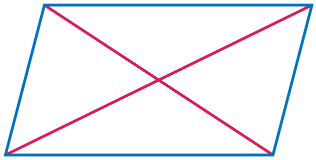

Figure 1: The two diagonals of a parallelogram.

*I would probably never have written this paper without the support of my son Marcello. I warmly thank him, in particular, for helping me finding the formula for the three-dimensional case.

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Equivalently: given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, one has

$$\|\boldsymbol{a}+\boldsymbol{b}\|^2 + \|\boldsymbol{a}-\boldsymbol{b}\|^2 = 2(\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2)\,.$$

This identity holds in any inner product space, but, since the two vectors belong to the same plane, we can see it as being of a two-dimensional nature. The aim of this paper is to provide a generalization to higher dimensions.

The parallelogram law has a natural geometric interpretation, involving the areas of the squares constructed on the sides and on the diagonals of the parallelogram. In particular, when $\|\boldsymbol{a}+\boldsymbol{b}\| = \|\boldsymbol{a}-\boldsymbol{b}\|$, it reduces to the Pythagorean theorem. In this paper, however, we will look at the parallelogram law from a rather unusual point of view: writing it as

$$\frac{\|\boldsymbol{a}+\boldsymbol{b}\|^2 + \|\boldsymbol{a}-\boldsymbol{b}\|^2}{2} = 2\,\frac{\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2}{4}\,,$$

and taking the square roots, we can state it in the following equivalent form.

> *For any parallelogram, the ratio of the quadratic mean of the lengths of its diagonals to the quadratic mean of the lengths of its sides is equal to* $\sqrt{2}$.

Now, instead of a parallelogram, we will consider an N-dimensional parallelotope, and our goal will be to prove that the same type of proposition holds in this general case. Indeed, our result can be stated as follows.

Theorem 1.1. *For any N-dimensional parallelotope, the ratio of the quadratic mean of the $(N-1)$-dimensional measures of its diagonals to the quadratic mean of the $(N-1)$-dimensional measures of its faces is equal to $\sqrt{2}$.*

For $N = 2$, the 1-dimensional measure is the length, and we recover the parallelogram law. In the general case, we first need to specify what a *diagonal* should be, and indeed this will be clarified in the following sections. For example, if $N = 3$, the diagonals of a parallelepiped are precisely the parallelograms obtained joining the opposite edges of the parallelepiped (see Figure 2 below), so that the 2-dimensional measures of the diagonals are the areas of these parallelograms.

Notice that our definition of a diagonal is not the same as the one given in [1, 2], where a different generalization of the parallelogram law has been proposed; in the three-dimensional case, e.g., their diagonals are triangles. We believe that our definition is somewhat more natural, since here the diagonals share the same geometrical shape of the faces.

We provide the proof of our main theorem in Section 3. However, for the reader's convenience, we thought it useful to first explain its proof in detail in the more familiar three-dimensional case. This is what we are going to do next.

2 The three-dimensional case

To start with, let us consider a three-dimensional parallelepiped $\mathcal{P}$, and see how to extend the parallelogram law to this case. Instead of the lengths of the four sides of the parallelogram, we would like to take the areas of the *six faces* of the parallelepiped. On the other hand, the lengths of the two diagonals of the parallelogram should naturally be replaced by the areas of the *six diagonals* of the parallelepiped, i.e., the six parallelograms obtained joining the opposite edges of the parallelepiped. In this case, Theorem 1.1 can be rephrased as follows.

For any three-dimensional parallelepiped, the sum of the squares of the areas of its six diagonals is equal to twice the sum of the squares of the areas of its six faces.

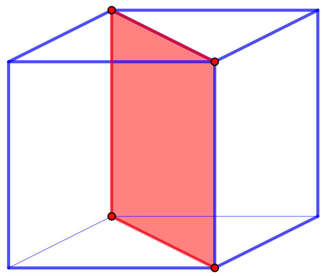

Figure 2: One of the six diagonals of a parallelepiped.

In order to prove this statement, assume the parallelepiped to be *generated* by the following three vectors:

$$\boldsymbol{a} = (a_1, a_2, a_3), \quad \boldsymbol{b} = (b_1, b_2, b_3), \quad \boldsymbol{c} = (c_1, c_2, c_3).$$

By this we mean that $\mathcal{P}$ is the set of points obtained as linear combinations of these three vectors, with coefficients in the interval $[0, 1]$:

$$\mathcal{P} = \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} : \alpha, \beta, \gamma \in [0, 1]\}.$$

The six faces of $\mathcal{P}$ are defined as

$$\begin{aligned}
\mathcal{F}_1^- &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \alpha = 0\}, \\
\mathcal{F}_1^+ &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \alpha = 1\}, \\
\mathcal{F}_2^- &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \beta = 0\}, \\
\mathcal{F}_2^+ &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \beta = 1\}, \\
\mathcal{F}_3^- &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \gamma = 0\}, \\
\mathcal{F}_3^+ &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \gamma = 1\}.
\end{aligned}$$

So,

$$\begin{aligned}
&\mathcal{F}_1^- \text{ is generated by } \boldsymbol{b} \text{ and } \boldsymbol{c}, \\
&\mathcal{F}_2^- \text{ is generated by } \boldsymbol{a} \text{ and } \boldsymbol{c}, \\
&\mathcal{F}_3^- \text{ is generated by } \boldsymbol{a} \text{ and } \boldsymbol{b},
\end{aligned}$$

while $\mathcal{F}_k^+$ is congruent to $\mathcal{F}_k^-$, for each $k = 1, 2, 3$.

The six diagonals of $\mathcal{P}$ are defined as

$$\begin{aligned}
\mathcal{D}^1_{1,2} &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \alpha = \beta\},\\
\mathcal{D}^2_{1,2} &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \alpha + \beta = 1\},\\
\mathcal{D}^1_{1,3} &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \alpha = \gamma\},\\
\mathcal{D}^2_{1,3} &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \alpha + \gamma = 1\},\\
\mathcal{D}^1_{2,3} &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \beta = \gamma\},\\
\mathcal{D}^2_{2,3} &= \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \in \mathcal{P} : \beta + \gamma = 1\}.
\end{aligned}$$

So,

$$\begin{aligned}
&\mathcal{D}^1_{1,2} \text{ is generated by } \boldsymbol{a} + \boldsymbol{b} \text{ and } \boldsymbol{c},\\
&\mathcal{D}^1_{1,3} \text{ is generated by } \boldsymbol{a} + \boldsymbol{c} \text{ and } \boldsymbol{b},\\
&\mathcal{D}^1_{2,3} \text{ is generated by } \boldsymbol{b} + \boldsymbol{c} \text{ and } \boldsymbol{a},
\end{aligned}$$

while

$$\begin{aligned}
&\mathcal{D}^2_{1,2} \text{ is congruent to the set generated by } \boldsymbol{a} - \boldsymbol{b} \text{ and } \boldsymbol{c},\\
&\mathcal{D}^2_{1,3} \text{ is congruent to the set generated by } \boldsymbol{a} - \boldsymbol{c} \text{ and } \boldsymbol{b},\\
&\mathcal{D}^2_{2,3} \text{ is congruent to the set generated by } \boldsymbol{b} - \boldsymbol{c} \text{ and } \boldsymbol{a}.
\end{aligned}$$

Our proposition is thus translated into the following identity:

$$\begin{aligned}
&\|(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c}\|^2 + \|(\boldsymbol{a} - \boldsymbol{b}) \times \boldsymbol{c}\|^2\\
&+ \|(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b}\|^2 + \|(\boldsymbol{a} - \boldsymbol{c}) \times \boldsymbol{b}\|^2\\
&+ \|(\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a}\|^2 + \|(\boldsymbol{b} - \boldsymbol{c}) \times \boldsymbol{a}\|^2 = 4(\|\boldsymbol{b} \times \boldsymbol{c}\|^2 + \|\boldsymbol{a} \times \boldsymbol{c}\|^2 + \|\boldsymbol{a} \times \boldsymbol{b}\|^2).
\end{aligned}$$

Here, we have used the vector product, so that, e.g.,

$$\begin{aligned}
\|\boldsymbol{a} \times \boldsymbol{b}\|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2\\
&= (a_2b_3 - b_2a_3)^2 + (a_3b_1 - b_3a_1)^2 + (a_1b_2 - b_1a_2)^2.
\end{aligned}$$

In order to prove the above identity, we just notice that, by the parallelogram law,

$$\begin{aligned}
&\|(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c}\|^2 + \|(\boldsymbol{a} - \boldsymbol{b}) \times \boldsymbol{c}\|^2 =\\
&\qquad = \|(\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{c})\|^2 + \|(\boldsymbol{a} \times \boldsymbol{c}) - (\boldsymbol{b} \times \boldsymbol{c})\|^2\\
&\qquad = 2(\|\boldsymbol{a} \times \boldsymbol{c}\|^2 + \|\boldsymbol{b} \times \boldsymbol{c}\|^2),
\end{aligned}$$

and similarly

$$\begin{aligned}
\|(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b}\|^2 + \|(\boldsymbol{a} - \boldsymbol{c}) \times \boldsymbol{b}\|^2 &= 2(\|\boldsymbol{a} \times \boldsymbol{b}\|^2 + \|\boldsymbol{c} \times \boldsymbol{b}\|^2),\\
\|(\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a}\|^2 + \|(\boldsymbol{b} - \boldsymbol{c}) \times \boldsymbol{a}\|^2 &= 2(\|\boldsymbol{b} \times \boldsymbol{a}\|^2 + \|\boldsymbol{c} \times \boldsymbol{a}\|^2).
\end{aligned}$$

Summing up the three formulas, our identity is proved.

Remark 2.1. There surely are several ways to extend the parallelogram law to higher dimensions. Just to mention one of these, in the three-dimensional case we have

$$\begin{aligned}&\|\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}\|^2+\|\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}\|^2\\ &+\|\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}\|^2+\|\boldsymbol{a}-\boldsymbol{b}-\boldsymbol{c}\|^2=4(\|\boldsymbol{a}\|^2+\|\boldsymbol{b}\|^2+\|\boldsymbol{c}\|^2).\end{aligned}$$

We acknowledge the referee for pointing out this identity. It is proved directly (by the use of the classical parallelogram law) and can be easily extended to any dimension.

3 Proof of the main theorem

We now provide a proof for the general N-dimensional case. Let $\mathcal{P}$ be the parallelotope generated by the vectors $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N$, i.e.,

$$\mathcal{P}=\left\{\sum_{k=1}^N c_k\boldsymbol{a}_k : c_k\in[0,1],\ \text{for } k=1,\ldots,N\right\}.$$

Its $2N$ faces are defined by

$$\mathcal{F}_n^-=\left\{\sum_{k=1}^N c_k\boldsymbol{a}_k\in\mathcal{P} : c_n=0\right\},\quad \mathcal{F}_n^+=\left\{\sum_{k=1}^N c_k\boldsymbol{a}_k\in\mathcal{P} : c_n=1\right\},$$

with $n=1,\ldots,N$. Each $\mathcal{F}_n^-$ is generated by the vectors $\boldsymbol{a}_1,\ldots,\widehat{\boldsymbol{a}_n},\ldots,\boldsymbol{a}_N$, where, as usual, $\widehat{\boldsymbol{a}_n}$ means that $\boldsymbol{a}_n$ is missing, while $\mathcal{F}_n^+$ is a translation of $\mathcal{F}_n^-$, for every $n=1,\ldots,N$.

Concerning the diagonals, they are defined as

$$\mathcal{D}_{i,j}^1=\left\{\sum_{k=1}^N c_k\boldsymbol{a}_k\in\mathcal{P} : c_i=c_j\right\},\quad \mathcal{D}_{i,j}^2=\left\{\sum_{k=1}^N c_k\boldsymbol{a}_k\in\mathcal{P} : c_i+c_j=1\right\},$$

with indices $i<j$ varying from 1 to N. There are $N(N-1)$ of them. Hence, we have that

$$\mathcal{D}_{i,j}^1 \text{ is generated by } \boldsymbol{a}_i+\boldsymbol{a}_j \text{ and } \boldsymbol{a}_1,\ldots,\widehat{\boldsymbol{a}_i},\ldots,\widehat{\boldsymbol{a}_j},\ldots,\boldsymbol{a}_N,$$

while

$$\mathcal{D}_{i,j}^2 \text{ is a translation of the set generated by } \boldsymbol{a}_i-\boldsymbol{a}_j \text{ and } \boldsymbol{a}_1,\ldots,\widehat{\boldsymbol{a}_i},\ldots,\widehat{\boldsymbol{a}_j},\ldots,\boldsymbol{a}_N.$$

In order to compute the $(N-1)$-dimensional measures of the faces and the diagonals of our parallelotope, we make use of the following proposition involving the exterior product of vectors in $\mathbb{R}^N$. (See, e.g., [3] for the definition and the main properties of the exterior product.)

Proposition 3.1. *The M-dimensional measure of a parallelotope generated by M vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_M$ in $\mathbb{R}^N$, with $1\le M\le N$, is given by $\|\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_M\|$.*

Proof. If $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_M$ are linearly dependent, the M-dimensional measure of the parallelotope generated by $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_M$ is equal to zero, hence coincides with $\|\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_M\|$.

Assume now that the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$ are linearly independent, and let V be the subspace generated by them. Choose an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_M$ of V, and write

$$\begin{aligned} \boldsymbol{v}_1 &= v_{11}\boldsymbol{e}_1 + \cdots + v_{1M}\boldsymbol{e}_M, \\ &\vdots \\ \boldsymbol{v}_M &= v_{M1}\boldsymbol{e}_1 + \cdots + v_{MM}\boldsymbol{e}_M. \end{aligned}$$

Then,

$$\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_M = \det \begin{pmatrix} v_{11} & \cdots & v_{1M} \\ \vdots & & \vdots \\ v_{M1} & \cdots & v_{MM} \end{pmatrix} \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_M,$$

so that

$$\|\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_M\| = \left| \det \begin{pmatrix} v_{11} & \cdots & v_{1M} \\ \vdots & & \vdots \\ v_{M1} & \cdots & v_{MM} \end{pmatrix} \right|,$$

which is indeed the M-dimensional measure of the parallelotope generated by the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$. □

Hence, the $(N-1)$-dimensional measures of the faces $\mathcal{F}_n^{\pm}$ are given by

$$\|\boldsymbol{a}_1 \wedge \cdots \wedge \widehat{\boldsymbol{a}_n} \wedge \cdots \wedge \boldsymbol{a}_N\|,$$

while the $(N-1)$-dimensional measures of the diagonals $\mathcal{D}^1_{i,j}$ are equal to

$$\|(\boldsymbol{a}_i + \boldsymbol{a}_j) \wedge \textstyle\bigwedge_{k\neq i,j} \boldsymbol{a}_k\|,$$

and those of the diagonals $\mathcal{D}^2_{i,j}$ are equal to

$$\|(\boldsymbol{a}_i - \boldsymbol{a}_j) \wedge \textstyle\bigwedge_{k\neq i,j} \boldsymbol{a}_k\|.$$

Choosing any couple $i < j$, by the parallelogram law we have that

$$\begin{aligned} &\|(\boldsymbol{a}_i + \boldsymbol{a}_j) \wedge \textstyle\bigwedge_{k\neq i,j} \boldsymbol{a}_k\|^2 + \|(\boldsymbol{a}_i - \boldsymbol{a}_j) \wedge \textstyle\bigwedge_{k\neq i,j} \boldsymbol{a}_k\|^2 = \\ &\qquad = 2\Big(\|\boldsymbol{a}_1 \wedge \cdots \wedge \widehat{\boldsymbol{a}_j} \wedge \cdots \wedge \boldsymbol{a}_N\|^2 + \|\boldsymbol{a}_1 \wedge \cdots \wedge \widehat{\boldsymbol{a}_i} \wedge \cdots \wedge \boldsymbol{a}_N\|^2\Big). \end{aligned}$$

We now want to take the sum of all these equalities, with $i < j$ varying form 1 to N. We claim that, for any $n = 1, \ldots, N$, when performing such a sum, in the right hand side,

the term $2\|\boldsymbol{a}_1 \wedge \cdots \wedge \widehat{\boldsymbol{a}_n} \wedge \cdots \wedge \boldsymbol{a}_N\|^2$ will appear $N-1$ times.

Indeed, this term may appear with $j = n$, while i varies from 1 to $n-1$, or with $i = n$, while j varies from $n+1$ to N, and there are exactly $N-1$ of such possibilities. Hence, summing all the equalities, we have that

$$\begin{aligned} &\sum_{i<j} \Big(\|(\boldsymbol{a}_i + \boldsymbol{a}_j) \wedge \textstyle\bigwedge_{k\neq i,j} \boldsymbol{a}_k\|^2 + \|(\boldsymbol{a}_i - \boldsymbol{a}_j) \wedge \textstyle\bigwedge_{k\neq i,j} \boldsymbol{a}_k\|^2\Big) = \\ &\qquad\qquad = (N-1) \sum_{n=1}^{N} 2\|\boldsymbol{a}_1 \wedge \cdots \wedge \widehat{\boldsymbol{a}_n} \wedge \cdots \wedge \boldsymbol{a}_N\|^2. \end{aligned}$$

So, we have proved the following.

For any N-dimensional parallelotope, the sum of the squares of the $(N-1)$-dimensional measures of its $N(N-1)$ diagonals is equal to $N-1$ times the sum of the squares of the $(N-1)$-dimensional measures of its $2N$ faces.

The proof of the theorem is now easily completed, dividing each of the two sums by the number of their addends and taking the square roots.

Remark 3.2. Since our result is valid in any dimension N, it would be interesting to investigate whether it could be extended also to some infinite-dimensional vector spaces. This seems to be a remarkable problem which could lead to further insight on the nature of these identities.

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