Generalising the Pari-Mutuel Model

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Abstract. We introduce two models for imprecise probabilities which generalise the Pari-Mutuel Model while retaining its simple structure. Their consistency properties are investigated, as well as their capability of formalising an assessor's different attitudes. It turns out that one model is always coherent, while the other is (occasionally coherent but) generally only 2-coherent, and may elicit a conflicting attitude towards risk.

1 Introduction

The term imprecise probability incorporates a large variety of uncertainty models. While being well suited for assessing imprecise, uncertain or vague beliefs, general models, like coherent lower probabilities, may be less manageable for other purposes, like inference or merely checking their coherence. Some special models are nimbler with respect to these issues. In particular, the Pari-Mutuel Model (PMM) is assessed once a reference precise probability P_0 and a parameter are given, and is guaranteed to be coherent [W, PVZ, MMD]. $P_0(A)$ may also be interpreted as the 'true' probability of event A in the assessor's or agent's mind, but unlike its derived PMM may not correspond, in a behavioural or betting scheme, to the agent's selling or buying prices for A.

Our purpose in this paper is to explore further models that generalise the PMM while retaining its simple features. We also focus on what sort of beliefs they can express. After recalling some preliminary matters in Sect. 2, we lay down the general framework for the new models, i.e., the family of what we term Nearly-Linear (NL) models, in Sect. 3. Then, notable instances of NL models, the Vertical Barrier PMM (VB-PMM) and the Horizontal Barrier PMM (HB-PMM), are investigated in Sects. 4 and 5, respectively. The VB-PMM is coherent (even 2-monotone as a lower probability) and may express, so to say, an agent's greedier attitude than the PMM. The HB-PMM is always at least 2-coherent, but may be coherent subject to certain (restrictive) conditions. Its coherence is characterised in a finite setting: for upper probabilities, it is equivalent to subadditivity. Behaviourally, the HB-PMM elicits an agent's conflicting (and partly irrational) beliefs towards risk. Section 6 contains some comparisons with related models in the literature, while Sect. 7 concludes the paper. Due to space limitations, proofs of the results are omitted.

2 Preliminaries

Let $\underline{P}$ ($\overline{P}$) be a lower (upper) probability, i.e., a map from a set $\mathcal{D}$ of events into $\mathbb{R}$.

$\underline{P}$ is *coherent* on $\mathcal{D}$ iff, $\forall n \in \mathbb{N}$, $\forall A_0, \ldots, A_n \in \mathcal{D}$, $\forall s_0, \ldots, s_n \geq 0$, defining $\underline{G} = \sum_{i=1}^{n} s_i(I_{A_i} - \underline{P}(A_i)) - s_0(I_{A_0} - \underline{P}(A_0))$, it holds that $\max \underline{G} \geq 0$ [W].

$\underline{P}$ is *2-coherent* on $\mathcal{D}$ if either $n = 2$ and $s_0 = 0$ or $n \in \{0, 1\}$ in the above definition [W, PV].

In a behavioural interpretation, $\underline{P}(A)$ ($\overline{P}(A)$) is an agent's supremum buying price (infimum selling price) for A or its indicator I_A [W]. In the gain $\underline{G}$ above, $I_{A_i} - \underline{P}(A_i)$ is the agent's elementary gain from exchanging event A_i at the price $\underline{P}(A_i)$; coherence and 2-coherence require $\max \underline{G} \geq 0$, i.e., that a finite linear combination of bets on events in $\mathcal{D}$ with certain constraints on the coefficients does not produce a sure loss.

In this paper, $\mathcal{D}$ will be the set $\mathcal{A}(\mathbb{P})$ of events logically dependent on a given partition $\mathbb{P}$ (the powerset of $\mathbb{P}$). Given $\underline{P}$ and $\overline{P}$ on $\mathcal{A}(\mathbb{P})$, they are *conjugate* if $\underline{P}(A) = 1 - \overline{P}(\neg A)$, $\forall A \in \mathcal{A}(\mathbb{P})$.

$\overline{P}$ is *coherent*, alternatively *2-coherent* on $\mathcal{A}(\mathbb{P})$, if its conjugate $\underline{P}$ is.

It is necessary for coherence of $\overline{P}$, $\underline{P}$ that [W, Sect. 2.7.4]:

$(c1)$ $\overline{P}(A) + \overline{P}(B) \geq \overline{P}(A \vee B)$ (*subadditivity*),
$(c2)$ if $A \wedge B = \emptyset$, $\quad \underline{P}(A) + \underline{P}(B) \leq \underline{P}(A \vee B)$ (*superadditivity*).

Definition 1. *$\underline{P}_{\mathrm{PMM}} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ is a* Pari-Mutuel lower probability *if $\underline{P}_{\mathrm{PMM}}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}$, $\forall A \in \mathcal{A}(\mathbb{P})$, where P_0 is a given probability and $\delta \in \mathbb{R}^+$. Its conjugate upper probability is $\overline{P}_{\mathrm{PMM}}(A) = \min\{(1+\delta)P_0(A), 1\}$. ($\underline{P}_{\mathrm{PMM}}, \overline{P}_{\mathrm{PMM}}$) constitute a* Pari-Mutuel Model *(PMM).*

$\underline{P}_{\mathrm{PMM}}$ and $\overline{P}_{\mathrm{PMM}}$ are coherent. $\underline{P}_{\mathrm{PMM}}$ is also *2-monotone*: $\forall A, B \in \mathcal{A}(\mathbb{P})$, $\underline{P}_{\mathrm{PMM}}(A \vee B) + \underline{P}_{\mathrm{PMM}}(A \wedge B) \geq \underline{P}_{\mathrm{PMM}}(A) + \underline{P}_{\mathrm{PMM}}(B)$ (while $\overline{P}_{\mathrm{PMM}}$ is 2-alternating) [W, PVZ].

2-coherence is a weaker consistency requirement than coherence (cf. [PV] for details). On $\mathcal{A}(\mathbb{P})$, a still weaker condition is that μ ($\mu = \underline{P}$ or $\mu = \overline{P}$) is a *capacity*: it is requested only that $\forall A, B \in \mathcal{A}(\mathbb{P}) : A \Rightarrow B$, it is $\mu(A) \leq \mu(B)$ (*monotonicity*), and that $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$ (*normalisation*).

3 Nearly-Linear Imprecise Probability Models

The Pari-Mutuel Model and the models we shall investigate in the next sections belong to the broader family of *Nearly-Linear Models*, which we define next.

Let for this $\mu : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be either a lower or an upper probability.

Definition 2. *$\mu : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ is a* Nearly-Linear (NL) *imprecise probability iff $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$ and, given a probability P_0 on $\mathcal{A}(\mathbb{P})$, $a \in \mathbb{R}$, $b > 0$, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}$,*

$$\mu(A) \stackrel{\text{def}}{=} \min\{\max\{bP_0(A) + a, 0\}, 1\} = \max\{\min\{bP_0(A) + a, 1\}, 0\}. \quad (1)$$

Lemma 1. *A NL μ is a capacity.*

If μ is given by Definition 2, we shall say shortly that μ is NL(a, b).

An interesting feature of NL models is that they are *self-conjugate*: if μ is NL(a, b), also its conjugate $\mu^c(A) = 1 - \mu(\neg A)$, $\forall A \in \mathcal{A}(\mathbb{P})$, is NL$(a', b')$:

Proposition 1. *If μ is NL(a, b), then μ^c is NL(a', b'), with*

$$a' = 1 - (a + b), \quad b' = b. \tag{2}$$

Example 1. In the PMM, $\overline{P}_{\mathrm{PMM}}$ is NL$(0, 1 + \delta)$, $\underline{P}_{\mathrm{PMM}}$ is NL$(-\delta, 1 + \delta)$, hence here $a = -\delta < 0$, $b = 1 + \delta > 1$, $a + b = 1$.

A NL model typically gives extreme evaluations to a number of events whose probability P_0 is strictly between 0 and 1. We may keep track of this defining:

Definition 3. *Given μ, NL(a, b), define:*

- $\mathcal{N} = \{A \in \mathcal{A}(\mathbb{P}) : \mu(A) = 0\} = \left\{A \in \mathcal{A}(\mathbb{P}) : P_0(A) \leq -\frac{a}{b}\right\} \cup \{\emptyset\}$,
- $\mathcal{U} = \{A \in \mathcal{A}(\mathbb{P}) : \mu(A) = 1\} = \left\{A \in \mathcal{A}(\mathbb{P}) : P_0(A) \geq \frac{1-a}{b}\right\} \cup \{\Omega\}$,
- $\mathcal{E} = \mathcal{A}(\mathbb{P}) \setminus (\mathcal{N} \cup \mathcal{U}) = \left\{A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\} : -\frac{a}{b} < P_0(A) < \frac{1-a}{b}\right\}$.

$\mathcal{N}$ is the set of *null events* according to μ, $\mathcal{U}$ the set of *universal events*.

If a generic NL measure μ includes a known model μ^* as a special case, μ is interpreted as either a lower or upper probability if μ^* is so. In general, we may apply the *maximum consistency principle*: μ is a lower probability if it determines a model with a higher degree of consistency than interpreting μ as an upper probability.

In the next two sections, we analyse the two major NL submodels.[1] They relax the PMM condition $a + b = 1$ (cf. Example 1) to, respectively, $a + b \leq 1$ and $a + b \geq 1$, while both keeping $a \leq 0$.

4 The Vertical Barrier Pari-Mutuel Model

To introduce our first model, let μ be a NL(a, b) measure such that

$$0 < a + b \leq 1, \quad a \leq 0. \tag{3}$$

Then, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$, $bP_0(A) + a \leq a + b \leq 1$, and μ in (1) simplifies to $\mu(A) = \max\{bP_0(A) + a, 0\}$. ($\mu(\emptyset)$ is also computed with this formula.)

Note that, when $a + b \leq 0$, μ reduces to the vacuous lower probability $\underline{P}_V(A) = 0$, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$. Hence the constraint $a + b > 0$ rules out (only) this case.

Recalling Example 1, when $a + b = 1$ and $a = -\delta < 0$, μ is the lower probability of a PMM (Definition 1).[2]

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[1] It can be shown that a third, less relevant, submodel completes the family of NL models.

[2] When $b = 1$, $a = 0$, μ is a probability. We shall hereafter neglect this subcase.

Putting $a = 0$, $b = \varepsilon < 1$, we obtain the lower probability of the ε-*contamination model* (also termed linear-vacuous mixture in [W]):

$$\underline{P}(A) = \varepsilon P_0(A), \quad A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\} \quad (\underline{P}(\Omega) = 1).$$

Clearly, requiring conditions (3) μ is a lower probability. Its conjugate upper probability is easily obtained using (2). Summing up, we define

Definition 4. *A* Vertical Barrier Pari-Mutuel Model (VB-PMM) *is a NL model where $\underline{P}$ and its conjugate $\overline{P}$ are given by:*

$$\underline{P}(A) = \max\{bP_0(A) + a, 0\}, \quad \forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\} \quad (\underline{P}(\Omega) = 1), \tag{4}$$

$$\overline{P}(A) = \min\{bP_0(A) + c, 1\}, \quad \forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset\} \quad (\overline{P}(\emptyset) = 0), \tag{5}$$

with a, b satisfying (3) and $c = 1 - (a + b) \geq 0$.

A VB-PMM offers very good consistency properties:

Proposition 2. *In a VB-PMM, $\underline{P}$ and $\overline{P}$ are coherent. Further, $\underline{P}$ is 2-monotone, $\overline{P}$ is 2-alternating.*

To justify the name and significance of a VB-PMM, take its upper probability $\overline{P}$, given by (5). Then:

(*i*) $\overline{P}(A) \geq P_0(A), \forall A$. Obvious when $\overline{P}(A) = 1$ or $A = \emptyset$; otherwise, use Definition 4 to get $\overline{P}(A) = bP_0(A) + 1 - (a+b) \geq P_0(A)$ iff $(1-b)P_0(A) \leq 1-(a+b)$. If $1 - b \leq 0$, this inequality holds trivially $(1 - (a + b) \geq 0)$; if $1 - b > 0$, it is equivalent to $P_0(A) \leq \frac{1-b-a}{1-b}$, true because $\frac{1-b-a}{1-b} \geq 1$ for $a \leq 0$;
(*ii*) $\overline{P}(A) \to c \geq 0$ as $P_0(A) \to 0$ (for $P_0(A)$ low enough, $\overline{P}(A) = bP_0(A) + c \to c$);
(*iii*) $\overline{P}(A) = 1$ iff $P_0(A) \geq \frac{1-c}{b} = \frac{b+a}{b}$.

Now compare $\overline{P}$ with its special case $c = 0$, i.e., $a + b = 1$ (and $b > 1$), which specialises $\overline{P}$ into $\overline{P}_{\mathrm{PMM}}(A) = \min\{bP_0(A), 1\}$. In the behavioural interpretation, both (a generic) $\overline{P}$ and $\overline{P}_{\mathrm{PMM}}$ imply that the agent is essentially unwilling to sell events whose reference or 'true' probability P_0 is too high, by (*iii*), and in any case her/his selling price is not less than the 'fair' price P_0, by (*i*). $\overline{P}$ adds a further barrier regarding low probability events: by (*ii*), if $c > 0$ the agent is not willing to sell (too) low probability events for less than c, whilst $\overline{P}_{\mathrm{PMM}}$ enforces no such barrier. We may deduce that, *ceteris paribus*, the $\overline{P}$-agent is, loosely speaking, greedier than the $\overline{P}_{\mathrm{PMM}}$-agent. This can be easily justified in real-world situations: if the agent is a bookmaker or an insurer, for instance, $c > 0$ may take account of the agent's fixed costs in managing any bet/contract.

While c measures the agent's advantage at $P_0 = 0$, b determines how it varies with P_0 growing. In fact, the advantage is unchanged, decreasing or increasing according to whether it is, respectively, $b = 1$, $b < 1$, $b > 1$.

These features of the VB-PMM can be visualised in a $(P_0, \overline{P})$ plot, as in Fig. 1, 1): the VB-PMM additional barrier is the dotted segment on the $\overline{P}$-axis. The interpretation of $\underline{P}$, defined by (4), is similar. It is easy to check that:

(i') $\underline{P}(A) \leq P_0(A)$, $\forall A$;
(ii') $\underline{P}(A) \to a + b \leq 1$ as $P_0(A) \to 1$;
(iii') $\underline{P}(A) = 0$ iff $P_0(A) \leq -\frac{a}{b}$.

Now the agent using $\underline{P}$ acts as a buyer, but by (ii') does not want to pay more than $a + b$ for any event, even those whose probability P_0 is very high. If $a + b < 1$, this amounts to requiring that the maximum gain $\underline{G}_{\text{MAX}}$ from buying any A for $\underline{P}(A)$ (achieved when A occurs) is $1 - (a + b) > 0$. By contrast, $\underline{G}_{\text{MAX}} \to 0$ as $P_0(A) \to 1$ if $a + b = 1$, as in the PMM. Thus, $\underline{P}$ in the typical VB-PMM (i.e., such that $a + b < 1$) introduces an additional barrier, of width $1 - (a + b)$, with respect to $\underline{P}_{\text{PMM}}$: the dotted segment in the $P_0 = 1$ line of Fig. 1, 1).

5 The Horizontal Barrier Pari-Mutuel Model

Let now μ be a NL(a, b) measure, with the conditions

(k) $a + b > 1, \quad 2a + b < 1$.

Note that conditions (k) imply $a < 0$, $b > 1$. It can be shown that

Proposition 3. *μ is a 2-coherent lower probability, whilst it is not a 2-coherent upper probability.*

From Proposition 3, and by the maximum consistency principle stated at the end of Sect. 3, μ is conveniently viewed as a lower probability. We define then:

Definition 5. *A* Horizontal Barrier Pari-Mutuel Model (HB-PMM) *is a NL model where $\underline{P}$ and its conjugate $\overline{P}$ satisfy $\underline{P}(\emptyset) = \overline{P}(\emptyset) = 0$, $\underline{P}(\Omega) = \overline{P}(\Omega) = 1$, $c = 1 - (a + b) < 0$, a, b are as in (k)*[3] *and, for all $A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}$,*

$$\underline{P}(A) = \min\{\max\{bP_0(A) + a, 0\}, 1\}, \tag{6}$$

$$\overline{P}(A) = \max\{\min\{bP_0(A) + c, 1\}, 0\}. \tag{7}$$

Let us discuss the basic features of this model, referring to $\underline{P}$ given by (6). It is easy to check that (in particular, (jj) and (jjj) follow simply from Definition 3):

(j) $\underline{P}(A) > P_0(A)$ iff $1 > P_0(A) > -\frac{a}{b-1}$;
(jj) $\underline{P}(A) = 0$ iff $P_0(A) \leq -\frac{a}{b}$;
(jjj) $\underline{P}(A) = 1$ iff $P_0(A) \geq \frac{1-a}{b}$.

It follows easily from (k) that conditions (j), (jj), (jjj) are not vacuous, i.e., may be satisfied by some events. As for (j), for instance, $-\frac{a}{b-1} < 1$ iff $-a < b - 1$ (by (k)) iff $a + b > 1$, which is true, and it is always $-\frac{a}{b-1} > 0$.

Conditions (j), (jj), (jjj) point out an interesting feature of the HB-PMM: the beliefs it represents may be conflicting and, partly, irrational. In fact, assuming again that P_0 is the 'true' probability for the events in $\mathcal{A}(\mathbb{P})$, by (j) the agent

[3] $a + b > 1$ in (k) could be relaxed to $a + b \geq 1$, thus including the PMM as a special HB-PMM. We left out this case to focus on the 'proper' HB-PMMs.

is willing to buy some events for less, others for more than their probability P_0. In the extreme situations, by (jj) and (jjj), the agent would not buy events whose probability is too low, whilst would certainly buy a high probability event A at the price of 1, gaining from the transaction at most 0 (if A occurs). Thus the agent underestimates the riskiness of a transaction regarding high probability events, but overestimates the risk with low probability events. She/he may be both risk averse and not. Which attitude prevails? In a sense, the prudential one. To see this, note that by (jj) and (jjj) the HB-PMM sets up two horizontal barriers in the $(P_0, \underline{P})$ plane (cf. Fig. 1, 2)). The lower (prudential) barrier is a segment with measure $-\frac{a}{b}$, the upper barrier (in the imprudent area) a segment measuring $1 - \frac{1-a}{b}$, and is narrower: $-\frac{a}{b} > 1 - \frac{1-a}{b}$ iff $2a + b < 1$, true by (k). Similarly, the boundary probability P_0 between the opposite attitudes is set at $-\frac{a}{b-1}$, larger than $\frac{1}{2}$ (by (k)). In this sense, the prudent behaviour prevails.

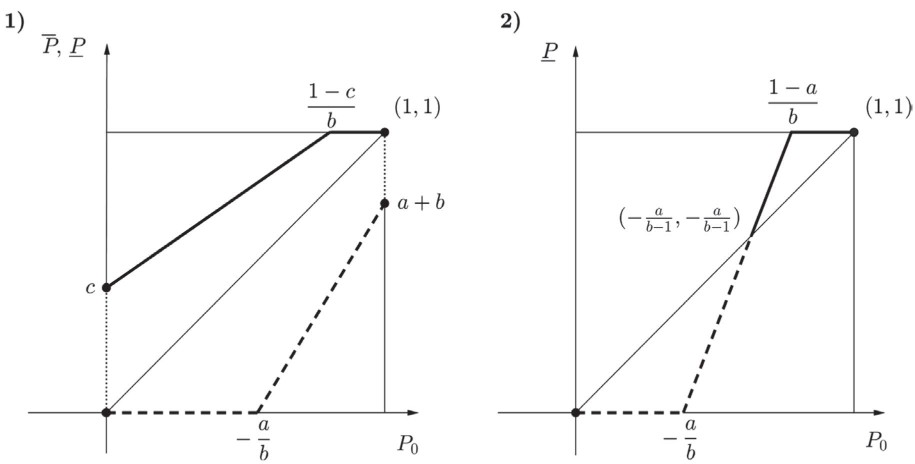

Fig. 1. Plots of $\underline{P}$ or $\overline{P}$ against P_0. (1) A VB-PMM $\overline{P}$ (continuous bold line) and a (non-conjugate) VB-PMM $\underline{P}$ (dashed bold). (2) A HB-PMM $\underline{P}$ (dashed bold line in the prudential part, continuous bold otherwise).

For $\overline{P}$, defined by (7), we can get to specular conclusions. Again the HB-PMM agent is subject to conflicting moods: she/he is unwilling to sell high probability events, but would give away for free low probability events. The lower barrier represents now the imprudent behaviour at its utmost degree, and is narrower than the upper barrier - that emphasising the cautious attitude.

Given this and Proposition 3, one would be tempted to conclude that the HB-PMM can be no more than 2-coherent. While this is true for the 'typical' HB-PMM model, coherence is compatible with some HB-PMM. Even more, there are instances of some HB-PMM $\underline{P}$ (or also $\overline{P}$) which are (0–1 valued) precise probabilities, as in the following example.

Example 2. Let $\mathbb{P} = \{\omega_1, \omega_2, \omega_3\}$, and define $\underline{P}$ by (6), with $a = -0.15$, $b = 1.25$ (a, b comply with (k)). The starting probability P_0 takes the following values on $\mathbb{P}$: $P_0(\omega_1) = P_0(\omega_2) = 0.02$, $P_0(\omega_3) = 0.96$. The resulting HB-PMM lower probability $\underline{P}$ is 0–1 valued: $\underline{P}(A) = 0$ if $A \in \{\emptyset, \omega_1, \omega_2, \omega_1 \vee \omega_2\}$, $\underline{P}(A) = 1$ otherwise. Clearly, $\underline{P}$ is a probability on $\mathcal{A}(\mathbb{P})$.

However, these instances are more an exception, rather than the rule. As for $\underline{P}$ being a precise probability, it holds that:

Proposition 4. *If $\underline{P}$ in the HB-PMM is a precise probability, then it is necessarily 0–1 valued. Conversely, if $\underline{P}$ is 0–1 valued, it may be a probability or a lower probability, coherent or only 2-coherent.*

Coherence of $\underline{P}$, or of $\overline{P}$, is subject to rather restrictive conditions. To see this, we suppose in the next two results that $\mathbb{P}$ is finite. We refer to an upper probability $\overline{P}$, because the conditions for coherence are more straightforwardly described in this case. We present first necessary conditions for $\overline{P}$ to be subadditive, which on its turn is necessary for coherence of $\overline{P}$ (Sect. 2, ($c1$)), then state (Proposition 6) that subadditivity alone is also sufficient for coherence of a HB-PMM $\overline{P}$.

Proposition 5. *Let $\overline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be defined by (7), $\mathbb{P}$ finite. Suppose $\overline{P}$ is subadditive. Then (referring to the sets $\mathcal{E}$, $\mathcal{N}$ of Definition 3), for $A \in \mathcal{A}(\mathbb{P})$:*

(a) *$A \in \mathcal{E}$ iff $A = \omega^* \vee \bigvee_{h=1}^{k} \omega_{i_h}$, with $\omega^* \in \mathbb{P} \cap \mathcal{E}$, $\omega_{i_h} \in \mathbb{P} \cap \mathcal{N}$, $P_0(\omega_{i_h}) = 0$, $h = 1, \ldots, k$, $k \in \mathbb{N}$;*
(b) *$A \in \mathcal{N}$ iff $A = \bigvee_{h=1}^{k} \omega_{i_h}$, with $\omega_{i_h} \in \mathbb{P} \cap \mathcal{N}$, $h = 1, \ldots, k$, $k \in \mathbb{N}$;*
(c) *if $A \in \mathcal{E}$, then $\overline{P}(A) = \overline{P}(\omega^*)$, with $\omega^* \in \mathbb{P} \cap \mathcal{E}$, $\omega^* \Rightarrow A$.*

By Proposition 5(a), (c), if $\overline{P}$ is coherent (hence subadditive), its value on any event A in $\mathcal{E}$ is that of the one atom in $\mathbb{P}$, among those implying A, that belongs to $\mathcal{E}$. Put differently, $\overline{P}(A)$ depends on a single atom only, ω^*. It ensues that, on the whole $\mathcal{A}(\mathbb{P})$, $\overline{P}$ may take up at most $n + 2$ distinct values including 0 and 1, if $|\mathbb{P}| = n$. Clearly, these are severe constraints. As for subadditivity, it holds that

Proposition 6. *Let $\overline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be defined by (7), $\mathbb{P}$ finite. Then $\overline{P}$ is coherent iff it is subadditive.*

A corresponding condition for $\underline{P}$ is less immediate, since superadditivity is not the conjugate property of subadditivity. However, superadditivity is necessary for 2-coherence of $\underline{P}$ in the HB-PMM, whatever the cardinality of $\mathbb{P}$:

Proposition 7. *Let $\underline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be defined by (6), with* $\mathbb{P}$ arbitrary *(finite or not). Then $\underline{P}$ is superadditive (i.e., satisfies* ($c2$) *in Sect. 2).*

6 Similar Models

Despite the simplicity of NL models, there are not so many similar or partly overlapping models in the literature, to the best of our knowledge.

In a paper focused on statistical robustness issues, Rieder [R] introduces a specific VB-PMM and proves the 2-monotonicity of $\underline{P}$. His model is a special case of ours, since he requires (using our parametrisation) conditions (3), and the extra condition $a \geq -1$.

Neo-additive capacities, introduced in [CEG], are somewhat similar to NL models, because $\mu(A) = bP_0(A) + a$ there, when $A \in \mathcal{E}$. Yet, the approach is radically different: the sets $\mathcal{N}$, $\mathcal{E}$, $\mathcal{U}$ are fixed a priori, and it is required that $A \in \mathcal{N}$ iff $\neg A \in \mathcal{U}$. This condition is unduly restrictive, in our view, for measures that are not precise probabilities. It is usually not met by NL models, not even by the PMM (just think that if $\mu = \underline{P}_{\mathrm{PMM}}$, then $\mathcal{U} = \{\Omega\}$, while generally $\mathcal{N} = \{A \in \mathcal{A}(\mathbb{P}) : P_0(A) \leq -\frac{a}{b}\}$ is larger than $\{\emptyset\}$). Further, μ is only required to be a capacity, while our models ensure at least 2-coherence. Interestingly, neo-additive capacities were introduced to describe both optimistic and pessimistic attitudes towards uncertainty at the same time. This is similar to the agent's waving attitude towards risky contracts expressed by the HB-PMM.

7 Conclusions

In this paper we introduced two models of imprecise probabilities, both generalising the PMM, and studied their basic features and consistency properties. While the VB-PMM is always coherent, the HB-PMM is generally not, but may formalise a conflicting behaviour of the agent towards risk. Further work is needed to complete the analysis of NL models and to explore their connections with other models, for instance probability intervals that were shown to be closely related with the PMM in [MMD]. We also plan to study conditioning with the VB-PMM, its natural extension and relationships with risk measures; this should generalise the analogous work in [PVZ] for the PMM.

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