Enhanced arc-flow formulations to minimize weighted completion time on identical parallel machines

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Abstract

We consider the problem of scheduling a set of jobs on a set of identical parallel machines, with the aim of minimizing the total weighted completion time. The problem has been solved in the literature with a number of mathematical formulations, some of which require the implementation of tailored branch-and-price methods. In our work, we solve the problem instead by means of new arc-flow formulations, by first representing it on a capacitated network and then invoking a mixed integer linear model with a pseudo-polynomial number of variables and constraints. According to our computational tests, existing formulations from the literature can solve to proven optimality benchmark instances with up to 100 jobs, whereas our most performing arc-flow formulation solves all instances with up to 400 jobs and provides very low gap for larger instances with up to 1000 jobs.

1 Introduction

We are given a set $J = \{1, 2, \ldots, n\}$ of jobs to be scheduled on a set $M = \{1, 2, \ldots, m\}$ of identical parallel machines. Each job $j \in J$ has a processing time p_j and a penalty weight w_j. A schedule is feasible if each job is assigned to a unique machine and processed without preemption, and each machine processes at most one job at a time. Let C_j define the completion time of job j, our goal is to find a feasible schedule for which the total weighted completion time, $\sum_{j=1}^{n} w_j C_j$, is a minimum. The problem is denoted as $P||\sum w_j C_j$ as in the three-field classification of Graham *et al.* (1979). In the following, we suppose that processing times and penalty weights take integer values.

The $P||\sum w_j C_j$ was proven to be $\mathcal{NP}$-hard by Bruno *et al.* (1974). Despite being a classical production scheduling problem, with real-world applications, it has not received much attention in the literature and cannot be considered a well solved problem. To the best of our knowledge, state-of-the-art exact methods for the $P||\sum w_j C_j$ are the branch-and-bound algorithms developed by Azizoglu and Kirca (1999), Chen and Powell (1999) and Van den Akker *et al.* (1999), the last two of which make use of column generation techniques to solve the relaxed problem at each node. Aside from these works, the $P||\sum w_j C_j$ can be solved by adapting mathematical formulations originally developed for similar one machine or unrelated parallel machines problems. Among these formulations, we mention the *time indexed* (TI) *mixed integer linear programming* (MILP) model by Sousa and Wolsey (1992), the convex integer quadratic programming model by Skutella

(2001) and the preemptive TI model by Bülbül and Şen (2017). According to our tests, these methods fail in solving some $P||\sum w_jC_j$ benchmark instances involving just 100 jobs.

In this paper, we solve exactly large-size instances of the $P||\sum w_jC_j$ by focusing on the development of *arc-flow* (AF) formulations. AF formulations represent the problem as a capacitated network with side constraints, and consist of a MILP model with a pseudo-polynomial number of variables and constraints. AF formulations have been used to model many combinatorial optimization problems (see, e.g., Wolsey 1977 and Valério de Carvalho 1999), and have recently obtained successful results on important areas such as bin packing and cutting stock problems (see, e.g., the recent survey by Delorme *et al.* 2016). For the area of scheduling, we are only aware of a very recent publication by Mrad and Souayah (2018) that presents an AF formulation for the problem of minimizing makespan on identical parallel machines. In our work, we first propose a straight AF formulation, and then enhance it through a set of techniques that aim at reducing the number of variables and constraints by combining established reduction procedures from the literature with some specific features of the $P||\sum w_jC_j$. This results in a powerful method that solves to proven optimality large instances and provide low optimality gaps for very large instances.

The remainder of this paper is organized as follows. In the next Section 2, we review the main literature, whereas in Section 3 we adapt to the $P||\sum w_jC_j$ some mathematical formulations from the literature. In Section 4, we present the straight and enhanced AF formulations. In Section 5, we provide the outcome of extensive computational experiments and finally, in Section 6, we present some concluding remarks.

2 Literature review

The $P||\sum w_jC_j$ is a generalization of $1||\sum C_j$, $1||\sum w_jC_j$ and $P||\sum C_j$, which are all solvable in polynomial time using the well known *shortest processing time* (SPT) rule, or the *weighted shortest processing time* (WSPT) rule of Smith (1956). The WSPT rule sorts jobs according to non-increasing w_j/p_j. The $P||\sum w_jC_j$, on the contrary, is a difficult problem, and was proven to be $\mathcal{NP}$-hard even with just two machines ($P2||\sum w_jC_j$, see Bruno *et al.* 1974).

The literature on the $P||\sum w_jC_j$ focused on the development of early heuristic methods and exact *branch-and-bound* (B&B) algorithms. Eastman *et al.* (1964) proposed a heuristic that uses a variant of the WSPT rule. Kawaguchi and Kyan (1986) showed that such heuristic guarantees a solution whose total weighted completion time is not worse than $(\sqrt{2}+1)/2$ times the optimal solution value. Elmaghraby and Park (1974) proposed a B&B based on the use of lower bounds and properties of optimal solutions. Sarin *et al.* (1988) improved the work by Elmaghraby and Park (1974), by proposing a new branching scheme that substantially reduces the number of schedules to be evaluated in the B&B

tree. They solved instances with up to 30 jobs and 5 machines. Belouadah and Potts (1994) incorporated Lagrangian relaxation in a B&B based on a TI formulation, solving instances with up to 30 jobs and 8 machines. Another B&B was designed by Azizoglu and Kirca (1999), who used the same branching scheme of Sarin *et al.* (1988), but enriched it with the lower bound of Webster (1995), solving instances with up to 35 jobs and 5 machines.

Chen and Powell (1999) tackled the $P||\sum w_jC_j$, the $Q||\sum w_jC_j$ and the $R||\sum w_jC_j$ by means of a *set covering* (SC) formulation where each column corresponds to a single machine schedule. To deal with the large number of feasible schedules, they developed a *branch-and-price* (B&P) method in which at each node of an enumeration tree a valid lower bound was obtained by column generation. In the same year, Van den Akker *et al.* (1999) independently developed a similar B&P. They focused only on the $P||\sum w_jC_j$, and obtained slightly better results than Chen and Powell (1999) by branching on completion times instead of branching on variables that indicate whether a certain job is processed immediately after another job. Their B&P solved $P||\sum w_jC_j$ instances with up to 100 jobs and 10 machines. Very recently, Kowalczyk and Leus (2018) extended the method of Van den Akker *et al.* (1999) by investigating the use of stabilization techniques, a generic branching rule (see Ryan and Foster 1981), and a zero-suppressed binary decision diagram approach (see Minato 1993) for solving the pricing subproblem.

Sousa and Wolsey (1992) proposed a TI formulation to solve single machine scheduling problems with general objective function. Their formulation originates from early works on scheduling (see, e.g., Bowman 1959 and Pritsker *et al.* 1969) and is easily adaptable to multiple machine problems.

Recent literature focused on the related $R||\sum w_jC_j$, which considers unrelated parallel machines. Skutella (2001) proposed a *convex integer quadratic programming* (CIQP) relaxation as the basis for an approximation algorithm. In his approach, the problem is formulated as an integer quadratic programming model with $n \times m$ assignment variables. Then, integrality is relaxed and the objective function is convexified to obtain a CIQP relaxation that can be solved in polynomial time. Finally, the relaxed solution is transformed in a feasible solution by applying a randomized rounding method. Later on, Plateau and Rios-Solis (2010) embedded the CIQP relaxation by Skutella (2001) in a branch-and-bound algorithm to obtain an exact approach. Bülbül and Şen (2017) proposed a Benders decomposition method based on a TI formulation. Their formulation accepts preemptive solutions, but is proven to yield non-preemptive optimal solutions because of the use of tailored coefficients in the objective function. Their method obtained better results than those produced by the CIQP formulation. We also highlight the recent works by Rodriguez *et al.* (2012, 2013), who developed metaheuristic algorithms based on GRASP and iterated greedy paradigms, and tested them on instances having either unrelated or uniform machines. Note that we cannot compare with the literature on these instances, as our methods are specifically tailored for the case of identical machines.

We conclude this section by referring the interested reader to the review by Li and Yang (2009) on models, relaxations and algorithms for minimizing weighted completion times on parallel machines. Another review of models for parallel machines scheduling problems, which includes a computational evaluation of MILP models, was proposed by Unlu and Mason (2010). They classified the formulations into four different types according to the characteristics of their variables (TI variables, network variables, assignment variables, and positional date variables), and concluded that TI formulations tend to perform better than the others. A very recent review on preemptive models for scheduling problems with controllable processing times has been presented by Shioura *et al.* (2018), who also included a section on methods based on flow computations.

3 Existing mathematical formulations

In this section, we provide an overview of mathematical formulations for the $P||\sum w_jC_j$ that we obtained by adapting models originally presented for related problems.

3.1 Sousa and Wolsey's time indexed formulation

The TI formulation by Sousa and Wolsey (1992) was originally designed to deal with the problem of sequencing jobs over the time on a single machine subject to resource constraints. As previously highlighted, this formulation can be modified to consider parallel machines and weighted completion time as follows:

$$\text{(TI)} \quad \min \sum_{j\in J}\sum_{t=0}^{T-p_j} w_j t x_{jt} + \sum_{j\in J} w_j p_j \tag{1}$$

$$\text{st.} \sum_{t=0}^{T-p_j} x_{jt} = 1 \qquad j \in J \tag{2}$$

$$\sum_{j\in J}\sum_{s=\max\{0,t+1-p_j\}}^{\min\{t,T+1-p_j\}} x_{is} \leq m \qquad t = 0,\ldots,T-1 \tag{3}$$

$$x_{jt} \in \{0,1\} \qquad j \in J, t = 0,\ldots,T-p_j \tag{4}$$

where x_{jt} is a binary decision variable taking value 1 if job j starts its processing at time t, 0 otherwise. The time horizon is defined by T and should be sufficiently large to ensure optimality and as short as possible to avoid the creation of unnecessary variables. The objective function (1) seeks the minimization of the total weighted completion time. Note that we expressed the completion time of a job as the sum of starting time and processing time, formally using $C_j = \sum_t t x_{jt} + p_j$. The term $\sum_j w_j p_j$ is a constant and is thus irrelevant for the formulation. Constraints (2) ensure that each job is processed exactly once. Constraints (3) forbid overlapping among the jobs by imposing that at most m jobs are executed in parallel at any time. Constraints (4) define the variables' domain. Model

(1)–(4) contains a pseudo-polynomial number of variables, a common characteristic of TI formulations, which amounts to $\mathcal{O}(nT)$.

3.2 Skutella's convex integer quadratic programming formulation

The idea behind the method of Skutella (2001) is to formulate the $R||\sum w_jC_j$ as an integer quadratic program and then convexify the objective function. His formulation uses $n \times m$ integer assignment variables and can be adapted to the $P||\sum w_jC_j$ as follows:

$$\text{(CIQP)} \quad \min \sum_{j\in J} w_jC_j \tag{5}$$

$$\text{st.} \sum_{k\in M} x_j^k = 1 \qquad j \in J \tag{6}$$

$$C_j = \sum_{k\in M} x_j^k \left(\frac{1+x_j^k}{2} p_j + \sum_{i\in J, i\prec j} x_i^k p_i \right) \qquad j \in J \tag{7}$$

$$x_j^k \in \{0,1\} \qquad j \in J, k \in M \tag{8}$$

where $x_j^k = 1$ if job j is scheduled on machine k, 0 otherwise. The notation $i \prec j$ in (7) means that either $(w_i/p_i > w_j/p_j)$ or $(w_i/p_i = w_j/p_j$ and $i < j)$. This is used to take into account that the jobs are scheduled on each machine by non-increasing order of w_j/p_j, i.e., by following the WSPT rule. The relaxation obtained by dropping integrality constraints from (8) can be solved easily. Indeed, Skutella (2001) showed that $x_j^k = 1/m \ \forall j \in J$ and $k \in M$ is an optimal solution to this relaxation when the machines are identical. The optimal integer solution of model (5)–(8) can be obtained by invoking a commercial CIQP solver such as CPLEX or Gurobi (as done by Plateau and Rios-Solis 2010 for the $R||\sum w_jC_j$).

3.3 Bülbül and Şen's preemptive time indexed formulation

Bülbül and Şen (2017) modeled the preemptive version of the $R||\sum w_jC_j$ by means of a *preemptive time indexed* (PTI) formulation making use of $n \times m \times T$ continuous variables and $n \times m$ binary variables. Then, they proved that it is always possible to devise a non-preemptive solution having the same objective value of the optimal PTI preemptive one, concluding that the PTI model is optimal also for the (non-preemptive) $R||\sum w_jC_j$. Their formulation can be adapted to the $P||\sum w_jC_j$ as follows:

$$\text{(PTI)} \quad \min \sum_{j\in J} \sum_{t=1}^{T} \sum_{k\in M} \frac{w_j}{p_j} \left(t + \frac{p_j - 1}{2} \right) x_{jkt} \tag{9}$$

$$\text{s.t.} \sum_{t=1}^{T} x_{jkt} = p_j y_{jk} \qquad j \in J, k \in M \tag{10}$$

$$\sum_{j\in J} x_{jkt} \leq 1 \qquad k \in M, t = 1, \ldots, T \tag{11}$$

$$\sum_{k \in M} y_{jk} = 1 \qquad j \in J \tag{12}$$

$$x_{jkt} \geq 0 \qquad j \in J, k \in M, t = 1, \ldots, T \tag{13}$$

$$y_{jk} \in \{0, 1\} \qquad j \in J, k \in M \tag{14}$$

where x_{jkt} are continuous variables representing the quantity of job j, i.e., the number of unit-length parts of job j, which is finished at time t on machine k, and y_{jk} takes value 1 if job j is assigned to machine k, 0 otherwise. Constraints (10) state that all unit-length parts of job j must be processed on the same machine. Constraints (11) ensure that each machine processes at most one job at a time. Constraints (12) guarantee that each job is assigned to exactly one machine. Constraints (13) and (14) give the variables' domains.

To tackle the pseudo-polynomial number of variables and constraints in the aforementioned model, the authors proposed a Benders decomposition approach. Their idea is to start by first solving a master problem composed by variables y and constraints (12) and (14), and obtain a solution $\bar{y}$. Then, solving a set of m subproblems, each for a machine, that use the $\bar{y}$ solution but only involve variables x and constraints (10), (11) and (13). The subproblems either deliver an optimal $P||\sum w_j C_j$ solution, or some optimality Benders cuts to be added to the master problem. This process is reiterated until proof of optimality or some stopping criteria are met.

3.4 Set covering formulation

Set covering (SC) formulations are widely used to model combinatorial optimization problems as covering problems. Van den Akker *et al.* (1999) followed this idea and modeled the $P||\sum w_j C_j$ by using an SC formulation having an exponential number of variables. Let S be a set containing all feasible schedules for a single machine, let a_{js} be a binary coefficient indicating whether job $j \in J$ is included or not in schedule s, and let x_s be a binary variable assuming value 1 if schedule $s \in S$ is selected, 0 otherwise. The $P||\sum w_j C_j$ can be modeled as:

$$\text{(SC)} \quad \min \sum_{s \in S} c_s x_s \tag{15}$$

$$\text{s.t.} \sum_{s \in S} x_s = m \tag{16}$$

$$\sum_{s \in S} a_{js} x_s = 1 \qquad j \in J \tag{17}$$

$$x_s \in \{0, 1\} \qquad s \in S \tag{18}$$

Constraints (16) state that exactly m schedules are selected. Constraints (17) ensure that each job is processed once, and constraints (18) impose variables to be binary. As model (15)–(18) has an exponential number of variables, the authors solved it with a branch-and-price algorithm. In particular, they solved each node of an enumeration tree

by means of a column generation method, which looks for negative cost schedules by invoking a tailored *dynamic programming* (DP) algorithm. They performed branching by considering the minimum completion time of a fractional job. Let $\bar{x}$ be the current solution and $\bar{S} \subseteq S$ the set of schedules associated with positive $\bar{x}$ values. A fractional job is defined as a job j for which $\sum_{s \in \bar{S}} C_j(s)\bar{x}_s > \min\{C_j(s)|\bar{x}_s > 0\}$, where $C_j(s)$ is the completion time of job j in schedule s. Note that Chen and Powell (1999) also proposed a branch-and-price algorithm to solve model (15)–(18), but, differently from Van den Akker *et al.* (1999), they performed branching directly on the x variables.

4 Arc-flow formulations

We first present a straight formulation, and then enhance it with reduction procedures.

4.1 Straight arc-flow formulation

AF formulations are an established combinatorial optimization technique that models problems by using flows on a capacitated network (see, e.g., Wolsey 1977). When applied to machine scheduling problems, the flow obtained by solving an AF formulation can be easily decomposed into paths (see Ahuja *et al.* 1993), so that each path corresponds to a schedule of activities on a machine. AF formulations make use of a pseudo-polynomial number of variables and constraints, and are thus related to the TI formulations that we previously described. The research effort behind AF is, however, to try to reduce as much as possible the required number of variables and constraints, thus keeping the size of the model as small as possible while preserving optimality. In our work, we follow the recent literature on AF, that, starting from Valério de Carvalho (1999), used these techniques to obtain good computational results on cutting and packing problems (see, e.g., Delorme *et al.* 2016 for an updated survey), but we take into account issues that are typical of the scheduling field.

Our AF formulation models the $P||\sum w_j C_j$ as the problem of finding m independent paths that start from a source node 0, end at a destination node T, and cover all the jobs. For the sake of clarity, we start by presenting the very basic model, and focus later (in Algorithm 1) on a first reduction of variables and constraints that is based on the WSPT sorting. Our very basic AF formulation uses a direct acyclic multigraph $G = (N, A)$. The set of vertices $N \subseteq \{0, 1, \ldots, T\}$ can be initially considered as the set of *normal patterns* (for which we refer to the seminal papers by Herz 1972 and Christofides and Whitlock 1977, and to the recent discussion in Côté and Iori 2018), i.e., the set of all the feasible combinations of jobs' processing times whose resulting value is between 0 and T. Let $J_+ = J \cup \{0\}$ include the original set of jobs plus a dummy job 0 having $p_0 = 0$ and $w_0 = 0$. The set of arcs is partitioned as $A = \cup_{j \in J_+} A_j$. Each A_j represents the set of *jobs arcs* associated with job $j \in J$, and, for the moment, let us define it as $A_j = \{(q, r, j) : r - q = p_j \text{ and } q \in N\}$. In addition, A_0 represents the set of *loss arcs*,

that are used to model the amount of idle time between the end of activities and T on a machine, and is defined as $A_0 = \{(q, T, 0) : q \in N\}$. Let us also use $\delta^+(q) \subseteq A$, respectively $\delta^-(q) \subseteq A$, to define the subset of arcs that emanate from, respectively enter, a given node $q \in N$. A feasible $P||\sum w_j C_j$ solution can be represented as a set of m paths in G, each corresponding to a machine schedule that start in 0 and make use of jobs arcs and of possibly one last loss arc to reach T.

To formulate the $P||\sum w_j C_j$ as an AF, we associate with each job arc $(q, r, j) \in A$ a variable x_{qrj} that has a twofold meaning: for jobs arcs $(q, r, j) \in A_j$, x_{qrj} takes value 1 if job j is scheduled at start time q, 0 otherwise; for loss arcs $(q, T, 0) \in A_0$, x_{qT0} gives the number of paths that end with arc $(q, T, 0)$, i.e., that contain activities that finish at time q. The $P||\sum w_j C_j$ can then be modeled as:

$$\text{(AF)} \quad \min \sum_{(q,r,j)\in A} w_j q x_{qrj} + \sum_{j\in J} w_j p_j \tag{19}$$

$$\sum_{(q,r,j)\in\delta^+(q)} x_{qrj} - \sum_{(p,q,j)\in\delta^-(q)} x_{pqj} = \begin{cases} m, & \text{if } q = 0 \\ -m, & \text{if } q = T \\ 0, & \text{otherwise} \end{cases} \qquad q \in N \tag{20}$$

$$\sum_{(q,r,j)\in A} x_{qrj} \geq 1 \qquad j \in J \tag{21}$$

$$x_{qrj} \in \{0, 1\} \qquad (q, r, j) \in A \setminus A_0 \tag{22}$$

$$0 \leq x_{qT0} \leq m \qquad (q, T, 0) \in A_0 \tag{23}$$

The objective function (19) minimizes the sum of the weighted completion times. Constraints (20) impose both flow conservation at each node and the use of exactly m paths. Constraints (21) impose all jobs to be scheduled, whereas constraints (22) and (23) give the variables' domains. Note that variables associated to loss arcs do not need to be defined as integers.

A first, simple but very important rule can be used to decrease the number of variables and constraints in the model. As previously discussed, in any optimal solution of the $P||\sum w_j C_j$ the jobs are sequenced on each machine by following the WSPT rule. Consequently, only arcs fulfilling this sorting rule can be considered (the first job in the order can only start in 0, the second job can start in 0 or right after the first job, and so on). The procedure that we implemented to build the underlying AF multi-graph, given in Algorithm 1, takes this fact into consideration, producing a first reduction of the size of sets N and A, and hence of the number of variables and constraints in the formulation. The procedure initializes the set of nodes and arcs to the empty set. It then considers one job j at a time, according to the WSPT rule, to create the A_j sets at steps 4–7. The sets of nodes and of loss arcs are constructed at steps 8–9, and the overall set of arcs at step 10.

To ease comprehension, we present a simple example with 4 jobs, 2 machines, and

Algorithm 1 Construction of the AF multi-graph

```
1: procedure CREATE_PATTERNS_AND_ARCS(T)
2:     initialize P[0...T] ← 0;                          ▷ P: array of size T + 1
3:     initialize N ← ∅; A[0...n] ← ∅;                   ▷ N: set of vertices; A: set of arcs
4:     P[0] ← 1;
5:     for j ∈ J according to the WSPT rule do
6:         for t ← T − p_j down to 0 do
7:             if P[t] = 1 then P[t + p_j] ← 1; A[j] ← A[j] ∪ {(t, t + p_j, j)};   ▷ A[j]: set of job arcs of j
8:     for t ← 0 to T do
9:         if P[t] = 1 then N ← N ∪ {t}; A[0] ← A[0] ∪ {(t, T, 0)}   ▷ A[0]: set of loss arcs
10:    A ← ∪_{j∈J+} A[j]
11:    return N, A
```

$T = 8$ (details on how to compute a strict value of T are given in Section 4.2 below). The characteristics of the 4 jobs and their sorting according to the WSPT rule are given in Figure 1a. The AF multi-graph built by Algorithm 1 is given in Figure 1b, and contains 9 vertices, 11 job arcs, and 7 loss arcs. An optimal solution is provided in Figure 1c, where we highlight the 2 paths corresponding to the machine schedules.

We now notice a relevant property of the AF formulation.

Proposition 1. *The AF formulation* (19)–(23) *is equivalent to the TI formulation* (1)–(4).

Proof. Our proof uses arguments similar to that adopted in Valério de Carvalho (2002) for the proof of his Proposition 5.1 (which shows the equivalence of the AF and TI formulations for the cutting stock problem). To simplify the reasoning, we consider a basic AF formulation in which the vertices set N contains all nodes from 0 to T (i.e., we do not apply the reduction due to the normal patterns) and the arc set contains all the job and loss arcs. Moreover, we substitute the '$\geq$' sign in (21) with the '=' sign, without loosing optimal solutions because any solution selecting more than one arc for the same job can be improved by choosing only one of these arcs.

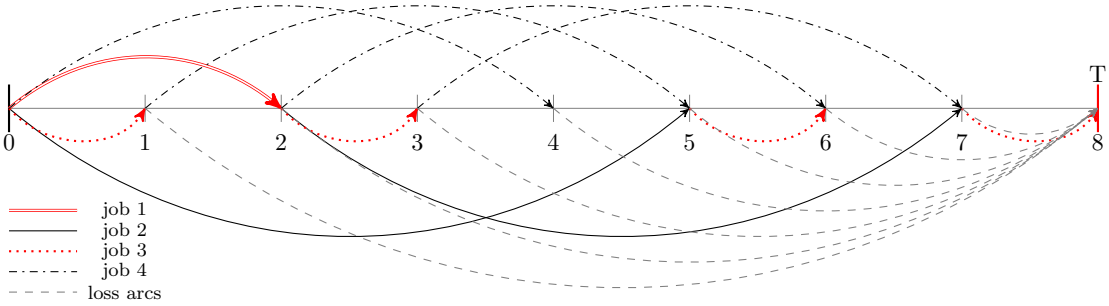
Remind that the three indices of the x variables in AF are introduced to simplify the writing of the model, but only two indices are necessary. Indeed, job arc (q, r, j) is introduced only when $r = q + p_j$, and hence x_{qrj} is set to one, in AF, if job j starts at time q, as x_{jt} is set to one, in TI, when job j starts at time t. Using this observation one can see that the objective functions (19) and (1) are equivalent.

Both constraints (21) in AF (with the '=' sign) and constraints (2) in TI impose that a single starting time for each job j is chosen.

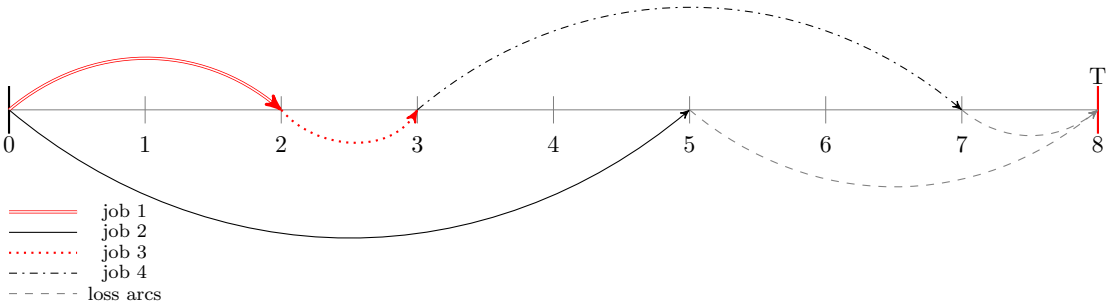
We conclude the proof by showing that the remaining constraints in AF and TI are equivalent, by using an unimodular transformation, similar to the one in Valério de Carvalho (2002). To better understand this transformation we refer to the example of Figure 1, having two machines, four jobs and an upper bound $T = 8$ for the completion of all jobs. In Figure 2a we report the constraint matrix of constraints (3) in TI, while in Figure 2b we report the constraint matrix of the flow conservation constraints (20) in AF (note that

j	p_j	w_j	w_j/p_j
1	2	4	2.00
2	5	7	1.40
3	1	1	1.00
4	4	3	0.75

(a) Input data ($T = 8$)



(b) AF formulation



(c) Optimal solution value = 67

Figure 1: Example of AF formulation

in the TI matrix we added an empty row corresponding to $t = T$, for easier comparison with the AF matrix). In the TI matrix we have a column for each starting time of each job. In the AF formulation the first part of the matrix refers to the job arcs, while the last part report on the loss arcs used to model the empty space in the machines. In the first part of the two matrices there is a column by column correspondence: variable x_{qrj} in AF defines a possible starting time of job j at time q, exactly as variable x_{jq} do in TI.

Consider the AF constraint matrix and, for each $t = 0, \ldots, T$, let us substitute the t-th flow conservation constraint with the sum of the first $t+1$ constraints (20), thus obtaining an equivalent constraint matrix. The resulting matrix is depicted in Figure 2c.

One can note that the first part of the new constraint matrix is identical to the first part of the TI matrix. In the second part we have the same r.h.s. as in TI, but equal sign instead of '$\leq$' sign. However, each constraint t is completed with the sum of the first $t + 1$ loss arcs: $\sum_{q=0}^{t} x_{qT0}$, thus giving at least one slack variable available for each constraint. It follows that the new matrix is equivalent to the TI matrix, which conclude the proof. □

t	x_{10}	x_{20}	x_{30}	x_{40}	x_{11}	x_{21}	x_{31}	x_{41}	x_{12}	x_{22}	x_{32}	x_{42}	x_{13}	x_{23}	x_{33}	x_{43}	x_{14}	x_{24}	x_{34}	x_{44}	x_{15}	x_{25}	x_{35}	x_{45}	x_{16}	x_{26}	x_{36}	x_{46}	x_{17}	x_{27}	x_{37}	x_{47}			
0	1	1	1	1																													$\leq$	2
1	1	1		1	1	1	1	1																									$\leq$	2
2		1		1	1	1		1	1	1	1	1																					$\leq$	2
3		1		1		1		1	1	1		1	1	1	1	1																	$\leq$	2
4		1				1		1		1		1	1	1		1	1		1	1													$\leq$	2
5						1				1		1		1		1	1			1	1		1										$\leq$	2
6										1				1		1				1	1				1		1						$\leq$	2
7														1						1					1						1		$\leq$	2
8																																	$\leq$	0

(a) Time-indexed formulation

q	x_{021}	x_{052}	x_{013}	x_{044}	x_{131}	x_{162}	x_{123}	x_{154}	x_{241}	x_{272}	x_{233}	x_{264}	x_{351}	x_{382}	x_{343}	x_{374}	x_{461}	x_{492}	x_{453}	x_{484}	x_{571}	x_{5102}	x_{563}	x_{594}	x_{681}	x_{6112}	x_{673}	x_{6104}	x_{791}	x_{7122}	x_{783}	x_{7114}	x_{080}	x_{180}	x_{280}	x_{380}	x_{480}	x_{580}	x_{680}	x_{780}		
0	1	1	1	1																													1								=	2
1			-1		1	1	1	1																										1							=	0
2	-1						-1		1	1	1	1																							1						=	0
3					-1						-1		1	1	1	1																				1					=	0
4				-1					-1						-1		1		1	1																	1				=	0
5		-1						-1					-1						-1		1		1																1			=	0
6						-1						-1					-1						-1		1		1													1		=	0
7										-1						-1					-1						-1				1										1	=	0
8														-1						-1					-1						-1		-1	-1	-1	-1	-1	-1	-1	-1	=	-2

(b) Arc-flow formulation

q	x_{021}	x_{052}	x_{013}	x_{044}	x_{131}	x_{162}	x_{123}	x_{154}	x_{241}	x_{272}	x_{233}	x_{264}	x_{351}	x_{382}	x_{343}	x_{374}	x_{461}	x_{492}	x_{453}	x_{484}	x_{571}	x_{5102}	x_{563}	x_{594}	x_{681}	x_{6112}	x_{673}	x_{6104}	x_{791}	x_{7122}	x_{783}	x_{7114}	x_{080}	x_{180}	x_{280}	x_{380}	x_{480}	x_{580}	x_{680}	x_{780}		
0	1	1	1	1																													1								=	2
1	1	1		1	1	1	1	1																									1	1							=	2
2		1		1	1	1		1	1	1	1	1																					1	1	1						=	2
3		1		1		1		1	1	1		1	1	1	1	1																	1	1	1	1					=	2
4		1				1		1		1		1	1	1		1	1		1	1													1	1	1	1	1				=	2
5						1				1		1		1		1	1			1	1		1										1	1	1	1	1	1			=	2
6										1				1		1				1	1				1		1						1	1	1	1	1	1	1		=	2
7														1						1					1						1		1	1	1	1	1	1	1	1	=	2
8																																									=	0

(c) Arc-flow formulation after unimodular transformation

Figure 2: Illustration of the equivalence between TI and AF on the example of Figure 1

4.2 Enhanced arc-flow formulation

In this section, we show how to further reduce the number of variables and constraints required by the AF formulation (19)–(23), improving its computational behavior while preserving optimality.

The size of the AF formulation linearly depends from the horizon T, thus, a proper time horizon estimation is necessary. To this aim, we notice that Van den Akker *et al.* (1999) considered the properties of an optimal schedule, originally developed by Elmaghraby and Park (1974), and remarked that there exists at least an optimal solution for which "the last job on any machine is completed between time $H_{\min} = \frac{1}{m}\sum_{j\in J} p_j - \frac{(m-1)}{m}p_{\max}$ and $H_{\max} = \frac{1}{m}\sum_{j\in J} p_j + \frac{(m-1)}{m}p_{\max}$", where $p_{\max} = \max_{j\in J} p_j$. On the basis of this statement, we can use $H_{\max}$ to set T as

$$T = \left\lfloor \frac{1}{m}\sum\nolimits_{j\in J} p_j + \frac{(m-1)}{m}p_{\max} \right\rfloor \tag{24}$$

We can then use the value of $H_{\min}$ to limit the number of loss arcs. As the last job on any machine is completed at or after $H_{\min}$, we can create only loss arcs starting from vertices $q \in N$ with $q \geq \lceil H_{\min} \rceil$. We further increase this bound by considering Property 1 in Azizoglu and Kirca (1999), thus obtaining

$$T' = \left\lceil \frac{1}{m}\sum\nolimits_{j\in J} p_j - \frac{\sum_{k=1}^{m-1} \bar{p}_k}{m} \right\rceil \tag{25}$$

where $\bar{p}$ is an array containing the processing times of all jobs $j \in J$ in non-increasing order. Since $\sum_{k=1}^{m-1} \bar{p}_k \leq (m-1)p_{\max}$ holds, the value in (25) is not less than $H_{\min}$.

Our next enhancement relies on the creation of a so-called *time window* $[a_j, b_j]$ for each job $j \in J$. By using once more the properties of an optimal schedule in Van den Akker *et al.* (1999), we derive an earliest possible start time a_j and a latest possible start time b_j that guarantee the existence of an optimal solution. The values of a_j and b_j are based on the property that, if $w_j \geq w_k$ and $p_j \leq p_k$ for a certain pair of jobs j and k, then there exists an optimal solution in which j starts not later than k. The time windows are computed as follows. For each $j \in J$, we first define $\mathcal{P}_j = \{k \in J : k < j, w_k \geq w_j, p_k \leq p_j\}$ and $\mathcal{L}_j = \{k \in J : k > j, w_k \leq w_j, p_k \geq p_j\}$. Following the aforementioned property, there exists an optimal schedule in which all jobs in $\mathcal{P}_j$ start no later than j, so, if $\mathcal{P}_j$ contains at least m elements, one may conclude that at least $|\mathcal{P}_j| - m + 1$ jobs in $\mathcal{P}_j$ are finished before j starts being processed. Consequently, if $|\mathcal{P}_j| < m$ we set $a_j = 0$, otherwise we set $a_j = \lceil \rho_j/m \rceil$, where ρ_j is the sum of the $|\mathcal{P}_j| - m + 1$ smallest processing times. In an analogous mode, one can note that there is an optimal solution in which j starts no later than the jobs in $\mathcal{L}_j$. Consequently, for each job $j \in J$ a maximum starting time can be set as $b_j = T - \left\lceil \left(\sum_{k\in\mathcal{L}_j} p_k + p_j\right)/m \right\rceil$. In addition, if $\mathcal{L}_j = \emptyset$ then, as stated in Belouadah and Potts (1994), one can set $b_j = \lceil \left(\sum_{k\in J} p_k - p_j\right)/m \rceil$.

The next procedure that we propose attempts to reduce the number of arcs by grouping

together identical jobs. To this aim, we merge together all jobs $j \in J$ having identical p_j and w_j values into *job types*. Let $J' = \{1, 2, \ldots, n'\}$ be the resulting set of job types, and d_j be the number of jobs contained in each job type. With respect to the original AF formulation, this change involve creating a different set of arcs A' and replacing the original binary variables with integer variables, as shown next. This allows to reduce consistently the number of symmetries in the model. The time windows for each job type $j \in J'$ are simply obtained by setting $a_j = \min\{a_k : k \in J, p_k = p_j, w_k = w_j\}$ and $b_j = \max\{b_k : k \in J, p_k = p_j, w_k = w_j\}$. We then create only arcs that start in a time $q \in [a_j, b_j]$, for each job type $j \in J'$.

Our *enhanced arc flow formulation* (EAF) is then:

$$\text{(EAF)} \quad \min \sum_{(q,r,j)\in A'} w_j q x_{qrj} + \sum_{j\in J'} w_j p_j \tag{26}$$

$$\sum_{(q,r,j)\in\delta'^{+}(q)} x_{qrj} - \sum_{(p,q,j)\in\delta'^{-}(q)} x_{pqj} = \begin{cases} m, & \text{if } q = 0 \\ -m, & \text{if } q = T \\ 0, & \text{otherwise} \end{cases} \qquad q \in N' \tag{27}$$

$$\sum_{(q,r,j)\in A'} x_{qrj} \geq d_j \qquad j \in J' \tag{28}$$

$$x_{qrj} \in \{0, \ldots, d_j\} \qquad (q, r, j) \in A' \setminus A'_0 \tag{29}$$

$$0 \leq x_{qT0} \leq m \qquad (q, T, 0) \in A'_0 \tag{30}$$

The EAF model (26)–(30) is based on a reduced multigraph $G' = (N', A')$, in which both sets of nodes and arcs are obtained by applying the above reductions criteria from the original graph G used for AF. The EAF model considers the set of job types J' instead of that of jobs J in AF, and consequently adopts an integer variable x_{qrj} giving the number of jobs of type j that are scheduled from q to $r = q + p_j$. Each variable of this type might take a value at most d_j, as stated in constraints (29). Constraints (27) impose flow conservation on the m paths, and constraints (28) impose demand to be satisfied.

The way in which the EAF multi-graph G' is built is shown in Algorithm 2, which updates the previous Algorithm 1 used for AF. The procedure initializes the sets N' and A' of nodes and arcs, respectively, to the empty set. It then considers at steps 4–9 one job type j at a time, according to the WSPT rule, and creates the A'_j sets keeping into account that each job type j contains d_j identical jobs and should start at a $q \in [a_j, b_j]$.

5 Computational experiments

The discussed models have been coded in C++ and solved using Gurobi Optimizer 7.0. The experiments were performed by using a single thread on a PC equipped with an Intel Xeon E5530 2.40 GHz quad-core processor and 20GB of RAM, running under Ubuntu 14.04.5 LTS. We first discuss the benchmark instances used for the experiments, then

Algorithm 2 Construction of the EAF multi-graph

```
1: procedure CreatePatterns_and_Arcs(T)
2:     initialize P[0...T] ← 0;                          ▷ P: array of size T + 1
3:     initialize N' ← ∅; A'[0...n] ← ∅;                 ▷ N': set of vertices; A': set of arcs
4:     P[0] ← 1;
5:     for j ∈ J' according to the WSPT rule do
6:         for t ← b_j down to a_j do
7:             if P[t] = 1 then
8:                 for q ← 1 to d_j do
9:                     if t + qp_j ≤ b_j then P[t + qp_j] ← 1; A'[j] ← A'[j] ∪ {(t, t + qp_j, j)};
10:    for t ← 0 to T do
11:        if P[t] = 1 then
12:            N' ← N' ∪ {t};
13:            if T' ≤ t< T then A'[0] ← A'[0] ∪ {(t, T, 0)};   ▷ A'[0]: set of loss arcs
14:    A' ← ∪_{j∈J+} A'[j]
15:    return N', A'
```

present an upper bounding procedure devised to speed up the convergence of the models, and finally we present an extensive computational evaluation.

5.1 Benchmark instances

In our experiments, we considered two benchmark sets of instances.

The first set is derived from the one proposed by Bülbül and Şen (2017) for the $R||\sum w_jC_j$. Their set is made by instances with $n \in \{30, 100, 400, 1000\}$ and $m \in \{2, 4, 6, 8, 16, 30\}$. Processing times p_j^k (i.e., processing time of job j in machine k) were drawn according to a uniform distribution $U[1, p_{\max}]$, where $p_{\max} \in \{20, 100\}$, and penalty weights w_j were created using a uniform distribution $U[1, 20]$. For each combination of $(n, m, p_{\max})$, except when $n = 30$ and $m \in \{16, 30\}$, 10 instances were created, resulting in a set of 440 instances which is now available at `http://people.sabanciuniv.edu/bulbul/papers/Bulbul_Sen_Rm_TWCT_data-results_JoS_2016.rar`. We adapted these instances to the $P||\sum w_jC_j$ by imposing the processing time of each job j to be equal to its processing time on the first unrelated machine in the $R||\sum w_jC_j$ (i.e., $p_j = p_j^1$ $\forall j$). To better evaluate the performance of the models on large instances, we used the procedure adopted by Bülbül and Şen (2017) to create a new additional set with $n = 700$, obtaining in this way a total of 560 instances.

The second set has been proposed by Kowalczyk and Leus (2018) and consists of 2400 instances with $n \in \{20, 50, 100, 150\}$ and $m \in \{3, 5, 8, 10, 12\}$. The instances are divided into six different classes according to the distribution of processing times and weights. For each class and each combination of (n, m), 20 instances were created.

5.2 Upper bound by iterated local search

To start the model with a valid upper bound, we developed a modified version of the *iterated local search* (ILS) based metaheuristic of Kramer and Subramanian (2017). The original method consists of a multi-start ILS for general earliness-tardiness scheduling

problems on unrelated machines, and incorporates special structures to reduce the complexity for exploring the neighborhoods.

In general words, the ILS by Kramer and Subramanian (2017) is composed by constructive, local search and perturbation phases. We modified the construction and local search phases to take into account that in the $P||\sum w_j C_j$ all machine schedules follow the WSPT rule. That resulted in a speed up of the algorithm. The initial solutions, which are obtained either randomly or by a greedy randomized adaptive search procedure (GRASP), are now sorted according to the WSPT rule on each machine. The two main differences with respect to Kramer and Subramanian (2017) regard the GRASP construction procedure and the local search phase. For the GRASP, we initially sort jobs according to the WSPT rule instead of performing a random sorting. In addition, at the end of the procedure we consider each machine in turn and sort the jobs that have been assigned to it by using once more the WSPT rule.

Regarding the local search, Kramer and Subramanian (2017) employed a randomized variable neighborhood descent (RVND) procedure (see Mladenović and Hansen 1997). In our modified version, the intra-machine neighborhood structures have been replaced by a simple WSPT sorting procedure, which is invoked at the end of each inter-machine neighborhood search, as depicted in Algorithm 3.

The ILS was adopted to provide an initial feasible solution for all our methods below as follows. It was not executed for small-size instances having $n \leq 100$ jobs. It was instead executed for 100 seconds for medium-size instances with $100 < n < 400$ jobs, and for 300 seconds for large-size instances with $n \geq 400$ jobs.

Algorithm 3 RVND

```
1: procedure RVND(π)                                   ▷ π is the input solution
2:     initialize L = L_0;          ▷ L_0: list containing all inter-machines neighborhood structures
3:     while L ≠ ∅ do
4:         select a neighborhood N ∈ L at random;
5:         find π' ∈ N, the best neighbor solution of π;
6:         for k ← 1 to m do sort jobs in π'[k] according to WSPT;
7:         if f(π') < f(π) then                        ▷ f(π) represents the cost of solution π
8:             π ← π';
9:             reinitialize L;
10:        else
11:            L ← L \ {N};                                    ▷ remove N from L
12:    return π
```

5.3 Computational results on benchmark set 1

In Tables 1 and 2 we compare the performance of formulations CIQP (model (5)–(8)), PTI (model (9)–(14)), TI (model (1)–(4)), SC (model (15)–(18)), AF (model (19)–(23)) and EAF (model (26)–(30)). To solve SC, we reimplemented the B&P by Van den Akker *et al.* (1999). For instances with more than 100 jobs, each method received as initial cutoff the upper bound produced by the ILS algorithm that we developed using the time limit

detailed at the end of Section 5.2.

Table 1 summarizes the results that we obtained for the instances with $p_{\max} = 20$, whereas Table 2 focuses on the case where $p_{\max} = 100$. For each group of 10 instances defined by the couple (n, m) and for each attempted method, we report the number of instances for which at least the root node of the model was solved, #root (not reported for CIPQ and PTI) and the number of optimal solutions found, #opt. In columns t(s) we report the average execution time in seconds for the 10 instances in the line. If for some of these instances either time or memory limit has been reached, then we consider the entire time limit in the computation of the average t(s) value. Note that we directly write *t.lim*, respectively *m.lim*, when the time limit, respectively memory limit, has been reached on all the 10 instances in the line. Note also that, to facilitate direct comparison among the methods, t(s) does not contain the time required for running the ILS.

A "-" indicates that the value in the entry is not available because the model was not run on that group of instances. For AF and EAF we also report the average gap per million, computed as $\text{gap}_{\text{pm}}=10^6(U - L)/U$, with U and L being, respectively, the best upper and lower bound value obtained in the run (a "-" is reported when no valid L is obtained, i.e., when even the LP relaxation of the model was not solved due to time or memory limits).

Table 1: Results for set 1 instances with $p_{\max} = 20$ (time limit = 300 seconds, ILS time not included)

		Existing formulations										New formulations							
		CIQP		PTI		SC			TI			AF				EAF			
n	m	#opt	t(s)	#opt	t(s)	#root	#opt	t(s)	#root	#opt	t(s)	#root	#opt	t(s)	gap_{pm}	#root	#opt	t(s)	gap_{pm}
30	2	10	1.1	10	11.1	10	10	0.2	10	10	1.2	10	10	0.1	0.0	10	10	0.0	0.0
	4	1	278.1	8	165.9	10	10	0.1	10	10	0.5	10	10	0.1	0.0	10	10	0.0	0.0
	6	0	t.lim	3	240.1	10	10	0.0	10	10	0.2	10	10	0.0	0.0	10	10	0.0	0.0
	8	0	t.lim	3	243.9	10	10	0.0	10	10	0.2	10	10	0.0	0.0	10	10	0.0	0.0
100	2	0	t.lim	0	t.lim	9	0	t.lim	10	10	199.8	10	10	1.6	0.0	10	10	0.9	0.0
	4	0	t.lim	0	t.lim	10	3	276.5	10	10	65.3	10	10	2.1	0.0	10	10	0.4	0.0
	6	0	t.lim	0	t.lim	10	10	101.3	10	10	29.5	10	10	0.5	0.0	10	10	0.4	0.0
	8	0	t.lim	0	t.lim	10	10	15.9	10	10	19.3	10	10	0.4	0.0	10	10	0.3	0.0
	16	0	t.lim	0	t.lim	10	10	2.2	10	10	3.9	10	10	0.1	0.0	10	10	0.1	0.0
	30	0	t.lim	0	t.lim	10	10	0.4	10	10	1.2	10	10	0.1	0.0	10	10	0.1	0.0
400	2	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	10	15.0	0.0	10	10	4.8	0.0
	4	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	10	29.3	0.0	10	10	17.0	0.0
	6	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	10	32.3	0.0	10	10	13.2	0.0
	8	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	10	33.7	0.0	10	10	10.6	0.0
	16	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	10	16.2	0.0	10	10	3.5	0.0
	30	0	t.lim	0	t.lim	10	0	t.lim	10	3	272.9	10	10	3.9	0.0	10	10	0.9	0.0
700	2	-	-	-	-	-	-	-	-	-	-	10	6	191.3	0.1	10	10	66.8	0.0
	4	-	-	-	-	-	-	-	-	-	-	10	7	220.2	0.4	10	10	56.9	0.0
	6	-	-	-	-	-	-	-	-	-	-	10	4	240.6	1.7	10	10	55.4	0.0
	8	-	-	-	-	-	-	-	-	-	-	10	6	185.0	1.9	10	10	86.4	0.0
	16	-	-	-	-	-	-	-	-	-	-	10	9	202.3	0.2	10	10	20.3	0.0
	30	-	-	-	-	-	-	-	-	-	-	10	10	39.9	0.0	10	10	3.1	0.0
1000	2	-	-	-	-	-	-	-	-	-	-	10	0	t.lim	0.3	10	10	105.9	0.0
	4	-	-	-	-	-	-	-	-	-	-	10	0	t.lim	1.5	10	10	89.6	0.0
	6	-	-	-	-	-	-	-	-	-	-	10	3	287.2	2.1	10	10	139.6	0.0
	8	-	-	-	-	-	-	-	-	-	-	10	3	278.7	1.8	10	10	91.6	0.0
	16	-	-	-	-	-	-	-	-	-	-	10	0	300.0	8.6	10	10	64.2	0.0
	30	-	-	-	-	-	-	-	-	-	-	10	4	264.3	8.2	10	10	24.0	0.0
total/avg		11	288.5	24	266.3	109	83	156.0	110	103	54.0	280	212	105.2	1.0	280	280	30.6	0.0

The results show that EAF clearly outperforms all other methods on the attempted instances. It solves to proven optimality all instances with $p_{\max} = 20$ and all instances

Table 2: Results for set 1 instances with $p_{\max} = 100$ (time limit = 300 seconds, ILS time not included)

n	m	Existing formulations										New formulations							
		CIQP		PTI		SC			TI			AF				EAF			
		#opt	t(s)	#opt	t(s)	#root	#opt	t(s)	#root	#opt	t(s)	#root	#opt	t(s)	gap_{pm}	#root	#opt	t(s)	gap_{pm}
30	2	10	0.8	10	115.0	10	10	0.2	10	10	73.3	10	10	0.5	0.0	10	10	0.3	0.0
	4	0	t.lim	0	t.lim	10	10	0.1	10	10	18.6	10	10	0.3	0.0	10	10	0.2	0.0
	6	0	t.lim	0	t.lim	10	10	0.0	10	10	6.9	10	10	0.2	0.0	10	10	0.2	0.0
	8	0	t.lim	0	t.lim	10	10	0.0	10	10	4.4	10	10	0.2	0.0	10	10	0.1	0.0
100	2	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	10	10.7	0.0	10	10	7.6	0.0
	4	0	t.lim	0	t.lim	10	3	257.2	0	0	m.lim	10	10	36.5	0.2	10	10	25.9	0.2
	6	0	t.lim	0	t.lim	10	4	224.5	0	0	m.lim	10	10	14.5	0.0	10	10	8.3	0.0
	8	0	t.lim	0	t.lim	10	3	239.7	0	0	t.lim	10	10	8.1	0.0	10	10	10.3	0.0
	16	0	t.lim	0	t.lim	10	8	86.9	9	6	264.0	10	10	2.5	0.0	10	10	1.5	0.0
	30	0	t.lim	0	t.lim	10	10	1.7	10	10	49.3	10	10	0.6	0.0	10	10	0.5	0.0
400	2	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	0	t.lim	0.6	10	10	264.6	0.0
	4	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	2	292.5	2.6	10	10	180.7	0.0
	6	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	7	272.2	2.6	10	10	176.6	0.0
	8	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	9	233.3	1.0	10	10	177.5	0.0
	16	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	8	181.4	4.6	10	10	111.8	0.0
	30	0	t.lim	0	t.lim	0	0	t.lim	0	0	m.lim	10	10	94.6	0.0	10	10	64.1	0.0
700	2	-	-	-	-	-	-	-	-	-	-	0	0	t.lim	-	9	0	t.lim	0.8
	4	-	-	-	-	-	-	-	-	-	-	6	0	t.lim	3.2	9	0	t.lim	3.3
	6	-	-	-	-	-	-	-	-	-	-	6	0	t.lim	5.5	10	0	t.lim	5.5
	8	-	-	-	-	-	-	-	-	-	-	10	0	t.lim	8.4	10	0	t.lim	8.4
	16	-	-	-	-	-	-	-	-	-	-	10	0	t.lim	16.0	10	0	t.lim	16.0
	30	-	-	-	-	-	-	-	-	-	-	10	0	t.lim	28.5	10	0	t.lim	28.4
1000	2	-	-	-	-	-	-	-	-	-	-	0	0	m.lim	-	0	0	m.lim	-
	4	-	-	-	-	-	-	-	-	-	-	0	0	m.lim	-	0	0	m.lim	-
	6	-	-	-	-	-	-	-	-	-	-	0	0	m.lim	-	0	0	m.lim	-
	8	-	-	-	-	-	-	-	-	-	-	7	0	t.lim	6.4	9	0	t.lim	6.4
	16	-	-	-	-	-	-	-	-	-	-	10	0	t.lim	13.5	10	0	t.lim	13.5
	30	-	-	-	-	-	-	-	-	-	-	10	0	t.lim	23.0	10	0	t.lim	23.0
total/avg		10	289.3	10	288.4	90	68	181.9	53	53	102.4	229	136	165.9	4.8	247	160	149.2	4.2

with $p_{\max} = 100$ and $n \leq 400$, so 440 out of the 560 tested instances. The version without enhancements, AF, solves all the instances with up to 400 jobs for $p_{\max} = 20$ and up to 100 jobs for $p_{\max} = 100$, for a total of 348 out of 560 instances. This proves that the enhancements presented in Section 4.2 are very effective. For the unsolved instances, the gaps are extremely small, amounting to just a few units per million on average.

The B&P implemented to solve SC fails in solving some instances with just 100 jobs, which is coherent with the results in Van den Akker *et al.* (1999). The same happens for TI, which fails in solving instances with 100 jobs when $p_{\max} = 100$ and m is small. The performance of both CIPQ and PTI is very poor, but that could be explained by the fact that the two models were originally developed for the $R||\sum w_j C_j$, and hence do not exploit the symmetries induced by the identical machines. The main advantage of CIPQ is related to its polynomial size, but it optimally solves only few instances with just 30 jobs. For what concerns PTI, it is worth mentioning that it was solved by means of a Benders decomposition method in Bülbül and Şen (2017), as discussed in Section 3.3. The authors provided us with the results obtained by their method on our $P||\sum w_j C_j$ instances. Unfortunately, these were worse on average that those obtained by the direct solution of the PTI model by means of the Gurobi solver. This, once more, can be imputed to the fact that their method was developed for the case of unrelated machines.

We conducted further experiments by running the models under larger time limits up to one hour. The summary of the results that we obtained is presented in Table 3, which

Table 3: Number of optimal solutions found under different time limits

(a) $p_{\max} = 20$ (280 instances)

time limit	Formulation					
	CQIP	PTI	SC	TI	AF	EAF
300	11	24	83	103	212	280
600	12	28	86	110	247	280
900	12	31	89	110	254	280
1200	12	32	90	110	260	280
1500	13	32	90	110	265	280
1800	14	32	90	110	268	280
2100	14	32	90	110	272	280
2400	15	32	92	110	275	280
2700	15	32	95	110	276	280
3000	15	32	98	110	276	280
3300	15	32	98	110	277	280
3600	15	32	99	110	278	280

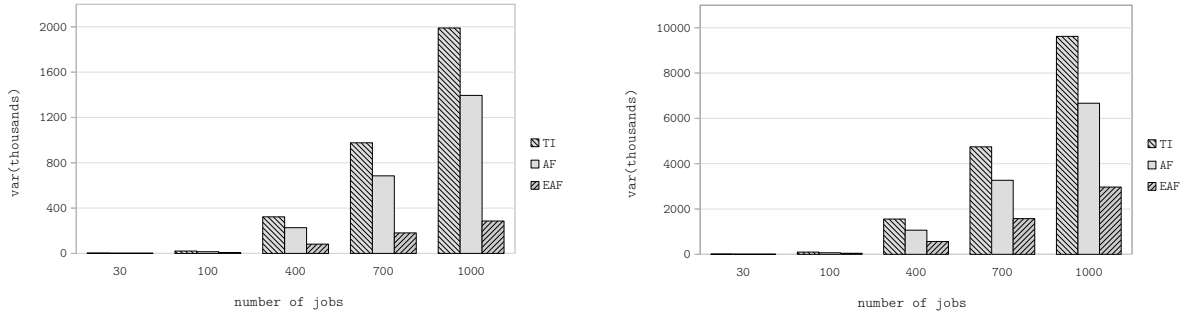
(b) $p_{\max} = 100$ (280 instances)

time limit	Formulation					
	CQIP	PTI	SC	TI	AF	EAF
300	10	10	68	53	136	160
600	11	10	70	60	150	169
900	12	10	73	61	168	178
1200	13	10	73	61	173	180
1500	13	10	74	62	179	192
1800	14	10	76	64	180	200
2100	15	10	77	65	183	206
2400	15	10	77	66	184	210
2700	15	10	78	68	188	213
3000	15	10	78	68	192	216
3300	15	10	79	68	194	219
3600	15	10	80	69	197	222

shows, for each model and for each time limit, the number of optimal solutions found under the different time limits. Once more, it can be observed that AF and EAF clearly outperform the other methods. It can also be seen that the enhancements discussed in Section 4.2 are indeed effective, because EAF solved to proven optimality all instances with $p_{\max} = 20$ in less than 300 seconds, whereas AF did not solve two of them within 1 hour. Concerning the instances with $p_{\max} = 100$, by increasing the time limit from 300 to 3600 seconds, AF was able to solve 61 instances more, including all the unsolved instances with $n = 400$, whereas EAF solved 62 more, including 48 out of 60 instances with $n = 700$ that where not solved within 300 seconds. In total, EAF was able to optimally solve 502 out of 560 instances.

Time indexed formulations, such as PTI, TI, AF and EAF, have a pseudo-polynomial size, and hence may require large amounts of memory when the time horizon grows. It can be noticed indeed that all of them do not solve some instances due to memory limit. In this sense, TI starts to run out of memory for instances with only 100 jobs, PTI is able to deal with instances with up to 400 jobs (although cannot optimally solve them due to time limit), whereas AF and EAF deal with instances with up to 1000 jobs.

This better behavior can be justified by the difference in the number of variables required by models. This fact is graphically highlighted in Figure 3, which presents the average number of variables in thousands, per group of instances having the same n. The reduction of AF and even more EAF with respect to the plain TI model is evident. This effect can be observed more in details in Table 4, which reports, for each group of instances having same n and m, the number of variables in thousands, var(thousands), and the percentage reduction of variables from one model to the next, red(%). From Table 4, it is possible to notice that for the instances with $p_{\max} = 20$ reductions of about 30% are obtained by AF over TI, and of even 80% by EAF over AF. This allows to move from an average of about 7×10^5 TI variables to just 1×10^5 EAF variables. Concerning the instances with $p_{\max} = 100$, EAF formulates the problem using, on average, 50% less variables than AF, which in turns uses 30% less variables than TI. The results also indicates that, as the instance grows, the reductions become larger.



(a) $p_{\max} = 20$ (b) $p_{\max} = 100$

Figure 3: Impact of the proposed enhancements on the number of variables

Table 4: Variables (in thousands) required by the main pseudo-polynomial formulations

		$p_{\max} = 20$					$p_{\max} = 100$				
n	m	var(thousands)			red(%)		var(thousands)			red(%)	
		TI	AF	EAF	AF *vs* TI	EAF *vs* AF	TI	AF	EAF	AF *vs* TI	EAF *vs* AF
30	2	4.8	3.0	1.8	37.5	41.0	20.8	11.8	7.6	43.2	35.5
	4	2.6	1.8	1.3	29.2	29.4	11.9	7.8	5.7	34.4	27.4
	6	1.8	1.3	1.0	24.8	28.6	8.4	5.8	4.5	31.3	21.2
	8	1.4	1.1	0.8	22.2	24.8	6.7	4.6	3.7	32.4	19.6
100	2	51.9	31.4	17.0	39.5	46.0	253.9	150.7	89.7	40.7	40.5
	4	27.6	20.1	10.8	27.0	46.5	128.0	90.5	58.3	29.2	35.6
	6	17.7	13.6	7.6	23.5	44.2	90.1	69.8	44.2	22.5	36.6
	8	13.7	11.1	6.7	19.3	39.9	66.4	51.9	33.7	21.9	35.0
	16	7.5	6.4	4.1	13.9	36.3	36.1	29.7	21.8	17.8	26.5
	30	4.4	3.8	2.7	12.1	30.4	21.7	18.7	14.1	13.8	24.4
400	2	843.2	509.4	185.7	39.6	63.5	4088.9	2409.8	1291.0	41.1	46.4
	4	424.3	309.1	108.0	27.2	65.1	2017.3	1418.7	726.3	29.7	48.8
	6	282.6	220.6	75.8	21.9	65.7	1360.9	1034.0	530.2	24.0	48.7
	8	213.2	172.6	60.8	19.0	64.8	1040.4	827.6	429.4	20.5	48.1
	16	106.9	92.6	34.4	13.4	62.9	533.0	456.4	247.8	14.4	45.7
	30	60.2	54.6	22.4	9.4	59.0	290.0	256.2	151.4	11.7	40.9
700	2	2539.6	1520.0	418.7	40.1	72.5	12613.7	7437.6	3720.1	41.0	50.0
	4	1282.9	930.0	240.6	27.5	74.1	6114.2	4336.5	2073.8	29.1	52.2
	6	874.3	680.5	170.7	22.2	74.9	4161.0	3191.1	1464.8	23.3	54.1
	8	654.0	528.4	132.2	19.2	75.0	3148.3	2514.4	1140.7	20.1	54.6
	16	326.9	284.2	75.2	13.1	73.5	1582.9	1355.0	639.9	14.4	52.8
	30	177.4	160.9	45.7	9.3	71.6	859.6	766.9	386.1	10.8	49.6
1000	2	5216.2	3129.4	671.1	40.0	78.6	25320.3	14942.6	7020.6	41.0	53.0
	4	2626.0	1902.1	378.1	27.6	80.1	12725.0	9083.5	3980.6	28.6	56.2
	6	1757.3	1369.4	267.0	22.1	80.5	8445.0	6673.8	2853.3	21.0	57.2
	8	1314.7	1063.1	207.7	19.1	80.5	6327.7	5051.4	2090.2	20.2	58.6
	16	664.4	579.4	115.4	12.8	80.1	3212.8	2742.8	1154.8	14.6	57.9
	30	359.4	326.5	69.6	9.2	78.7	1711.3	1531.0	688.2	10.5	55.1
total/avg		709.2	497.4	119.0	29.9	76.1	3435.6	2365.3	1098.6	31.2	53.6

5.4 Computational results on benchmark set 2

Very recently, Kowalczyk and Leus (2018) improved the branch-and-price method of Van den Akker *et al.* (1999) by introducing the use of stabilization techniques, generic branching, and a *zero-suppressed binary decision diagram* (ZDDs) for solving the pricing subproblem. By combining these techniques, they devised three main methods: the first, named VHV-DP, uses the branching scheme and the DP in Van den Akker *et al.* (1999), but includes stabilization; the second, VHV-ZDD, also uses the branching scheme of Van den Akker *et al.* (1999) and stabilization, but solves the pricing subproblem with the the ZDDs technique; the third, RF-ZDD, differs from the second by the fact that the branching decisions follow the generic scheme of Ryan and Foster (1981). The three methods were computationally tested on the benchmark set 2 described in Section 5.1.

We performed experiments by running for 600 seconds our best mathematical formulation, namely, EAF, on the same instances and compared our results with those obtained by Kowalczyk and Leus (2018). As in the previous section, for instances involving more than 100 jobs EAF used the ILS of Section 5.2 to obtain an initial solution. According to the single thread results in `https://www.cpubenchmark.net/`, the processor used by Kowalczyk and Leus (2018), an Intel Core i7-3770 3.40 GHz, is about 1.9 times faster than our Intel Xeon E5530 2.40 GHz processor.

The results that we obtained are presented in Tables 5, 6 and 7. Columns #opt and t(s) represent, for each method, class and group of 20 instances, the number of instances solved to proven optimality and the average computational time, respectively. Concerning RF-ZDD, VHV-ZDD and VHV-DP, the t(s) values report the average times in Kowalczyk and Leus (2018) (i.e., not multiplied by 1.9), but now include in the computation of the average the entire time limit value (3600 seconds) for those instances that were not solved to proven optimality. The three methods by Kowalczyk and Leus (2018) obtained good results but could not solve all instances to proven optimality. In general, VHV-DP was able to solve more instances than RF-ZDD and VHV-ZDD, whereas RF-ZDD proved to be less time consuming on average. We can observe from the tables that our ILS+EAF algorithm generally outperformed the other methods. Indeed, it could solve all 2400 instances within the time limit and usually very quickly (only 10 instances required more than 300 seconds, and the slowest case required 395 seconds). Instances with up to 100 jobs were solved in a matter of seconds. Instances with 150 jobs required longer times, also due to the 100 seconds allowed for the ILS execution. The instances from class VI represent the most challenging testbed for ILS+EAF, nevertheless they were all solved in about 50 seconds on average.

Table 5: Results on set 2 instances – Classes I and II

		Class I								Class II							
n	m	RF-ZDD		VHV-ZDD		VHV-DP		ILS+EAF		RF-ZDD		VHV-ZDD		VHV-DP		ILS+EAF	
		#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)
20	3	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.1
	5	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.1
	8	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	10	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	12	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
50	3	20	1.9	20	1.2	20	1.1	20	0.0	20	0.7	20	0.6	20	0.7	20	0.9
	5	20	0.9	20	0.5	20	0.6	20	0.0	20	0.6	19	180.6	20	0.6	20	0.7
	8	20	0.4	20	0.2	20	0.2	20	0.0	20	0.4	19	180.3	19	180.3	20	0.6
	10	20	0.3	20	0.1	20	0.1	20	0.0	20	0.3	20	0.2	20	0.2	20	0.5
	12	20	0.1	20	0.1	20	0.1	20	0.0	20	0.1	20	0.1	20	0.1	20	0.3
100	3	20	62.7	20	59.8	20	64.5	20	0.2	20	23.0	20	37.6	20	34.0	20	9.4
	5	19	210.3	20	31.5	20	32.5	20	0.2	20	56.0	19	286.4	18	415.1	20	8.4
	8	19	194.4	20	7.0	20	7.1	20	0.1	20	53.2	20	133.6	20	101.0	20	5.0
	10	19	189.3	20	3.7	20	3.8	20	0.1	20	41.4	20	18.1	20	17.9	20	4.6
	12	20	7.3	20	2.2	20	2.1	20	0.1	20	50.8	20	74.8	20	75.0	20	2.9
150	3	18	873.8	20	1162.0	20	1123.4	20	100.4	20	362.3	15	1590.9	17	1147.7	20	127.0
	5	20	323.8	20	813.9	20	813.6	20	100.2	16	1438.2	16	1485.2	15	1656.8	20	137.7
	8	18	475.6	20	275.9	20	261.5	20	100.2	17	1035.5	16	1641.0	16	1624.3	20	126.0
	10	20	85.5	20	117.4	20	132.1	20	100.1	16	1273.3	16	1183.7	17	1168.7	20	110.0
	12	20	52.8	20	53.0	20	54.8	20	100.1	18	599.4	20	348.9	20	356.4	20	105.8
total/avg		393	124.0	400	126.4	400	124.9	400	25.1	387	246.8	380	358.1	382	338.9	400	32.0

Table 6: Results on set 2 instances – Classes III and IV

n	m	Class III								Class IV							
		RF-ZDD		VHV-ZDD		VHV-DP		ILS+EAF		RF-ZDD		VHV-ZDD		VHV-DP		ILS+EAF	
		#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)
20	3	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	5	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	8	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	10	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	12	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
50	3	20	2.1	20	4.0	20	4.2	20	0.1	20	0.7	20	0.7	20	1.0	20	0.5
	5	20	1.1	20	1.9	20	2.0	20	0.1	20	1.2	20	2.1	20	2.6	20	0.2
	8	20	0.6	20	0.6	20	0.6	20	0.1	20	0.4	20	0.5	20	0.6	20	0.1
	10	20	0.4	20	0.4	20	0.5	20	0.1	20	0.4	20	0.5	20	0.6	20	0.1
	12	20	0.2	20	0.2	20	0.2	20	0.1	20	0.1	20	0.1	20	0.2	20	0.1
100	3	20	41.5	20	158.8	20	162.3	20	1.0	20	69.7	20	186.7	20	169.1	20	17.5
	5	20	22.4	20	109.5	20	103.7	20	0.8	20	33.3	20	147.8	20	159.3	20	11.6
	8	20	10.7	20	40.8	20	39.3	20	0.4	20	14.8	20	44.3	20	47.1	20	1.2
	10	20	7.1	20	20.5	20	20.0	20	0.3	20	9.7	20	31.5	20	34.8	20	1.0
	12	20	6.1	20	12.2	20	12.4	20	0.2	19	186.9	20	18.4	20	20.4	20	0.4
150	3	20	430.2	20	1508.3	20	1517.5	20	100.6	20	973.5	16	2529.2	20	2108.4	20	108.7
	5	20	166.6	20	965.5	20	1063.9	20	100.5	20	383.7	20	1281.3	20	1239.4	20	104.0
	8	20	71.9	20	504.8	20	517.1	20	100.3	20	164.3	20	612.9	20	648.8	20	101.9
	10	20	51.1	20	360.9	20	361.6	20	100.4	20	93.3	20	358.6	20	429.6	20	101.8
	12	20	36.8	20	224.4	20	245.2	20	100.2	20	69.0	19	458.2	20	287.1	20	101.0
total/avg		400	42.4	400	195.6	400	202.5	400	25.3	399	100.1	395	283.6	400	257.4	400	27.5

Table 7: Results on set 2 instances – Classes V and VI

n	m	Class V								Class VI							
		RF-ZDD		VHV-ZDD		VHV-DP		ILS+EAF		RF-ZDD		VHV-ZDD		VHV-DP		ILS+EAF	
		#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)	#opt	t(s)
20	3	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	5	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	8	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	10	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
	12	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0	20	0.0
50	3	20	1.5	20	1.8	20	1.9	20	1.2	20	2.3	20	3.4	20	2.7	20	2.3
	5	20	1.5	20	2.3	20	3.0	20	0.7	20	1.4	20	1.8	20	1.8	20	1.2
	8	20	0.3	20	0.6	20	0.7	20	0.2	20	1.0	20	0.7	20	0.9	20	0.8
	10	20	0.4	20	0.4	20	0.6	20	0.1	20	1.1	20	0.7	20	0.7	20	0.4
	12	20	0.2	20	0.2	20	0.2	20	0.1	20	1.1	20	1.0	20	0.8	20	0.4
100	3	19	314.0	20	359.5	20	290.4	20	46.8	20	78.7	20	242.5	20	200.1	20	73.0
	5	20	55.0	20	180.4	20	200.7	20	17.2	20	37.8	20	126.7	20	112.8	20	22.8
	8	19	197.8	18	497.5	18	408.9	20	2.3	20	25.2	20	58.0	20	44.6	20	24.9
	10	20	12.9	20	35.2	20	39.5	20	3.6	20	22.7	20	32.7	20	32.2	20	8.1
	12	18	368.1	18	379.3	19	209.4	20	1.0	20	19.0	20	20.8	20	18.8	20	17.8
150	3	20	1538.0	14	3086.7	20	2508.5	20	120.9	20	800.0	18	3028.9	20	2662.2	20	233.1
	5	20	584.1	20	1488.4	19	1467.4	20	107.2	20	398.4	20	1751.4	20	1434.7	20	178.5
	8	19	498.7	18	1053.2	18	1022.9	20	106.4	20	257.7	20	691.6	20	633.1	20	145.3
	10	20	164.4	20	422.1	20	541.1	20	103.1	20	210.4	20	439.3	20	406.3	20	151.8
	12	19	272.2	20	291.8	20	402.0	20	101.7	20	192.7	20	303.6	20	265.7	20	135.1
total/avg		394	200.5	388	390.0	394	354.9	400	30.6	400	102.5	398	335.2	400	290.9	400	49.8

6 Conclusions

In this work, we have proposed pseudo-polynomial arc-flow (AF) formulations to solve the problem of scheduling a set of jobs on a set of identical parallel machines by minimizing the total weighted completion time. A first straight AF model already benefits from the fact that schedules follow a weighted shortest processing time (WSPT) rule on each machine. A second enhanced AF model (EAF) improves AF by embedding further reduction techniques. EAF needs on average less than 50% of the variables required by AF, and, in some cases, this number drops to less than 20%. Computational experiments showed that EAF is very effective and solves instances with up to 1000 jobs and 30 machines, performing much better than direct time-indexed formulations and even advanced branch-and-price methods. Still, troubles might arise when the processing times of the jobs assume large values, and more research is envisaged to solve large instances of this type, possibly involving heuristics and column generation methods.

An interesting future research direction also involves the application of the discussed techniques to problems with release dates and/or setup times. In such cases, optimal solutions might nor respect the WSPT sorting, and thus these problems might be very challenging for AF models. In general, in the scheduling area much work has been done on time-indexed formulations, but the application of AF models is new and could lead to large computational benefits in the solution of many problems.

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