Multiplicity of periodic solutions for systems of weakly coupled parametrized second order differential equations

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Abstract. We prove a multiplicity result of periodic solutions for a system of second order differential equations having asymmetric nonlinearities. The proof is based on a recent generalization of the Poincaré–Birkhoff fixed point theorem provided by Fonda and Ureña.

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1. Introduction and main result

In this paper we study periodic solutions of a weakly coupled parametrized system of second order differential equations.

The study of existence of periodic solutions for *scalar* second order differential equations presents a wide literature: we refer to the survey by Mawhin [24] and the references therein for an overview on this topic. In particular, we focus our attention on the classical result of Lazer and McKenna [23] dated 1987 and its generalizations due to Del Pino et al. [6] in 1992 and Fonda and Ghirardelli [9] in 2010. We also refer to [3,10,17,27] for related results and to [28,30] and references therein for a comprehensive introduction to this topic.

Let us quote here, for the reader's convenience [6, Theorem 1.2].

Theorem 1.1. (Del Pino et al. [6]) *Consider the differential equation* $u'' + g(u) = s(1 + h(t))$, *where* $g : \mathbb{R} \to \mathbb{R}$ *is of class* C^1, $h : \mathbb{R} \to \mathbb{R}$ *is continuous and* 2π*-periodic and* s *is a real parameter. Assume the existence of the limits* $\nu = \lim_{x\to-\infty} g'(x)$ *and* $\mu = \lim_{x\to+\infty} g'(x)$ *satisfying*

$$(k-1)^2 < \nu < k^2 \le m^2 < \mu < (m+1)^2 \tag{1}$$

for some positive integers k and m. Moreover μ and ν are such that

$$\chi = 2\frac{\sqrt{\mu\nu}}{\sqrt{\mu}+\sqrt{\nu}} \text{ is not an integer.} \tag{2}$$

Denote by n the integer part of χ. Then, there exist two positive constants h_0 and s_0 such that if $\|h\|_\infty \leq h_0$ and $|s| > s_0$ then the equation has at least $2(m-n)+1$ solutions for positive s, and $2(n-k+1)+1$ solutions for negative s.

In this paper we are interested in possible extensions of this multiplicity result to the case of a system of differential equations: we have in mind in particular the physical model of *coupled oscillators*, see e.g. [1,2,4,22,25,26,29] and references therein. In our main result, Theorem 1.3 below, we prove the existence of "many" periodic solutions if some non-resonance hypotheses hold. In the trivial case of a system consisting of a unique equation we will recover Theorem 1.1 and its generalization due to Fonda and Ghirardelli [9, Theorem 2] for continuous nonlinearities. More in detail, we investigate periodic solutions of systems in $\mathbb{R}^N$ of the type

$$\begin{cases} x_1'' + g_1(t, x_1, \dots, x_N) = sw_1(t), \\ x_2'' + g_2(t, x_1, \dots, x_N) = sw_2(t), \\ \quad\vdots \\ x_N'' + g_N(t, x_1, \dots, x_N) = sw_N(t), \end{cases} \tag{S}$$

where $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $w : \mathbb{R} \to \mathbb{R}^N$ are continuous functions and s is a real parameter. Such functions are T-periodic in the time variable. In the following we will denote by $x \in \mathbb{R}^N$ the vector $x = (x_1, \dots, x_N)$.

In the proof of our main result, we apply the *higher dimensional* Poincaré–Birkhoff theorem proved by Fonda and Ureña in [18]. In [19] a simplified version of such a theorem is presented for smooth functions, see also [11]. Recently, some applications of the results of Fonda and Ureña to systems of ordinary differential equations have been presented in [8,15,16].

We underline that the main results of [18] have been obtained without assuming the uniqueness of solutions to the Cauchy problems. For this reason we can drop Lipschitz regularity assumptions on the function g, which we will assume to be merely continuous. A more general framework can be treated introducing Carathéodory type of regularity, cf. Remark 1.7 below.

We collect here for convenience all the assumptions of Theorem 1.3. At first, notice that Poincaré–Birkhoff theorem applies for area-preserving maps, so that we ask a Hamiltonian structure for system (S). Hence, we assume the following.

(H0) There is a continuous function $H : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, continuously differentiable in x, satisfying

$$g_i(t, x) = \frac{\partial}{\partial x_i} H(t, x),$$

for every index i.

The previous assumption permits to rewrite system (S) as

$$x'' + \nabla_x H(t, x) = sw(t). \tag{S'}$$

In addition, we assume that the following set of hypotheses holds for every index $i = 1, \ldots, N$. Here and in the sequel, given $x \in \mathbb{R}^N$ we denote by $\check{x}^i$ the vector $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$, obtained removing the i-th component.

($\mathrm{H1}^i$) There are positive numbers ν_1^i, ν_2^i such that

$$\nu_1^i \leq \liminf_{x_i \to -\infty} \frac{g_i(t,x)}{x_i} \leq \limsup_{x_i \to -\infty} \frac{g_i(t,x)}{x_i} \leq \nu_2^i$$

uniformly for every $t \in [0, T]$ and $\check{x}^i \in \mathbb{R}^{N-1}$.

($\mathrm{H2}^i$) There is a function $a_i(t)$ such that

$$\lim_{x_i \to +\infty} \frac{g_i(t,x)}{x_i} = a_i(t)$$

uniformly for every $t \in [0, T]$ and $\check{x}^i \in \mathbb{R}^{N-1}$.

($\mathrm{H3}^i$) There are positive numbers μ_1^i, μ_2^i and an integer $m_i \geq 0$ such that, for every $t \in [0, T]$,

$$\left(\frac{2\pi m_i}{T}\right)^2 < \mu_1^i \leq a_i(t) \leq \mu_2^i < \left(\frac{2\pi(m_i+1)}{T}\right)^2. \tag{3}$$

Moreover, the only solution of the scalar differential equation

$$\begin{cases} \zeta'' + a_i(t)\zeta = w_i(t), \\ \zeta(0) = \zeta(T), \quad \zeta'(0) = \zeta'(T) \end{cases} \tag{4}$$

is strictly positive.

($\mathrm{H4}^i$) There is an integer $n_i \geq 0$ such that

$$\frac{T}{n_i + 1} < \frac{\pi}{\sqrt{\mu_2^i}} + \frac{\pi}{\sqrt{\nu_2^i}} \leq \frac{\pi}{\sqrt{\mu_1^i}} + \frac{\pi}{\sqrt{\nu_1^i}} < \frac{T}{n_i}. \tag{5}$$

Notice that, by convention, we set $\frac{T}{0} = \infty$ in (5).

Remark 1.2. Assumptions ($\mathrm{H1}^i$) and ($\mathrm{H2}^i$) are fulfilled if $g_i(t,x) = \check{g}_i(t, x_i) + p_i(t,x)$ where $\check{g}_i$ satisfies the inequalities in ($\mathrm{H1}^i$) and ($\mathrm{H2}^i$), and p_i is any bounded function. Assumption ($\mathrm{H3}^i$) holds if $a_i(t) \equiv a_i \in \mathbb{R}$ satisfies (3) and $w_i(t) \simeq 1$.

The assumptions in (3) and (5) are known as *non-resonance conditions*. In particular assumption (3), which is related to (1) in Theorem 1.1, is the typical non-resonance condition for the Hill's equation $x'' + a_i(t)x = e(t)$ and (5), which is related to (2) in Theorem 1.1, is the typical non-resonance condition for asymmetric nonlinearities. It is well-known that a scalar second order differential equation with a nonlinearity satisfying this type of conditions admits at least one periodic solution and the results date back to the pioneering works by Dolph [5], Dancer [7] and Fučík [20,21]. See [13,24] for details. An extension to weakly coupled systems has been recently provided by Fonda and the second author in [14]. We will need [14, Theorem 2.4] in order to prove the following theorem, which is our main result:

Theorem 1.3. *Assume the validity of* (H0) *and* ($H1^i$)–($H4^i$), *for every index* $i = 1, \ldots, N$. *Then, there exists* $s_0 > 0$ *such that, for every* $s \geq s_0$, *the Hamiltonian system* (S) *has at least*

$$1 + \left[(N+1) \prod_{i=1}^{N} |m_i - n_i| \right] \tag{6}$$

periodic solutions.

We state here our result only for positive s for clarity. Let us stress that a corresponding result for negative value of s can be achieved adding two conditions analogous to ($H2^i$)–($H3^i$) concerning the behavior of g at $-\infty$. Moreover we can consider different parameters $s_1, \ldots, s_N$ in each component. We will explain briefly such possibilities in Sect. 4, see in particular Theorem 4.1.

Let us now sketch the structure of the proof of Theorem 1.3. We first prove, for large values of the parameter s, the existence of a *pivot* solution. A crucial property of such a solution is that all its components are positive. Then by a change of coordinates we find a system equivalent to (S) having a *twist-structure*. This allows us to apply the higher dimensional Poincaré–Birkhoff theorem. In particular we find other periodic solutions by estimating the rotation number of the components of every solution to (S), when such components have either *large amplitude* or are *near the components of the pivot solution*. Unfortunately, such a procedure does not allow us to give additional informations on nodal properties of these solutions.

Remark 1.4. Notice that, in the scalar case $N = 1$, Theorem 1.3 leads to the existence of $1 + 2|m_1 - n_1|$ periodic solutions, cf. [9, Theorem 2].

Remark 1.5. In [18], for Hamiltonian system (S′) in $\mathbb{R}^{2N}$, the higher dimensional Poincaré–Birkhoff theorem gives a better result when the Hamiltonian function H is twice continuously differentiable with respect to x and the T-periodic solutions are known to be non-degenerate *a priori*: in this case we find at least 2^N (instead of $N + 1$) T-periodic solutions. Such a condition is not easy to be verified in general; by the way, adding such an assumption we would find $1 + \left[2^N \prod_{i=1}^{N} |m_i - n_i| \right]$ T-periodic solutions.

Remark 1.6. In [10, Theorem 1.1] the corresponding result of [9, Theorem 2] for general Hamiltonian systems in the plane is provided. Following [10] one can generalize Theorem 1.3 to a Hamiltonian system in $\mathbb{R}^{2N} = (\mathbb{R}^2)^N$ studying the behavior of the solutions in every planar component. We do not enter in details to avoid technicalities.

We conclude with the following remark on the possibility of treating Carathéodory functions in (S).

Remark 1.7. In the applications, nonlinearities which are discontinuous in time are sometimes treated. We wish to underline that our main result applies also for nonlinearities having a L^r-Carathéodory regularity (with $r > 1$). In fact, the higher dimensional Poincaré–Birkhoff theorem can be applied also in this setting, cf. [18, Section 8].

The paper is organized as follows: in Sect. 2 we introduce some preliminary lemmas which are necessary in order to prove our main result in Sect. 3. Then in Sect. 4 we present some possible variants and improvements of our main theorem with some examples.

2. Some preliminary lemmas

In this section we provide some preliminary lemmas needed to prove our main theorem. We follow the main ideas of [9] and sometimes we will take advantage of some computations just proved there. We will always implicitly assume the hypotheses of Theorem 1.3 to be satisfied.

The following lemma is a direct consequence of ($H3^i$) and follows easily by the continuation principle. We omit the proof referring to [30, Theorem 2.1] or [9, Lemma 1] for details.

Lemma 2.1. *There are three positive constants ε_0, c_0 and C_0 such that, for every index $i \in \{1, \dots, N\}$, if $\eta, \gamma : [0, T] \to \mathbb{R}$ satisfy $\|\eta\|_\infty \le \varepsilon_0$ and $\|\gamma - a_i\|_\infty \le \varepsilon_0$, then the scalar linear problem*

$$\begin{cases} \zeta'' + \gamma(t)\zeta = w_i(t) + \eta(t), \\ \zeta(0) = \zeta(T), \quad \zeta'(0) = \zeta'(T) \end{cases} \tag{7}$$

has a unique solution ζ such that $c_0 \le \zeta(t) \le C_0$ for every $t \in [0, T]$.

From now on, we will assume without loss of generality that, for all the indexes i,

$$\frac{T}{n_i + 1} < \frac{\pi}{\sqrt{\mu_2^i + \varepsilon_0}} + \frac{\pi}{\sqrt{\nu_2^i + \varepsilon_0}} \le \frac{\pi}{\sqrt{\mu_1^i - \varepsilon_0}} + \frac{\pi}{\sqrt{\nu_1^i - \varepsilon_0}} < \frac{T}{n_i},$$
$$\left(\frac{2\pi m_i}{T}\right)^2 < \mu_1^i - \varepsilon_0 \le \mu_2^i + \varepsilon_0 < \left(\frac{2\pi(m_i + 1)}{T}\right)^2,$$

and $\nu_1^i - \varepsilon_0 > 0$ hold, where ε_0 is given by Lemma 2.1.

Remark 2.2. Let ε_0 be as above. Each component g_i, $i = 1, \dots, N$, can be written as follows, cf. [9, Lemma 2]:

$$g_i(t, x) = \tilde{a}_i(t, x)x_i^+ - b_i(t, x)x_i^- + r_i(t, x),$$

where $\tilde{a}, b, r : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous functions such that, for every $t \in [0, T]$, all $x \in \mathbb{R}^N$ and $i = 1, \dots, N$,

$$a_i(t) - \varepsilon_0 \le \tilde{a}_i(t, x) \le a_i(t) + \varepsilon_0,$$
$$\nu_1^i(t) - \varepsilon_0 \le b_i(t, x) \le \nu_2^i(t) + \varepsilon_0$$

and there is $\tilde{r} \in \mathbb{R}$ such that $|r_i(t, x)| \le \tilde{r}$.

Following [9], we introduce in (S) the variable $z = (z_1, \dots, z_N)$ as $z = x/s$ thus obtaining the equivalent system

$$\begin{cases} z_i'' + \dfrac{g_i(t, sz)}{s} = w_i(t), \quad i = 1, \dots, N, \\ z(0) = z(T), \quad z'(0) = z'(T). \end{cases} \tag{8}$$

Lemma 2.3. *There is a $\bar{s} > 1$ such that, for every $s \geq \bar{s}$, problem (8) has a solution $z = z(s, \cdot)$ whose components satisfy $c_0 \leq z_i(s,t) \leq C_0$ for every $t \in [0,T]$ and $i \in \{1, \dots, N\}$, where c_0, C_0 are the positive constants given by Lemma 2.1.*

Proof. By Remark 2.2, we can write the differential equations in system (8) as

$$z_i'' + \tilde{a}_i(t, sz) z_i^+ - b_i(t, sz) z_i^- = w_i(t) - \frac{r_i(t, sz)}{s}, \quad i = 1, \dots, N. \tag{9}$$

In particular, if $z : \mathbb{R} \to \mathbb{R}^N$, satisfies $z_i > 0$ for every i, then it solves (9) if and only if it solves

$$z_i'' + \tilde{a}_i(t, sz) z_i = w_i(t) - \frac{r_i(t, sz)}{s}, \quad i = 1, \dots, N. \tag{10}$$

Notice that the inequalities

$$\left(\frac{2\pi m_i}{T}\right)^2 < \mu_1^i - \varepsilon_o \leq \tilde{a}_i(t, sz) \leq \mu_2^i + \varepsilon_0 < \left(\frac{2\pi(m_i + 1)}{T}\right)^2 \tag{11}$$

hold for every $t \in [0, T]$, all $s \geq 1$, $z \in \mathbb{R}^N$ and $i \in \{1, \dots, N\}$.

The non-resonance condition (11) permits us to apply successfully [14, Theorem 2.4] (cf. [14, Corollary 5.1]) yielding to the existence of a T-periodic solution $z = z(s, \cdot)$ for system (10) for any $s \geq 1$.

We show now that, for s sufficiently large, such a periodic solution $z(s, \cdot)$ must have positive components. In particular $z(s, \cdot)$ solves (8), in view of the above equivalence.

We fix a component $i \in \{1, \dots, N\}$ and put $\zeta = z_i(s, \cdot)$. Then, ζ solves the scalar linear equation

$$\zeta'' + \tilde{a}_i(t, sz(s,t))\zeta = w_i(t) - \frac{r_i(t, sz(s,t))}{s}. \tag{12}$$

Let $\bar{s} = \tilde{r}/\varepsilon_0$, then setting $\gamma = \tilde{a}_i(\cdot, sz(s, \cdot))$ and $\eta = \frac{1}{s} r_i(\cdot, sz(s, \cdot))$ we have $\|\gamma - a_i\|_\infty \leq \varepsilon_0$ and $\|\eta\|_1 \leq \varepsilon_0$, for every $s \geq \bar{s}$. Applying Lemma 2.1, for $s \geq \bar{s}$, the scalar equation (12) has a unique T-periodic solution ζ, which is positive. Thus, $z_i(s, \cdot)$ is positive, and the assertion follows. □

In the previous lemma we have proved the existence of the *pivot solution* $z = z(s, \cdot)$ which, in turn, gives the previously mentioned pivot solution $x = x(s, \cdot) = sz(s, \cdot)$ of system (S). Observe that $x(s, \cdot)$ is also positive in every component. We now introduce the variable $y = (y_1, \dots, y_N)$ as $y = z - z(s, \cdot)$. We obtain the system

$$\begin{cases} y_i'' + \tilde{g}_i(s, t, y) = 0, \quad i = 1, \dots, N, \\ y(0) = y(T), \quad y'(0) = y'(T), \end{cases} \tag{13}$$

where, for every index i,

$$\tilde{g}_i(s,t,y) = \frac{g_i(t, s(y + z(s,t))) - g_i(t, sz(s,t))}{s}. \tag{14}$$

In this way, the pivot solution corresponds to the trivial solution $y \equiv 0$ of (13).

Lemma 2.4. *For $i \in \{1,\dots,N\}$, the limit*

$$\lim_{s\to+\infty} \tilde{g}_i(s,t,y) = a_i(t)y_i$$

exists uniformly for every $t \in [0,T]$ and $y \in \mathbb{R}^N$ with $|y_i| \le \frac{1}{2}c_0$.

Proof. Fix $i \in \{1,\dots,N\}$. Being $c_0 \le z_i(s,t) \le C_0$, by (H2^i), we can find for every $\epsilon > 0$ the following estimate for s sufficiently large

$$\begin{aligned}
|\tilde{g}_i(s,t,y) - a_i(t)y_i| &= \left| \frac{g_i(t, s(y + z(s,t))) - g_i(t, sz(s,t))}{s} - a_i(t)y_i \right| \\
&\le \left| \frac{g_i(t, s(y{+}z(s,t))){-}a_i(t)\, s(y_i + z_i(s,t))}{s(y_i + z_i(s,t))} \right| \cdot |y_i{+}z_i(s,t)| \\
&\quad + \left| \frac{g_i(t, sz(s,t)) - a_i(t)\, sz_i(s,t)}{sz_i(s,t)} \right| \cdot |z_i(s,t)| \\
&\le \epsilon(c_0/2 + C_0) + \epsilon C_0,
\end{aligned}$$

for every $t \in [0,T]$ and $y \in \mathbb{R}^N$ with $|y_i| \le \frac{1}{2}c_0$. The assertion follows. □

For $i \in \{1,\dots,N\}$ we set

$$\tilde{a}_i(s,t,y) = \tilde{a}_i(t, s(y + z(s,t))), \quad b_i(s,t,y) = b_i(t, s(y + z(s,t))),$$

and we define

$$r_i(s,t,y) = \tilde{g}_i(s,t,y) - \tilde{a}_i(s,t,y)y_i^+ + b_i(s,t,y)y_i^-.$$

As a straightforward consequence of Remark 2.2 we have

$$a_i(t) - \varepsilon_0 \le \tilde{a}_i(s,t,y) \le a_i(t) + \varepsilon_0, \quad \nu_1^i - \varepsilon_0 \le b_i(s,t,y) \le \nu_2^i + \varepsilon_0 \tag{15}$$

for every $t \in [0,T]$, all $y \in \mathbb{R}^N$ and every index i. We can find also an upper bound on $r_i(s,\cdot,\cdot)$:

$$\begin{aligned}
|r_i(s,t,y)| = |\tilde{a}_i(s,t,y)[(y_i + z_i(s,t))^+ {-} y_i^+] - b_i(s,t,y)[(y_i + z_i(s,t))^- {-} y_i^-] \\
-\tilde{a}_i(t, sz(s,t))z_i(s,t) + \frac{1}{s}[r_i(t, s(y + z(s,t))) - r_i(t, sz(s,t))]| \\
\le (\mu_i + 2\varepsilon_0)C_0 + (\nu_i + 2\varepsilon_0)C_0 + (\mu_i + 2\varepsilon_0)C_0 + 2\tilde{r} \le \tilde{C},
\end{aligned}$$

independently of $s \ge 1$, for every $t \in [0,T]$, all $y \in \mathbb{R}^N$ and $i \in \{1,\dots,N\}$ for a suitable $\tilde{C} \in \mathbb{R}$. In particular, for $s \ge 1$ we have, for every index i,

$$\|\tilde{g}_i(s,t,y)\|_\infty \le C\|y\|_\infty + \tilde{C}, \tag{16}$$

for every $t \in [0,T]$ and all $y \in \mathbb{R}^N$, where $C = \max_{i=1,\dots,N}\{\mu_2^i, \nu_2^i\} + \varepsilon_0$.

Let us consider, for every $(\alpha,\beta) \in \mathbb{R}^{2N}$, the Cauchy problem

$$\begin{cases} y_i'' + \tilde{g}_i(s,t,y) = 0, \quad i = 1,\dots,N, \\ y(0) = \alpha, \quad y'(0) = \beta. \end{cases} \tag{17}$$

Observe in particular that, by (16), all the solutions of (17) are globally defined, even if the uniqueness of the solutions is not guaranteed. Given a solution of (17) such that, for some i, we have $(y_i(t), y_i'(t)) \neq (0,0)$ for every $t \in [0,T]$, we can introduce polar coordinates in the i-th component

$$(y_i(t), y_i'(t)) = \rho_i(t)(\cos\theta_i(t), \sin\theta_i(t)),$$

thus obtaining the following equations for the radial and angular velocities of the i-th component

$$\begin{aligned} \rho_i' &= \rho_i \cos\theta_i \sin\theta_i - \tilde{g}_i(s,t,\rho_i\cos\theta_i, \check{y}^i)\sin\theta_i, \\ -\theta_i' &= \tilde{g}_i(s,t,\rho_i\cos\theta_i,\check{y}^i)\cos\theta_i/\rho_i + \sin^2\theta_i. \end{aligned} \tag{18}$$

Lemma 2.5. *It is possible to find δ, R_1, s_0, with $0 < \delta < R_1 < \frac{1}{2}c_0$ and $s_0 > \bar{s}$, where c_0 and $\bar{s}$ are given by Lemma 2.3, with the following property: for every $s \geq s_0$, if y is a solution of (17), with $(\alpha,\beta) \in \mathbb{R}^{2N}$ satisfying $\alpha_i^2 + \beta_i^2 = R_1^2$ for a certain index i, then one has $\delta \leq \rho_i(t) \leq \frac{1}{2}c_0$ for every $t \in [0,T]$.*

Proof. Set $R_1 = \frac{1}{8}c_0\,\mathrm{e}^{-(1+\|a\|_\infty)T} < \frac{1}{2}c_0$, $\delta = \frac{1}{4}R_1\mathrm{e}^{-(1+\|a\|_\infty)T}$ and $\epsilon \leq R_1/T$. Consider (α,β) as in the statement. Suppose that there exists $\bar{t} \in [0,T]$ such that $\rho_i(\bar{t}) = \frac{1}{2}c_0$ and $\rho_i(t) < \frac{1}{2}c_0$ for every $t \in [0,\bar{t})$. By Lemma 2.4 we can find $s_\epsilon > \bar{s}$ such that $|\tilde{g}_i(s,t,y) - a_i(t)y_i| \leq \epsilon$ for every $s > s_\epsilon$, $t \in [0,T]$ and $y \in \mathbb{R}^N$ with $|y_i| \leq \frac{1}{2}c_0$. By (18) we find $|\rho_i'| \leq (1 + \|a\|_\infty)\rho_i + \epsilon$ so that by a Gronwall argument we have $\rho_i(t) \leq (R_1 + \epsilon\bar{t})\mathrm{e}^{(1+\|a\|_\infty)t}$ thus giving $\rho_i(\bar{t}) \leq 2R_1\mathrm{e}^{(1+\|a\|_\infty)T} \leq \frac{1}{4}c_0$. We get a contradiction. Arguing similarly as above we can also prove that $\rho_i(t) > \delta$ for every $t \in [0,T]$. $\square$

As a consequence of the previous lemma, it follows that all the solutions of (17), such that $\alpha_i^2 + \beta_i^2 \geq R_1^2$ for every index i, can be parametrized in polar coordinates (ρ_i, θ_i) in every component. For such solutions it will be crucial to estimate the *rotation number* of the i-th component.

Let us recall that, given a solution y to (17) such that, for some i, we have $(y_i(t), y_i'(t)) \neq (0,0)$ for every $t \in [0,T]$, the rotation number of its i-th component y_i is given by

$$\mathrm{rot}^i(y) = -\frac{\theta_i(T) - \theta_i(0)}{2\pi}.$$

We now provide some estimates on the rotation number. Let $(\alpha,\beta) \in \mathbb{R}^{2N}$ be such that $\alpha_i^2 + \beta_i^2 = R_1^2$ for a certain index i. Then, for every $s \geq s_0$ Lemma 2.5 above guarantees that the solution y^s to (17) can be expressed in polar coordinates (ρ_i^s, θ_i^s) in the i-th component. By Lemma 2.4, using the estimate of the angular velocity in (18), we have

$$\lim_{s\to\infty} \theta_i^s(t) = \vartheta_i(t),$$

where ϑ_i satisfies $-\vartheta_i'(t) = a_i(t)\cos^2\vartheta_i(t) + \sin^2\vartheta_i(t)$ and $\vartheta_i(0) = \theta_i^s(0)$. By a standard *non-resonance argument*, assumption (H3^i) provides (cf. e.g. [7, 9, 12–14] and the references therein)

$$m_i < -\frac{\vartheta_i(T) - \vartheta_i(0)}{2\pi} < m_i + 1,$$

thus giving $m_i < \mathrm{rot}^i(y^s) < m_i + 1$ for s sufficiently large. We have thus proved, enlarging s_0 if necessary, the following.

Lemma 2.6. *For every $s \geq s_0$, any solution to the Cauchy problem (17) associated to an initial datum (α, β) such that $\alpha_i^2 + \beta_i^2 = R_1^2$ for a certain index i, satisfies $m_i < \mathrm{rot}^i(y) < m_i + 1$.*

Arguing similarly as in the proof of Lemma 2.5 we can prove the following.

Lemma 2.7. *For every $\chi > 0$ there exists $R_\chi > \chi$ with the following property: every solution of (17) with $s \geq 1$ and (α, β) such that $\alpha_i^2 + \beta_i^2 = R_\chi$, for a certain index i, satisfies $\rho_i(t) > \chi$, for every $t \in [0, T]$.*

Proof. The proof is similar to the one of Lemma 2.5. In this case, by (16), we get the estimate $|\rho_i'| \leq (1 + C)\rho_i + \tilde{C}$, then again by a Gronwall argument the proof easily follows. □

Now, by a standard non-resonance argument, we can prove that there exists $\chi_0 > 0$ large enough to guarantee that every solution to (17), such that $\rho_i(t) > \chi_0$ for every $t \in [0, T]$ and a certain index i, satisfies $n_i < \mathrm{rot}^i(y) < n_i + 1$.

In fact, if we fix an index i, by (18) we have $-\theta_i' = \Theta_i(s, t, \rho_i, \theta_i, \check{y}_i)$ with

$$\Theta_i(s, t, \varrho, \vartheta, \check{y}_i) = \left(\tilde{a}_i(s, t, \upsilon)(\cos\vartheta)^+ - b_i(s, t, \upsilon)(\cos\vartheta)^- + \frac{r_i(s, t, \upsilon)}{\varrho} \right) \cos\vartheta + \sin^2\vartheta,$$

where $\check{y}_i = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N)$
and $\upsilon = (y_1, \ldots, y_{i-1}, \varrho\cos\vartheta, y_{i+1}, \ldots, y_N)$.

Following, for example, the main ideas of the proof of [14, Theorem 4.1] (see also [13, Theorem 3.10], or [9] for an alternative proof) we define

$$\psi_{i,1}(\vartheta) = \begin{cases} (\mu_1^i - \epsilon)\cos^2\vartheta + \sin^2\vartheta & \vartheta \in [-\pi/2, \pi/2], \\ (\nu_1^i - \epsilon)\cos^2\vartheta + \sin^2\vartheta & \vartheta \in [\pi/2, 3\pi/2], \end{cases}$$

$$\psi_{i,2}(\vartheta) = \begin{cases} (\mu_2^i + \epsilon)\cos^2\vartheta + \sin^2\vartheta & \vartheta \in [-\pi/2, \pi/2], \\ (\nu_2^i + \epsilon)\cos^2\vartheta + \sin^2\vartheta & \vartheta \in [\pi/2, 3\pi/2]. \end{cases}$$

By (15), we have

$$\psi_{i,1}(\vartheta) \leq \liminf_{\varrho\to\infty} \Theta_i(s, t, \varrho, \vartheta, \check{y}_i) \leq \limsup_{\varrho\to\infty} \Theta_i(s, t, \varrho, \vartheta, \check{y}_i) \leq \psi_{i,2}(\vartheta),$$

uniformly in $t \in [0, T]$, $s \geq 1$, $\check{y}_i \in \mathbb{R}^{N-1}$. Then, by a simple computation we get, by ($\mathrm{H4}^i$), $n_i < \mathrm{rot}^i(y) < n_i + 1$ if $\rho_i(t) > \chi_0$ for every $t \in [0, T]$, with χ_0 large enough. As an immediate consequence, setting $R_2 = R_{\chi_0}$, given by Lemma 2.7, we have the following lemma.

Lemma 2.8. *For every $s > 1$, there exists $R_2 > 0$ such that all the solutions to the Cauchy problem (17) associated to the initial data (α, β) with $\alpha_i^2 + \beta_i^2 = R_2^2$ for a certain index i, satisfy $n_i < \mathrm{rot}^i(y) < n_i + 1$.*

3. Proof of the main result

The proof of Theorem 1.3 follows from the application of Theorem 3.1 below. This is a simplified version of a higher dimensional version of the Poincaré–Birkhoff theorem, recently obtained by Fonda and Ureña. We refer in particular to [18, Theorem 1.2].

Let $R_2 > R_1 > 0$ be given. We denote by $\Omega = \left(\overline{B}_{R_2} \setminus B_{R_1}\right)^N$ the N-annulus in $\mathbb{R}^{2N}$, where B_r is the ball in $\mathbb{R}^2$ of radius r, centered at the origin.

Theorem 3.1. *Assume that every solution y of the Cauchy problem (17), departing from $(\alpha, \beta) \in \partial\Omega$, is defined on $[0, T]$ and, using its polar coordinates, satisfies*

$$\rho_i(t) > 0, \quad \textit{for every } t \in [0, T] \textit{ and } i = 1, \dots, N. \tag{19}$$

Assume moreover that there are positive integers $l_1, \dots, l_N$ such that, for each index i,

$$\mathrm{rot}^i(y) < l_i \textit{ if } \rho_i(0) = R_1, \quad \textit{and} \quad \mathrm{rot}^i(y) > l_i \textit{ if } \rho_i(0) = R_2. \tag{20}$$

Then, the problem (13) has at least $N+1$ distinct T-periodic solutions y, with $y(0) \in \Omega$, such that $\mathrm{rot}^i(y) = l_i$, for every $i = 1, \dots, N$.

Proof of Theorem 1.3. If $m_i = n_i$ for a certain index i then Theorem 1.3 easily follows: the *pivot* solution is the only required solution, cf. (6). Hence, we suppose $m_i \neq n_i$ for every index i. In order to prove the existence of the required number of solutions, we apply many times Theorem 3.1 by choosing different values of $l_1, \dots, l_N$ satisfying (20). For every $s \geq s_0$, Lemma 2.5 ensures the validity of the non-vanishing condition (19), while Lemmas 2.6 and 2.8 give the validity of (20) in the following way: we can choose for every index i an integer $l_i \in L_i = \{m_i + 1, \dots, n_i\}$ if $m_i < n_i$, or $l_i \in L_i = \{n_i + 1, \dots, m_i\}$ if $m_i > n_i$, where m_i and n_i are provided respectively by ($\mathrm{H3}^i$) and ($\mathrm{H4}^i$). The number of possible choices of the values $l_1, \dots, l_N$ is given by the number of elements of $L = L_1 \times \cdots \times L_N$ which is $\prod_{i=1}^N |m_i - n_i|$. Hence, we can apply Theorem 3.1 for every element $(l_1, \dots, l_N) \in L$, so that we obtain $(N+1)\prod_{i=1}^N |m_i - n_i|$ periodic solutions, which, together to the *pivot* solution, provide the required number of periodic solutions (6). Theorem 1.3 is thus proved. □

4. Further results and applications

Following [6,9] we introduce the change of coordinates $\hat{x}_i = -x_i$, thus obtaining the following assumptions specular to ($\mathrm{H1}^i$)–($\mathrm{H4}^i$):

($\mathrm{J1}^i$) There are positive numbers μ_1^i, μ_2^i such that

$$\mu_1^i \leq \liminf_{x_i \to +\infty} \frac{g_i(t, x)}{x_i} \leq \limsup_{x_i \to +\infty} \frac{g_i(t, x)}{x_i} \leq \mu_2^i$$

uniformly for every $t \in [0, T]$ and $\check{x}^i \in \mathbb{R}^{N-1}$.

(J2^i) There is a function $b_i(t)$ such that

$$\lim_{x_i \to -\infty} \frac{g_i(t,x)}{x_i} = b_i(t)$$

uniformly for every $t \in [0,T]$ and $\check{x}^i \in \mathbb{R}^{N-1}$.
(J3^i) There are positive numbers ν_1^i, ν_2^i and an integer $m_i \geq 0$ such that, for every $t \in [0,T]$,

$$\left(\frac{2\pi m_i}{T}\right)^2 < \nu_1^i \leq b_i(t) \leq \nu_2^i < \left(\frac{2\pi(m_i+1)}{T}\right)^2. \tag{21}$$

Moreover, the only solution of the scalar differential equation

$$\begin{cases} \zeta'' + b_i(t)\zeta = w_i(t), \\ \zeta(0) = \zeta(T), \quad \zeta'(0) = \zeta'(T) \end{cases} \tag{22}$$

is strictly positive.
(J4^i) There is an integer $n_i \geq 0$ such that

$$\frac{T}{n_i+1} < \frac{\pi}{\sqrt{\mu_2^i}} + \frac{\pi}{\sqrt{\nu_2^i}} \leq \frac{\pi}{\sqrt{\mu_1^i}} + \frac{\pi}{\sqrt{\nu_1^i}} < \frac{T}{n_i}. \tag{23}$$

Replacing assumptions (H1^i)–(H4^i) with (J1^i)–(J4^i) in Theorem 1.3 we obtain a multiplicity result of periodic solutions for negative values of the parameter s.

Moreover, we stress that the proofs of the lemmas in Sect. 2 work by components so that the statements of such lemmas can be relaxed: e.g., in Lemma 2.3 we can find positive $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_N$ such that the conclusion follows for every $s_1, s_2, \ldots, s_N$ satisfying $s_i \geq \bar{s}_i$ for every i. Similarly, instead of the values δ, R_1, s_0 introduced in Lemma 2.5 one can find for every index i different values δ^i, R_1^i, s_0^i. Again a similar remark is valid for Lemma 2.7. Consequently, it is possible to apply a slightly more general version of Theorem 3.1, that is the higher dimensional Poincaré–Birkhoff theorem on N-annuli of the type $\prod_{i=1}^N (\overline{B}_{R_2^i} \setminus B_{R_1^i})$. We have chosen to not enter in such details in the previous sections for the clarity of the proofs. We observe that, for a general nonlinearity, it might be difficult to evaluate all these constants: we can only guarantee their existence.

Taking into account all the previous remarks we can state a slightly more general version of our main theorem, which can be proved along the line of Theorem 1.3.

Theorem 4.1. *Assume* (H0) *and, for every index* $i = 1, \ldots, N$, *suppose that either conditions* (H1^i)–(H4^i) *or* (J1^i)–(J4^i) *hold. Then, there exists* $\boldsymbol{s_0} = (s_0^1, \ldots, s_0^N) \in \mathbb{R}^N$ *with* $s_0^i > 0$ *for every index* i, *such that for every* $\boldsymbol{s} = (s^1, \ldots, s^N) \in \mathbb{R}^N$, *satisfying either* $s^j \geq s_0^j$ *if we have supposed* (H1^j)–(H4^j)

or $s^j \leq -s_0^j$ *if we have supposed* $(\mathrm{J}1^j)$–$(\mathrm{J}4^j)$*, the Hamiltonian system*

$$\begin{cases} x_1'' + g_1(t, x_1, \ldots, x_N) = s_1 w_1(t), \\ x_2'' + g_2(t, x_1, \ldots, x_N) = s_2 w_2(t), \\ \qquad \vdots \\ x_N'' + g_N(t, x_1, \ldots, x_N) = s_N w_N(t), \end{cases} \tag{24}$$

has at least

$$1 + \left[(N+1) \prod_{i=1}^{N} |m_i - n_i| \right]$$

periodic solutions.

Observe that, in this framework, the pivot solution is positive for those components satisfying $(\mathrm{H}1^j)$–$(\mathrm{H}4^j)$, negative if $(\mathrm{J}1^j)$–$(\mathrm{J}4^j)$ holds for them.

Finally, we can also consider nonlinearities satisfying both $(\mathrm{J}1^i)$–$(\mathrm{J}4^i)$ and $(\mathrm{H}1^i)$–$(\mathrm{H}4^i)$, with different constants m_i and n_i, thus permitting us to apply many times Theorem 4.1. Let us give an example of application.

Example 4.2. Consider the 2π-periodic system

$$\begin{cases} x_1'' + 20x_1^+ - 200x_1^- + \cos(t + x_1 + x_2) = s_1, \\ x_2'' + 50x_2^+ - 500x_2^- + \cos(t + x_1 + x_2) = s_2. \end{cases}$$

Notice that the system is already written in the form given by Remark 2.2. Let us show the existence of $s_0 > 0$ such that there exist at least

19 2π-periodic solutions for $s_1 > s_0$ and $s_2 > s_0$,
73 2π-periodic solutions for $s_1 > s_0$ and $s_2 < -s_0$,
73 2π-periodic solutions for $s_1 < -s_0$ and $s_2 > s_0$,
289 2π-periodic solutions for $s_1 < -s_0$ and $s_2 < -s_0$.

In fact, in the notation of the previous sections, we have $a_1(t) \equiv 20$, $b_1(t) \equiv 200$, $a_2(t) \equiv 50$, $b_2(t) \equiv 500$. Hence,

$$\begin{aligned} (H1^1)–(H4^1) &\text{ holds with } m_1 = 4, n_1 = 6, |m_1 - n_1| = 2, \\ (J1^1)–(J4^1) &\text{ holds with } m_1 = 14, n_1 = 6, |m_1 - n_1| = 8, \\ (H1^2)–(H4^2) &\text{ holds with } m_2 = 7, n_2 = 10, |m_2 - n_2| = 3, \\ (J1^2)–(J4^2) &\text{ holds with } m_2 = 22, n_2 = 10, |m_2 - n_2| = 12, \end{aligned}$$

so that Theorem 4.1 applies four times giving

$$\begin{aligned} &(H1^1)–(H4^1), (H1^2) - (H4^2) \Rightarrow 1 + 3 \cdot 2 \cdot 3 = 19 \text{ solutions}, \\ &(H1^1)–(H4^1), (J1^2) - (J4^2) \Rightarrow 1 + 3 \cdot 2 \cdot 12 = 73 \text{ solutions}, \\ &(J1^1)–(J4^1), (H1^2) - (H4^2) \Rightarrow 1 + 3 \cdot 8 \cdot 3 = 73 \text{ solutions}, \\ &(J1^1)–(J4^1), (J1^2) - (J4^2) \Rightarrow 1 + 3 \cdot 8 \cdot 12 = 289 \text{ solutions}. \end{aligned}$$

Notice that we can consider other bounded functions instead of $\cos(t+x_1+x_2)$, but preserving the Hamiltonian structure of the system, cf. Example 4.4 below.

We are going now to discuss the number of solutions provided by Theorem 1.3. For a system which is *totally uncoupled*, that is, $g_i(t,x) = g_i(t,x_i)$ for every index i, we expect to find $\prod_{i=1}^N (1+2d_i)$ periodic solutions, where $d_i = |m_i - n_i|$, simply applying N times the corresponding scalar result, e.g. Theorem 1.1, see also [9,23]. Now, by the distributive property we have

$$\prod_{i=1}^{N}(1+2d_i) = \sum_{\sigma\in\{0,1\}^N} \prod_{i=1}^{N}(2d_i)^{\sigma_i} = \sum_{\sigma\in\{0,1\}^N} 2^{\ell(\sigma)} \prod_{i=1}^{N} d_i^{\sigma_i}, \tag{25}$$

where, given $\sigma \in \{0,1\}^N$, we set $\ell(\sigma) = \sum_{i=1}^N \sigma_i$. Notice in particular that the choice $\sigma = (0,\dots,0)$ corresponds to $2^{\ell(\sigma)} \prod_{i=1}^N d_i^{\sigma_i} = 1$, while the choice $\sigma = (1,\dots,1)$ gives $2^{\ell(\sigma)} \prod_{i=1}^N d_i^{\sigma_i} = 2^N \prod_{i=1}^N |m_i - n_i|$. Hence, Theorem 1.3 provides the number of periodic solutions corresponding to these two choices of $\sigma \in \{0,1\}^N$, if they are *a priori* known to be non-degenerate, cf. Remark 1.5. Under our assumptions (H1^i)–(H4^i), we cannot obtain necessarily a better result because we do not have, in (13), for every index i,

$$\tilde{g}_i(s,t,y) = 0, \quad \begin{array}{l}\text{for every } s \geq s_0 \text{ and } t \in [0,T], \\ \text{for every } y \in \mathbb{R}^N \text{ such that } y_i = 0.\end{array} \tag{26}$$

For pure academic purpose let us assume the validity of (26), that is an "extremely weak coupling condition". Let us show that, following the procedure adopted in [15], we can recover all the expected periodic solutions. In fact, for every $\sigma \in \{0,1\}^N$, we can consider the system of equations

$$\begin{cases} y_i'' + \tilde{g}_i(s,t,y) = 0, & \text{if } \sigma_i = 1, \\ y_i \equiv 0, & \text{if } \sigma_i = 0, \end{cases} \qquad i = 1,\dots,N; \tag{27}$$

which is essentially a $\ell(\sigma)$-dimensional system. Notice that a T-periodic solution of (27) is a solution of (13) too, and so it corresponds to a T-periodic solution of (S) such that its i-th component coincides with the i-th component of the pivot solution, for every index satisfying $\sigma_i = 0$. Fix $\sigma \in \{0,1\}^N$: repeating the reasoning of the proof of Theorem 1.3, we can choose, for every index i, a value $l_i \in L_i$, where now we set $L_i = \{0\}$ if $\sigma_i = 0$. So, $L = L_1 \times \cdots \times L_N$ has $\prod_{i=1}^N d_i^{\sigma_i}$ elements (where $0^0 = 1$ by convention). The application of Theorem 3.1 in dimension $\ell(\sigma)$ gives the existence of $2^{\ell(\sigma)}$ periodic solutions of (27)—*if they are* a priori *known to be non-degenerate*—(in fact we have to consider only nontrivial coordinates in order to determine the rotation number). Hence we find, for every $\sigma \in \{0,1\}^N$, at least $2^{\ell(\sigma)} \prod_{i=1}^N d_i^{\sigma_i}$ periodic solutions.

Summing up we have the following result.

Corollary 4.3. *Let the assumptions of Theorem 1.3 hold. Assume moreover (26) for every index i, and that all the periodic solutions of (S) are a priori known to be non-degenerate. Then the system (S) admits the number (25) of periodic solutions expected for a totally uncoupled system.*

We conclude with an example of a weakly coupled system satisfying (26) for some particular values of the parameter s. In correspondence of such values one can find more periodic solutions.

Example 4.4. Consider the 2π-periodic Hamiltonian system

$$\begin{cases} x_1'' + 20x_1^+ - 200x_1^- + 20\sin(20x_1)\cos(50x_2)p(t) = s_1, \\ x_2'' + 50x_2^+ - 500x_2^- + 50\cos(20x_1)\sin(50x_2)p(t) = s_2. \end{cases}$$

In this system, we have different parameters s_1 and s_2. In such a situation the change of coordinates $z = x/s$ is replaced by $z_1 = x_1/|s_1|$ and $z_2 = x_2/|s_2|$. In particular condition (26) has to be rewritten as

$$\tilde{g}_i(s_1, s_2, t, y_1, y_2) = 0, \quad \begin{array}{l} \text{for every } |s_1| \geq s_0,\ |s_2| \geq s_0 \text{ and } t \in [0, T], \\ \text{for every } y \in \mathbb{R}^N \text{ such that } y_i = 0, \end{array} \tag{28}$$

where (14) is replaced by

$$\begin{aligned} \tilde{g}_i(s_1, s_2, t, y_1, y_2) \ :=\ & \frac{1}{|s_i|}[g_i(t, |s_1|(y_1 + z_1(s_1, s_2, t)), |s_2|(y_2 + z_2(s_1, s_2, t))) \\ & - g_i(t, |s_1|z_1(s_1, s_2, t), |s_2|z_2(s_1, s_2, t))]. \end{aligned} \tag{29}$$

If we fix $s_1, s_2 \in \pi\mathbb{Z}$, then there exists a constant solution (x_1, x_2) where $x_1 = s_1/20$ if $s_1 > 0$ or $x_1 = s_1/200$ if $s_1 < 0$, and $x_2 = s_2/50$ if $s_2 > 0$ or $x_2 = s_2/500$ if $s_2 < 0$. Let us provide explicitly the computation only for the case $s_1, s_2 > 0$: by (29), we obtain

$$\begin{aligned} \tilde{g}_1(s_1, s_2, t, y_1, y_2) = \frac{1}{s_1}\Bigg[& 20\left(s_1y_1 + \frac{s_1}{20}\right)^+ - 200\left(s_1y_1 + \frac{s_1}{20}\right)^- \\ & + 20\sin\left(20\left(s_1y_1 + \frac{s_1}{20}\right)\right)\cos\left(50\left(s_2y_2 + \frac{s_2}{50}\right)\right)p(t) \\ & - 20\left(\frac{s_1}{20}\right) - 20\sin\left(20\left(\frac{s_1}{20}\right)\right)\cos\left(50\left(\frac{s_2}{50}\right)\right)p(t)\Bigg]. \end{aligned}$$

Hence we have $\tilde{g}_1(k\pi, s_2, t, 0, y_2) = 0$ for every $k > 0$, $s_2 > 0$, $t \in [0, 2\pi]$ and $y_2 \in \mathbb{R}$. Arguing similarly we can compute that $\tilde{g}_1(k\pi, s_2, t, 0, y_2) = 0$ for every $k \in \mathbb{Z}$, $s_2 \in \mathbb{R}$, $t \in [0, 2\pi]$ and $y_2 \in \mathbb{R}$. Similarly we get $\tilde{g}_2(s_1, k\pi, t, y_1, 0) = 0$ for every $k \in \mathbb{Z}$, $s_1 \in \mathbb{R}$, $t \in [0, 2\pi]$ and $y_1 \in \mathbb{R}$. Hence, (28) holds if we require $s_1, s_2 \in \pi\mathbb{Z}$. In correspondence of such values we get a larger number of periodic solutions:

$$\begin{aligned} & \sum_{\sigma \in \{0,1\}^2} (\ell(\sigma) + 1)|m_1 - n_1|^{\sigma_1} |m_2 - n_2|^{\sigma_2} \\ & = 1 + 2|m_1 - n_1| + 2|m_2 - n_2| + 3|m_1 - n_1||m_2 - n_2| \end{aligned}$$

(we cannot replace $(\ell(\sigma) + 1)$ with $2^{\ell(\sigma)}$, as in (25), because we do not know if such solutions are non-degenerate). So, arguing as in Example 4.2, there exists $s_0 > 0$ large enough such that, taking $s_1, s_2 \in \pi\mathbb{Z}$, we have the existence of at least

$29 = 1+4+6+18$ 2π-periodic solutions if $s_1 > s_0$ and $s_2 > s_0$,
$101 = 1+4+24+72$ 2π-periodic solutions if $s_1 > s_0$ and $s_2 < -s_0$,
$95 = 1+16+6+72$ 2π-periodic solutions if $s_1 < -s_0$ and $s_2 > s_0$,
$329 = 1+16+24+288$ 2π-periodic solutions if $s_1 < -s_0$ and $s_2 < -s_0$.

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References

[1] Andronov, A.A., Vitt, A.A., Khaikin, S.E.: Theory of Oscillators, Reprint of the 1966 Translation. Dover Publications, New York (1987)

[2] Arioli, G., Ruf, B.: Periodic solutions for a system of forced and nonlinearly coupled oscillators with applications to electrical circuits. Dynam. Syst. Appl. **4**, 87–102 (1995)

[3] Binding, P.A., Rynne, B.P.: Half-eigenvalues of periodic Sturm–Liouville problems. J. Differ. Equ. **206**, 280–305 (2004)

[4] Chicone, C.: Periodic solutions of coupled oscillators near resonance. SIAM J. Math. Anal. **26**, 1257–1283 (1995)

[5] Dancer, E.N.: Boundary-value problems for weakly nonlinear ordinary differential equations. Bull. Aust. Math. Soc. **15**, 321–328 (1976)

[6] Del Pino, M.A., Manásevich, R.F., Murua, A.: On the number of 2π-periodic solutions for $u + g(u) = s(1 + h(t))$ using the Poincaré–Birkhoff theorem. J. Differ. Equ. **95**, 240–258 (1992)

[7] Dolph, C.L.: Nonlinear integral equations of the Hammerstein type. Trans. Am. Math. Soc. **66**, 289–307 (1949)

[8] Fonda, A., Garrione, M., Gidoni, P.: Periodic perturbations of Hamiltonian systems. Adv. Nonlinear Anal. **5**, 367–382 (2016)

[9] Fonda, A., Ghirardelli, L.: Multiple periodic solutions of scalar second order differential equations. Nonlinear Anal. **72**, 4005–4015 (2010)

[10] Fonda, A., Ghirardelli, L.: Multiple periodic solutions of Hamiltonian systems in the plane. Topol. Methods Nonlinear Anal. **36**, 27–38 (2010)

[11] Fonda, A., Gidoni, P.: An avoiding cones condition for the Poincaré–Birkhoff Theorem. J. Differ. Equ. **262**, 1064–1084 (2017)

[12] Fonda, A., Mawhin, J.: Planar differential systems at resonance. Adv. Differ. Equ. **11**, 1111–1133 (2006)

[13] Fonda, A., Sfecci, A.: A general method for the existence of periodic solutions of differential equations in the plane. J. Differ. Equ. **252**, 1369–1391 (2012)

[14] Fonda, A., Sfecci, A.: Periodic solutions of a system of coupled oscillators with one-sided superlinear retraction forces. Differ. Integral Equ. **25**, 993–1010 (2012)

[15] Fonda, A., Sfecci, A.: Periodic solutions weakly coupled superlinear systems. J. Differ. Equ. **260**, 2150–2162 (2016)

[16] Fonda, A., Sfecci, A.: Multiple periodic solutions of Hamiltonian systems confined in a box. Discrete Contin. Dyn. Syst. **(to appear)**

[17] Fonda, A., Sfecci, A.: On a singular periodic Ambrosetti–Prodi problem. Nonlinear Anal. **149**, 146–155 (2017)

[18] Fonda, A., Ureña, A.J.: A higher dimensional Poincaré–Birkhoff Theorem for Hamiltonian flows. Ann. Inst. Henri Poincaré Anal. Non Linéaire **(to appear)**. doi:10.1016/j.anihpc.2016.04.002

[19] Fonda, A., Ureña, A.J.: A higher-dimensional Poincaré–Birkhoff theorem without monotone twist. C. R. Math. Acad. Sci. Paris **354**, 475–479 (2016)

[20] Fučík, S.: Boundary value problems with jumping nonlinearities. Časopis Pèst. Mat. **101**, 69–87 (1976)

[21] Fučík, S.: Solvability of Nonlinear Equations and Boundary Value Problems. Reidel, Boston (1980)

[22] Gaines, E., Mawhin, J.: Lecture Notes in Mathematics. Coincidence degree and nonlinear differential equations, vol. 568. Springer, Berlin (1977)

[23] Lazer, A.C., McKenna, P.J.: Large scale oscillatory behavior in loaded asymmetric systems. Ann. Inst. H. Poincaré Anal. Non Linéaire **4**, 243–274 (1987)

[24] Mawhin, J.: Resonance and nonlinearity: a survey. Ukranian Math. J. **59**, 197–214 (2007)

[25] Mawhin, J., Willem, M.: Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences, vol. 74. Springer, Berlin (1989)

[26] Mickens, R.E.: A regular perturbation technique for nonlinearly coupled oscillators in resonance. J. Sound Vibr. **81**, 307–310 (1982)

[27] Rebelo, C.: Multiple periodic solutions of second order equations with asymmetric nonlinearities. Discrete Contin. Dyn. Syst. **3**, 25–34 (1997)

[28] Rebelo, C., Zanolin, F.: Multiplicity result for periodic solutions of second order ODEs with asymmetric nonlinearities. Trans. Am. Math. Soc. **348**, 2349–2389 (1996)

[29] Struwe, M.: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Variational methods, 4th edn, p. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Berlin (2008)

[30] Zanini, C., Zanolin, F.: A multiplicity result of periodic solutions for parameter dependent asymmetric non-autonomous equations. Dyn. Cont. Discrete Impuls. Syst. Ser. A Math. Anal. **12**, 343–361 (2005)

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