SILTING MODULES AND RING EPIMORPHISMS

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Dedicated to Alberto Facchini on the occasion of his 60th birthday

ABSTRACT. There are well-known constructions relating ring epimorphisms and tilting modules. The new notion of silting module provides a wider framework for studying this interplay. To every partial silting module we associate a ring epimorphism which we describe explicitly as an idempotent quotient of the endomorphism ring of the Bongartz completion. For hereditary rings, this assignment is used to parametrise homological ring epimorphisms by silting modules. We further show that homological ring epimorphisms of a hereditary ring form a lattice which completes the poset of noncrossing partitions in the case of finite dimensional algebras.
KEYWORDS: tilting module; silting module; ring epimorphism; universal localisation; torsion pair; hereditary ring.

1. INTRODUCTION

There is a close relationship between ring epimorphisms and tilting theory which goes back to [GL] and was further studied in [AS1, AA]. In fact, ring epimorphisms with nice homological properties can be used to construct tilting modules. Here we have to deal with large tilting modules, since even the ring epimorphisms $A \longrightarrow B$ of a finite dimensional algebra A usually involve infinite dimensional algebras B. Tilting modules arising from ring epimorphisms play an important role in classification results. For example, over a Dedekind domain, all tilting modules are of this form, and over the Kronecker algebra, this applies to all but one up to equivalence ([AS1, AS2, M]).

In [AMV], we developed the new notion of silting module as a common generalisation of (possibly large) tilting modules and support τ-tilting modules. In this paper, we show that silting theory is an appropriate context to study the phenomena above, as it provides a wider framework for the interplay with ring epimorphisms. This idea is supported by parallel work in [MS], where it is shown that for finite dimensional algebras of finite representation type, silting modules are in bijection with universal localisations. Here we prove a similar result for hereditary rings, and we use this approach to shed some new light on combinatorial aspects of silting theory.

Silting modules over an arbitrary ring A capture some of the main features of tilting and support τ-tilting modules. As shown in [AMV], they generate torsion classes in the module category which provide left approximations, they are the 0^{th}-cohomologies of 2-term silting complexes, and they correspond bijectively to certain t-structures and co-t-structures in the derived category.

Of particular relevance to this paper is the existence of a suitable notion of partial silting module. In [AMV], we proved that every partial silting module admits an analogue of the Bongartz complement and, thus, can be completed to a silting module. Here, we associate to a given partial silting module a full subcategory of $Mod(A)$ which can be understood as its perpendicular category. We show that this subcategory is bireflective, that is, its inclusion functor admits both left and right adjoints. Such subcategories are known to correspond bijectively to equivalence classes of ring epimorphisms starting in A ([GdP]). We can thus assign a ring epimorphism $A \longrightarrow B$ to every partial silting module.

This assignment extends results proved in the context of tilting ([GL, CTT2]) or support τ-tilting modules ([J]). In particular, it is shown in [J] that the ring B associated with a τ-rigid (that is, a finitely generated partial silting) module T_1 over a finite dimensional algebra A is a support algebra of the endomorphism ring $End_A(T)$ of the Bongartz completion T of T_1. We prove (Theorem 3.5) that all ring epimorphisms arising from partial silting modules can be described in a similar way as idempotent quotients of $End_A(T)$.

Working with large modules implies, however, that we have to replace idempotent elements by idempotent ideals, and the proof requires a detailed analysis of the functors involved in our construction. We further investigate properties of the ring epimorphism associated with a partial silting module. It turns out that even finite dimensional partial tilting modules over finite dimensional algebras can give rise to non-injective or non-homological ring epimorphisms.

Later, we focus on the case when A is a hereditary ring. Here all ring epimorphisms arising from partial silting modules are homological. Conversely, every homological ring epimorphism $f : A \longrightarrow B$ gives rise to a silting A-module $T = B \oplus Coker(f)$, and the map f can be regarded as a minimal left $Add(T)$-approximation of the regular module A. We thus restrict our attention to minimal silting modules, that is, silting modules T providing a minimal $Add(T)$-approximation sequence $A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$. Using that A is hereditary, it follows that T_1 is partial silting and, thus, we can assign to T a well-defined ring epimorphism. We show that this assignment establishes a bijection between minimal silting modules and homological ring epimorphisms (Theorem 5.8), where the minimal tilting modules correspond to injective homological ring epimorphisms. Combining this bijection with results from [Sch3, KSt], where it is shown that homological ring epimorphisms of hereditary rings are parametrised by wide, i.e. abelian and extension-closed, subcategories of finitely presented modules, we obtain bijections as follows (Corollary 5.17).

$$\begin{tikzcd}[column sep=small]
\left\{\begin{array}{c}\text{equivalence classes of}\\ \text{minimal silting } A\text{-modules}\end{array}\right\} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \left\{\begin{array}{c}\text{epiclasses of homological}\\ \text{ring epimorphisms of } A\end{array}\right\} \arrow[dl, leftrightarrow] \\
& \left\{\begin{array}{c}\text{wide subcategories}\\ \text{of } mod(A)\end{array}\right\} &
\end{tikzcd}$$

Over a finite dimensional hereditary algebra A, our correspondence restricts to a bijection between finite dimensional support tilting modules and homological ring epimorphisms $A \longrightarrow B$ with finite dimensional B, and we recover results from [IT, M]. Of course, the combinatorial interpretation of finite dimensional support tilting modules in terms of noncrossing partitions or clusters is lost when working in our general setting. However, a ring theoretic counterpart is provided by the poset of all homological ring epimorphisms. The advantage is that this poset is indeed a complete lattice (Proposition 5.13). Notice that if we restrict to ring epimorphisms with finite dimensional target, we obtain the poset of exceptional antichains from [R] (see also [HK]) which is not a lattice in general.

Finally, we show that our classification of minimal tilting modules over hereditary rings fits in a number of classification results over further rings that reveal a deep connection between tilting theory and localisation.

In a forthcoming paper, we will explore the connections between silting theory and categorical localisation of module categories and derived categories. An important role will be played by the explicit description of the ring epimorphism associated with a partial silting module mentioned above.

The structure of the paper is as follows. Section 2 recalls some facts on silting modules and ring epimorphisms. In Section 3, we construct the bireflective subcategory associated with a partial silting module and we describe explicitly the associated ring epimorphism. In Section 4, we investigate the homological properties of these ring epimorphisms. Finally, Section 5 discusses silting modules over hereditary rings and the classification of homological ring epimorphisms.

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2. Preliminaries

Throughout, A is a unitary ring and by an A-module we mean a right A-module, unless otherwise stated. The category of all A-modules will be denoted by $Mod(A)$ and its full subcategory of projective A-modules by $Proj(A)$. The category $mod(A)$ denotes the category of finitely presented modules. In some contexts, we will consider finite dimensional algebras over an algebraically closed field $\mathbb{K}$, usually denoted by Λ.

For a given A-module M, we denote by M° the subcategory of $Mod(A)$ consisting of the objects N such that $Hom_A(M,N)=0$, and by $M^{\perp_1}$ the subcategory of $Mod(A)$ consisting of the objects N such that $Ext^1_A(M,N)=0$. We set $M^{\perp}$ to be $M^{\circ}\cap M^{\perp_1}$. Further, $Add(M)$ denotes the additive closure of M consisting of all modules isomorphic to a direct summand of an arbitrary direct sum of copies of M, while $Gen(M)$ is the subcategory of M-generated modules (that is, all epimorphic images of modules in $Add(M)$).

2.1. **Silting modules.** Recall from [CT] that an A-module T is called **partial tilting**, if $T^{\perp_1}$ is a torsion class containing T. Moreover, T is **tilting**, if $Gen(T)=T^{\perp_1}$. In order to introduce the notion of (partial) silting module, consider, for a morphism σ in $Proj(A)$, the class of A-modules

$$\mathcal{D}_\sigma := \{X \in Mod(A) | Hom_A(\sigma, X) \text{ is surjective}\}.$$

Definition 2.1. [AMV, Definition 3.10] We say that an A-module T is

- **partial silting** if there is a projective presentation σ of T such that
  - (S1) $\mathcal{D}_\sigma$ is a torsion class.
  - (S2) T lies in $\mathcal{D}_\sigma$.
- **silting** if there is a projective presentation σ of T such that $Gen(T)=\mathcal{D}_\sigma$.

We will then say that T is (partial) silting **with respect to** σ. The class $\mathcal{D}_\sigma$ associated to a silting module is called a **silting class**.

It follows easily from the definition that (partial) tilting modules are (partial) silting (see [AMV, Proposition 3.13(1)]). Recall that an object M in a class $\mathcal{C}$ of A-modules is said to be **Ext-projective in** $\mathcal{C}$, if $\mathcal{C}$ is contained in $M^{\perp_1}$. The following facts on silting modules will be useful later.

Proposition 2.2. [AMV, Lemma 3.4 and 3.7, Proposition 3.5 and 3.13] *Let T be a silting A-module. The following statements hold.*

(1) *T is tilting over $A/Ann(T)$.*
(2) *$Add(T)$ is the class of Ext-projective modules in $Gen(T)$.*
(3) *There is an exact sequence*

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

such that T_0 and T_1 lie in $Add(T)$ and ϕ is a left $Gen(T)$-approximation.

In particular, from (2) above it follows that two silting modules have the same additive closure if and only if they generate the same torsion class, in which case we say that they are **equivalent**. Moreover, from (3) it is easy to see that every A-module admits a left $Gen(T)$-approximation.

2.2. **Ring epimorphisms.** Recall that a **ring epimorphism** is an epimorphism in the category of rings with unit. Two ring epimorphisms $f:A\longrightarrow B$ and $g:A\longrightarrow C$ are said to be **equivalent** if there is a ring isomorphism $h:B\longrightarrow C$ such that $g=h\circ f$. We then say that B and C lie in the same **epiclass** of A. Epiclasses of a ring A can be classified by suitable subcategories of $Mod(A)$. For a ring epimorphism $f:A\longrightarrow B$ we denote by $\mathcal{X}_B$ the essential image of the associated restriction functor f_*.

Theorem 2.3. [GdP, Theorem 1.2][GL] *There is a bijection between:*

(1) *epiclasses of ring epimorphisms $A\longrightarrow B$;*
(2) *bireflective subcategories $\mathcal{X}_B$ of $Mod(A)$, i.e., full subcategories of $Mod(A)$ closed under products, coproducts, kernels and cokernels.*

Bireflective subcategories $\mathcal{X}$ are precisely those for which the inclusion functor $\mathcal{X}\longrightarrow Mod(A)$ admits both a left and a right adjoint. As a consequence, the unit of the adjunction given by the left adjoint of the inclusion functor yields left $\mathcal{X}$-approximations $\psi_M : M\longrightarrow X_M$, for all M in $Mod(A)$, such that $Hom_A(\psi_M, X)$

is an isomorphism for all X in $\mathcal{X}$. These approximations are called $\mathcal{X}$-**reflections** and they are left-minimal, i.e., any endomorphism θ of X_M with $\theta \circ \psi_M = \psi_M$ is an isomorphism. In fact, if $f : A \longrightarrow B$ is the ring epimorphism associated with $\mathcal{X}$, then ψ_M is the natural map $M \longrightarrow M \otimes_A B$. Dually, there are right $\mathcal{X}$-approximations with analogous properties called $\mathcal{X}$-**coreflections**.

The following two results will be frequently used throughout.

Proposition 2.4. [Sch1, Theorem 4.8] *Let $A \longrightarrow B$ be a ring epimorphism. The following are equivalent.*

(1) $Tor_1^A(B,B) = 0$;

(2) $Ext_A^1(M,N) \cong Ext_B^1(M,N)$ *for all B-modules M and N.*

Proposition 2.5. *Let $\pi : A \longrightarrow \bar{A}$ be a surjective ring epimorphism with kernel I. The following holds.*

(1) *The subcategory $\mathcal{X}_{\bar{A}}$ is closed under quotients and subobjects in $Mod(A)$.*

(2) *I is idempotent if and only if $\mathcal{X}_{\bar{A}}$ is closed under extensions in $Mod(A)$. In this case, we have $\mathcal{X}_{\bar{A}} = I^{\circ}$.*

Proof. (1) This follows easily from the fact that $\mathcal{X}_{\bar{A}} = \{X \in Mod(A) \mid XI = 0\}$.

(2) First observe that, by applying the functor $- \otimes_A A/I$ to the short exact sequence induced by the inclusion of I in A, we get $Tor_1^A(\bar{A},\bar{A}) = Tor_1^A(A/I,A/I) \cong I \otimes_A A/I \cong I/I^2$. The first part of the statement follows then from Proposition 2.4. Assume now that $I = I^2$. By applying the functor $Hom_A(-,X)$, for an A-module X, to the short exact sequence induced by π, we get the exact sequence

$$0 \longrightarrow Hom_A(\bar{A},X) \xrightarrow{\pi_X} Hom_A(A,X) \longrightarrow Hom_A(I,X) \longrightarrow Ext_A^1(\bar{A},X) \longrightarrow 0$$

If $X \in \mathcal{X}_{\bar{A}}$, then π_X is an isomorphism and, since $Ext_A^1(\bar{A},X) = 0$ by Proposition 2.4, it follows that $X \in I^{\circ}$. Conversely, if $Hom_A(I,X) = 0$, then π_X is an isomorphism turning X into an $\bar{A}$-module. $\square$

A ring epimorphism $f : A \longrightarrow B$ is said to be **homological** if for all $i > 0$ we have $Tor_i^A(B,B) = 0$ or, equivalently, if $Ext_B^i(M,N) \cong Ext_A^i(M,N)$ for all M and N in $Mod(B)$ (see [GL, Theorem 4.4]). Certain homological ring epimorphisms of A induce tilting modules.

Definition 2.6. A tilting A-module T is said to **arise from a ring epimorphism**, if there is an injective ring epimorphism $A \longrightarrow B$ such that $B \oplus B/A$ is a tilting A-module equivalent to T.

In this definition, the ring epimorphism is unique up to equivalence. Moreover, the canonical sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow B/A \longrightarrow 0.$$

is an approximation sequence as in Proposition 2.2(3).

The following theorem relates ring epimorphisms and tilting modules.

Theorem 2.7. [AS1, Theorem 3.5, Theorem 3.10]

(1) *Let $A \longrightarrow B$ be an injective homological ring epimorphism such that the A-module B has projective dimension at most one. Then $B \oplus B/A$ is a tilting A-module and $\mathcal{X}_B$ equals $(B/A)^{\perp}$.*

(2) *Let T be a tilting A-module. Then T arises from a ring epimorphism if and only if there is an $Add(T)$-approximation sequence*

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

such that $Hom_A(T_1,T_0) = 0$.

3. Ring epimorphisms arising from partial silting modules

We start by generalising some ideas from [CTT2] on partial tilting modules. We fix a partial silting A-module T_1 with associated torsion class $\mathcal{D}_\sigma$ given by a projective presentation σ of T_1. Since $Gen(T_1)$ is a torsion class by [AMV, Lemma 2.3], there are two torsion pairs associated with T_1:

$$(\mathcal{D},\mathcal{R}) := (\mathcal{D}_\sigma, \mathcal{D}_\sigma^{\circ}) \text{ and } (\mathcal{T},\mathcal{F}) := (Gen(T_1), T_1^{\circ}).$$

We are interested in the full subcategory $\mathcal{Y} := \mathcal{D} \cap \mathcal{F}$ of $Mod(A)$. Note that, by definition,

$$\mathcal{F} = \{X \in Mod(A) | Hom_A(\sigma, X) \text{ is injective}\}$$

and, therefore,

$$\mathcal{Y} = \{X \in Mod(A) | Hom_A(\sigma, X) \text{ is bijective}\}.$$

We will show that $\mathcal{Y}$ is a bireflective subcategory of $Mod(A)$ and, thus, we can associate a ring epimorphism $A \longrightarrow B$ such that $\mathcal{Y} = \mathcal{X}_B$.

Remark 3.1. Given a ring epimorphism $f : A \longrightarrow B$ such that $\mathcal{X}_B = \mathcal{Y}$, it follows that $\sigma \otimes_A B$ is an isomorphism (compare to [Sch2, Theorem 5.2] and [M, Proposition 3.3(1)]). Indeed, since $Hom_A(\sigma, X)$ is an isomorphism for all X in $\mathcal{Y}$, so is $Hom_B(\sigma \otimes_A B, Y)$ for every B-module Y by the adjunction $(- \otimes_A B, f_*)$. Hence, $Hom_B(T_1 \otimes_A B, Y) = 0$ for all Y in $Mod(B)$, showing that $T_1 \otimes_A B = 0$ and that $\sigma \otimes_A B$ is surjective. Now, if we write $\sigma : P \longrightarrow Q$, then $Hom_B(\sigma \otimes_A B, P \otimes_A B)$ is an isomorphism by the above. Consequently, the identity map on $P \otimes_A B$ factors through $\sigma \otimes_A B$, proving that $\sigma \otimes_A B$ is also injective, as wanted. Note that, in case σ is a map between finitely generated projective modules, B is the universal localisation of A at $\{\sigma\}$ (see Theorem 5.14).

The following arguments mimic the approach taken in [CTT2, Proposition 1.4].

Lemma 3.2. *Consider an A-module M in $\mathcal{Y}$ together with a short exact sequence in $Mod(A)$*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then we have the following equivalent conditions

$$L \in \mathcal{Y} \Leftrightarrow L \in \mathcal{D} \Leftrightarrow N \in \mathcal{F} \Leftrightarrow N \in \mathcal{Y}.$$

Proof. Since M belongs to $\mathcal{Y} = \mathcal{D} \cap \mathcal{F}$, we know that L lies in $\mathcal{F}$ and N lies in $\mathcal{D}$. This proves the two outer equivalences. For the remaining one consider the following commutative diagram induced by $\sigma : P \longrightarrow Q$

$$\begin{tikzcd}
0 \arrow[r] & Hom_A(Q,L) \arrow[r] \arrow[d, "Hom_A(\sigma,L)"] & Hom_A(Q,M) \arrow[r] \arrow[d, "Hom_A(\sigma,M)", "\cong"'] & Hom_A(Q,N) \arrow[r] \arrow[d, "Hom_A(\sigma,N)"] & 0 \\
0 \arrow[r] & Hom_A(P,L) \arrow[r] & Hom_A(P,M) \arrow[r] & Hom_A(P,N) \arrow[r] & 0.
\end{tikzcd}$$

By the Snake Lemma, $Hom_A(\sigma, L)$ is surjective if and only if $Hom_A(\sigma, N)$ is injective. □

We can construct left $\mathcal{Y}$-approximations for any A-module X. Take a $\mathcal{D}$-approximation sequence

$$X \xrightarrow{\phi_X} M_X \longrightarrow T_1^{(I)} \longrightarrow 0, \tag{3.1}$$

which is constructed as in [AMV, Theorem 3.15]. Recall that $T := T_1 \oplus M_A$ is a silting module with $Gen(T) = \mathcal{D}$. Now, consider the composition

$$\psi_X : X \xrightarrow{\phi_X} M_X \xrightarrow{q_X} M_X / \tau_{T_1}(M_X) =: \overline{M}_X$$

where τ_{T_1} denotes the trace of T_1, which is the torsion-radical with respect to $\mathcal{T}$. Note that it is clear by construction that $\overline{M}_X$ lies in $\mathcal{Y}$ for any A-module X and, moreover, since ψ_X is the composition of a left $\mathcal{D}$-approximation and a left $\mathcal{F}$-approximation, it is a left $\mathcal{Y}$-approximation.

Proposition 3.3. *The full subcategory $\mathcal{Y}$ of $Mod(A)$ is bireflective and extension-closed. Moreover, the $\mathcal{Y}$-reflection of an A-module X is given by ψ_X.*

Proof. We have to show that $\mathcal{Y} = \{X \in Mod(A) | Hom_A(\sigma, X) \text{ is bijective}\}$ is closed under products, coproducts, kernels, cokernels and extensions. Clearly, $\mathcal{Y}$ is closed under extensions and coproducts, since so are $\mathcal{D}$ and $\mathcal{F}$. Note that $\mathcal{Y}$ is also closed under products, since $Hom_A(\sigma, \prod X_i) = \prod Hom_A(\sigma, X_i)$ for X_i in $Mod(A)$ and products are exact. Finally, take a map $\omega : M \longrightarrow N$ in $\mathcal{Y}$. Clearly, $Im(\omega)$ belongs to $\mathcal{Y}$, since it is a quotient of M (thus, in $\mathcal{D}$) and a submodule of N (thus, in $\mathcal{F}$). Now the claim follows by Lemma 3.2.

It remains to show that $Hom_A(\psi_X, Y)$ is an isomorphism for all Y in $\mathcal{Y}$. It is clearly a surjection, since ψ_X is a left $\mathcal{Y}$-approximation. Moreover, we have $Ker(Hom_A(\psi_X, Y)) = Hom_A(Coker(\psi_X), Y)$. Since Y lies in $\mathcal{F}$, it suffices to show that $Coker(\psi_X)$ is in $\mathcal{T} = Gen(T_1)$. The following commutative diagram with surjective vertical maps finishes the proof

$$\begin{tikzcd} X \arrow[r, "\phi_X"] \arrow[d, equal] & M_X \arrow[r] \arrow[d] & T_1^{(I)} \arrow[r] \arrow[d] & 0 \\ X \arrow[r, "\psi_X"] & \overline{M}_X \arrow[r] & Coker(\psi_X) \arrow[r] & 0. \end{tikzcd}$$

□

Hence, it follows from Theorem 2.3 and Proposition 2.4 that we can associate to every partial silting module a ring epimorphism $A \longrightarrow B$ with $Tor_1^A(B, B) = 0$. Moreover, we can choose B to be $End_A(M_A/\tau_{T_1}(M_A))$ (see, for example, [AS1] for details). These ring epimorphisms are closely related to the ring epimorphisms built in [CTT1], as the following lemma shows.

Lemma 3.4. *Let T_1 be a partial silting module and $T = T_1 \oplus M_A$ its Bongartz completion to a silting module as above. Let $f : A \longrightarrow B$ be the ring epimorphism associated with T_1 and $\pi : A \longrightarrow \bar{A} := A/Ann(T)$ the canonical projection. Then there is a ring epimorphism $g : \bar{A} \longrightarrow B$ satisfying $f = g \circ \pi$ and such that*

$$Im(g_*) = Ker(Hom_{\bar{A}}(T_1, -)) \cap Ker(Ext^1_{\bar{A}}(T_1, -)).$$

Proof. Recall that $\mathcal{Y} = Gen(T) \cap T_1{}^{\circ}$. Since $Gen(T)$ is contained in $Im(\pi_*)$, we can naturally identify $\mathcal{Y}$ with a bireflective subcategory of $Mod(\bar{A})$. The associated ring epimorphism $g : \bar{A} \longrightarrow B$ is the one occurring in the factorisation $f = g \circ \pi$, since the restriction functors clearly satisfy $f_* = \pi_* \circ g_*$. Now, the approximation sequence (3.1) for $X = A$ induces a short exact sequence in $Mod(\bar{A})$ of the form

$$0 \longrightarrow \bar{A} \xrightarrow{\overline{\phi_A}} M_A \longrightarrow T_1^{(I)} \longrightarrow 0.$$

Since, by Proposition 2.2(1), T is a tilting $\bar{A}$-module, it follows that $Gen(T) = Ker(Ext^1_{\bar{A}}(T_1, -))$ in $Mod(\bar{A})$. By the definition of $\mathcal{Y}$, it then follows that $Im(g_*) = Ker(Hom_{\bar{A}}(T_1, -)) \cap Ker(Ext^1_{\bar{A}}(T_1, -))$, as wanted. □

The following theorem describes explicitly the ring epimorphism arising from a partial silting module.

Theorem 3.5. *Let T_1 be a partial silting module, $T = T_1 \oplus M_A$ its Bongartz completion to a silting module as above and $f : A \longrightarrow B$ the ring epimorphism associated with T_1. Then there is an isomorphism of rings between B and $End_A(T)/I$, where I is the two-sided ideal given by the endomorphisms of T factoring through an object in $Add(T_1)$. Furthermore, the ideal I is idempotent.*

Proof. For simplicity, throughout this proof we denote M_A by M. According to the discussion above, we fix the ring B to be $End_A(\overline{M})$, where $\overline{M} = M/\tau_{T_1}(M)$. We prove our theorem in several steps. First, we define a surjective ring homomorphism $p : End_A(T) \longrightarrow End_A(\overline{M})$, thus yielding an isomorphism of rings $\bar{p} : End_A(T)/Ker(p) \longrightarrow End(\overline{M})$. In a second step, we will show that $Ker(p)$ coincides with the ideal I of endomorphisms of T factoring through an object in $Add(T_1)$. Steps 3 and 4 show that I is idempotent.

Step 1: A surjective ring homomorphism. Consider the short exact sequence

$$0 \longrightarrow \tau_{T_1}(M) \xrightarrow{i} M \xrightarrow{q} \overline{M} \longrightarrow 0.$$

Note that the trace of T_1 in M will be preserved by any endomorphism of M. Consequently, for all a in $End_A(M)$ there is a unique endomorphism $\bar{a}$ in $End_A(\overline{M})$ such that $\bar{a} \circ q = q \circ a$. Endomorphisms of M lie in *a corner* of the endomorphism ring of T, which can be written as a 2×2-matrix as follows

$$End_A(T) = \begin{pmatrix} End_A(M) & Hom_A(T_1, M) \\ Hom_A(M, T_1) & End_A(T_1) \end{pmatrix}.$$

We define the following map

$$p : End_A(T) \longrightarrow End_A(\overline{M}) = B \ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \bar{a}.$$

It is straightforward to check that p is a ring homomorphism. To show that it is surjective, it is enough to see that endomorphisms of $\overline{M}$ lift to endomorphisms of M. Indeed, by applying the functor $Hom_A(M, -)$ to the short exact sequence above, we get the exact sequence

$$End_A(M) \longrightarrow Hom_A(M, \overline{M}) \longrightarrow Ext^1_A(M, \tau_{T_1}(M)).$$

Since $M \in Add(T)$ and $\tau_{T_1}(M) \in Gen(T)$, by Proposition 2.2(2), we know that $Ext^1_A(M, \tau_{T_1}(M)) = 0$. Hence, for any δ in $End_A(\overline{M})$ there is a morphism a in $End_A(M)$ such that $\delta \circ q = q \circ a$, showing that p is surjective.

Step 2: Computing the kernel. We now show that an endomorphism γ of T belongs to the kernel of p if and only if it factors through a module in $Add(T_1)$. Write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and suppose first that γ factors through $Add(T_1)$. In particular, the image of γ lies in the trace of T_1 in T. This shows that $Im(a) \subseteq \tau_{T_1}(M)$ and, thus, $\bar{a} \circ q = q \circ a = 0$. Since q is surjective, $p(\gamma) = \bar{a} = 0$, as wanted. Conversely, if γ lies in the kernel of p, we get $q \circ a = 0$, meaning that the image of a lies in the trace of T_1. Hence, there is a map $\omega : M \longrightarrow \tau_{T_1}(M)$ making the following diagram commute

$$\begin{tikzcd} & & M \arrow[dl, "\omega"'] \arrow[d, "a"] & & \\ 0 \arrow[r] & \tau_{T_1}(M) \arrow[r, "i"] & M \arrow[r, "q"] & \overline{M} \arrow[r] & 0. \end{tikzcd}$$

Let J be the set $Hom_A(T_1, \tau_{T_1}(M))$ and consider the short exact sequence induced by the universal map μ

$$0 \longrightarrow Ker(\mu) \longrightarrow T_1^{(J)} \xrightarrow{\mu} \tau_{T_1}(M) \longrightarrow 0.$$

Using the presentation $\sigma : P \longrightarrow Q$ of T_1 defining $\mathcal{D}_\sigma = Gen(T)$, we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Hom_A(T_1, Ker(\mu)) \arrow[r] \arrow[d] & Hom_A(T_1, T_1^{(J)}) \arrow[r, "\kappa"] \arrow[d] & Hom_A(T_1, \tau_{T_1}(M)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Hom_A(Q, Ker(\mu)) \arrow[r] \arrow[d, "\eta"] & Hom_A(Q, T_1^{(J)}) \arrow[r] \arrow[d] & Hom_A(Q, \tau_{T_1}(M)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Hom_A(P, Ker(\mu)) \arrow[r] & Hom_A(P, T_1^{(J)}) \arrow[r] \arrow[d] & Hom_A(P, \tau_{T_1}(M)) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $\kappa = Hom_A(T_1, \mu)$ is surjective by the definition of μ. By the Snake Lemma, it follows that η is surjective and $Ker(\mu)$ lies in $\mathcal{D}_\sigma$. Consequently, since M is in $Add(T)$, we get $Ext_A^1(M, Ker(\mu)) = 0$, using Proposition 2.2(2). Hence, there is a map $\nu : M \longrightarrow T_1^{(J)}$ such that $\omega = \mu \circ \nu$. We conclude that $a = i \circ \omega = i \circ \mu \circ \nu$ factors through $Add(T_1)$. Now, γ can be decomposed as the sum of four endomorphisms of T induced naturally from the maps a, b, c and d. It is clear that the endomorphisms induced by b, c and d factor through $Add(T_1)$ and by the arguments above so does the endomorphism induced by a. Since the endomorphisms factoring through $Add(T_1)$ form an ideal of $End_A(T)$, it follows that γ lies in that ideal, as wanted.

Step 3: A commutative diagram of functors. The steps above show that the canonical projection $End_A(T) \longrightarrow End_A(T)/Ker(p)$ lies in the same epiclass as $p : End_A(T) \longrightarrow B$. Hence, in order to show that the ideal $I = Ker(p)$ is idempotent, it is enough to check that $Im(p_*)$ is extension-closed in $Mod(End_A(T))$ (see Proposition 2.5(2)). In this step, we show that the restriction functors p_* and f_* satisfy the relation $p_* \cong Hom_A(T, f_*(-))$, i.e., there is a commutative diagram of functors

$$\begin{tikzcd}
 & Mod(A) \arrow[dd, "Hom_A(T,-)"] \\
Mod(B) \arrow[ur, "f_*"] \arrow[dr, "p_*"'] & \\
 & Mod(End_A(T)).
\end{tikzcd}$$

Recall that p_* and f_* can be rewritten as $Hom_B(B, -)$, where B is regarded, respectively, as a left $End_A(T)$-module via p and as a left A-module via f. Hence, using the adjunction

$$Hom_A(T, f_*(-)) \cong Hom_B(T \otimes_A B, -)$$

it is enough to check that B and $T \otimes_A B$ are isomorphic as left $End_A(T)$-modules. Since $Hom_A(T_1, \mathcal{X}_B) = 0$ by construction, the canonical epimorphism $T = M \oplus T_1 \longrightarrow M \xrightarrow{q} \overline{M}$ yields an isomorphism $s : T \otimes_A B \longrightarrow M \otimes_A B \longrightarrow \overline{M} \otimes_A B$ of right A-modules given by $s((m, t_1) \otimes x) = q(m) \otimes x$, for $m \in M$, $t_1 \in T_1$ and $x \in B$. We check that s is a map of left $End_A(T)$-modules. On one hand, we have that

$$s(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(m, t_1) \otimes x) = s((a(m) + b(t_1), c(m) + d(t_1)) \otimes x) = q(a(m)) \otimes x.$$

On the other hand, using the left action of $End_A(T)$ on $\overline{M}$ via p we have, as wanted,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(q(m)\otimes x) = \bar{a}q(m)\otimes x = q(a(m))\otimes x.$$

It remains to show that $\overline{M}\otimes_A B \cong B$ as left $End_A(T)$-modules, or equivalently, as left B-modules. To this end, consider the composition $\widetilde{\psi}_A : A \longrightarrow \overline{M} \longrightarrow \overline{M}\otimes_A B$, where the first map is ψ_A as in Proposition 3.3 and the second map is the natural isomorphism of right A-modules. Clearly, the maps $\widetilde{\psi}_A : A \longrightarrow \overline{M}\otimes_A B$ and $f : A \longrightarrow B$ are both $\mathcal{X}_B$-reflections of A. Hence, there is a unique isomorphism of right A-modules $\theta : B \longrightarrow \overline{M}\otimes_A B$ such that $\theta f = \widetilde{\psi}_A$. It is easy to check that θ is also left B-linear as it is given by $\theta(b) = b\cdot(\psi_A(1)\otimes 1)$.

Step 4: Reducing to the tilting case. Consider now the ring $\bar{A} = A/Ann(T)$ and the canonical projection $\pi : A \longrightarrow \bar{A}$. By Lemma 3.4 and Step 3, we get that $p_* \cong Hom_{\bar{A}}(T, \bar{f}_*(-))$, where $\bar{f} : \bar{A} \longrightarrow B$ is the ring epimorphism given by the factorisation $f = \bar{f}\circ\pi$. Since T is an $\bar{A}$-tilting module (see Proposition 2.2(1)), we may assume without loss of generality that T is a tilting A-module. Then the following useful facts hold true for T.

(1) [B, Proposition 3.4] The functor $Tor_1^{End_A(T)}(Hom_A(T,-),T)$ is identically zero.
(2) [CT, Corollary 2.18] The endofunctor $Hom_A(T,-)\otimes_{End_A(T)} T$ of $Mod(A)$ acts as the identity functor on $Gen(T)$.

We now show that $Im(p_*)$ is extension-closed in $Mod(End_A(T))$, which completes the proof by Proposition 2.5. Let X and Z lie in $Mod(B)$ and consider a short exact sequence in $Mod(End_A(T))$ of the form

$$\varepsilon : \quad 0 \longrightarrow p_*(X) \longrightarrow Y \longrightarrow p_*(Z) \longrightarrow 0.$$

Since $p_* \cong Hom_A(T, f_*(-))$, it follows from (1) above that we get a short exact sequence in $Mod(A)$

$$\varepsilon\otimes_{End_A(T)} T : \quad 0 \longrightarrow Hom_A(T, f_*(X))\otimes_{End_A(T)} T \longrightarrow Y\otimes_{End_A(T)} T \longrightarrow Hom_A(T, f_*(Z))\otimes_{End_A(T)} T \longrightarrow 0.$$

By (2) above, the two outer terms of this sequence are isomorphic to $f_*(X)$ and $f_*(Y)$. Since $Im(f_*)$ is extension-closed in $Mod(A)$ (see Proposition 3.3), also $Y\otimes_{End_A(T)} T$ is in $Im(f_*)$. By Proposition 2.2(2), the functor $Hom_A(T,-)$ is exact for sequences in $Gen(T)$, so $Hom_A(T, \varepsilon\otimes_{End_A(T)} T)$ is a short exact sequence in $Mod(End_A(T))$. Now, the unit of the adjunction $(-\otimes_{End_A(T)} T, Hom_A(T,-))$ yields a map of short exact sequences

$$\varepsilon \longrightarrow Hom_A(T, \varepsilon\otimes_{End_A(T)} T)$$

which, again by fact (2), induces an isomorphism

$$Y \cong Hom_A(T, Y\otimes_{End_A(T)} T).$$

Hence, Y lies in $Im(Hom_A(T, f_*(-))) = Im(p_*)$. $\square$

Remark 3.6. If every element of the ideal I is an endomorphism of T factoring through $add(T_1)$, then I is generated by the idempotent element of $End_A(T)$ corresponding to the summand T_1 of T. This holds, for example, if T is a finitely generated silting module over a finite dimensional algebra. This case has been explored in [J].

We finish this section with a reduction result for torsion classes that, in the context of partial tilting modules, was first proved in [CTT1, Theorem 4.4]. A similar result was shown for τ-rigid modules and their completion to support τ-tilting modules over finite dimensional algebras in [J, Theorems 3.12 and 3.13]. Given two full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $Mod(A)$, we denote by $\mathcal{X}\star\mathcal{Y}$ the full subcategory containing the A-modules M such that there are X in $\mathcal{X}$ and Y in $\mathcal{Y}$ and a short exact sequence of the form

$$0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0.$$

Recall that an A-module M is called **finendo** if it is a finitely generated module over its endomorphism ring. Equivalently, by [ATT, Proposition 1.2], every A-module has a left $Gen(M)$-approximation. A class $\mathcal{T}$ of A-modules such that every module has a left $\mathcal{T}$-approximation is called **preenveloping**.

Theorem 3.7. *Let T_1 be a partial silting A-module with associated ring epimorphism $f : A \longrightarrow B$ and let T be the completion of T_1 to a silting A-module as above. Then the following holds.*

(1) *There is a bijection between*

$$\left\{ \begin{array}{c} \textit{torsion classes } \mathcal{T} \textit{ in } Mod(A) \\ \textit{with } Gen(T_1) \subseteq \mathcal{T} \subseteq Gen(T) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{torsion classes} \\ \textit{in } Mod(B) \end{array} \right\}$$

where a torsion class $\mathcal{T}$ in $Mod(A)$ is mapped to $\mathcal{T} \otimes_A B$. Conversely, a torsion class $\mathcal{G}$ in $Mod(B)$ is mapped to $Gen(T_1) \star f_*(\mathcal{G})$.

(2) *The assignment $\mathcal{T} \mapsto \mathcal{T} \otimes_A B$ in (1) sends preenveloping torsion classes in $Mod(A)$ to preenveloping torsion classes in $Mod(B)$. Moreover, if the A-module T_1 is finendo, then the bijection in (1) restricts to a bijection between*

$$\left\{ \begin{array}{c} \textit{preenveloping torsion classes } \mathcal{T} \\ \textit{in } Mod(A) \textit{ with } Gen(T_1) \subseteq \mathcal{T} \subseteq Gen(T) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{preenveloping torsion} \\ \textit{classes in } Mod(B) \end{array} \right\}.$$

Proof. Let $\mathcal{T}$ be a torsion class in $Mod(A)$ fulfilling $Gen(T_1) \subseteq \mathcal{T} \subseteq Gen(T)$. Since $\mathcal{X}_B = Gen(T) \cap T_1^\circ$, the $\mathcal{X}_B$-reflection of a module in $\mathcal{T}$ is in fact its T_1°-reflection. This shows that $\mathcal{T} \otimes_A B = (\mathcal{T} \cap T_1^\circ) \otimes_A B$, which can be identified with $\mathcal{T} \cap T_1^\circ$ in $Mod(A)$ via the restriction functor. Now the first statement follows by exactly the same arguments used in the proof of [J, Theorem 3.12]. Indeed, following the argument in [J], one shows that the assignments are well-defined by checking that $\mathcal{T} \otimes_A B$ is a torsion class in $Mod(B)$ (see [J, Proposition 3.23]), and that $Gen(T_1) \star f_*(\mathcal{G})$ is a torsion class in $Mod(A)$ (see [J, Proposition 3.26]) which clearly lies between $Gen(T_1)$ and $Gen(T)$. To conclude, one argues as in the proof of [J, Theorem 3.12], showing that the assignments are inverse to each other.

For statement (2), take a preenveloping torsion class $\mathcal{T} \subseteq Mod(A)$ with $Gen(T_1) \subseteq \mathcal{T} \subseteq Gen(T)$. Since T_1° is a torsion-free class and $\mathcal{T}$ is a preenveloping torsion class in $Mod(A)$, every A-module admits a left $\mathcal{T} \cap T_1^\circ$-approximation, as argued before Proposition 3.3. In particular, the torsion class $\mathcal{T} \otimes_A B$ in $Mod(B)$ is preenveloping. Suppose now that T_1 is finendo and let $\mathcal{G}$ be a preenveloping torsion class in $Mod(B)$. We have to show that every A-module has a left $Gen(T_1) \star f_*(\mathcal{G})$-approximation. Since T_1 is finendo, using [GT, Theorem 1.1], it suffices to check that every A-module admits a left $f_*(\mathcal{G})$-approximation. By assumption, every B-module has a left $\mathcal{G}$-approximation and, since $\mathcal{X}_B$ is a bireflective subcategory of $Mod(A)$, the result follows. ☐

4. Examples

In this section, we focus on finite dimensional $\mathbb{K}$-algebras. We investigate homological properties of the ring epimorphism associated with a partial silting module.

Example 4.1. Let Λ be the quotient of the path algebra over $\mathbb{K}$ for the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

by the ideal generated by $\alpha\beta\alpha$ and $\beta\alpha\beta$. The simple Λ-module S_1 is partial silting with respect to its minimal projective presentation $\sigma : P_2 \longrightarrow P_1$. In fact, we have that $\mathcal{D}_\sigma = Add(P_1 \oplus P_1/rad^2 P_1 \oplus S_1)$ and the corresponding completion of S_1 to a silting module (in the sense of the previous section) is given by $T := S_1 \oplus P_1^{\oplus 2}$. Moreover, the associated bireflective subcategory of $Mod(\Lambda)$ is given by $Add(P_1/rad^2(P_1))$ yielding the ring epimorphism $\Lambda \longrightarrow B$ with $B \cong End_A(T)/\langle e_{S_1} \rangle \cong M_2(\mathbb{K})$ (see Theorem 3.5 and Remark 3.6). Since the Λ-module $P_1/rad^2(P_1)$ has infinite projective dimension and it is periodic with respect to the syzygy, there is some $d > 1$ with $Ext^d_\Lambda(B,B) \neq 0$. Thus, the ring epimorphism $\Lambda \longrightarrow B$ is not homological.

The example above illustrates the fact that ring epimorphisms associated with partial silting modules (even if simple ones) are in general not homological. In the tilting case, however, one can provide sufficient conditions for this property to hold. The following result is motivated by [GL, Theorem 4.16]. For simplicity we use the convention $M^{\oplus 0} := 0$ for any Λ-module M.

Proposition 4.2. *Let Λ be a finite dimensional $\mathbb{K}$-algebra and T_1 be a finite dimensional, non-projective, partial tilting Λ-module with $End_\Lambda(T_1) \cong \mathbb{K}$. Then the associated ring epimorphism $f : \Lambda \longrightarrow B$ has kernel $\tau_{T_1}(\Lambda)$, and it is homological if and only if $Ker(f) \cong T_1^{\oplus n}$ for some $n \geq 0$. Moreover, if f is homological, then B has projective dimension at most 2 as a right Λ-module and, in fact, its projective dimension is less or equal than 1 if and only if f is injective.*

Proof. Choose Λ and T_1 as above and let σ be a monomorphic projective presentation of T_1. Let $f : \Lambda \longrightarrow B$ be the ring epimorphism associated with the bireflective subcategory $\mathcal{D}_\sigma \cap T_1^\circ = T_1^\perp$. Since T_1 is finite dimensional and indecomposable, we can choose a Bongartz complement T_0 of T_1 such that there is a short exact sequence

$$\mu_1 : \quad 0 \longrightarrow \Lambda \xrightarrow{\ \phi\ } T_0 \xrightarrow{\ \xi\ } T_1^{\oplus k} \longrightarrow 0,$$

for some $k \geq 1$, where the map ϕ is a minimal left $T_1^{\perp_1}$-approximation of Λ. Recall that f is the $\mathcal{X}_B$-reflection of Λ, which we know to be the composition of ϕ with the quotient map $T_0 \longrightarrow T_0/\tau_{T_1}(T_0)$ (see Proposition 3.3). It follows that $Ker(f) \cong \tau_{T_1}(T_0) \cap \phi(\Lambda)$. We will show that ϕ induces an isomorphism $\tau_{T_1}(\Lambda) \cong \tau_{T_1}(T_0)$. For that, it is enough to show that the monomorphism $Hom_\Lambda(T_1,\phi) : Hom_\Lambda(T_1,\Lambda) \longrightarrow Hom_\Lambda(T_1,T_0)$ is an isomorphism. Given $\eta : T_1 \longrightarrow T_0$, if the composition $\xi \circ \eta : T_1 \longrightarrow T_1^{\oplus k}$ is non-zero, then it is a split monomorphism since $End_\Lambda(T_1) \cong \mathbb{K}$. Therefore, η is a split monomorphism and $\xi_{|\eta(T_1)}$ is an isomorphism between a direct summand of T_0 and a direct summand of $T_1^{\oplus k}$, contradicting the minimality of ϕ. This shows that $\xi \circ \eta = 0$ and η factors through ϕ. We conclude that $Ker(f) = \tau_{T_1}(\Lambda) \cong \tau_{T_1}(T_0)$.

Suppose now that $Ker(f) \cong T_1^{\oplus n}$, for some $n \geq 0$. The isomorphisms $Ker(f) \cong \tau_{T_1}(T_0)$ and $B \cong T_0/\tau_{T_1}(T_0)$ give then rise to a short exact sequence of the form

$$0 \longrightarrow T_1^{\oplus n} \longrightarrow T_0 \longrightarrow B \longrightarrow 0.$$

Since $Tor_1^\Lambda(B,B) = 0$, by applying the functor $- \otimes_\Lambda B$ to the sequence above, we see that $Tor_i^\Lambda(B,B) = 0$, for all $i > 2$ (since both T_0 and $T_1^{\oplus n}$ have projective dimension at most 1). Moreover, since $\sigma \otimes_\Lambda B$ is an isomorphism by Remark 3.1, we have that $Tor_1^\Lambda(T_1^{\oplus n}, B) = 0$ and, thus, $Tor_2^\Lambda(B,B) = 0$, proving that f is homological.

Conversely, suppose that f is a homological ring epimorphism and let n be the dimension of the $\mathbb{K}$-vector space $Hom_\Lambda(T_1,\Lambda)$. Choosing a basis of $Hom_\Lambda(T_1,\Lambda)$, let $\varepsilon : T_1^{\oplus n} \longrightarrow \Lambda$ denote the induced universal map and consider the short exact sequence induced by it

$$\mu_2 : \quad 0 \longrightarrow Ker(\varepsilon) \longrightarrow T_1^{\oplus n} \xrightarrow{\ \varepsilon\ } \tau_{T_1}(\Lambda) \longrightarrow 0.$$

It follows, by construction, that $Ker(\varepsilon)$ lies in $T_1^\perp = \mathcal{X}_B$. We now observe that also $Ker(\varepsilon) \otimes_\Lambda B = 0$. Applying $- \otimes_\Lambda B$ to the sequence μ_1 we see that $Tor_1^\Lambda(T_0, B) = 0$. Applying the same functor to the sequence

$$0 \longrightarrow \tau_{T_1}(T_0) \longrightarrow T_0 \longrightarrow B \longrightarrow 0$$

we conclude that $Tor_1(\tau_{T_1}(T_0), B) = 0$, because f is homological. Since $\tau_{T_1}(T_0) \cong \tau_{T_1}(\Lambda)$, and $T_1 \otimes_\Lambda B = 0$, by applying once again $- \otimes_\Lambda B$ to the sequence μ_2 we see that $Ker(\varepsilon) \otimes_\Lambda B = 0$. Since $Ker(\varepsilon)$ lies in $\mathcal{X}_B$, this means that $Ker(\varepsilon) = 0$ and $T_1^{\oplus n} \cong \tau_{T_1}(\Lambda) = Ker(f)$.

Finally, if f is homological, the previous assertions show that, as a right Λ-module, B is isomorphic to the quotient $T_0/T_1^{\oplus n}$, from which it follows that the projective dimension of B_Λ is less or equal than 2. If $n = 0$ (i.e., f is injective), then the projective dimension of B is less or equal than 1. Conversely, if the projective dimension of B is less or equal than 1, then $T_1^{\oplus n}$ is either zero or projective. Since T_1 is by assumption not projective, it follows that $n = 0$ and $B \cong T_0$ as a right Λ-module, thus finishing the proof. $\square$

Note that the proposition shows in particular that if the above ring epimorphism $f : \Lambda \longrightarrow B$ is injective, then it is homological. Examples of injective homological ring epimorphisms occur very naturally in the context of hereditary rings, as we will see in the next section. We finish this section with two examples where f arises from a partial tilting module with trivial endomorphism ring and such that in one case f is homological but not injective and in the other case f is not homological.

Example 4.3. Let Λ be the quotient of the path algebra over $\mathbb{K}$ for the quiver

$$\begin{tikzcd}
 & & 1 & \\
5 \arrow[r, "\varepsilon"] & 3 \arrow[ur, "\gamma"] & & 2 \arrow[ul, "\alpha"'] \\
 & & 4 \arrow[ul, "\delta"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

by the ideal generated by $\alpha\beta - \gamma\delta$ and $\gamma\varepsilon$. The Auslander-Reiten quiver of Λ is given by

$$\begin{tikzcd}
 & P_2 \arrow[dr] & & I_5 \arrow[dr] & & & & \\
P_4 \arrow[ur] \arrow[dr] & & M_1 \arrow[ur] \arrow[dr] & & S_3 \arrow[dr] & & I_2 \arrow[dr] & \\
 & P_3 \arrow[ur] \arrow[dr] & & M_3 \arrow[ur] \arrow[r] \arrow[dr] & P_1 \arrow[r] & M_4 \arrow[ur] \arrow[dr] & & I_1 \\
P_5 \arrow[ur] & & M_2 \arrow[ur] & & S_2 \arrow[ur] & & I_3 \arrow[ur] &
\end{tikzcd}$$

Given a finite dimensional partial tilting module Λ-module T_1 we denote, as before, by $f : \Lambda \longrightarrow B$ the associated ring epimorphism.

Case 1: The epimorphism f is homological but not injective. Let $T_1 := M_2$. The associated bireflective subcategory is described by $T_1^{\perp} = Add(P_2 \oplus P_3 \oplus P_4 \oplus I_2 \oplus M_1 \oplus I_5 \oplus S_2 \oplus I_1)$. It is clear that $\tau_{T_1}(\Lambda) = \tau_{M_2}(P_1) \cong M_2$ and, thus, by Proposition 4.2, we conclude that f is homological and non-injective. In particular, the projective dimension of B as an Λ-module is exactly 2. This can also be computed directly from the description of $T_1^{\perp}$ by observing that I_2 is a projective B-module which has projective dimension 2 as a Λ-module. In fact, since $P_2 \oplus P_3 \oplus P_4 \oplus I_2$ is a projective generator in $\mathcal{X}_B$, it follows that B is Morita equivalent to the quotient of the path algebra of the quiver

$$\bullet \xleftarrow{\mu} \bullet \xrightarrow{\nu} \bullet \xrightarrow{\omega} \bullet$$

by the ideal generated by the path $\omega\nu$.

Case 2: The epimorphism f is not homological. Consider the partial tilting Λ-module M_1. The associated bireflective subcategory is described by $M_1^{\perp} = Add(P_2 \oplus P_3 \oplus P_5 \oplus M_2 \oplus I_1)$. It is clear, however, that $\tau_{T_1}(\Lambda) = \tau_{T_1}(P_1) \cong M_3$. Therefore, by Proposition 4.2, f is not homological. This can also be seen by observing that B is Morita equivalent to $\mathbb{K}Q \times \mathbb{K} \times \mathbb{K}$, where Q is the quiver $\bullet \longrightarrow \bullet$, and, thus, it is hereditary. However, one checks that $Ext^2_{\Lambda}(I_1, P_5) \neq 0$, showing that f cannot be homological.

5. MINIMAL SILTING MODULES OVER HEREDITARY RINGS

In this section, we study silting modules over hereditary rings that turn out to play the role of generalised support tilting modules. Afterwards, we define minimal silting modules. This definition allows us to associate a unique ring epimorphism to every such silting module. We then use this assignment to establish a bijection between minimal silting modules and homological ring epimorphisms.

Recall that a ring A is (two-sided) **hereditary** if all right and all left ideals of A are projective. We have the following useful lemma.

Lemma 5.1. *Let A be a hereditary ring and $\mathcal{T}$ be a subcategory of $Mod(A)$ such that $\mathcal{T} = Add(\mathcal{T})$. If $\phi : A \longrightarrow T_0$ is a left $\mathcal{T}$-approximation with $Ext^1_A(T_0,T_0) = 0$, then $Ker(\phi) = Ann(\mathcal{T})$ is a two-sided idempotent ideal.*

Proof. It is easy to see that $J := Ker(\phi)$ equals $Ann(\mathcal{T})$ and, thus, it is a two-sided ideal (and projective both as left and as right A-module). Since A is hereditary, ${}^{\perp_1}T_0$ is closed for subobjects. As $A/J \cong \mathrm{Im}(\phi) \subseteq T_0$, it follows that $Ext^1_A(A/J,T_0) = 0$. Applying the functor $Hom_A(-,T_0)$ to the short exact sequence induced by the inclusion of J in A and using the fact that any map from A to T_0 factors through ϕ (and thus through the quotient A/J), we conclude that $Hom_A(J,T_0) = 0$. Since ϕ is a $\mathcal{T}$-approximation, we have $\mathcal{T} \subseteq Gen(T_0)$. Hence, using the fact that J is a projective A-module, we get that $Hom_A(J,\mathcal{T}) = 0$. Consider now the monomorphism $\bar{\phi} : A/J \longrightarrow T_0$ induced by ϕ. Applying the functor $-\otimes_A J$ to the short exact sequence induced by $\bar{\phi}$, since $Tor^A_1(-,J) = 0$, there is a monomorphism $A/J \otimes_A J \longrightarrow T_0 \otimes_A J$. Now, let $f : A^{(I)} \longrightarrow T_0$ be an epimorphism, for some set I. Then it follows that there is a surjection $f \otimes_A J : J^{(I)} \longrightarrow T_0 \otimes_A J$. Since J is projective and $\mathcal{T} = Add(\mathcal{T})$, $T_0 \otimes_A J$ lies in $Add(T_0) \subseteq \mathcal{T}$ and, therefore, $f \otimes_A J = 0$, which implies that $T_0 \otimes_A J = 0$. This shows that $0 = A/J \otimes_A J = J/J^2$ and, thus, J is idempotent. □

Proposition 5.2. *Let A be a hereditary ring.*

(1) *An A-module T is silting if and only if T is tilting over $A/Ann(T)$ and the ideal $Ann(T)$ is idempotent. In other words, silting A-modules are support tilting.*
(2) [BS, Lemma 4.5] *Let $f : A \longrightarrow B$ be a homological ring epimorphism. Then the kernel of f is an idempotent ideal. In particular, f can be written as the composition of two homological ring epimorphisms: $A \longrightarrow A/Ker(f)$ and $A/Ker(f) \longrightarrow B$.*

Proof. (1) Assume that T is silting. Thus, by Proposition 2.2(1), T is tilting over the quotient ring $\bar{A} := A/Ann(T)$. Moreover, there is a left $Gen(T)$-approximation $\phi : A \longrightarrow T_0$ with T_0 in $Add(T)$ and $Ker(\phi) = Ann(T) = Ann(Gen(T))$. Since T is silting, T_0 has no self-extensions and, thus, by Lemma 5.1, $Ann(T)$ is idempotent. Conversely, suppose that T is a tilting $\bar{A}$-module with $Ann(T)$ idempotent. Consider the projective A-presentation σ of T given as the direct sum of a monomorphic presentation of T with the trivial map $Ann(T) \longrightarrow 0$. Since $Ann(T)$ is idempotent, it follows from Proposition 2.5(2) that

$$\mathcal{D}_\sigma = T^{\perp_1} \cap Ann(T)^\circ = Ker(Ext^1_{\bar{A}}(T,-)) = Gen(T).$$

Consequently, T is a silting A-module. □

Note that a similar statement does not hold without the hereditary assumption.

Example 5.3. Let T be a sincere finitely generated silting module over a finite dimensional $\mathbb{K}$-algebra Λ that is not tilting. Such modules T are just non-faithful τ-tilting modules over Λ (see [AMV] and [AIR]). Since T is not faithful, $Ann(T) \neq 0$ and since T is sincere, $Ann(T)$ cannot contain any idempotent $e \neq 0$ of Λ. In particular, it is not an idempotent ideal. Moreover, Example 4.3 (Case 1) provides an example of a homological ring epimorphism whose kernel is not idempotent.

In the following, we wish to assign a ring epimorphism to a silting module T, using the construction of Section 3. To this end, we need a canonical choice of a partial silting module T_1 associated with T. Therefore we consider the following class of silting modules.

Definition 5.4. Let A be a hereditary ring and T be a silting A-module. Then T is called **minimal**, if A_A admits a minimal left $Add(T)$-approximation.

Clearly, the definition of minimal silting modules also applies to tilting modules. Note that already in the setting of tilting modules, we obtain many non-trivial examples.

Example 5.5. Let A be a hereditary ring.

(1) Let T be an endofinite silting A-module, i.e., T has finite length over its endomorphism ring. Therefore, by [KS, Theorem 4.1], $Add(T)$ is closed for products and, thus, by [KS, Theorem 3.1], every A-module admits a minimal left $Add(T)$-approximation. In particular, finitely generated silting modules over hereditary Artin algebras are minimal.

(2) Let A be noetherian and consider the minimal injective coresolution of the free module of rank one

$$0 \longrightarrow A \longrightarrow E_1 \longrightarrow E_2 \longrightarrow 0.$$

It follows that $T := E_1 \oplus E_2$ is a tilting A-module where $Gen(T)$ is given by the class of injective A-modules. Since injective envelopes are left-minimal, T is a minimal tilting module.

(3) Let $T = B \oplus B/A$ be a tilting A-module arising from an injective homological ring epimorphism $f : A \longrightarrow B$ as in Theorem 2.7. Then T is minimal. In fact, we have the following canonical $Add(T)$-approximation sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow B/A \longrightarrow 0.$$

Since f is a reflection map, it is clearly left-minimal.

Minimal silting modules are motivated by the following construction. Let A be hereditary, and let T be a minimal silting A-module with associated torsion class $\mathcal{D}_\sigma = Gen(T)$. Consider the minimal $Add(T)$-approximation sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0.$$

By Proposition 5.2(1), T is a tilting module over the quotient ring $\bar{A} := A/Ann(T)$ and $Ann(T)$ is idempotent. We get the induced minimal $Add(T)$-approximation sequence in $Mod(\bar{A})$

$$0 \longrightarrow \bar{A} \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0,$$

from which we infer that an $\bar{A}$-module X belongs to $Gen(T)$ if and only if $\mathrm{Ext}^1_{\bar{A}}(T_1, X) = 0$. Since the ideal $Ann(T)$ is idempotent we get from Proposition 2.5(2) that

$$Gen(T) = T_1^{\perp_1} \cap \mathcal{X}_{\bar{A}} = T_1^{\perp_1} \cap Ann(T)^\circ. \tag{5.1}$$

We claim that T_1 is a partial silting A-module with respect to a projective presentation σ_1, given as the direct sum of a monomorphic presentation μ of T_1 with the trivial map $Ann(T) \longrightarrow 0$. In fact, we get the equality $\mathcal{D}_{\sigma_1} = \mathcal{D}_\mu \cap Ann(T)^\circ = T_1^{\perp_1} \cap Ann(T)^\circ = Gen(T) = \mathcal{D}_\sigma$. Following Proposition 3.3, we consider the bireflective subcategory

$$\mathcal{Y} = T_1^{\perp} \cap \mathcal{X}_{\bar{A}} \tag{5.2}$$

associated with T_1. The corresponding ring epimorphism will be denoted by $A \longrightarrow B_T$. Since the approximation ϕ was chosen minimal and, hence, the module T_1 is uniquely determined, we obtain a well-defined map from (equivalence classes of) minimal silting modules to (epiclasses of) ring epimorphisms by mapping T to the ring epimorphism $A \longrightarrow B_T$. We need the following technical proposition motivated by the results in [IT, Section 2].

Proposition 5.6. *Let A be a hereditary ring and T be a minimal silting A-module with associated ring epimorphism $A \longrightarrow B_T$. Then $\mathcal{X}_{B_T}$ coincides with*

$$\mathfrak{a}(Gen(T)) := \{X \in Gen(T) \mid \forall (g : Y \longrightarrow X) \in Gen(T), Ker(g) \in Gen(T)\}.$$

Moreover, if $A \longrightarrow T_0$ is the minimal left $Add(T)$-approximation, then we have $Proj(\mathcal{X}_{B_T}) = Add(T_0)$.

Proof. Let $\phi : A \longrightarrow T_0$ be the minimal left $Add(T)$-approximation and $T_1 = Coker(\phi)$. Let $\bar{A} = A/Ann(T)$ and observe that, by Lemma 3.4, we have

$$\mathcal{X}_{B_T} = \{X \in \mathcal{X}_{\bar{A}} \mid Hom_{\bar{A}}(T_1, X) = 0 = Ext^1_{\bar{A}}(T_1, X)\}.$$

We first prove that $\mathcal{X}_{B_T} \subseteq \mathfrak{a}(Gen(T))$. Take X in $\mathcal{X}_{B_T}$. Since $Ext^1_{\bar{A}}(T_1, X) = 0$ and T is a tilting module over $\bar{A}$, the module X lies in $Gen(T)$ (see also equation (5.1) above). Now consider a test map $g : Y \longrightarrow X$ with Y in $Gen(T)$. Without loss of generality, we may assume g to be surjective, since $Im(g)$ also lies in $\mathcal{X}_{B_T}$. Moreover, note that $Ker(g)$ belongs to $\mathcal{X}_{\bar{A}}$, since so do X and Y. By applying the functor $Hom_{\bar{A}}(T_1, -)$ to the short exact sequence induced by g, we obtain the exact sequence

$$Hom_{\bar{A}}(T_1, Y) \longrightarrow Hom_{\bar{A}}(T_1, X) \longrightarrow Ext^1_{\bar{A}}(T_1, Ker(g)) \longrightarrow Ext^1_{\bar{A}}(T_1, Y).$$

Since, by assumption, $Hom_{\bar{A}}(T_1, X) = 0$ and Y lies in $Gen(T)$ (showing that $Ext^1_{\bar{A}}(T_1, Y) = 0$), it follows that $Ext^1_{\bar{A}}(T_1, Ker(g)) = 0$. This proves that $Ker(g)$ lies in $Gen(T)$ and, thus, X lies in $\mathfrak{a}(Gen(T))$.

Conversely, since $\mathfrak{a}(Gen(T)) \subseteq Gen(T)$, it is enough to show that $Hom_{\bar{A}}(T_1, \mathfrak{a}(Gen(T))) = 0$. By definition, $\mathfrak{a}(Gen(T))$ is closed for subobjects in $Gen(T)$. In particular, the image of any morphism from an object in $Gen(T)$ to an object in $\mathfrak{a}(Gen(T))$ is itself in $\mathfrak{a}(Gen(T))$. Thus, to prove our claim, it is enough to show that there are no surjections from T_1 to any object C in $\mathfrak{a}(Gen(T))$. Let $\omega : T_1 \longrightarrow C$ be such a surjection and consider the commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "1"] \arrow[d, "d"] & A \arrow[d, "\phi"] & & \\
0 \arrow[r] & K \arrow[r, "a"] \arrow[d, "b"] & T_0 \arrow[r, "\omega\circ\psi"] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & X \arrow[r, "c"] \arrow[d] & T_1 \arrow[r, "\omega"] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. Since C lies in $\mathfrak{a}(Gen(T))$, K lies in $Gen(T)$. Thus, d factors through ϕ, i.e., there is a map $e : T_0 \longrightarrow K$ such that $e \circ \phi = d$. This shows that $a \circ e \circ \phi = a \circ d = \phi$ which by minimality of ϕ yields that $a \circ e$ is an isomorphism. In particular, a is an epimorphism and $C = 0$.

Finally, let us show that $Proj(\mathcal{X}_{B_T}) = Add(T_0)$ by observing that T_0 is a projective generator in $\mathfrak{a}(Gen(T))$. Since A is hereditary, by [IT, Proposition 2.15], $\mathfrak{a}(Gen(T))$ coincides with

$$\{X \in Gen(T) \mid \forall (g : Y \twoheadrightarrow X) \in Gen(T), Ker(g) \in Gen(T)\}.$$

First we see that T_0 lies in $\mathfrak{a}(Gen(T))$. Given Y in $Gen(T)$ and an epimorphism $\omega : Y \longrightarrow T_0$, since A is projective there is $\beta : A \longrightarrow Y$ such that $\omega \circ \beta = \phi$. Moreover, since T_0 is a left $Gen(T)$-approximation, there is $\gamma : T_0 \longrightarrow Y$ such that $\beta = \gamma \circ \phi$. Therefore, we have that $\omega \circ \gamma \circ \phi = \omega \circ \beta = \phi$, which by minimality of ϕ shows that ω is a split epimorphism and, thus, $Ker(\omega)$ lies in $Gen(T)$. Next, we know from Proposition 2.2(2) that T_0 is Ext-projective in $Gen(T)$. Thus, the functor $Hom_A(T_0, -)$ is exact for short exact sequences in $Gen(T)$, and in particular, T_0 is a projective object in $\mathfrak{a}(Gen(T))$. Moreover, it is a generator of $Gen(T)$ and, hence, also a generator for $\mathfrak{a}(Gen(T))$, thus proving our claim. ☐

Remark 5.7. The first part of the above proof shows the following. Let A be any ring and T be a silting A-module such that A admits a minimal left $Gen(T)$-approximation $\phi : A \longrightarrow T_0$. Then the bireflective subcategory $Gen(T) \cap Coker(\phi)^{\circ}$ coincides with $\mathfrak{a}(Gen(T))$. However, notice that, in contrast to the hereditary setting, $Hom_A(Coker(\phi), T_0)$ may not vanish and T_0 does not generally lie in $\mathfrak{a}(Gen(T))$.

Now we are able to state the main result of this section.

Theorem 5.8. *Let A be a hereditary ring. Then the assignment $\alpha : T \mapsto (f : A \longrightarrow B_T)$ yields a bijection between*

(1) *equivalence classes of minimal silting A-modules;*
(2) *epiclasses of homological ring epimorphisms of A.*

Moreover, α restricts to a bijection between

(1) *equivalence classes of minimal tilting A-modules;*
(2) *epiclasses of injective homological ring epimorphisms of A.*

Proof. First, we check that α is well-defined. By construction, the ring epimorphism f is uniquely determined by T. Moreover, by Proposition 3.3, the subcategory $\mathcal{X}_{B_T}$ is closed under extensions in $Mod(A)$ and, therefore, by Proposition 2.4, $Tor_1^A(B_T, B_T) = 0$. Since A is hereditary, all higher Tor-groups vanish showing that the ring epimorphism f is homological.

The injectivity of α follows from Proposition 5.6: if T and T' are minimal silting modules with $\alpha(T) = \alpha(T')$, then we have that $Add(T_0) = Add(T_0')$ and, thus, T and T' are equivalent.

Next, we prove the surjectivity of α. Let $f : A \longrightarrow B$ be a homological ring epimorphism. By Proposition 5.2(2), we get a commutative diagram of homological ring epimorphisms

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & A/Ker(f) \arrow[ur, "f'"] & \end{tikzcd}$$

where f' is injective and the quotient ring $\bar{A} := A/Ker(f)$ is again hereditary. Thus, by Theorem 2.7(1), $T := B \oplus B/\bar{A}$ is a tilting module over $\bar{A}$ and it is minimal as argued in Example 5.5(3). By Proposition 5.2(1), T becomes a minimal silting A-module with respect to a projective A-presentation σ of T that is given as the direct sum of a monomorphic presentation of T and the trivial map $Ker(f) \longrightarrow 0$. It remains to check that $\alpha(T)$ lies in the same epiclass as the ring epimorphism $f : A \longrightarrow B$. Since the minimal left $Add(T)$-approximation of A is given by the module map $f : A \longrightarrow B$, by construction (see also equation (5.2) above), we have

$$\mathcal{X}_{B_T} = (B/\bar{A})^{\perp} \cap \mathcal{X}_{\bar{A}}.$$

But this coincides with $\mathcal{X}_B$ by Theorem 2.7(1).

It follows from the previous arguments that the inverse α^{-1} of α assigns to a homological ring epimorphism $f : A \longrightarrow B$ the minimal silting A-module $B \oplus Coker(f)$. Therefore, in case f is injective, the module $\alpha^{-1}(f)$ is actually a tilting module, by Theorem 2.7(1). It remains to check the restriction of the map α. Let T be a minimal tilting A-module with a monomorphic minimal $Add(T)$-approximation $\phi : A \longrightarrow T_0$. Using Proposition 5.6, it follows that T_0 lies in $\mathcal{X}_{B_T}$. Since $\mathcal{X}_{B_T} \subset Gen(T)$, the map ϕ is then also a minimal left $\mathcal{X}_{B_T}$-approximation, hence it is the $\mathcal{X}_{B_T}$-reflection of A. In particular, the ring epimorphism $A \longrightarrow B_T$, which coincides with ϕ as an A-module homomorphism, is injective. □

We have the following immediate corollary of Theorem 5.8.

Corollary 5.9. *Let A be a hereditary ring. Then a tilting A-module T arises from a ring epimorphism if and only if T is minimal.*

Example 5.10. Let Λ be the Kronecker algebra, i.e., the path algebra of the quiver $\bullet \rightrightarrows \bullet$ over a field $\mathbb{K}$. First observe that a non-zero support tilting module which is not tilting is equivalent to a simple Λ-module. These are clearly minimal silting modules. From [M] and [AS2] we know that all tilting modules except the Lukas tilting module (see [L1] and [L2]) arise from ring epimorphisms. Combining this fact with Theorem 5.8 and Proposition 5.2(1), we obtain a classification of all minimal silting Λ-modules. They are precisely the simple modules and all but one tilting module; the unique silting module which is not minimal is the Lukas tilting module.

Let us briefly analyse the Lukas tilting module L in more detail. The tilting class $Gen(L)$ is given by the Λ-modules without any indecomposable preprojective summands. Since $Add(L') = Add(L)$ for all non-zero modules $L' \in Add(L)$ (see [L1, Theorem 6.1] and [L2, Theorem 3.1]), we conclude that for all $Add(L)$-approximation sequences

$$0 \longrightarrow \Lambda \longrightarrow L_0 \longrightarrow L_1 \longrightarrow 0$$

the bireflective subcategory $L_1^{\perp}$ only contains the zero-module. Indeed, any module $X \in L_1^{\perp}$ is both contained in $L_1^{\perp_1} = Gen(L) = Gen(L_1)$ and in L_1°.

Following [IT], we refer to full, abelian and extension-closed subcategories of finitely presented modules as **wide**, and we call them **finitely generated** if they contain a generator.

Remark 5.11. Over a finite dimensional algebra Λ, a wide subcategory $\mathcal{W}$ of $mod(\Lambda)$ has a generator if and only if every finite dimensional Λ-module admits both a left and a right $\mathcal{W}$-approximation. Indeed, a left $\mathcal{W}$-approximation of Λ clearly yields a generator for $\mathcal{W}$. Conversely, by [AuS, Theorem 4.5], if $\mathcal{W}$ has a generator, then every finite dimensional Λ-module has a left $\mathcal{W}$-approximation, which can be chosen minimal. Since $\mathcal{W}$ is closed for kernels, it is easy to see that minimal $\mathcal{W}$-approximations are in fact reflections. As a consequence, the reflection of Λ yields a projective generator for $\mathcal{W}$, and $\mathcal{W}$ is equivalent to $mod(B)$ for some finite dimensional algebra B. In particular, $\mathcal{W}$ then contains a cogenerator which, again by [AuS, Theorem 4.5], shows that every finite dimensional Λ-module admits a right $\mathcal{W}$-approximation.

Corollary 5.12. [IT, Section 2],[M, Theorem 4.2] *If Λ is a finite dimensional hereditary algebra, the map α from Theorem 5.8 restricts to a bijection between*

(1) *equivalence classes of finite dimensional support tilting Λ-modules;*
(2) *epiclasses of homological ring epimorphisms $\Lambda \longrightarrow B$ with B finite dimensional;*

and there is a further bijection with

(3) *finitely generated wide subcategories of $mod(\Lambda)$*

by assigning to a ring epimorphism $\Lambda \longrightarrow B$ the class $\mathcal{X}_B \cap mod(\Lambda) \cong mod(B)$.

Proof. The assignment α establishes a bijection between (1) and (2) as a direct consequence of Theorem 5.8. Indeed, if T is a finite dimensional support tilting Λ-module, then also the module T_0 appearing in the minimal left $Add(T)$-approximation $A \longrightarrow T_0$ is finite dimensional. Hence, the algebra B is finite dimensional since T_0 and B are isomorphic right Λ-modules. Finally, the bijection between (2) and (3) follows from [Iy, Theorem 1.6.1(2)] and Proposition 2.4, using Remark 5.11. □

In [IT, R] further bijections are established, providing a combinatorial interpretation of finitely generated support tilting modules in terms of noncrossing partitions, clusters, or antichains. The poset of exceptional antichains, which is isomorphic to the poset of generalised noncrossing partitions ([R, Theorem 3.6.6]), is defined via the partial order on finitely generated wide subcategories given by inclusion. By Corollary 5.12, the latter induces a partial order on the epiclasses of homological ring epimorphisms $\Lambda \longrightarrow B$ where B is a finite dimensional algebra.

In fact, there is a natural partial order on the set of epiclasses of any ring A. Given two ring epimorphisms $f_1 : A \longrightarrow B_1$ and $f_2 : A \longrightarrow B_2$, we set

$$f_1 \geq f_2$$

if there is a ring homomorphism $g : B_1 \longrightarrow B_2$ such that $g \circ f_1 = f_2$ or, equivalently, if $\mathcal{X}_{B_1} \supseteq \mathcal{X}_{B_2}$.

Recall that a partially ordered set P is a **lattice** if all finite subsets of P admit a meet and a join. Furthermore, P is a **complete** lattice if any subset of P has a meet and a join.

Proposition 5.13. *Let A be a hereditary ring. Then the epiclasses of homological ring epimorphisms starting in A form a complete lattice with respect to $\geq$.*

Proof. By Theorem 2.3 and Proposition 2.4, the essential images of the restriction functors associated to homological ring epimorphisms starting in A are characterised by closure properties: these are precisely the subcategories of $Mod(A)$ closed for kernels, cokernels, products, coproducts and extensions. Consequently, by definition of the partial order $\geq$, it follows that, given a family of ring epimorphisms, we can construct their meet by intersecting the associated subcategories of $Mod(A)$, and their join by closing the union of the associated subcategories under the operations above. $\square$

If we restrict the partial order $\geq$ to homological ring epimorphisms between finite dimensional hereditary algebras, we obtain the poset of exceptional antichains. This is known to be a lattice for hereditary algebras of finite representation type (compare [IT, R]). Indeed, this also follows from Proposition 5.13 since every ring epimorphism with representation-finite domain has a finite dimensional codomain ([GdP, Corollary 2.3]). In general, however, the poset of exceptional antichains is not a lattice, as remarked in [R, p.65] and illustrated in Example 5.18 below. But before discussing examples, let us introduce the concept of universal localisation which provides an alternative point of view on homological ring epimorphisms for hereditary rings.

Theorem 5.14. [Sch1, Theorem 4.1] *Let A be a ring and Σ be a class of morphisms between finitely generated projective right A-modules. Then there is a ring homomorphism $f : A \longrightarrow A_\Sigma$ such that*

(1) *f is* Σ-inverting, *i.e. if σ belongs to Σ, then $\sigma \otimes_A A_\Sigma$ is an isomorphism of right A_Σ-modules, and*
(2) *f is* universal Σ-inverting, *i.e. for any Σ-inverting morphism $f' : A \longrightarrow B$ there exists a unique ring homomorphism $g : A_\Sigma \longrightarrow B$ such that $g \circ f = f'$.*

*The homomorphism $f\colon A \longrightarrow A_\Sigma$ is a ring epimorphism with $Tor_1^A(A_\Sigma, A_\Sigma) = 0$, called the **universal localisation** of A at Σ.*

Let now $\mathcal{U}$ be a set of finitely presented modules of projective dimension at most one. For each $U \in \mathcal{U}$, we fix a projective resolution $0 \longrightarrow P \xrightarrow{\sigma_U} Q \longrightarrow U \longrightarrow 0$ in $mod(A)$ and we set $\Sigma = \{\sigma_U \mid U \in \mathcal{U}\}$. We denote by $f_\mathcal{U} : A \longrightarrow A_\mathcal{U}$ the universal localisation of A at Σ. Note that $A_\mathcal{U}$ does not depend on the chosen class Σ (compare [C, Theorem 0.6.2]). Moreover, the ring epimorphism $f_\mathcal{U}$ corresponds to the bireflective subcategory $\mathcal{X}_{A_\mathcal{U}} = \mathcal{U}^\perp$ by [AA, Proposition 2.7]. Finally, if A is hereditary, then $f_\mathcal{U}$ is injective if and only if the modules in $\mathcal{U}$ are **bound**, i.e. they are finitely presented modules U such that $Hom_A(U,A) = 0$ (compare [Sch3] and [M, Lemma 4.1]).

Theorem 5.15. [Sch3, Theorem 2.3],[KSt, Theorem 6.1] *Let A be a hereditary ring. Then a ring epimorphism starting in A is homological if and only if it is a universal localisation. Moreover, the assignment $\gamma\colon \mathcal{W} \mapsto (f_\mathcal{W} : A \longrightarrow A_\mathcal{W})$ defines a bijection between*

(1) *wide subcategories of $mod(A)$;*
(2) *epiclasses of universal localisations of A*

which restricts to a bijection between

(1) *wide subcategories of bound A-modules;*
(2) *epiclasses of injective universal localisations of A.*

Remark 5.16. (1) Notice the difference between the bijections in Corollary 5.12 and Theorem 5.15: in the first case the wide subcategory associated with $f : A \longrightarrow B$ is $\mathcal{X}_B \cap mod(A)$, while in the second case we are taking the wide subcategory $\mathcal{W}$ of all modules U with a projective resolution σ_U which is inverted by the functor $- \otimes_A B$, or in other words, $\mathcal{W} = {}^{\perp}\mathcal{X}_B \cap mod(A)$, and $\mathcal{X}_B = \mathcal{W}^{\perp}$.

(2) By [Sch3, Theorem 2.5], the wide subcategories $\mathcal{W}$ consisting of bound A-modules correspond bijectively to Hom-perpendicular sets of finitely presented bound A-modules, that is, antichains of non-projective modules in the terminology of [R].

Corollary 5.17. *Let A be a hereditary ring. Then there is a commutative triangle of bijections*

$$\begin{tikzcd}
\left\{\begin{array}{c}\textit{equivalence classes of}\\ \textit{minimal silting } A\textit{-modules}\end{array}\right\} \arrow[rr, "\alpha"] & & \left\{\begin{array}{c}\textit{epiclasses of universal}\\ \textit{localisations of } A\end{array}\right\} \\
& \left\{\begin{array}{c}\textit{wide subcategories}\\ \textit{of } mod(A)\end{array}\right\} \arrow[ul, dashed, "\beta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

where α is defined in Theorem 5.8, γ is defined in Theorem 5.15 and β assigns to a wide subcategory $\mathcal{W}$ the silting class $\mathcal{W}^{\perp_1} \cap \mathcal{X}_{\bar{A}}$, where $\bar{A} = A/Ker(f_{\mathcal{W}})$. Moreover, γ is an anti-isomorphism of lattices.

Proof. We show that $\beta = \alpha^{-1} \circ \gamma$. From [AS1, Corollary 4.13] and [Sch3, Theorem 2.6] we know that $A_{\mathcal{W}} \oplus A_{\mathcal{W}}/\bar{A}$ is a tilting $\bar{A}$-module with tilting class $Gen(A_{\mathcal{W}}) = KerExt^1_{\bar{A}}(\mathcal{W}, -) = \mathcal{W}^{\perp_1} \cap \mathcal{X}_{\bar{A}}$. Hence, we have $\beta(\mathcal{W}) = Gen(A_{\mathcal{W}})$, which is the silting class of the silting module $A_{\mathcal{W}} \oplus Coker f_{\mathcal{W}} = \alpha^{-1}(f_{\mathcal{W}}) = \alpha^{-1}(\gamma(\mathcal{W}))$ (see the proof of Theorem 5.8).

Now, if $\mathcal{W}_1, \mathcal{W}_2$ are wide subcategories such that $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then $\mathcal{W}_1{}^{\perp} \supseteq \mathcal{W}_2{}^{\perp}$, and for the corresponding homological ring epimorphisms $f_i : A \longrightarrow B_i$ with $f_i = \gamma(\mathcal{W}_i)$ it follows from Theorem 5.15 and Remark 5.16(1) that $\mathcal{X}_{B_1} \supseteq \mathcal{X}_{B_2}$, that is, $f_1 \geq f_2$.

Note that, in particular, the join of two homological ring epimorphisms f_1 and f_2 is given by the universal localisation at the wide subcategory $\mathcal{W}_1 \cap \mathcal{W}_2$, and the meet is given by the universal localisation at $\mathcal{W}_1 \cup \mathcal{W}_2$ (or equivalently, by Theorem 5.15, at the smallest wide subcategory containing $\mathcal{W}_1 \cup \mathcal{W}_2$). $\square$

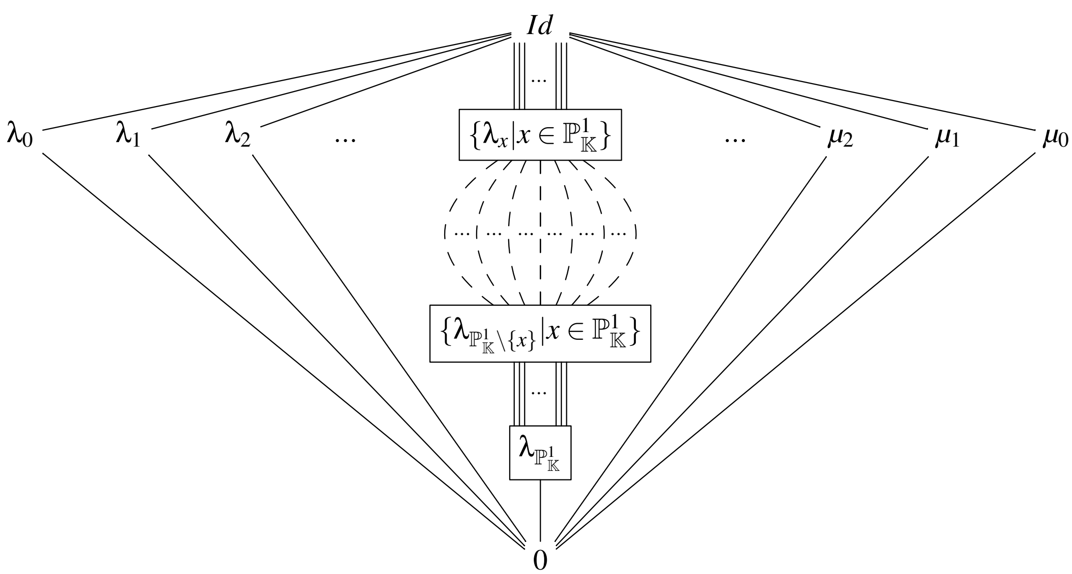
The following examples illustrate the behaviour of the lattice of homological ring epimorphisms for hereditary rings.

Example 5.18. The meet of two homological ring epimorphisms $f_1 : A \longrightarrow B_1$ and $f_2 : A \longrightarrow B_2$ can be infinite dimensional even when B_1 and B_2 are finite dimensional algebras. Indeed, if Λ is a finite dimensional tame hereditary algebra and $\mathcal{W}$ is a non-homogeneous tube with simple regular modules $S_1, \dots, S_r$, then $f_1 : \Lambda \longrightarrow \Lambda_{\{S_1\}}$ and $f_2 : \Lambda \longrightarrow \Lambda_{\{S_2,\dots,S_r\}}$ have finite dimensional targets, but their meet $f : \Lambda \longrightarrow \Lambda_{\mathcal{W}}$ has an infinite dimensional target (see [CB, Section 4] and [AS2, Proposition 1.10]). A specific instance of this phenomenon is provided in [R, Example 3.1.4].

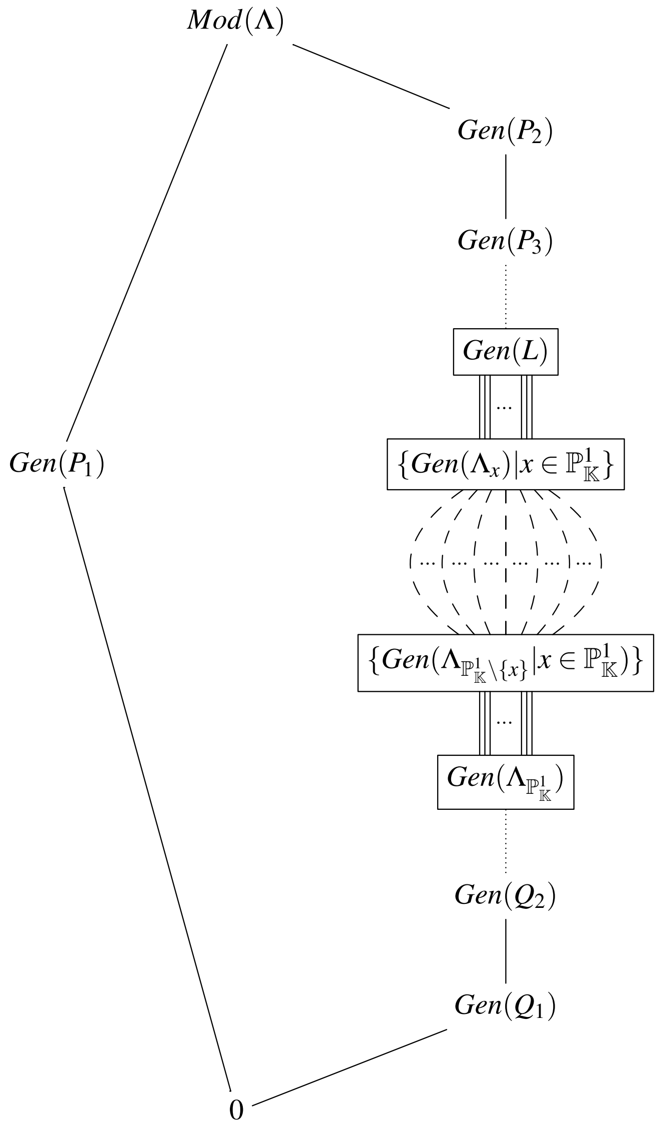
Example 5.19. We compute the lattice of homological ring epimorphisms for the Kronecker algebra Λ (see Example 5.10). Let us denote by P_i (respectively Q_i), with $i \in \mathbb{N}$, the (finite dimensional) indecomposable preprojective (respectively, preinjective) modules, indexed such that $dim_{\mathbb{K}} Hom_A(P_i, P_{i+1}) = 2$ (respectively, $dim_{\mathbb{K}} Hom_A(Q_{i+1}, Q_i) = 2$). Also, we identify below the quasi-simple regular Λ-modules with points in the projective line $\mathbb{P}^1_{\mathbb{K}}$. Following Example 5.10, we can list all minimal silting Λ-modules and all homological ring epimorphisms of Λ (together with their associated bireflective subcategories) as follows.

Silting module	Homological ring epimorphism	Bireflective subcategory of $Mod(\Lambda)$
0	$\Lambda \longrightarrow 0$	$\mathcal{X}_0 = 0$
$\Lambda = P_1 \oplus P_2$	Id	$\mathcal{X}_{Id} = Mod(\Lambda)$
P_1	$\lambda_0 : \Lambda \longrightarrow \Lambda/\Lambda e_2 \Lambda$	$\mathcal{X}_{\lambda_0} = Add(P_1)$
Q_1	$\mu_0 : \Lambda \longrightarrow \Lambda/\Lambda e_1 \Lambda$	$\mathcal{X}_{\mu_0} = Add(Q_1)$
$(P_i \oplus P_{i+1})_{i \geq 2}$	$(\lambda_i : \Lambda \longrightarrow \Lambda_{\{P_{i+1}\}})_{i \in \mathbb{N}}$	$(\mathcal{X}_{\lambda_i} = Add(P_i))_{i \in \mathbb{N}}$
$(Q_{i+1} \oplus Q_i)_{i \in \mathbb{N}}$	$(\mu_i : \Lambda \longrightarrow \Lambda_{\{Q_i\}})_{i \in \mathbb{N}}$	$(\mathcal{X}_{\mu_i} = Add(Q_{i+1}))_{i \in \mathbb{N}}$
$(\Lambda_{\mathcal{U}} \oplus \Lambda_{\mathcal{U}}/\Lambda)_{\emptyset \neq \mathcal{U} \subseteq \mathbb{P}^1_{\mathbb{K}}}$	$(\lambda_{\mathcal{U}} : \Lambda \longrightarrow \Lambda_{\mathcal{U}})_{\emptyset \neq \mathcal{U} \subseteq \mathbb{P}^1_{\mathbb{K}}}$	$(\mathcal{X}_{\lambda_{\mathcal{U}}} = \mathcal{U}^{\perp})_{\emptyset \neq \mathcal{U} \subseteq \mathbb{P}^1_{\mathbb{K}}}$

The lattice of homological ring epimorphisms for the Kronecker algebra is then as follows



where the interval between Id and $\lambda_{\mathbb{P}^1_{\mathbb{K}}}$ represents the dual poset of subsets of $\mathbb{P}^1_{\mathbb{K}}$. The ring epimorphisms with infinite dimensional target are those in frames, i.e., those of the form $\lambda_{\mathcal{U}}$ with $\emptyset \neq \mathcal{U} \subseteq \mathbb{P}^1_{\mathbb{K}}$. The poset obtained by excluding these elements is precisely the poset of exceptional antichains from [R]. Note that the above lattice is dual to the one of wide subcategories in $mod(\Lambda)$. The poset of silting classes is, however, completely different. As before, let L denote the Lukas tilting module (which is not minimal and, thus, does not appear above). The silting classes corresponding to infinite dimensional silting modules are framed.



Next, we focus on the case of a Dedekind (i.e. commutative and hereditary) domain. Here there is an interesting connection with Gabriel topologies. Indeed, over any coherent ring A, there is a bijective correspondence assigning to every Serre subcategory $\mathcal{U}$ of $mod(A)$ a Gabriel topology of finite type $\mathcal{L}_{\mathcal{U}}$ on A (see [Ste, Theorem VI.5.1],[H, Theorem 2.8],[K, Corollary 2.10]). Notice that every Serre subcategory is wide, and over commutative noetherian rings also the converse is true by [T, Theorem A]. Also, recall that every Gabriel topology $\mathcal{L}$ induces a ring homomorphism $A \longrightarrow Q_{\mathcal{L}}$. We say that a ring epimorphism $A \longrightarrow B$ is non-trivial if $B \neq 0$, and we consider only non-trivial Gabriel topologies, i.e. consisting of non-zero ideals.

Example 5.20. If A is a Dedekind domain, all non-trivial homological ring epimorphism are injective and the maps α, β and γ from Corollary 5.17 define bijections between:

(a) equivalence classes of tilting A-modules;
(b) wide subcategories of bound A-modules;
(c) Gabriel topologies on A;
(d) epiclasses of non-trivial universal localisations of A;
(e) epiclasses of non-trivial homological ring epimorphisms of A.

Indeed, we know that all tilting modules are minimal from [AS1, Corollary 6.12]. Further, all Gabriel topologies are of finite type, and over commutative semihereditary rings the latter coincide with perfect Gabriel topologies by [Ste, Chapter XI, Proposition 3.3]. This means that the localisations $A \longrightarrow Q_{\mathcal{L}}$ of A induced by Gabriel topologies are (flat) ring epimorphisms, and in fact, they are precisely the non-trivial universal localisations of A by [BS, Theorem 7.8]. So, it only remains to check that every non-trivial universal localisation $f_{\mathcal{W}}$ is injective. But this is true over any commutative semihereditary domain. Indeed, if $\mathcal{W}$ contains a non-bound module, then it contains a non-zero projective module P, which must vanish under $-\otimes_A A_{\mathcal{W}}$. Then also the trace of P in A vanishes under $-\otimes_A A_{\mathcal{W}}$. But since A has non non-trivial idempotents, it follows from [Lam, Theorem 2.44] that the trace coincides with A, so $A_{\mathcal{W}} = 0$.

Let us give an alternate description of the maps β and γ. To this end, we employ the Auslander-Bridger transpose Tr. On bound modules, Tr coincides with the functor $\operatorname{Ext}^1_A(-,A)$, and it is therefore exact, and in particular, it maps Serre subcategories to Serre subcategories. So, given a wide subcategory $\mathcal{W}$ of bound modules, we can consider the Serre subcategory $\mathcal{U} = Tr(\mathcal{W})$ together with the associated (perfect) Gabriel topology $\mathcal{L}_{\mathcal{U}}$. By the formula in [APST, Lemma 2.9 (iii)], the tilting class $\beta(\mathcal{W}) = \mathcal{W}^{\perp_1}$ then consists of the $\mathcal{L}_{\mathcal{U}}$-divisible modules (as defined in [Ste, p.155]). Moreover, the universal localisation $\gamma(\mathcal{W}) = f_{\mathcal{W}}$ can be interpreted as the localisation $A \to Q_{\mathcal{L}_{\mathcal{U}}}$ induced by $\mathcal{L}_{\mathcal{U}}$. Indeed, the latter corresponds to the bireflective subcategory $\mathcal{U}^{\perp} = \mathcal{X}_{A_{\mathcal{U}}}$, and thus it is in the same epiclass as the universal localisation $f_{\mathcal{U}}$ at $\mathcal{U}$, which in turn is in the same epiclass as $f_{\mathcal{W}}$, as shown in [Sch1, pages 51-52].

Finally, we observe that the poset of non-trivial homological ring epimorphisms of a Dedekind domain A is dual to the poset of subsets of maximal ideals of A (see [AS1, Corollary 6.12]).

We finish the paper by placing our classification of minimal tilting modules over hereditary rings in the context of further classification results for some special classes of rings.

Remark 5.21. If A is a Prüfer domain (i.e. a domain for which every finitely generated ideal is projective), the maps β and γ above define bijections between:

(a) equivalence classes of tilting A-modules;
(b) wide subcategories of bound A-modules;
(c) perfect Gabriel topologies on A;
(d) epiclasses of non-trivial universal localisations of A.

Moreover, all non-trivial universal localisations are injective homological ring epimorphisms, but the converse is not true in general (see [BS, Section 8]).

Indeed, the assignment γ is a bijection between (b) and (d): one can check that the arguments in [Sch3] also hold for semihereditary rings. The correspondence between (c) and (d) follows as in Example 5.20. In particular, we can deduce that wide and Serre subcategories of bound modules coincide. Finally, the bijection between (a) and (c) is [BET, Theorem 5.3].

Remark 5.22. If A is a commutative noetherian ring, there are bijections between:

(a) equivalence classes of tilting A-modules;
(b) wide subcategories of bound A-modules;
(c) faithful Gabriel topologies of A.

A Gabriel topology is called **faithful** if the localisation $A \longrightarrow Q_{\mathcal{L}}$ is injective. The bijection between (a) and (c) is [APST, Theorem 2.11]. Again, to a wide subcategory of bound modules $\mathcal{U}$, we associate the Gabriel topology $\mathcal{L}_{\mathcal{U}}$ and the tilting class consisting of the $\mathcal{L}_{\mathcal{U}}$-divisible modules.

Our results for hereditary rings thus share some common features with further classifications over other classes of rings. It would be nice to have a general statement encompassing all these cases.

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