

Combinatorial properties of the G-degree

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Abstract

A strong interaction is known to exist between edge-colored graphs (which encode PL pseudo-manifolds of arbitrary dimension) and random tensor models (as a possible approach to the study of Quantum Gravity). The key tool is the *G-degree* of the involved graphs, which drives the $1/N$ expansion in the tensor models context. In the present paper - by making use of combinatorial properties concerning Hamiltonian decompositions of the complete graph - we prove that, in any even dimension $d \geq 4$, the G-degree of all bipartite graphs, as well as of all (bipartite or non-bipartite) graphs representing singular manifolds, is an integer multiple of $(d - 1)!$. As a consequence, in even dimension, the terms of the $1/N$ expansion corresponding to odd powers of $1/N$ are null in the complex context, and do not involve colored graphs representing singular manifolds in the real context. In particular, in the 4-dimensional case, where the G-degree is shown to depend only on the regular genera with respect to an arbitrary pair of “associated” cyclic permutations, several results are obtained, relating the G-degree or the regular genus of 5-colored graphs and the Euler characteristic of the associated PL 4-manifolds.

Keywords: edge-colored graph; PL-manifold; singular manifold; colored tensor model; regular genus; Gurau-degree.

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1 Introduction

It is well-known that regular edge-colored graphs may encode PL-pseudomanifolds, giving rise to a combinatorial representation theory (*crystallization theory*) for singular PL-manifolds of arbitrary dimension (see Section 2).

In the last decade, the strong interaction between the topology of edge-colored graphs and random tensor models has been deeply investigated, bringing insights in both research fields.

The colored tensor models theory arises as a possible approach to the study of Quantum Gravity: in some sense, its aim is to generalize to higher dimension the matrix models theory which, in dimension two, has shown to be quite useful at providing a framework for Quantum Gravity. The key generalization is the recovery of the so called $1/N$ expansion in the tensor models context. In matrix models, the $1/N$ expansion is driven by the genera of the surfaces represented by Feynman graphs; in the higher dimensional setting of tensor models the $1/N$ expansion is driven by the *G-degree* of these graphs (see Definition 3), that equals the genus of the represented surface in dimension two.

If $(\mathbb{C}^N)^{\otimes d}$ denotes the d -tensor product of the N -dimensional complex space \mathbb{C}^N , a $(d + 1)$ -dimensional colored tensor model is a formal partition function

$$\mathcal{Z}[N, \{\alpha_B\}] := \int_{\mathbf{f}} \frac{dT d\bar{T}}{(2\pi)^{Nd}} \exp(-N^{d-1} \bar{T} \cdot T + \sum_B \alpha_B B(T, \bar{T})),$$

where T belongs to $(\mathbb{C}^N)^{\otimes d}$, \bar{T} to its dual and $B(T, \bar{T})$ are *trace invariants* obtained by contracting the indices of the components of T and \bar{T} . In this framework, colored graphs naturally arise as Feynman graphs encoding tensor trace invariants.

As shown in [4], the *free energy* $\frac{1}{Nd} \log \mathcal{Z}[N, \{t_B\}]$ is the formal series

$$\frac{1}{Nd} \log \mathcal{Z}[N, \{t_B\}] = \sum_{\omega_G \geq 0} N^{-\frac{2}{(d-1)!} \omega_G} F_{\omega_G}[\{t_B\}] \in \mathbb{C}[[N^{-1}, \{t_B\}]], \quad (1)$$

where the coefficients $F_{\omega_G}[\{t_B\}]$ are generating functions of connected bipartite $(d + 1)$ -colored graphs with fixed G-degree ω_G .

The $1/N$ expansion of formula (1) describes the rôle of colored graphs (and of their G-degree ω_G) within colored tensor models theory and explains the importance of trying to understand which are the manifolds and pseudomanifolds represented by $(d + 1)$ -colored graphs with a given G-degree.

A more detailed description of these relationships between Quantum Gravity via tensor models and topology of colored graphs may be found in [4], [16], [15], [7].

A parallel tensor models theory, involving *real* tensor variables $T \in (\mathbb{R}^N)^{\otimes d}$, has been developed, taking into account also non-bipartite colored graphs (see [18]): this is why both bipartite and non-bipartite colored graphs will be considered within the paper.

Section 2 contains a quick review of crystallization theory, including the idea of *regular embedding* of edge-colored graphs into surfaces, which is crucial for the definitions of *G-degree* and *regular genus* of graphs (Definition 3).

In Section 3, combinatorial properties concerning Hamiltonian decompositions of the complete graph allow to prove the main results of the paper.

Theorem 1 *For each bipartite $(d + 1)$ -colored graph (Γ, γ) , with d even, $d \geq 4$,*

$$\omega_G(\Gamma) \equiv 0 \pmod{(d - 1)!}$$

Theorem 2 *For each (bipartite or non-bipartite) $(d + 1)$ -colored graph (Γ, γ) representing a singular d -manifold, with d even, $d \geq 4$,*

$$\omega_G(\Gamma) \equiv 0 \pmod{(d - 1)!}$$

Note that the above results turn out to have specific importance in the tensor models framework. In fact Theorem 1 implies that, in the d -dimensional complex context, with d even and $d \geq 4$, the only non-null terms in the $1/N$ expansion of formula (1) are the ones corresponding to even (integer) powers of $1/N$.

On the other hand, Theorem 2 ensures that in the real tensor models framework, where also non-bipartite graphs are involved, the $1/N$ expansion contains colored graphs representing (orientable or non-orientable) singular manifolds - and, in particular, closed manifolds - only in the terms corresponding to even (integer) powers of $1/N$. Both Theorems extend to arbitrary even dimension a result proved in [7, Corollary 23] for graphs representing singular 4-manifolds.

Section 4 is devoted to the 4-dimensional case: in this particular situation, the general results of Section 3 allow to obtain interesting properties relating the G-degree with the topology of the associated PL 4-manifolds. In fact, the G-degree of a 5-colored graph is shown to depend only on the regular genera with respect to an arbitrary pair of “associated” cyclic permutations (Proposition 10). This fact yields relations between these two genera and the Euler characteristic of the associated PL 4-manifold (Proposition 13 and Proposition 14); moreover, two interesting classes of crystallizations arise in a natural way, whose intersection consists in the known class of semi-simple crystallizations, introduced in [2] (see Remark 7).

2 Edge-colored graphs and G-degree

A *singular d -manifold* is a compact connected d -dimensional polyhedron admitting a simplicial triangulation where the links of vertices are closed connected $(d - 1)$ -manifolds, while the link of any h -simplex, for each $h > 0$, is a PL $(d - h - 1)$ -sphere. A vertex whose link is not a PL $(d - 1)$ -sphere is called *singular*.

Remark 1 The class of singular d -manifolds includes the class of closed d -manifolds: in fact, a closed d -manifold is a singular d -manifold without singular vertices. Moreover, if N is a singular d -manifold, then a compact PL d -manifold \check{N} is obtained by deleting small open neighbourhoods of its singular vertices. Obviously, $N = \check{N}$ if and only if N is a closed manifold; otherwise, \check{N} has a non-empty boundary without spherical components. Conversely, given a compact PL d -manifold M , a singular d -manifold \widehat{M} can be obtained by capping off each component of ∂M by a cone over it.

Note that, in virtue of the above correspondence, a bijection is defined between singular d -manifolds and compact PL d -manifolds with no spherical boundary components.

Definition 1 A $(d + 1)$ -colored graph ($d \geq 2$) is a pair (Γ, γ) , where $\Gamma = (V(\Gamma), E(\Gamma))$ is a regular $d + 1$ valent multigraph (i.e. multiple edges are allowed, while loops are forbidden) and $\gamma : E(\Gamma) \rightarrow \Delta_d = \{0, \dots, d\}$ is a map injective on adjacent edges, called *coloration*.

For every $\mathcal{B} \subseteq \Delta_d$ let $\Gamma_{\mathcal{B}}$ be the subgraph obtained from (Γ, γ) by deleting all the edges colored by $\Delta_d - \mathcal{B}$. The connected components of $\Gamma_{\mathcal{B}}$ are called *\mathcal{B} -residues* or, if $\#\mathcal{B} = h$, *h -residues* of Γ ; the symbol $g_{\mathcal{B}}$ denotes their number. In the following, if $\mathcal{B} = \{c_1, \dots, c_h\}$, its complementary set $\Delta_d - \mathcal{B}$ will be denoted by $\hat{c}_1 \dots \hat{c}_h$.

Given a $(d + 1)$ -colored graph (Γ, γ) , a d -dimensional pseudocomplex $K(\Gamma)$ can be associated by the following rules:

- for each vertex of Γ , let us consider a d -simplex and label its vertices by the elements of Δ_d ;
- for each pair of c -adjacent vertices of Γ ($c \in \Delta_d$), let us glue the corresponding d -simplices along their $(d - 1)$ -dimensional faces opposite to the c -labeled vertices, so that equally labeled vertices are identified.

$|K(\Gamma)|$ turns out to be a *d -pseudomanifold*¹, which is orientable if and only if Γ is bipartite, and (Γ, γ) is said to *represent* it.

Note that, by construction, $K(\Gamma)$ is endowed with a vertex-labeling by Δ_d that is injective on any simplex. Moreover, a bijective correspondence exists between the h -residues of Γ colored by any $\mathcal{B} \subseteq \Delta_d$ and the $(d - h)$ -simplices of $K(\Gamma)$ whose vertices are labeled by $\Delta_d - \mathcal{B}$.

In particular, for any color $c \in \Delta_d$, each connected component of $\Gamma_{\hat{c}}$ is a d -colored graph representing a pseudocomplex that is PL-homeomorphic to the link of a c -labeled vertex of $K(\Gamma)$ in its first barycentric subdivision. As a consequence, $|K(\Gamma)|$ is a singular d -manifold (resp. a closed d -manifold) iff for each color $c \in \Delta_d$, all \hat{c} -residues of Γ represent closed $(d - 1)$ -manifolds (resp. the $(d - 1)$ -sphere).

In virtue of the bijection described in Remark 1, a $(d + 1)$ -colored graph (Γ, γ) is said to *represent* a compact PL d -manifold M with no spherical boundary components if and only if it represents the associated singular manifold \widehat{M} .

Definition 2 A *crystallization* of a closed PL d -manifold M^d is a $(d + 1)$ -colored graph representing M^d , such that each d -residue is connected (i.e. $g_i = 1 \ \forall i \in \Delta_d$).

The following theorem extends to singular manifolds a well-known result - due to Pezzana ([17]) - founding the combinatorial representation theory for closed PL-manifolds of arbitrary dimension via colored graphs (the so called *crystallization theory*). See also [11] and [12] for the 3-dimensional case.

¹In fact, $|K(\Gamma)|$ is a *quasi-manifold*: see [13].

Theorem 3 ([9, Theorem 1]) *Any singular d -manifold - or, equivalently, any compact d -manifold with no spherical boundary components - admits a $(d + 1)$ -colored graph representing it.*

In particular, each closed PL d -manifold admits a crystallization.

It is well known the existence of a particular set of embeddings of a bipartite (resp. non-bipartite) $(d + 1)$ -colored graph into orientable (resp. non-orientable) surfaces.

Theorem 4 ([14]) *Let (Γ, γ) be a bipartite (resp. non-bipartite) $(d + 1)$ -colored graph of order $2p$. Then for each cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_d)$ of Δ_d , up to inverse, there exists a cellular embedding, called regular, of (Γ, γ) into an orientable (resp. non-orientable) closed surface $F_\varepsilon(\Gamma)$ whose regions are bounded by the images of the $\{\varepsilon_j, \varepsilon_{j+1}\}$ -colored cycles, for each $j \in \mathbb{Z}_{d+1}$. Moreover, the genus (resp. half the genus) $\rho_\varepsilon(\Gamma)$ of $F_\varepsilon(\Gamma)$ satisfies*

$$\chi(F_\varepsilon(\Gamma)) = 2 - 2\rho_\varepsilon(\Gamma) = \sum_{j \in \mathbb{Z}_{d+1}} g_{\varepsilon_j \varepsilon_{j+1}} + (1 - d)p.$$

No regular embeddings of (Γ, γ) exist into non-orientable (resp. orientable) surfaces.

The *Gurau degree* (often called *degree* in the tensor models literature) and the *regular genus* of a colored graph are defined in terms of the embeddings of Theorem 4.

Definition 3 *Let (Γ, γ) be a $(d + 1)$ -colored graph. If $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(\frac{d!}{2})}\}$ is the set of all cyclic permutations of Δ_d (up to inverse), $\rho_{\varepsilon^{(i)}}$ ($i = 1, \dots, \frac{d!}{2}$) is called the regular genus of Γ with respect to the permutation $\varepsilon^{(i)}$. Then, the Gurau degree (or G-degree for short) of Γ , denoted by $\omega_G(\Gamma)$, is defined as*

$$\omega_G(\Gamma) = \sum_{i=1}^{\frac{d!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma)$$

and the regular genus of Γ , denoted by $\rho(\Gamma)$, is defined as

$$\rho(\Gamma) = \min \left\{ \rho_{\varepsilon^{(i)}}(\Gamma) / i = 1, \dots, \frac{d!}{2} \right\}.$$

Note that, in dimension 2, any bipartite (resp. non-bipartite) 3-colored graph (Γ, γ) represents an orientable (resp. non-orientable) surface $|K(\Gamma)|$ and $\rho(\Gamma) = \omega_G(\Gamma)$ is exactly the genus (resp. half the genus) of $|K(\Gamma)|$. On the other hand, for $d \geq 3$, the G-degree of any $(d + 1)$ -colored graph (resp. the regular genus of any $(d + 1)$ -colored graph representing a closed PL d -manifold) is proved to be a non-negative integer, both in the bipartite and non-bipartite case: see [7, Proposition 7] (resp. [10, Proposition A]).

3 Proof of the general results

Within combinatorics, the problem of the existence of m -cycle decompositions of the complete graph K_n , or of the complete multigraph λK_n (i.e. the multigraph with n vertices and with λ edges joining each pair of distinct vertices) is long standing: a survey result, for general m , n and λ , is given in [6, Theorem 1.1].

Moreover, the following results hold, concerning Hamiltonian cycles (i.e. $m = n$) in K_n , both in the case n odd and in the case n even.

Proposition 5 [5, Theorem 1.3] *For all odd $n \geq 3$ there exists a partition of all Hamiltonian cycles of K_n into $(n - 2)!$ Hamiltonian cycle decompositions of K_n .*

Proposition 6 [19, Theorem 2.2] *For all even $n \geq 4$ there exists a partition of all Hamiltonian cycles of K_n into $\frac{(n-2)!}{2}$ classes, so that each edge of K_n appears in exactly two cycles belonging to the same class.*

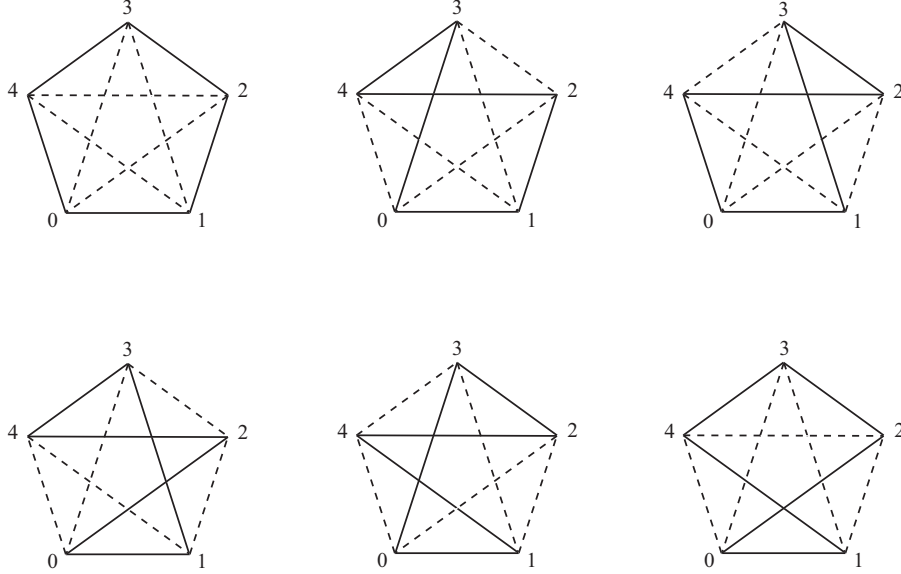


Figure 1: The six Hamiltonian cycle decompositions of K_5

Figure 1 describes - as an example of Proposition 5 - the six Hamiltonian cycle decompositions of K_5 , each of them containing a pair of disjoint Hamiltonian cycles (given by the dashed and continuous edges respectively). Note that, by labelling the vertices of K_n with the elements of Δ_{n-1} , each Hamiltonian cycle in K_n defines a cyclic permutation of Δ_{n-1} (together with its inverse), and viceversa.

On the other hand, the following statement regarding the G-degree has been recently proved.

Proposition 7 [7, Proposition 7] *If (Γ, γ) is a $(d+1)$ -colored graph of order $2p$ ($d \geq 3$), then*

$$\omega_G(\Gamma) = \frac{(d-1)!}{2} \cdot \left(d + p \cdot (d-1) \cdot \frac{d}{2} - \sum_{r,s \in \Delta_d} g_{rs} \right). \quad (2)$$

As a consequence, the G-degree of any $(d+1)$ -colored graph is a non-negative integer multiple of $\frac{(d-1)!}{2}$.

The result of Proposition 7, which was originally stated in the bipartite case (see [4]), suggested the definition, for $d \geq 3$, of the (integer) *reduced G-degree*

$$\omega'_G(\Gamma) = \frac{2}{(d-1)!} \cdot \omega_G(\Gamma),$$

which is used by many authors within tensor models theory (see for example [15]).²

Actually, we are able to prove that, if $d \geq 4$ is even, under rather weak hypotheses, the G-degree is multiple of $(d-1)!$ (or, equivalently, the reduced G-degree is even).

Proposition 8 *If $d \geq 4$ is even, and (Γ, γ) is a $(d+1)$ -colored graph such that each 3-residue is bipartite and each d -residue is either bipartite or non-bipartite with integer regular genus with respect to any permutation, then*

$$\omega_G(\Gamma) \equiv 0 \pmod{(d-1)!}$$

Proof. Let (Γ, γ) be a $(d+1)$ -colored graph, with $d \geq 4$, d even. Since $n = d+1$ is odd, Proposition 5 implies that all $\frac{d!}{2}$ cyclic permutations (up to inverse) of Δ_d can be partitioned in $(d-1)!$ classes, each containing $d/2$ cyclic permutations, $\bar{\varepsilon}^{(1)}, \bar{\varepsilon}^{(2)}, \dots, \bar{\varepsilon}^{(d/2)}$ say, so that

$$\sum_{i=1}^{d/2} \left[\sum_{j \in \mathbb{Z}_{d+1}} g_{\bar{\varepsilon}_j^{(i)}, \bar{\varepsilon}_{j+1}^{(i)}} \right] = \sum_{r,s \in \Delta_d} g_{r,s}.$$

²In fact, the exponents of N^{-1} in the $1/N$ expansion of formula (1) are all (non-negative) integers ω'_G .

Hence, by Theorem 4,

$$2 \cdot \frac{d}{2} - 2 \cdot \sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}} = \sum_{r,s \in \Delta_d} g_{rs} + p \cdot (1-d) \cdot \frac{d}{2},$$

and

$$2 \cdot \sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}} = d + p \cdot (d-1) \cdot \frac{d}{2} - \sum_{r,s \in \Delta_d} g_{rs}. \quad (3)$$

This proves that the quantity $\sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}}$ is constant, for each class of the above partition; then,

$$\omega_G(\Gamma) = (d-1)! \cdot \sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}}$$

immediately follows.³

Now, the hypotheses on the 3-residues and d -residues of Γ directly implies, in virtue of Theorem 4 and of [9, Lemma 2] (originally proved in [10, Lemma 4.2]), that $\sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}}$ is an integer, and hence $\omega_G(\Gamma) \equiv 0 \pmod{(d-1)!}$ (or equivalently, $\omega'_G(\Gamma) = 2 \cdot \frac{\omega_G(\Gamma)}{(d-1)!}$ is even). \square

Then, the results stated in Section 1 trivially follow.

Proof of Theorems 1 and 2.

Both in the case of (Γ, γ) bipartite and in the case of (Γ, γ) representing a singular d -manifold, the residues of Γ obviously satisfy the hypotheses of Proposition 8. Hence, if $d \geq 4$ is even, $\omega_G(\Gamma) \equiv 0 \pmod{(d-1)!}$ holds. \square

Another particular situation is covered by Proposition 8, as the following corollary explains.

Corollary 9 *Let (Γ, γ) be a $(d+1)$ -colored graph, with $d \geq 4$, d even. If (Γ, γ) is a non-bipartite $(d+1)$ -colored graph such that each d -residue is bipartite, then*

$$\omega_G(\Gamma) \equiv 0 \pmod{(d-1)!}$$

\square

Note that there exist $(d+1)$ -colored graphs, with d even, $d \geq 4$ and odd reduced G-degree: of course, in virtue of Theorems 1 and 2, the represented d -pseudomanifold must be non-orientable and it can't be a singular d -manifold. As an example, for each $d \geq 4$, the $(d+1)$ -colored graph (Γ, γ) of Figure 2 represents the $(d-2)$ -th suspension Σ of the real projective plane $\mathbb{R}\mathbb{P}^2$ and $\omega'_G(\Gamma) = d-1$. It is easy to check that Γ is non-bipartite and Σ is not a singular manifold, since all d -residues of Γ , with the exception of $\Gamma_{\hat{0}}$, do not represent a closed $(d-1)$ -manifold. Moreover, its (unique) $\{0, 1, c\}$ -residue, for any $c \in \{2, \dots, d\}$, is non-bipartite and represents the non-orientable genus one surface $\mathbb{R}\mathbb{P}^2$.

Remark 2 In the case $d \geq 3$ odd, Proposition 6 implies that all $\frac{d!}{2}$ cyclic permutations (up to inverse) of Δ_d can be partitioned in $\frac{(d-1)!}{2}$ classes, each containing d cyclic permutations, $\bar{\varepsilon}^{(1)}, \bar{\varepsilon}^{(2)}, \dots, \bar{\varepsilon}^{(d)}$ say, so that

$$\sum_{i=1}^d \left[\sum_{j \in \mathbb{Z}_{d+1}} g_{\bar{\varepsilon}_j^{(i)}, \bar{\varepsilon}_{j+1}^{(i)}} \right] = 2 \sum_{r,s \in \Delta_d} g_{r,s}.$$

³In this way, one can reobtain - for $d \geq 4$ even - relation (2).

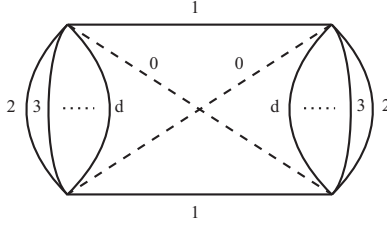


Figure 2: A $(d + 1)$ -colored graph representing the $(d - 2)$ -th suspension of $\mathbb{R}P^2$

Hence, a reasoning similar to the one used to prove Proposition 8 yields an alternative proof - for $d \geq 3$ odd - of relation (2): since

$$2 \cdot d - 2 \cdot \sum_{i=1}^d \rho_{\bar{\varepsilon}^{(i)}} = 2 \sum_{r,s \in \Delta_d} g_{rs} + p \cdot (1 - d) \cdot d$$

and

$$\sum_{i=1}^d \rho_{\bar{\varepsilon}^{(i)}} = d + p \cdot (d - 1) \cdot \frac{d}{2} - \sum_{r,s \in \Delta_d} g_{rs} \quad (4)$$

hold, then

$$\omega_G(\Gamma) = \frac{(d-1)!}{2} \cdot \sum_{i=1}^d \rho_{\bar{\varepsilon}^{(i)}} = \frac{(d-1)!}{2} \cdot \left(d + p \cdot (d-1) \cdot \frac{d}{2} - \sum_{r,s \in \Delta_d} g_{rs} \right)$$

directly follows.

Remark 3 It is worthwhile to stress that, for d even (resp. odd), formula (3) of Proposition 8 (resp. formula (4) of Remark 2) proves that the sum $\sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}}$ (resp. $\sum_{i=1}^d \rho_{\bar{\varepsilon}^{(i)}}$) of all regular genera with respect to the $d/2$ (resp. d) permutations belonging to the same class is half the reduced G-degree (resp. is the reduced G-degree)

$$\omega'_G(\Gamma) = d + p \cdot (d - 1) \cdot \frac{d}{2} - \sum_{r,s \in \Delta_d} g_{rs},$$

i.e. it is a constant which does not depend on the chosen partition class.

Hence, the regular genus $\rho(\Gamma)$ of the graph Γ is realized by the (not necessarily unique) permutation ε which maximizes the difference

$$\rho_{\bar{\varepsilon}}(\Gamma) - \rho_{\varepsilon}(\Gamma),$$

where $\rho_{\bar{\varepsilon}}(\Gamma)$ denotes the sum of the genera with respect to all other permutations of the same partition class.

4 The 4-dimensional case

In the 4-dimensional setting, the above combinatorial properties allow to prove further results about the G-degree.

In fact, it is easy to check that, for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ of Δ_4 , we have

$$\{(\varepsilon_j, \varepsilon_{j+1}) / j \in \mathbb{Z}_5\} \cup \{(\varepsilon_j, \varepsilon_{j+2}) / j \in \mathbb{Z}_5\} = \{(i, j) / i, j \in \Delta_5, i \neq j\}. \quad (5)$$

Let us denote by $\varepsilon' = (\varepsilon'_0, \varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4)$ the permutation $(\varepsilon_0, \varepsilon_2, \varepsilon_4, \varepsilon_1, \varepsilon_3)$ of Δ_4 , which may be said to be *associated* to ε .

Note that, when $d = 4$, the only partition of all 12 cyclic permutations of Δ_4 (up to inverse) is given by the six classes containing a given permutation ε and its associated ε' : see Figure 1, where each class of the Hamiltonian cycle decomposition of K_5 ensured by Proposition 5 is shown to correspond

to a pair $(\varepsilon_i, \varepsilon'_i)$ of associated permutations, for $i \in \mathbb{N}_6$. For example, the first class corresponds to the identical permutation $\varepsilon_1 = (0, 1, 2, 3, 4)$ and its associated permutation $\varepsilon'_1 = (0, 2, 4, 1, 3)$.

Then, the following result holds.

Proposition 10 *For each 5-colored graph (Γ, γ) , and for each pair $(\varepsilon, \varepsilon')$ of associated cyclic permutations of Δ_4 ,*

$$\omega_G(\Gamma) = 6\left(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma)\right).$$

Proof. Equality (5) directly yields

$$\sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}} + \sum_{j \in \mathbb{Z}_5} g_{\varepsilon'_j, \varepsilon'_{j+1}} = \sum_{i, j \in \Delta_5} g_{i, j}.$$

As a consequence, the sum of all regular genera of Γ with respect to the 12 cyclic permutations (up to inverse) of Δ_4 is six times the sum between the regular genera of Γ with respect to any pair $\varepsilon, \varepsilon'$ of associated permutations:

$$\omega_G(\Gamma) = 6\left(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma)\right).$$

□

Remark 4 By Proposition 10, for any 5-colored graph the sum between the regular genera of Γ with respect to any pair $\varepsilon, \varepsilon'$ of associated cyclic permutations is constant (see equality (3) and Remark 3, for $d = 4$):

$$\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma) = \frac{1}{2}\omega'_G(\Gamma) = 2 + 3p - \frac{1}{2} \sum_{i, j \in \Delta_5} g_{i, j}.$$

Hence, the regular genus $\rho(\Gamma)$ of the graph Γ is realized by the (not necessarily unique) permutation ε so that $\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma)$ is maximal.

Moreover:

Proposition 11 (a) *If (Γ, γ) is a 5-colored graph, then for each pair $(\varepsilon, \varepsilon')$ of associated cyclic permutations of Δ_4 ,*

$$2\left(\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma)\right) = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}} - \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}}$$

(b) *If (Γ, γ) is a 5-colored graph representing a singular 4-manifold M^4 , then for each pair $(\varepsilon, \varepsilon')$ of associated cyclic permutations of Δ_4 ,*

$$\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2}} - \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}, \varepsilon_{j+4}}$$

Proof. Statement (a) is an easy consequence of Theorem 4:

$$2 - 2\rho_\varepsilon(\Gamma) = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}} - 3p$$

and

$$2 - 2\rho_{\varepsilon'}(\Gamma) = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon'_j, \varepsilon'_{j+1}} - 3p = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}} - 3p.$$

On the other hand, relation $2g_{r, s, t} = g_{r, s} + g_{r, t} + g_{s, t} - p$ is known to be true for each order $2p$ 5-colored graph representing a singular 4-manifold (see [7, Lemma 21]). As a consequence we have:

$$2 \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2}} = \sum_{r, s \in \mathbb{Z}_5} g_{r, s} + \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}} - 5p$$

and

$$2 \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}, \varepsilon_{j+4}} = \sum_{r, s \in \mathbb{Z}_5} g_{r, s} + \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}} - 5p.$$

By making the difference,

$$2 \left(\sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2}} - \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}, \varepsilon_{j+4}} \right) = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}} - \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}}$$

is obtained; so, statement (b) follows, via statement (a). □

Proposition 11 enables to obtain the following improvement of [7, Proposition 29(a)].

Corollary 12 *Let (Γ, γ) be a 5-colored graph. Then:*

$$\omega_G(\Gamma) = 12 \cdot \rho(\Gamma) \iff$$

$$\sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}} = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}} \text{ for each cyclic permutation } \varepsilon \text{ of } \Delta_4.$$

Proof. From [7, Proposition 29 (a)], it is known that

$$\omega_G(\Gamma) = 12 \cdot \rho(\Gamma) \implies \sum_{j \in \mathbb{Z}_5} g_{\bar{\varepsilon}_j, \bar{\varepsilon}_{j+1}} = \sum_{j \in \mathbb{Z}_5} g_{\bar{\varepsilon}_j, \bar{\varepsilon}_{j+2}},$$

$\bar{\varepsilon}$ being the cyclic permutation of Δ_4 such that $\rho(\Gamma) = \rho_{\bar{\varepsilon}}(\Gamma)$. So, $\rho_{\bar{\varepsilon}'}(\Gamma) - \rho_{\bar{\varepsilon}}(\Gamma) = 0$ directly follows via Proposition 11 (a). Now, since $\rho_{\varepsilon}(\Gamma) + \rho_{\varepsilon'}(\Gamma)$ is constant for each pair $(\varepsilon, \varepsilon')$ of associated cyclic permutations of Δ_4 (see Remark 4), then:

$$|\rho_{\varepsilon'}(\Gamma) - \rho_{\varepsilon}(\Gamma)| \leq \rho_{\bar{\varepsilon}'}(\Gamma) - \rho_{\bar{\varepsilon}}(\Gamma).$$

Hence, $\rho_{\bar{\varepsilon}'}(\Gamma) - \rho_{\bar{\varepsilon}}(\Gamma) = 0$ implies $\sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+1}} = \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j, \varepsilon_{j+2}}$ (i.e. $\rho_{\varepsilon'}(\Gamma) - \rho_{\varepsilon}(\Gamma) = 0$, in virtue of Proposition 11 (a)) for each cyclic permutation ε of Δ_4 .

The reversed implication is straightforward, via Proposition 11(a). □

Proposition 13 *If (Γ, γ) is an order $2p$ 5-colored graph representing a singular 4-manifold M^4 , then, for each pair $(\varepsilon, \varepsilon')$ of associated cyclic permutations of Δ_4 :*

$$\chi(M^4) = \left(\rho_{\varepsilon}(\Gamma) + \rho_{\varepsilon'}(\Gamma) \right) - p + \sum_{i \in \Delta_4} g_i - 2.$$

Proof. It is sufficient to apply Proposition 10 to the third equality of [7, Proposition 22]. □

Let us now recall two particular types of crystallizations introduced and studied in [2] and [1]⁴: they are proved to be “minimal” with respect to regular genus, among all graphs representing the same PL 4-manifold.

Definition 4 A crystallization of a PL 4-manifold M^4 with $rk(\pi_1(M^4)) = m \geq 0$ is called a *semi-simple crystallization* if $g_{j,k,l} = 1 + m \quad \forall j, k, l \in \Delta_4$.

A crystallization of a PL 4-manifold M^4 with $rk(\pi_1(M^4)) = m$ is called a *weak semi-simple crystallization* if $g_{i,i+1,i+2} = 1 + m \quad \forall i \in \Delta_4$ (where the additions in subscripts are intended in \mathbb{Z}_5).

⁴Both semi-simple and weak semi-simple crystallizations generalize the notion of *simple crystallizations* for simply-connected PL 4-manifolds: see [3] and [8].

According to [2], for each order $2p$ crystallization (Γ, γ) of a closed PL 4-manifold M^4 , with $rk(\pi_1(M^4)) = m0$, let us set

$$g_{j,k,l} = 1 + m + t_{j,k,l}, \quad \text{with } t_{j,k,l} \quad \forall j, k, l \in \Delta_4.$$

Semi-simple (resp. weak semi-simple) crystallizations turn out to be characterized by $t_{j,k,l} = 0 \quad \forall j, k, l \in \Delta_4$ (resp. $t_{i,i+1,i+2} = 0 \quad \forall i \in \mathbb{Z}_5$).

In [2], the relation

$$p = 3\chi(M^4) + 5(2m - 1) + \sum_{j,k,l \in \Delta_4} t_{j,k,l} \quad (6)$$

is proved to hold; hence, $p = \bar{p} + q$ follows, where $q = \sum_{j,k,l \in \Delta_4} t_{j,k,l} \geq 0$ and $\bar{p} = 3\chi(M^4) + 5(2m - 1)$ is the minimum possible half order of a crystallization of M^4 , which is attained if and only if M^4 admits semi-simple crystallizations.

With the above notations, the following results can be obtained.

Proposition 14 *Let (Γ, γ) be an order $2p$ crystallization of a closed PL 4-manifold M^4 , with $rk(\pi_1(M^4)) = m$. Then, for each pair $(\varepsilon, \varepsilon')$ of associated cyclic permutations of Δ_4 , with $\rho_\varepsilon(\Gamma) \leq \rho_{\varepsilon'}(\Gamma)$:*

$$\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) = q - 2 \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \leq q. \quad (7)$$

Moreover, The following statements are equivalent:

- (a) a cyclic permutation ε of Δ_4 exists, such that $\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) = q$;
- (b) Γ is a weak semi-simple crystallization;
- (c) $\rho(\Gamma) = 2\chi(M^4) + 5m - 4$.⁵

Proof. In virtue of Proposition 11(b), an easy computation proves relation (7):

$$\begin{aligned} \rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) &= \sum_{i \in \mathbb{Z}_5} (m + 1 + t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}}) - \sum_{i \in \mathbb{Z}_5} (m + 1 + t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}) = \\ &= \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}} - \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = \\ &= \sum_{j,k,l \in \Delta_4} t_{j,k,l} - 2 \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = \\ &= q - 2 \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \leq q. \end{aligned}$$

Now, if Γ is a weak semi-simple crystallization, by definition itself a cyclic permutation ε of Δ_4 exists⁶, so that, for each $i \in \mathbb{Z}_5$, $g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = m + 1$, i.e. $t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = 0$. So, $\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) = q$ easily follows from relation (7).

On the other hand, if a cyclic permutation ε of Δ_4 exists, so that $\rho_{\varepsilon'}(\Gamma) - \rho_\varepsilon(\Gamma) = q$, relation (7) yields $t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = 0$, i.e. $g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} = m + 1$, which is exactly - up to color permutation - the definition of weak semi-simple crystallization. Hence, (a) and (b) are proved to be equivalent.

Then, by comparing Proposition 13 (with the assumption $g_i = 1 \quad \forall i \in \Delta_4$) and relation (6), we obtain:

$$\rho_{\varepsilon'}(\Gamma) + \rho_\varepsilon(\Gamma) = 2(2\chi(M^4) + 5m - 4) + q. \quad (8)$$

By making use of relation (7),

$$\rho_\varepsilon(\Gamma) = 2\chi(M^4) + 5m - 4 + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}} \quad (9)$$

⁵Note that, in this case, the permutation $\bar{\varepsilon}$ such that $\rho(\Gamma) = \rho_{\bar{\varepsilon}}(\Gamma)$ coincides with the permutation ε of point (a); moreover, $\rho(\Gamma) = \mathcal{G}(M^4)$ holds, in virtue of the inequality $\mathcal{G}(M^4) \geq 2\chi(M^4) + 5m - 4$, proved in [2].

⁶ ε turns out to be the permutation of Δ_4 associated to $\varepsilon' = (0, 1, 2, 3, 4)$.

easily follows, as well as

$$\rho_{\varepsilon'}(\Gamma) = 2\chi(M^4) + 5m - 4 + \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}}. \quad (10)$$

Relation (9) directly yields the co-implication between statements (b) and (c). \square

We conclude the paper with a list of remarks, which arise from the previous results.

Remark 5 With the above notations, formulae (9) and (10) immediately give

$$2 + \frac{\rho_\varepsilon(\Gamma)}{2} - \frac{5m}{2} - \frac{q}{4} \leq \chi(M^4) \leq 2 + \frac{\rho_{\varepsilon'}(\Gamma)}{2} - \frac{5m}{2} - \frac{q}{4}$$

for each crystallization of a closed PL 4-manifold M^4 . Another double inequality concerning the Euler characteristic of a closed PL 4-manifold M^4 and the regular genera of any order $2p$ crystallization of M^4 with respect to a pair of associated permutations may be easily obtained from Proposition 13, by making use of the assumptions $\sum_{i \in \Delta_4} g_i = 5$ and $\rho_\varepsilon(\Gamma) \leq \rho_{\varepsilon'}(\Gamma)$:

$$2\rho_\varepsilon(\Gamma) - p + 3 \leq \chi(M^4) \leq 2\rho_{\varepsilon'}(\Gamma) - p + 3.$$

Note that such double inequalities assume a specific relevance in case of “low” difference between $\rho_{\varepsilon'}(\Gamma)$ and $\rho_\varepsilon(\Gamma)$ (in particular if $\rho_{\varepsilon'}(\Gamma) = \rho_\varepsilon(\Gamma)$ occurs, possibly with $\rho(\Gamma) < \rho_\varepsilon(\Gamma)$).

On the other hand, Proposition 13 and formula (7) (resp. formula (8)) give

$$\begin{aligned} \chi(M^4) &= 2\rho_\varepsilon(\Gamma) - p + 3 + (q - 2 \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}) \\ (\text{resp. } \chi(M^4) &= 2 + \frac{\rho_\varepsilon(\Gamma)}{2} - \frac{5m}{2} - \frac{1}{2} \sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}). \end{aligned}$$

Hence, the following double inequalities arise, too, both involving the regular genus with respect to only one cyclic permutation:

$$\begin{aligned} 2\rho_\varepsilon(\Gamma) - p + 3 &\leq \chi(M^4) \leq 2\rho_\varepsilon(\Gamma) - p + q + 3; \\ 2 + \frac{\rho_\varepsilon(\Gamma)}{2} - \frac{5m}{2} - \frac{q}{4} &\leq \chi(M^4) \leq 2 + \frac{\rho_\varepsilon(\Gamma)}{2} - \frac{5m}{2}. \end{aligned}$$

Remark 6 Note that relation (8) exactly corresponds, via Proposition 10, to [7, Proposition 27]. Moreover, relation (9)⁷ directly implies the inequality $\rho_\varepsilon(\Gamma) \geq 2\chi(M^4) + 5m - 4$ (which is one of the upper bounds obtained in [2]) and ensures that, for each crystallization (Γ, γ) of a PL 4-manifold, the regular genus $\rho(\Gamma)$ is realized by the (not necessarily unique) permutation ε so that $\sum_{i \in \mathbb{Z}_5} t_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}$ (or, equivalently, $\sum_{i \in \mathbb{Z}_5} g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+4}}$) is minimal. Finally, the equivalence between items (b) and (c) in the above Proposition 14 gives a direct proof of [1, Theorem 2].

Remark 7 Semi-simple crystallizations turn out to be the intersection (characterized by $q = 0$) between the two classes of weak semi-simple crystallizations and of crystallizations satisfying $\omega_G(\Gamma) = 12 \cdot \rho(\Gamma)$ (and hence $\rho_\varepsilon(\Gamma) = 2\chi(M^4) + 5m - 4 + \frac{1}{2}q$, for each cyclic permutation ε of Δ_4).

Moreover, it is easy to check that

$$q \leq 2 \implies \Gamma \text{ is a weak semi-simple crystallization.}$$

In fact, if $q = \sum_{j,k,l \in \Delta_4} t_{j,k,l} \leq 2$, at most two triads (j, k, l) of distinct elements in Δ_4 exist, so that $g_{j,k,l} = 1 + m + t_{j,k,l} > 1 + m$. This ensures the existence of a cyclic permutation ε of Δ_4 so that, for each $i \in \mathbb{Z}_5$, $g_{\varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}} = m + 1$, which is exactly the requirement for a weak semi-simple crystallization.

⁷Actually, relation (9) corrects a trivial error in the proof (and statement) of [1, Lemma 7], not affecting the implications in order to prove the main result of that paper.

Remark 8 The formula obtained in [16, Lemma 4.2] for bipartite $(d+1)$ -colored graphs and extended to the general case in [7, Lemma 13] gives, if $d = 4$,

$$\omega_G(\Gamma) = 3\left(p + 4 - \sum_{i \in \Delta_4} g_i\right) + \sum_{i \in \Delta_4} \omega_G(\Gamma_{\hat{i}}),$$

where, for each $i \in \Delta_4$, $\omega_G(\Gamma_{\hat{i}})$ denotes the sum of the G-degrees of the connected components of $\Gamma_{\hat{i}}$. Hence, $\sum_{i \in \Delta_4} \omega_G(\Gamma_{\hat{i}})$ is always a multiple of 3 (recall Proposition 10 and Theorem 1).

Moreover, if (Γ, γ) represents a singular 4-manifold M^4 , Proposition 10 and Proposition 13 imply:

$$\sum_{i \in \Delta_4} \omega_G(\Gamma_{\hat{i}}) = 3\left(2\chi(M^4) + p - \sum_{i \in \Delta_4} g_i\right).$$

Note that, if (Γ, γ) is a crystallization of a closed PL 4-manifold M^4 with $rk(\pi_1(M^4)) = m$, relation (6) gives:

$$\sum_{i \in \Delta_4} \omega_G(\Gamma_{\hat{i}}) = 3\left[5\left(\chi(M^4) + 2m - 2\right) + q\right].$$

In particular, if (Γ, γ) is semi-simple (i.e. $q = 0$), $\sum_{i \in \Delta_4} \omega_G(\Gamma_{\hat{i}}) = 15\left(\chi(M^4) + 2m - 2\right)$ follows, as [2, Proposition 8] trivially implies.

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