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Simplified path integral for supersymmetric quantum mechanics and type-A trace anomalies

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Abstract: Particles in a curved space are classically described by a nonlinear sigma model action that can be quantized through path integrals. The latter require a precise regularization to deal with the derivative interactions arising from the nonlinear kinetic term. Recently, for maximally symmetric spaces, simplified path integrals have been developed: they allow to trade the nonlinear kinetic term with a purely quadratic kinetic term (linear sigma model). This happens at the expense of introducing a suitable effective scalar potential, which contains the information on the curvature of the space. The simplified path integral provides a sensible gain in the efficiency of perturbative calculations. Here we extend the construction to models with $N = 1$ supersymmetry on the worldline, which are applicable to the first quantized description of a Dirac fermion. As an application we use the simplified worldline path integral to compute the type-A trace anomaly of a Dirac fermion in d dimensions up to $d = 16$.

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Contents

1 Introduction

Recently, a simplified version of the path integral for the quantum mechanics of particles on maximally symmetric spaces has been constructed [1, 2]. It realizes an old proposal [3], which suggests a peculiar use of Riemann normal coordinates to trade the nonlinear kinetic term of the classical action of the particle with a purely quadratic kinetic term (linear sigma model) at the expense of introducing a suitable effective scalar potential. The conjecture was originally made for an arbitrary curved space, but the explicit proof presented in [1, 2] works only for spaces with maximal symmetry. The more subtle question of its validity on arbitrary geometries remains open, though a positive answer seems unlikely.

In the present paper we extend the construction to a $N = 1$ supersymmetric quantum mechanics, so that the resulting path integral can be used in a first quantized description of a Dirac fermion. In particular, we use the new path integral to compute the type-A trace anomalies of a Dirac field, up to $d = 16$ dimensions, extending analogous calculations performed in [1, 2] for the conformal scalar.

Other methods for identifying the type-A trace anomalies for the spin-1/2 field in higher dimensions are probably more efficient. One may use the zeta function approach as in [4, 5], or the AdS/CFT holographic paradigm as in [6], which extended to spin 1/2 the scalar case treated in [7, 8]. However, the path integral method has a clearer physical interpretation. It is a calculation from first principles, in which the particle producing the one-loop anomaly performs its virtual loop. This visualization makes the method more intuitive and flexible, allowing to study many other observables, as typical in the worldline formalism. The latter employs worldline path integral to represent and compute effective actions and scattering amplitudes, see [9] for a review in flat space, refs. [10–17] for recent applications to gauge theories, refs. [18–25] for extensions to curved spaces, and refs. [26–31] for applications to higher spin theories. Our main motivation for the present paper is to search for simpler methods for improving the efficiency of worldline calculations in curved spaces. The case of type-A trace anomalies is both a useful check as well as an interesting issue to investigate.

Thus, in section 2 we present a quick review of the scalar particle, which we then extended to the $N = 1$ supersymmetric version of the model. In section 3 we compute the perturbative expansion of the path integral for periodic (antiperiodic) boundary condition for worldline bosons (fermions), as appropriate for addressing one-loop quantities in worldline applications, and in section 4 we present an application of the simplified path integral to identify the type-A trace anomalies of a Dirac fermion (correcting a minor misprint for the spinor trace anomaly in $d = 12$ found in the literature). We verify our results for the anomalies with the zeta function and holographic formulas mentioned above. Eventually, we present our conclusions and outlook in section 5. To make the presentation self-contained we list in appendix A the relevant formulas for various geometrical objects of maximally symmetric spaces in Riemann normal coordinates, and in appendix B we report a list of the Wick contractions used in the main text.

2 Construction

A nonrelativistic particle of unit mass in a curved d-dimensional space has a lagrangian that takes the form of a nonlinear sigma model

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j \tag{2.1}$$

and corresponding hamiltonian

$$H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j \tag{2.2}$$

where $g_{ij}(x)$ is the metric in an arbitrary coordinate system, $\dot{x}^i = \frac{dx^i}{dt}$, and p_i the momenta conjugated to x^i. Canonical quantization produces a quantum hamiltonian

$$\hat{H}_\xi(\hat{x}, \hat{p}) = \frac{1}{2} g^{-\frac{1}{4}}(\hat{x}) \hat{p}_i g^{\frac{1}{2}}(\hat{x}) g^{ij}(\hat{x}) \hat{p}_j g^{-\frac{1}{4}}(\hat{x}) + \frac{\xi}{2} R(\hat{x}) \tag{2.3}$$

where hats denote operators. Ordering ambiguities have been partially fixed by requiring background coordinate invariance, leaving only a possible nonminimal coupling proportional to the scalar curvature R and parametrized by the coupling constant ξ. For simplicity we set the coupling $\xi = 0$ (minimal coupling). Other couplings, such as the conformal coupling $\xi = \frac{d-2}{4(d-1)}$ or the value $\xi = \frac{1}{4}$ that allows for a $N = 1$ supersymmetrization (it appears in the square of the Dirac operator), can be reintroduced by adding a scalar potential.

To obtain the simplified path integral, one starts by studying the evolution operator in euclidean time β (the heat kernel)

$$\hat{K}(\beta) = e^{-\beta \hat{H}} \tag{2.4}$$

where $\hat{H} \equiv \hat{H}_0$ is the hamiltonian operator with minimal coupling, which satisfies the heat equation

$$-\partial_\beta \hat{K}(\beta) = \hat{H}\hat{K}(\beta)\,, \qquad \hat{K}(0) = \mathbb{1}\,. \tag{2.5}$$

Using position eigenstates

$$\hat{x}^i|x\rangle = x^i|x\rangle\,, \qquad \langle x|x'\rangle = \frac{\delta^{(d)}(x-x')}{\sqrt{g(x)}}\,, \tag{2.6}$$

and corresponding resolution of the identity

$$\mathbb{1} = \int d^dx\sqrt{g(x)}\,|x\rangle\langle x|\,, \tag{2.7}$$

one constructs scalar wave functions $\psi(x) = \langle x|\psi\rangle$ for any vector $|\psi\rangle$ of the Hilbert space. In particular, the matrix element of the evolution operator

$$K(x,x';\beta) = \langle x|e^{-\beta\hat{H}}|x'\rangle \tag{2.8}$$

gives a biscalar under change of coordinates, and the heat equation takes the form

$$-\partial_\beta K(x,x';\beta) = -\frac{1}{2}\nabla^2_x\,K(x,x';\beta)\,, \qquad K(x,x';0) = \frac{\delta^{(d)}(x-x')}{\sqrt{g(x)}}\,, \tag{2.9}$$

where ∇^2_x is the scalar laplacian $\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$ acting on the x coordinates. Its solution has a well-defined path integral representation in terms of the nonlinear sigma model, see for example [32]. However, one can manipulate the heat equation to obtain a simplified equation admitting a path integral representation in terms of a *linear* sigma model. This is done as follows. One first transforms the transition amplitude into a bidensity

$$\overline{K}(x,x',\beta) = g^{\frac{1}{4}}(x)K(x,x',\beta)g^{\frac{1}{4}}(x')\,, \tag{2.10}$$

for which the heat equation takes the form

$$-\partial_\beta\overline{K}(x,x';\beta) = -\frac{1}{2}g^{\frac{1}{4}}(x)\nabla^2_x\Big(g^{-\frac{1}{4}}(x)\overline{K}(x,x';\beta)\Big)\,, \qquad \overline{K}(x,x';0) = \delta^{(d)}(x-x')\,. \tag{2.11}$$

Then, one evaluates the differential operator appearing on the right hand side of this equation to obtain the identity

$$-\frac{1}{2}g^{\frac{1}{4}}\nabla^2\,g^{-\frac{1}{4}} = -\frac{1}{2}\partial_i g^{ij}\partial_j + V_0\,, \tag{2.12}$$

with derivatives that act through, except in the effective scalar potential V_0 given by

$$V_0 = -\frac{1}{2}g^{-\frac{1}{4}}\partial_i\sqrt{g}g^{ij}\partial_j g^{-\frac{1}{4}}\,, \tag{2.13}$$

where all derivatives stop after acting on the last function. Thus the heat equation reads

$$-\partial_\beta\overline{K}(x,x';\beta) = \Big(-\frac{1}{2}\partial_i g^{ij}(x)\partial_j + V_0(x)\Big)\overline{K}(x,x';\beta)\,. \tag{2.14}$$

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At this stage one restricts to maximally symmetric spaces, uses Riemann normal coordinates centered at the initial point x', and realizes that the metric $g^{ij}(x)$, appearing in the term $\partial_i g^{ij}(x)\partial_j$, may be replaced by the constant metric δ^{ij}. This chain of steps brings one to consider the equivalent heat equation

$$- \partial_\beta \overline{K}(x,x';\beta) = \Big(-\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_0(x)\Big)\overline{K}(x,x';\beta)\,, \tag{2.15}$$

where the simplified hamiltonian operator $H = -\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_0(x)$ is interpreted as that of a particle on a flat space (in cartesian coordinates), interacting with an effective potential V_0 of quantum origin (it is proportional to $\hbar^2$ in arbitrary units). Evidently, this last equation admits a path integral representation in terms of a linear sigma model.

The replacement of $g^{ij}(x)$ with δ^{ij} is valid since by the hypothesis of maximal symmetry, and using Riemann normal coordinates with origin at x' (which then has a vanishing value of its coordinates, namely $x'^i = 0$), one deduces that $K(x,0;\beta)$ can only depend on the radial coordinate

$$r = \sqrt{\delta_{ij}x^i x^j} \tag{2.16}$$

since there is no other possible tensor that may be used to contract the indices of the coordinates x^i to form a scalar.[1] Then, using the explicit form of the metric (see appendix A, eqs. (A.4)–(A.6)), one indeed verifies that

$$\partial_i(g^{ij}(x)\partial_j \overline{K}(x,0;\beta)) = \delta^{ij}\partial_i\partial_j \overline{K}(x,0;\beta) \tag{2.17}$$

i.e. the validity of eq. (2.15) (again, we recall that in Riemann normal coordinates centered at x', the coordinates of the origin are $x'^i = 0$)—see refs. [1, 2] for more details.

Thus, one ends up with the linear sigma model

$$L(x,\dot{x}) = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + V_0(x) \tag{2.18}$$

that can be used instead of (2.1) in a path integral to evaluate the transition amplitude $\overline{K}(x,0;\beta)$ between an initial point $x'^i = 0$ (taken as the origin of the Riemann normal coordinates, that must be used in this set-up) and a final point x^i in euclidean time β. As the space is maximally symmetric, it is in particular homogeneous, and the origin can be chosen in any desired point of the manifold. This just to point out that the initial point of the heat kernel $K(x,0;\beta)$ can be kept arbitrary.

The effective potential can be explicitly evaluated on maximally symmetric spaces, and for a sphere of radius a and mass parameter $M = \frac{1}{a}$ one finds

$$\begin{aligned} V_0(x) &= -\frac{1}{2} g^{-\frac{1}{4}}\partial_i\sqrt{g}g^{ij}\partial_j g^{-\frac{1}{4}} \\ &= \frac{(d-1)}{8}\left[\frac{(d-5)}{4}\left(\frac{f'(r)}{1+f(r)}\right)^2 + \frac{1}{1+f(r)}\left(\frac{(d-1)}{r}f'(r) + f''(r)\right)\right] \\ &= \frac{d(1-d)}{12}M^2 + \frac{(d-1)(d-3)}{48}\frac{\Big(5(Mr)^2 - 3 + \big((Mr)^2+3\big)\cos(2Mr)\Big)}{r^2\sin^2(Mr)} \end{aligned} \tag{2.19}$$

[1] Note that in Riemann normal coordinates the metric takes the form given in (A.4), so that $r^2 \equiv \delta_{ij}x^i x^j = g_{ij}(x)x^i x^j$.

where we have used the explicit metric in (A.4) together with eq. (A.6). Note also that the potential is an even function of the radial coordinate $r = \sqrt{\delta_{ij}x^i x^j}$. The simplified path integral based on the linear sigma model (2.18) has been tested and used extensively in perturbative calculation in [1, 2], verifying its superior efficiency with respect to the equivalent path integral based on the nonlinear sigma model, used for example in [33].

Let us now turn to the supersymmetric version of the particle mechanics, identified by the (euclidean) lagrangian

$$L = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + \frac{1}{2}\psi^a(\dot{\psi}_a + \dot{x}^i\omega_{iab}(x)\psi^b) \tag{2.20}$$

where ψ^a are real Grassmann variables with flat indices, and ω_{iab} is the spin connection built from the vielbein e_i^a. The fermionic variables ψ^a are the supersymmetric partners of the coordinates x^i. Upon quantization they lead to operators that satisfy the anticommutation relations

$$\{\hat{\psi}^a, \hat{\psi}^b\} = \delta^{ab}\,, \tag{2.21}$$

a Clifford algebra which can either be represented by the usual Dirac gamma matrices ($\hat{\psi}^a = \frac{1}{\sqrt{2}}\gamma^a$, with $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$), or treated by a fermionic path integral — we refer to [19] and references therein for further details on this supersymmetric model, and on its use in worldline calculations for Dirac fermions in background gravity. Here we just recall that the spinning particle model was originally introduced in [34–36].

In the subsequent discussion we find it more useful to start our analysis using the gamma matrices. The conserved quantum supersymmetric charge of the model is proportional to the Dirac operator, and reads

$$\hat{Q} = -\frac{i}{\sqrt{2}}\nabla\!\!\!/(\omega) = -\frac{i}{\sqrt{2}}\gamma^a e_a^i(x)\left(\partial_i + \frac{1}{4}\omega_{iab}(x)\gamma^a\gamma^b\right) \tag{2.22}$$

while the related quantum hamiltonian becomes

$$\hat{H} = \hat{Q}^2 = -\frac{1}{2}\nabla\!\!\!/^2 = -\frac{1}{2}g^{ij}(x)\nabla_i(\omega,\Gamma)\nabla_j(\omega) + \frac{1}{8}R\,, \tag{2.23}$$

where we have indicated the connections present in the various covariant derivatives. Of course, all these operators act on a spinorial wave function (a Dirac spinor).

The heat kernel associated to this hamiltonian

$$\mathbb{K} = e^{-\beta\hat{H}} \tag{2.24}$$

has quantum mechanical matrix elements

$$\mathbb{K}_{\alpha\alpha'}(x,0;\beta) = \langle x,\alpha|e^{-\beta\hat{H}}|0,\alpha'\rangle$$

where α, α' are spinorial indices. In the following we will not show the spinorial indices explicitly, and just remember that $\mathbb{K}$ is matrix-valued. Now, using the fact that the space under consideration is maximally symmetric, one deduces that the heat kernel $\mathbb{K}(x,0;\beta)$

can only be a function of x^2, $\gamma_a\gamma^a \sim \mathbb{1}$ and $\delta_{ia}x^i\gamma^a$. In addition, as the gamma matrices appear only in even combinations (they are contained quadratically in the spin connections inside the hamiltonian (2.23)), one finds that the dependence on $\delta_{ia}x^i\gamma^a$ arises only through its square

$$(\delta_{ia}x^i\gamma^a)^2 = \mathbb{1}\,x^2 \tag{2.25}$$

which is again proportional to the identity matrix. Thus, the full heat kernel is proportional to the identity, and must be a function of $r = \sqrt{\delta_{ij}x^ix^j}$ only,

$$\mathbb{K}(x, 0; \beta) = \mathbb{1}\,U(r; \beta)\ . \tag{2.26}$$

Equipped with this result let us analyze the heat equation satisfied by the bidensity

$$\overline{\mathbb{K}}(x, 0; \beta) = g^{1/4}(x)\mathbb{K}(x, 0; \beta)g^{1/4}(0)\,, \tag{2.27}$$

(the value $g(0) = 1$ is actually irrelevant) which is

$$-\partial_\beta\overline{\mathbb{K}}(x, 0; \beta) = g^{1/4}(x)\hat{H}g^{-1/4}(x)\,\overline{\mathbb{K}}(x, 0; \beta)\ . \tag{2.28}$$

By expanding out the expression of the hamiltonian given in equation (2.23), we write

$$\begin{aligned} g^{1/4}\hat{H}g^{-1/4} =& -\frac{1}{2}g^{1/4}\nabla^2 g^{-1/4} \\ & -\frac{1}{8}(\partial_i\omega^i{}_{ab})\gamma^{ab} - \frac{1}{4}\omega^i{}_{ab}\gamma^{ab}\partial_i \\ & -\frac{1}{32}\omega_{iab}\,\omega^i{}_{cd}\,\gamma^{ab}\gamma^{cd} + \frac{1}{8}R\ . \end{aligned} \tag{2.29}$$

Using the explicit expression of the spin connection (A.15), which satisfies the Fock-Schwinger gauge (A.17), it is easy to check that the terms in the second line do not contribute when applied to $\bar{\mathbb{K}}$ (they give rise to terms proportional to $x^i\omega_{iab} \sim 0$, because of the Fock-Schwinger gauge), whereas the terms in the third line give expressions proportional to $\mathbb{1}$. In particular we find

$$-\frac{1}{32}\omega_{iab}\,\omega^i{}_{cd}\,\gamma^{ab}\gamma^{cd} = \frac{d-1}{8}M^2\left(\frac{1-\cos(Mr)}{\sin(Mr)}\right)^2\mathbb{1}\ . \tag{2.30}$$

Thus, recalling eq. (2.12) and the possibility of replacing $g^{ij}(x)$ with δ^{ij} in the first term of (2.12), we find that a simplified heat equation holds

$$-\partial_\beta\overline{\mathbb{K}}(x, 0; \beta) = \Big(-\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_{\frac{1}{2}}(x)\Big)\overline{\mathbb{K}}(x, 0; \beta) \tag{2.31}$$

with

$$V_{\frac{1}{2}}(x) = V_0(x) + \frac{d(d-1)M^2}{8} + \frac{d-1}{8}M^2\left(\frac{1-\cos(Mr)}{\sin(Mr)}\right)^2 \tag{2.32}$$

where the second addendum is just $\frac{1}{8}R$, and $V_0(x)$ is given in (2.19).

Expressions (2.31) and (2.32) are the crucial results. They allow to find a simplified path integral. The message they carry is that the heat kernel of a spinorial operator on maximally symmetric spaces, when written in Riemann normal coordinates, satisfies a flat heat equation with the information on the curvature fully encoded in an effective potential, just as it happens for the heat kernel of a scalar particle. As such it is straightforward to represent it as the path integral of a linear sigma model

$$\overline{\mathbb{K}}(x,0;\beta) = \mathbb{1} \int_{x(0)=0}^{x(\beta)=x} Dx\; e^{-S[x]}\,, \quad S[x] = \int_0^\beta dt \left(\frac{1}{2}\delta_{ij}\dot{x}^i(t)\dot{x}^j(t) + V_{\frac{1}{2}}(x(t)) \right) \tag{2.33}$$

with the effective potential $V_{\frac{1}{2}}$ given explicitly as a function of $r = \sqrt{\delta_{ij}x^i x^j}$ by

$$\begin{aligned} V_{\frac{1}{2}}(x) = {}& \frac{d(d-1)}{24}M^2 + \frac{(d-1)(d-3)}{48}\frac{\Big(5(Mr)^2 - 3 + \big((Mr)^2+3\big)\cos(2Mr)\Big)}{r^2\sin^2(Mr)} \\ & + \frac{(d-1)}{8}M^2 \left(\frac{1-\cos(Mr)}{\sin(Mr)} \right)^2 . \end{aligned} \tag{2.34}$$

Of course, one could reintroduce free worldline fermions ψ^a to represent the identity with a Grassmann path integral, so to have the full linear sigma model lagrangian

$$L = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \frac{1}{2}\psi_a\dot{\psi}^a + V_{\frac{1}{2}}(x) \tag{2.35}$$

which may be compared with the original nonlinear sigma model we started with in eq. (2.20). One could then use antiperiodic boundary conditions on the ψ's to produce the trace on the spinor indices, periodic boundary conditions to produce the trace with an insertion of γ^5, or more generally leave open boundary conditions. However, at this stage this is just an amusing observation, as the heat kernel remains trivial on the spinor indices, in particular traces are trivially computed.

In the following section we test the previous simplified path integral by computing its perturbative expansion. We then use it to obtain the type-A trace anomalies of a Dirac field coupled to gravity in dimensions $d \leq 16$.

3 Perturbative expansion

The short-time perturbative expansion of the kernel (2.33) can be formally written as a power series in β

$$\overline{\mathbb{K}}(x,0;\beta) = g^{1/4}(x)\frac{e^{-\frac{x^2}{2\beta}}}{(2\pi\beta)^{\frac{d}{2}}}\sum_{n=0}^{\infty} a_n(x,0)\beta^n\,, \tag{3.1}$$

where a_n are the so-called Seeley-DeWitt coefficients. In general they are matrix-valued, but as we have discussed they are proportional to the identity matrix on maximally symmetric spaces. In order to compute perturbatively the expansion with our simplified path integral, we find it convenient to use a rescaled time $\tau = t/\beta$, so that

$$S[x] = \int_0^1 d\tau \left(\frac{1}{2\beta}\delta_{ij}\dot{x}^i\dot{x}^j + \beta V_{\frac{1}{2}}(x(\tau)) \right), \tag{3.2}$$

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where the dot now indicates derivative with respect to τ. Then we Taylor expand the potential about the origin of the Riemann coordinates

$$V_{\frac{1}{2}}(x) = M^2 \sum_{l=0}^{\infty} k_{2l} \, (Mr)^{2l} \,, \quad \Rightarrow \quad S[x] = \int_0^1 d\tau \frac{1}{2\beta} \delta_{ij} \dot{x}^i \dot{x}^j + \sum_{l=0}^{\infty} S_{2l}[x] \tag{3.3}$$

and retain only the relevant "coupling constants" k_{2l} needed to carry out the expansion at the desired order. Explicitly,

$$S_{2l}[x] = \beta M^{2+2l} k_{2l} \int_0^1 d\tau \, (\delta_{ij} x^i x^j)^l \,. \tag{3.4}$$

The perturbative expansion is obtained by considering that the propagator associated to the free kinetic term is of order β, and reads

$$\langle x^i(\tau) x^j(\tau') \rangle = -\beta \delta^{ij} \Delta(\tau, \tau') \,, \quad \Delta(\tau, \tau') = \frac{1}{2}|\tau - \tau'| - \frac{1}{2}(\tau + \tau') + \tau\tau' \,, \tag{3.5}$$

while each vertex adds a power of β. Therefore, in order to carry out an expansion say to order β^m, one needs to retain couplings up to $k_{2(m-1)}$.

Specifically, we compute the expansion up to order β^8, which requires the following coupling constants extracted from $V_{\frac{1}{2}}$,[2]

$$\begin{aligned}
k_0 &= d(d-1)\left(-\frac{1}{12} + \frac{1}{8}\right) = \frac{d(d-1)}{24} \\
k_2 &= (d-1)\left((d-3)\frac{1}{120} + \frac{1}{32}\right) = \frac{(d-1)(4d+3)}{480} \\
k_4 &= (d-1)\left((d-3)\frac{1}{756} + \frac{1}{192}\right) = \frac{(d-1)(16d+15)}{12096} \\
k_6 &= (d-1)\left((d-3)\frac{1}{5400} + \frac{17}{23040}\right) = \frac{(d-1)(64d+63)}{345600} \\
k_8 &= (d-1)\left((d-3)\frac{1}{41580} + \frac{31}{322560}\right) = \frac{(d-1)(256d+255)}{10644480} \\
k_{10} &= (d-1)\left((d-3)\frac{691}{232186500} + \frac{691}{58060800}\right) = \frac{691(d-1)(1024d+1023)}{237758976000} \\
k_{12} &= (d-1)\left((d-3)\frac{1}{2806650} + \frac{5461}{3832012800}\right) = \frac{(d-1)(4096d+4095)}{11496038400} \\
k_{14} &= (d-1)\left((d-3)\frac{3617}{86837751000} + \frac{929569}{5579410636800}\right) = \frac{3617(d-1)(16384d+16383)}{1422749712384000} \,.
\end{aligned} \tag{3.6}$$

For simplicity, we consider the diagonal part of the heat kernel only by setting $x = 0$, which is relevant to obtain the trace anomalies or to compute the one-loop effective action

[2] The first addenda in the expressions below are the contributions from V_0, the others are the contribution from $\frac{1}{8}R$ and $\omega\omega$ term.

of a Dirac spinor. This involves the following correlators

$$
\begin{aligned}
\overline{\mathbb{K}}(0,0;\beta) = \frac{\mathbb{1}}{(2\pi\beta)^{\frac{d}{2}}} e^{-S_0} \exp\Big[& -\underbrace{\langle S_2\rangle}_{O(\beta^2)} - \underbrace{\langle S_4\rangle}_{O(\beta^3)} \underbrace{-\langle S_6\rangle + \frac{1}{2}\langle S_2^2\rangle_c}_{O(\beta^4)} \underbrace{-\langle S_8\rangle + \langle S_4 S_2\rangle_c}_{O(\beta^5)} \\
& \underbrace{-\langle S_{10}\rangle + \langle S_6 S_2\rangle_c + \frac{1}{2}\langle S_4^2\rangle_c - \frac{1}{3!}\langle S_2^3\rangle_c}_{O(\beta^6)} \underbrace{-\langle S_{12}\rangle + \langle S_8 S_2\rangle_c + \langle S_6 S_4\rangle_c - \frac{1}{2}\langle S_4 S_2^2\rangle_c}_{O(\beta^7)} \\
& \underbrace{-\langle S_{14}\rangle + \langle S_{10} S_2\rangle_c + \langle S_8 S_4\rangle_c + \frac{1}{2}\langle S_6^2\rangle_c - \frac{1}{2}\langle S_6 S_2^2\rangle_c - \frac{1}{2}\langle S_4^2 S_2\rangle_c + \frac{1}{4!}\langle S_2^4\rangle_c}_{O(\beta^8)} \\
& + O(\beta^9)\Big]
\end{aligned}
\tag{3.7}
$$

where the subscript "c" stands for "connected" correlation functions.

Previously, in refs. [1, 2] the same set of correlators for the scalar heat kernel was computed. The expression for the kernel (3.7) differs from that obtained in the scalar case only in the coupling constants, now given by (3.6). Hence, the final result for the fermion heat kernel at coinciding points can be obtained by plugging the new coupling constants into the expression of the scalar heat kernel, reported in appendix B for completeness. Thus we get

$$
\begin{aligned}
\overline{\mathbb{K}}(0,0;\beta) = \frac{\mathbb{1}}{(2\pi\beta)^{\frac{d}{2}}} \exp\Big[& -d(d-1)\frac{\beta M^2}{24} + d(d-1)\Big(-\frac{(\beta M^2)^2}{6!}\frac{4d+3}{4} \\
& - \frac{(\beta M^2)^3}{9!}(d+2)(16d+15) \\
& - \frac{(\beta M^2)^4}{10!}\frac{16d^3+257d^2+555d+315}{8} \\
& + \frac{(\beta M^2)^5}{11!}\frac{(d+2)\big(64d^3-333d^2-1341d-945\big)}{24} \\
& + \frac{(\beta M^2)^6}{13!}\frac{207744d^5+943595d^4-2652226d^3-18403426d^2-29381262d-14365890}{5040} \\
& + \frac{(\beta M^2)^7}{14!}\frac{(d+2)\big(16896d^5+243703d^4+213650d^3-2640054d^2-6680970d-4054050\big)}{720} \\
& - \frac{(\beta M^2)^8}{17!}\Big(\frac{3175680d^7-132047423d^6-1198310651d^5-2099217371d^4}{1440} \\
& \qquad + \frac{8069209407d^3+36235883583d^2+49125794355d+21995248275}{1440}\Big) \\
& + O(\beta^9)\Big)\Big]
\end{aligned}
\tag{3.8}
$$

which could equivalently be written in terms of the constant scalar curvature R. In this expression the exponential must be expanded, keeping terms up to order $O(\beta^8)$ included. This allows to read off the diagonal coefficients $a_n(0,0)$, with integer n up to $n = 8$. We use them in the next section to identify the type-A trace anomaly of a Dirac fermion in various dimensions.

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4 The type-A trace anomalies

The path integral calculation of the transition amplitude on a maximally symmetric space can be employed and tested to evaluate the type-A trace anomaly of a massless Dirac fermion coupled to gravity, in space-time dimensions $d \leq 16$. These anomalies are the ones proportional to the topological Euler density of the curved space [37], see also [38, 39] for their cohomological characterization.

In general, the trace anomaly of a Dirac fermion can be related to the transition amplitude of a $N = 1$ spinning particle in a curved space by

$$\left\langle T^m{}_m(x) \right\rangle_{\mathrm{QFT}} = -\lim_{\beta \to 0} \operatorname{tr} \mathbb{K}(x, x; \beta)\,, \tag{4.1}$$

where on the left hand side $T^m{}_m(x)$ is the trace of the stress tensor of the Dirac spinor in a curved background, obtained from the appropriate Dirac action S_D by $T_{ma}(x) = \frac{1}{e}\frac{\delta S_D}{\delta e^{ma}(x)}$ where $e^a_m(x)$ is the vielbein of the curved spacetime. The expectation value is performed in the corresponding quantum field theory. The right hand side can be viewed as the anomalous contribution arising from the QFT path integral measure, regulated *à la* Fujikawa [40], with the minus sign being the usual one due to the fermionic measure, and the trace being the trace on spinor indices. The regulator corresponds to the square of the Dirac operator, and is identified with the quantum hamiltonian $\hat{H}$ of the $N = 1$ spinning particle in a curved space

$$\hat{H} = -\frac{1}{2}(\nabla\!\!\!\!/\,)^2\,, \tag{4.2}$$

which appears in the heat kernel at coinciding points $\mathbb{K}(x, x; \beta)$. The latter can be evaluated with a path integral [41, 42]. It is understood that the $\beta \to 0$ limit in (4.1) picks up just the β-independent term, as divergent terms are removed by the QFT renormalization. This procedure selects the appropriate heat kernel coefficient $a_n(x, x)$ sitting in the expansion of $\mathbb{K}(x, x; \beta)$. It may be interpreted as the contribution to the anomaly of the regularized particle making its virtual loop, see for example [43], where a Pauli-Villars regularization gives rise to the Fujikawa regulator used above.

Expanding $\mathbb{K}(x, x; \beta)$ at the required order one can read off the trace anomalies in even d dimensions (odd dimensions support no anomaly if the space is boundaryless)

$$\left\langle T^m{}_m(x) \right\rangle_{\mathrm{QFT}} = -\frac{\operatorname{tr} a_{\frac{d}{2}}(x, x)}{(2\pi)^{\frac{d}{2}}} \tag{4.3}$$

that is, for even $d = 2n$ dimensions, the relevant coefficient is precisely $a_n(x, x)$. Of course, one may use Riemann normal coordinates centered at x, so that $\sqrt{g(x)} = 1$ and $\overline{\mathbb{K}}(x, x; \beta) = \mathbb{K}(x, x; \beta)$. This formula holds on a generic space. In the present maximally symmetric case, due to translational invariance, the choice of which point is the origin of the Riemann coordinates becomes irrelevant. Hence, $\overline{\mathbb{K}}(x, x; \beta) = \overline{\mathbb{K}}(0, 0; \beta)$, and the result obtained in the previous section is directly applicable. The trace in (4.3) reduces to the trace of the identity matrix, and counts the dimension of the spinor space, $2^{\frac{d}{2}}$ for even d dimensions.

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d	$\langle T^m{}_m \rangle$	$\langle T^m{}_m \rangle$
2	$\frac{R}{24\,\pi}$	$\frac{1}{12\,\pi a^2}$
4	$-\frac{11\,R^2}{34\,560\,\pi^2}$	$-\frac{11}{240\,\pi^2 a^4}$
6	$\frac{191\,R^3}{108\,864\,000\,\pi^3}$	$\frac{191}{4032\,\pi^3 a^6}$
8	$-\frac{2\,497\,R^4}{339\,880\,181\,760\,\pi^4}$	$-\frac{2\,497}{34\,560\,\pi^4 a^8}$
10	$\frac{14\,797\,R^5}{598\,615\,142\,400\,000\,\pi^5}$	$\frac{14\,797}{101\,376\,\pi^5 a^{10}}$
12	$-\frac{92\,427\,157\,R^6}{1\,330\,910\,037\,208\,675\,123\,200\,\pi^6}$	$-\frac{92\,427\,157}{251\,596\,800\pi^6 a^{12}}$
14	$\frac{36\,740\,617\,R^7}{219\,454\,597\,066\,612\,329\,676\,800\,\pi^7}$	$\frac{36\,740\,617}{33\,177\,600\,\pi^7 a^{14}}$
16	$-\frac{61\,430\,943\,169\,R^8}{173\,836\,853\,795\,629\,301\,760\,000\,000\,000\,\pi^8}$	$-\frac{61\,430\,943\,169}{15\,792\,537\,600\,\pi^8 a^{16}}$

Table 1. Type-A trace anomaly of a Dirac spinor in terms of the curvature scalar R, and in terms of the radius a, in various dimensions.

In table 1, we list the anomalies we obtain from the expansion (3.8), expressing the results both in terms of the scalar curvature R and in terms of the sphere radius $a = \frac{1}{M}$. These anomalies were listed up to $d = 12$ in terms of the radius a also in ref. [4] (however the value of the anomaly in $d = 12$ reported there is incorrect, their denominator differs from ours, presumably a misprint).

The type-A trace anomaly can also be obtained using the Riemann zeta-function associated to the differential operator (4.2)

$$\big\langle T^m{}_m(x) \big\rangle_{\rm QFT} = -\frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{\frac{d+1}{2}} a^d}\, \zeta_{\nabla\!\!\!/^2}(0)\,, \tag{4.4}$$

as discussed in [4, 5]. More recently, an efficient way of computing such trace anomalies within the AdS/CFT correspondence was proposed in [6]. In that reference, a simple formula for certain coefficients $c^{(d)}_{\nabla\!\!\!/^2}$ linked to the Riemann zeta function was found

$$c^{(d)}_{\nabla\!\!\!/^2} = \frac{4(-1)^{\frac{d}{2}}}{(8\pi)^{\frac{d}{2}} \left(\frac{d}{2}\right)! \left(\frac{d}{2}-1\right)!} \int_0^{\frac{1}{2}} d\nu \frac{\left(\frac{1}{2}+\nu\right)_{\frac{d}{2}} \left(\frac{1}{2}-\nu\right)_{\frac{d}{2}}}{\left(\frac{1}{2}\right)_{\frac{d}{2}}}\,, \tag{4.5}$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\dots(x+n-1)$ is the Pochhammer symbol (the raising factorial). We have checked that these coefficients are linked explicitly to the Riemann zeta function by

$$\zeta_{\nabla\!\!\!/^2}(0) = (4\pi)^{\frac{d}{2}} \left(\frac{d}{2}-1\right)!\; c^{(d)}_{\nabla\!\!\!/^2}\,. \tag{4.6}$$

which, in turn, allow to identify the anomaly in (4.4). One can check that both methods reproduce the same type-A trace anomalies, which indeed coincide with the ones computed by the simplified path integral in $d = 2, \dots, 16$ and listed in table 1.

Finally, it is convenient to summarize the type-A trace anomalies by presenting them in the form

$$\langle T^m{}_m(x)\rangle = (-1)^{n+1} a_{2n} \frac{E_{2n}}{(2\pi)^n} \tag{4.7}$$

where E_{2n} is the Euler density of the $d = 2n$ dimensional space defined by

$$E_{2n} = \frac{(2n)!}{2^n} R_{m_1 m_2}{}^{[m_1 m_2} \dots R_{m_{2n-1} m_{2n}}{}^{m_{2n-1} m_{2n}]} \tag{4.8}$$

(the square bracket denotes weighted antisymmetrization) with a_{2n} the constant anomaly coefficient. The stress tensors are normalized as usual, $T_{mn} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{mn}}$ for the scalar and $T_{ma} = \frac{1}{e} \frac{\delta S}{\delta e^{ma}}$ for the spinor, where g_{mn} and e_{ma} are the metric and vielbein, respectively. The sign $(-1)^{n+1}$ is conventionally inserted to make the coefficients a_{2n} positive, as we will check shortly. On spheres the Euler density evaluates to

$$E_{2n} = \frac{(2n)!}{(2n(2n-1))^n} R^n \tag{4.9}$$

and from the previous discussion we identify the following coefficients for a Dirac fermion

$$a_{2n}^{fermion} = \frac{2}{n!(2n)!} \int_0^{\frac{1}{2}} dx \prod_{i=0}^{n-1} \left(\left(i + \frac{1}{2} \right)^2 - x^2 \right) . \tag{4.10}$$

Similarly, we also identify the coefficients for a real conformal scalar, using formulas from [7],

$$a_{2n}^{scalar} = -\frac{(2n-1)!!}{((2n)!)^2} \int_0^1 dx \prod_{i=0}^{n-1} (i^2 - x^2) . \tag{4.11}$$

By inspection, one may notice that these coefficients are positive for every n, as the integrands are products of positive functions in the given range of integration (for the scalar, the explicit minus sign makes positive the contribution of the $i = 0$ term of the product). This positivity is not evident in the worldline method, and appears only at the end of our calculations. In general, these coefficients are expected to be positive, as they appear in conjectured higher dimensional extensions of the a-theorem and interpreted as a measure of the effective degrees of freedom at the fixed point. These conjectured a-theorems extend suitably the c-theorem of two dimensions [44] and the a-theorem of four dimensions [45], where indeed the coefficients have been proven to be positive for arbitrary unitary conformal field theories.

For $d = 2n = 2, 4, \dots, 16$ we report the values of these coefficients, as well as their ratio [5], in table 2.

5 Conclusions

We have considered the worldline path integral for the $N = 1$ supersymmetric quantum mechanics in curved space, which is characterized by a supersymmetric non-linear sigma model action. We have shown that, when the space has maximal symmetry, the nonlinear

$d = 2n$	$(2n+1)!\ a(\text{scalar})$	$(2n+1)!\ a(\text{fermion})$	$\frac{a(\text{fermion})}{a(\text{scalar})}$
2	$\frac{1}{2}$	$\frac{1}{2}$	1
4	$\frac{1}{12}$	$\frac{11}{12}$	11
6	$\frac{5}{72}$	$\frac{191}{72}$	$\frac{191}{5}$
8	$\frac{23}{240}$	$\frac{2497}{240}$	$\frac{2497}{23}$
10	$\frac{263}{1440}$	$\frac{14797}{288}$	$\frac{73985}{263}$
12	$\frac{133787}{302400}$	$\frac{92427157}{302400}$	$\frac{92427157}{133787}$
14	$\frac{157009}{120960}$	$\frac{36740617}{17280}$	$\frac{257184319}{157009}$
16	$\frac{16215071}{3628800}$	$\frac{61430943169}{3628800}$	$\frac{61430943169}{16215071}$

Table 2. The a coefficients of the type-A trace anomaly of a real conformal scalar and Dirac fermion. We have multiplied them by $(2n+1)!$ to make the numbers more readable.

sigma model can be traded with a purely bosonic linear sigma model and the curvature effects are taken care of by a suitable effective scalar potential, which extends the one studied in [1, 2]. We have tested our model by computing the type-A trace anomalies of a Dirac fermion in space-time dimension $d \leq 16$, showing that they match those obtained with other techniques. However, further checks can be performed. Firstly, since the full heat kernel for maximally symmetric spaces in Riemann normal coordinates is found to be proportional to the spinorial identity, the gravitational contribution to the chiral anomaly results proportional to $\operatorname{tr}\gamma^5$, and thus correctly vanishes. On the other hand, we can also verify that, for $d = 3, 5$, the expansions of the diagonal heat kernels have vanishing coefficients a_n for $n \geq 2, 3$ respectively, as predicted by Camporesi in his exact calculations of spinorial heat kernels in maximally symmetric spaces [46].

The present construction can presumably be extended also to the $N = 2$ supersymmetric quantum mechanics used in the description of differential p-forms and particles of spin 1 [20, 21, 47], as well as to the supersymmetric quantum mechanics at arbitrary N, which provide the degrees of freedom of the first quantized approach to higher spinning particles [48, 49]. The latter enjoy conformal symmetry [50, 51], and can be coupled to maximally symmetric spaces [52] (more generally, to conformally flat spaces [27]). The path integral for the spinning particle with N supersymmetries on curved spaces needs regularization schemes with suitable regularization-dependent counterterms [53]. A linear sigma model approach may simplify drastically the situation, at least on maximally symmetric spaces.

Finally, a direction worth looking at is the inclusion of boundaries in the maximally symmetric spaces, extending to curved spaces the worldline treatments of refs. [54, 55], and study in particular the possible trace anomalies supported by the boundaries, as discussed for example in [56–59].

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A Riemann normal coordinates in maximally symmetric spaces

Maximally symmetric spaces are spaces with a maximal number of isometries. Their curvature tensors can be expressed in terms of the metric as

$$R_{ijmn} = M^2(g_{im}g_{jn} - g_{in}g_{jm}) \tag{A.1}$$

$$R_{ij} = R_{mi}{}^{m}{}_{j} = M^2(d-1)g_{ij} \tag{A.2}$$

$$R = R_i{}^i = M^2(d-1)d\,, \tag{A.3}$$

where $M^2 = 1/a^2$ is a constant, which is positive for a sphere of radius a, vanishes for a flat space, and is negative for a real hyperbolic space. This exhausts the list of simply connected, maximally symmetric spaces. For simplicity we consider spheres, as real hyperbolic spaces can be obtained by a simple analytic continuation.

In the main text we use Riemann normal coordinates (for details see [60–62] and [63–65] for applications to nonlinear sigma models). On spheres the sectional curvature is positive, and we can take $M = \frac{1}{a} > 0$. One may then evaluate recursively all terms in the expansion of the metric and sum them up [33], to obtain

$$g_{ij}(x) = \delta_{ij} + f(r)P_{ij} = \delta_{ij} + f(r)P_{ij} \tag{A.4}$$

where x^i denote the Riemann normal coordinates centered around a point (the origin), P_{ij} indicates a projector given by

$$P_{ij} = \delta_{ij} - \hat{x}_i\hat{x}_j\,, \qquad \hat{x}^i = \frac{x^i}{r}\,, \qquad r = \sqrt{\vec{x}^{\,2}}\,, \tag{A.5}$$

and

$$f(r) = \frac{1 - 2(Mr)^2 - \cos(2Mr)}{2(Mr)^2}\,. \tag{A.6}$$

Note that the function $f(r)$ does not have poles and it is even in r, so that it depends only on $r^2 = \vec{x}^{\,2} = \delta_{ij}x^ix^j$. Note also that, because of the projector P_{ij} one has the equality $r^2 = g_{ij}(x)x^ix^j$. The inverse metric $g^{ij}(x)$ and metric determinant $g(x)$ are given by

$$g^{ij}(x) = \delta^{ij} - \frac{f(r)}{1+f(r)}P^{ij} \tag{A.7}$$

$$g(x) = (1+f(r))^{d-1}\,. \tag{A.8}$$

It is easy to check that the metric in (A.4) can be generated by the following choice of vielbein

$$e_i^a(x) = \delta_i^a + l(r)P_i^a(x) \tag{A.9}$$

where $x^a = \delta_i^a x^i$ and[3]

$$l(r) = -1 + \sqrt{1+f(r)} = -1 + \frac{\sin(Mr)}{Mr}\,. \tag{A.10}$$

[3] A priori, there are two independent solutions $l_\pm(r) = -1 \pm \sqrt{1+f(r)}$ of the quadratic equation that follows from $g_{ij} = \eta_{ab}e_i^a e_j^b$. However, only with the upper solution does the vielbein reduce to the flat vielbein when $M^2 \to 0$.

The inverse vielbein reads instead

$$e^{ai}(x) = \delta^{ai} + \left(-1 + \frac{1}{\sqrt{1+f(r)}}\right) P^{ai}(x) \; . \tag{A.11}$$

Thus, by using the relation

$$\omega_i{}^{ab}(x) = \frac{1}{2}e^{aj}\left(\partial_i e_j^b - \partial_j e_i^b\right) - \frac{1}{2}e^{bj}\left(\partial_i e_j^a - \partial_j e_i^a\right) - \frac{1}{2}e_i^c e^{aj} e^{bk}\left(\partial_j e_{ck} - \partial_k e_{cj}\right) \tag{A.12}$$

one can promptly compute the associated spin connection, which is found to be

$$\omega_i{}^{ab}(x) = \Omega(r)\frac{1}{2}x^j\left(\delta_j^a\delta_i^b - \delta_j^b\delta_i^a\right) \tag{A.13}$$

with

$$\Omega(r) = -\frac{2}{r}\left(l'(r) + \frac{l(r)}{r}\right) = 2M^2\frac{1-\cos(Mr)}{(Mr)^2} \, , \tag{A.14}$$

where the prime denotes the derivative with respect to the radial coordinate r. Equivalently, we can write the spin connection in the form

$$\omega_i{}^{ab}(x) = \frac{1}{M^2}\Omega(r)\,\frac{1}{2}x^j R_{ij}{}^{ab}(0) \tag{A.15}$$

where the prefactor reads

$$\begin{aligned}\frac{1}{M^2}\Omega(r) = 2\frac{1-\cos(Mr)}{(Mr)^2} &= \sum_{n=0}^{\infty}\frac{2(-)^n}{(2(n+1))!}(Mr)^{2n} \\ &= 1 - \frac{(Mr)^2}{12} + \frac{(Mr)^4}{360} - \frac{(Mr)^6}{20160} + \cdots \end{aligned} \tag{A.16}$$

and a power of M^2 is absorbed by $R_{ij}{}^{ab}(0)$.

Note that the vielbein (A.9) with (A.10) and the spin connection (A.15) satisfy the Fock-Schwinger gauge conditions

$$\begin{aligned} e_i^a(x)x^i &= \delta_i^a x^i \\ x^i\omega_i{}^{ab}(x) &= 0 \; . \end{aligned} \tag{A.17}$$

B Wick contractions

We collect here the perturbative contributions up to order β^8 involved in the transition amplitude (3.7), where $\langle\ldots\rangle_c$ indicates connected correlation functions. We use the abbreviation $\Delta(\tau_1,\tau_2) \equiv \Delta_{12}$ for the propagator, and $\int = \int_0^1 d\tau_1$, $\iint = \int_0^1 d\tau_1 \int_0^1 d\tau_2$, and so on, for multiple integrals. We also set $M=1$, as the dependence on M is easily restored.

$$S_0 = \beta k_0 \, , \tag{B.1}$$

$$\langle S_2\rangle = -\beta^2 k_2\, d \underbrace{\int \Delta_{11}}_{-\frac{1}{6}} \, , \tag{B.2}$$

$$\langle S_4\rangle = \beta^3 k_4\, d(d+2) \underbrace{\int \Delta_{11}^2}_{\frac{1}{30}}\,, \tag{B.3}$$

$$\langle S_6\rangle = -\beta^4 k_6\, d(d+2)(d+4) \underbrace{\int \Delta_{11}^3}_{-\frac{1}{140}}\,, \tag{B.4}$$

$$\frac{1}{2}\langle S_2^2\rangle_c = \beta^4 k_2^2\, d \underbrace{\iint \Delta_{12}^2}_{\frac{1}{90}}\,, \tag{B.5}$$

$$\langle S_8\rangle = \beta^5 k_8\, d(d+2)(d+4)(d+6) \underbrace{\int \Delta_{11}^4}_{\frac{1}{630}}\,, \tag{B.6}$$

$$\langle S_4 S_2\rangle_c = -\beta^5 k_4 k_2\, 4d(d+2) \underbrace{\iint \Delta_{12}^2 \Delta_{22}}_{-\frac{1}{420}}\,, \tag{B.7}$$

$$\langle S_{10}\rangle = -\beta^6 k_{10}\, d(d+2)(d+4)(d+6)(d+8) \underbrace{\int \Delta_{11}^5}_{-\frac{1}{2772}}\,, \tag{B.8}$$

$$\langle S_6 S_2\rangle_c = \beta^6 k_6 k_2\, 6d(d+2)(d+4) \underbrace{\iint \Delta_{12}^2 \Delta_{22}^2}_{\frac{1}{1890}}\,, \tag{B.9}$$

$$\frac{1}{2}\langle S_4^2\rangle_c = \frac{\beta^6}{2}\; k_4^2 \left(8d(d+2) \underbrace{\iint \Delta_{12}^4}_{\frac{1}{3150}} + 8d(d+2)^2 \underbrace{\iint \Delta_{11} \Delta_{12}^2 \Delta_{22}}_{\frac{13}{25200}} \right), \tag{B.10}$$

$$\frac{1}{3!}\langle S_2^3\rangle_c = -\frac{\beta^6}{3!} k_2^3\, 8d \underbrace{\iiint \Delta_{12}\Delta_{23}\Delta_{31}}_{-\frac{1}{945}}\,, \tag{B.11}$$

$$\langle S_{12}\rangle = \beta^7 k_{12}\, d(d+2)(d+4)(d+6)(d+8)(d+10) \underbrace{\int \Delta_{11}^6}_{\frac{1}{12012}}\,, \tag{B.12}$$

$$\langle S_8 S_2\rangle_c = -\beta^7 k_8 k_2\, 8d(d+2)(d+4)(d+6) \underbrace{\iint \Delta_{12}^2 \Delta_{11}^3}_{-\frac{1}{8316}}\,, \tag{B.13}$$

$$\begin{aligned}\langle S_6 S_4\rangle_c = -\beta^7 k_6 k_4 \Bigg(& 12d(d+2)^2(d+4) \underbrace{\iint \Delta_{11}^2 \Delta_{12}^2 \Delta_{22}}_{-\frac{2}{17325}} \\ & + 24d(d+2)(d+4) \underbrace{\iint \Delta_{11}\Delta_{12}^4}_{-\frac{1}{13860}} \Bigg),\end{aligned} \tag{B.14}$$

$$\frac{1}{2}\langle S_4 S_2^2\rangle_c = \beta^7 k_4 k_2^2\, 4d(d+2)\left(2\underbrace{\iiint \Delta_{12}\Delta_{13}\Delta_{23}\Delta_{33}}_{\frac{13}{56700}} + \underbrace{\iiint \Delta_{13}^2\Delta_{23}^2}_{\frac{1}{5670}}\right), \tag{B.15}$$

$$\langle S_{14}\rangle = -\beta^8 k_{14}\, d(d+2)(d+4)(d+6)(d+8)(d+10)(d+12)\underbrace{\int \Delta_{11}^7}_{-\frac{1}{51480}},$$

$$\langle S_{10} S_2\rangle_c = \beta^8 k_{10} k_2\, 10d(d+2)(d+4)(d+6)(d+8)\underbrace{\iint \Delta_{11}^4\Delta_{12}^2}_{\frac{1}{36036}}, \tag{B.16}$$

$$\begin{aligned}\langle S_8 S_4\rangle_c = \beta^8 k_8 k_4\Bigg(&16d(d+2)^2(d+4)(d+6)\underbrace{\iint \Delta_{11}^3\Delta_{12}^2\Delta_{22}}_{\frac{19}{720720}}\\ &+48d(d+2)(d+4)(d+6)\underbrace{\iint \Delta_{11}^2\Delta_{12}^4}_{\frac{1}{60060}}\Bigg),\end{aligned} \tag{B.17}$$

$$\begin{aligned}\frac{1}{2}\langle S_6^2\rangle_c = \frac{1}{2}\beta^8 k_6^2\Bigg(&18d(d+2)^2(d+4)^2\underbrace{\iint \Delta_{12}^2\Delta_{11}^2\Delta_{22}^2}_{\frac{491}{18918900}}\\ &+72d(d+2)(d+4)^2\underbrace{\iint \Delta_{12}^4\Delta_{11}\Delta_{22}}_{\frac{25}{1513512}}+48d(d+2)(d+4)\underbrace{\iint \Delta_{12}^6}_{\frac{1}{84084}}\Bigg),\end{aligned} \tag{B.18}$$

$$\frac{1}{2}\langle S_6 S_2^2\rangle_c = -\frac{1}{2}\beta^8 k_6 k_2^2\, 24d(d+2)(d+4)\left(\underbrace{\iiint \Delta_{11}^2\Delta_{12}\Delta_{13}\Delta_{23}}_{-\frac{8}{155925}} + \underbrace{\iiint \Delta_{11}\Delta_{12}^2\Delta_{13}^2}_{-\frac{1}{24948}}\right), \tag{B.19}$$

$$\begin{aligned}\frac{1}{2}\langle S_4^2 S_2\rangle_c = -\frac{1}{2}\beta^8 k_4^2 k_2\Bigg(&32d(d+2)^2\left(\underbrace{\iiint \Delta_{12}^2\Delta_{23}^2\Delta_{33}}_{-\frac{2}{51975}} + \underbrace{\iiint \Delta_{12}\Delta_{13}\Delta_{23}\Delta_{22}\Delta_{33}}_{-\frac{83}{1663200}}\right)\\ &+64d(d+2)\underbrace{\iiint \Delta_{12}\Delta_{13}\Delta_{23}^3}_{-\frac{1}{34650}}\Bigg),\end{aligned} \tag{B.20}$$

$$\frac{1}{4!}\langle S_2^4\rangle_c = \frac{1}{4!}\beta^8 k_2^4\, 48d\underbrace{\iiiint \Delta_{12}\Delta_{23}\Delta_{34}\Delta_{41}}_{\frac{1}{9450}}. \tag{B.21}$$

Inserting now the values of the coupling constants k_{2l} one obtains the final result in (3.8).

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