On further properties of fully zero-simple semihypergroups

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Abstract

Let $\mathfrak{F}_0$ the class of fully zero-simple semihypergroups. In this paper we study the main properties of residual semihypergroup $(H_+, \star)$ of a semihypergroup $(H, \circ)$ in $\mathfrak{F}_0$. We prove that the quotient semigroup $H_+/\beta^*_{H_+}$ is a completely simple and periodic semigroup. Moreover, we find the necessary and sufficient conditions for $(H_+, \star)$ to be a torsion group and, in particular, an Abelian 2-group.

Keywords: semihypergroups, simple semihypergroups, fully semihypergroups.

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1 Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Many authors have been working on this field and in [5] numerous applications are recalled on algebraic hyperstructures

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such as: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. The semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays some scholars have studied different aspects of semihypergroups [2, 3, 10, 11, 14, 17, 18, 21, 22, 23] and interesting problems arise in the study of their so called fundamental relations [1, 7, 8, 15, 20, 25], which lead to analyze conditions for their transitivity, and minimal cardinality problems. In particular, in [15] the fundamental relations in simple or zero-simple semihypergroups are considered and in [9, 12] the authors use the simple semihypergroups of size 3, whose relation β is not transitive, to characterize the fully simple semihypergroups. In [10] the number of isomorphism classes of fully simple semihypergroups of size $n+1$ is determined. This number is the $(n+1)$-th term of sequence $A000712$ [24], namely $\sum_{k=0}^{n} p(k)p(n-k)$, where $p(k)$ denotes the number of nonincreasing partitions of integer k. Moreover, in [11, 13] the authors use the zero-simple semihypergroups of size 3, whose relation β is not transitive, to characterize the class of fully zero-simple semihypergroups $(H, \circ)$ having all hyperproducts of size ≤ 2 and one of the following cases occurs:

1. the subset $\{0, x, y\}$ as subsemihypergroup, for all pairs (x, y) of distinct nonzero elements of H;

2. there exists an element $1 \in H - \{0\}$ such that $\{0, 1, x\}$ is a subsemihypergroup, for every element $x \in H - \{0, 1\}$.

This class of fully zero-simple semihypergroups consists of three subclasses, namely $\mathfrak{R}_0$, $\mathfrak{L}_0$ and $\mathfrak{G}_0$. The semihypergroups $(H, \circ)$ in $\mathfrak{R}_0$ satisfy the condition $\{y\} \subseteq x \circ y \subseteq \{0, y\}$, for all $x, y \in H$, instead the semihypergroups in $\mathfrak{L}_0$ are such that $\{x\} \subseteq x \circ y \subseteq \{0, x\}$. In this two cases (see [11]), if $n \geq 2$ and $|H| = n+1$ then the number of isomorphism classes is the $(n+1)$-th term of sequence $A000070$ [24], namely $\sum_{k=0}^{n} p(k)$. The class $\mathfrak{G}_0$ (see [13]) consists of semihypergroups whose hyperproduct tables can be regarded as superposition of the product table of a 0-semigroup and an elementary Abelian 2-group. The aim of the present paper is first, to analyze algebraic general properties of fully zero-simple semihypergroups in which the product of two elements can also have size greater than 2 ; in particular, after introducing some basic definitions and notations to be used throughout the paper, we consider the quotient semigroups H/β^*, $H_+/\beta^*_{H_+}$ of a fully zero-simple semihypergroup $(H,\circ)$ and of his residual semihypergroup $(H_+,\star)$. We prove that H/β^* is trivial, and we find that if $(H_+,\star)$ is a group then is a torsin group. Moreover, we have that $H_+/\beta^*_{H_+}$ is a periodic completely simple semigroup. This property is very useful to determine non-trivial examples of fully zero-simple semihypergroups that have hyperproducts of size ≥ 2, or fully zero-simple semihypergroups $(H,\circ)$ with $|x\circ y|\leq 2$, for all $x,y\in H$, which do not belong to $\mathfrak{R}_0$, $\mathfrak{L}_0$ or $\mathfrak{G}_0$. Finally, we find the necessary and sufficient conditions for $(H_+,\star)$ to be a torsion group and, in particular, an Abelian 2-group.

1.1 Basic definitions and results

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. A hyperoperation $\circ$ on H is a map from $H\times H$ to $P^*(H)$. For all $x,y\in H$, the subset $x\circ y$ is called the hyperproduct of x and y. If A,B are non-empty subsets of H then $A\circ B=\bigcup_{x\in A,y\in B} x\circ y$.

A *semihypergroup* is a non-empty set H endowed with an associative hyperproduct $\circ$, that is, $(x\circ y)\circ z=x\circ(y\circ z)$ for all $x,y,z\in H$.

A non-empty subset K of a semihypergroup $(H,\circ)$ is called a *subsemihypergroup* of $(H,\circ)$ if it is closed with respect to multiplication, that is, $x\circ y\subseteq K$ for all $x,y\in K$. If $(H,\circ)$ is a semihypergroup, then the intersection $\bigcap_{i\in I} S_i$ of a family $\{S_i\}_{i\in I}$ of subsemihypergroups of $(H,\circ)$, if it is non-empty, is again a subsemihypergroup of $(H,\circ)$. For every non-empty subset $A\subseteq H$ there is at least one subsemihypergroup of $(H,\circ)$ containing A, e.g., H itself. Hence the intersection of all subsemihypergroups of $(H,\circ)$ containing A is a subsemihypergroup. We denote it by $\widehat{A}$, and note that it is defined by two properties:

1. $A\subseteq\widehat{A}$;
2. if S is a subsemihypergroup of H and $A\subseteq S$, then $\widehat{A}\subseteq S$.

Furthermore, $\widehat{A}$ is characterized as the algebraic closure of A under the hyperproduct in $(H,\circ)$, namely we have $\widehat{A}=\bigcup_{n\geq 1}A^n$. Moreover, if H is finite, the set $\left\{r\in\mathbb{N}-\{0\}\mid \bigcup_{k=1}^{r}A^k=\bigcup_{k=1}^{r+1}A^k\right\}$ has minimum $m\leq|H|$ and then, it is known that

$$\widehat{A}=\bigcup_{k=1}^{m}A^k=\bigcup_{k=1}^{m+1}A^k=\ldots=\bigcup_{k=1}^{|H|}A^k. \tag{1}$$

If $x \in H$, we suppose $\circ x^1 = \{x\}$ and $\circ x^n = \underbrace{x \circ \ldots \circ x}_{\text{n times}}$ for all integer $n > 1$.
We refer to $\widehat{x} = \bigcup_{n \geq 1} \circ x^n$ as the *cyclic subsemihypergroup of* $(H, \circ)$ *generated by the element* x. It is the smallest subsemihypergroup containing x.

If K is a subsemihypergroup of $(H, \circ)$, it is said *hypercyclic* if there exists a hyperproduct P of elements in K such that $K = \widehat{P}$.

If $(H, \circ)$ is a semihypergroup, an element $0 \in H$ such that $x \circ 0 = \{0\}$ (resp., $0 \circ x = \{0\}$) for all $x \in H$ is called *right zero scalar element* or *right absorbing element* (resp., *left zero scalar element* or *left absorbing element*) of $(H, \circ)$. If 0 is both right and left scalar element, then 0 is called *zero scalar element* or *absorbing element*.

A semihypergroup $(H, \circ)$ is called *simple* if $H \circ x \circ H = H$, for all $x \in H$.

A semihypergroup $(H, \circ)$ with an absorbing element 0 is called *zero-simple* if $H \circ x \circ H = H$, for all $x \in H - \{0\}$.

Given a semihypergroup $(H, \circ)$, the relation β^* of H is the transitive closure of the relation $\beta = \cup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation in H and, for every integer $n > 1$, β_n is defined recursively as follows:

$$x \beta_n y \Longleftrightarrow \exists (z_1, \ldots, z_n) \in H^n \ : \ \{x, y\} \subseteq z_1 \circ z_2 \circ \ldots \circ z_n.$$

The relations β, β^* are called *fundamental relations* on H [25]. Their relevance in semihypergroup theory stems from the following facts [20]: The quotient set H/β^*, equipped with the operation $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$ for all $x, y \in H$ and $z \in x \circ y$, is a semigroup. Moreover, the relation β^* is the smallest strongly regular equivalence on H such that the quotient H/β^* is a semigroup.

The interested reader can find all relevant definitions, many properties and applications of fundamental relations, even in more abstract contexts, also in [4, 5, 6, 16, 20, 25].

A semihypergroup $(H, \circ)$ is said to be *fully zero-simple* if it fulfills the following conditions:

1. All subsemihypergroups of $(H, \circ)$ ($(H, \circ)$ itself included) are zero-simple.

2. The relation β in $(H, \circ)$ and the relation β_K in all subsemihypergroups $K \subset H$ of size ≥ 3 are not transitive.

Since in all semihypergroups of size ≤ 2 the relation β is transitive, it follows that every fully zero-simple semihypergroup has size ≥ 3.

We denote by $\mathfrak{F}_0$ the class of fully zero-simple semihypergroups. We use 0 to denote the zero scalar element of each semihypergroup $(H, \circ) \in \mathfrak{F}_0$. Moreover, we use the notation H_+ to indicate the set of nonzero elements in H, that is, $H_+ = H - \{0\}$. Finally, for reader's convenience, we collect in the following lemma some preliminary results from [11].

Lemma 1.1. *Let $(H, \circ) \in \mathfrak{F}_0$ then we have:*

1. *$H \circ H = H$;*
2. *if S is a subsemihypergroup of H such that $0 \notin S$, then $|S| = 1$. Moreover, if $|S| \geq 2$ then the zero element of S is 0;*
3. *there exist $x, y \in H_+$ such that $0 \in x \circ y$;*
4. *for every sequence $z_1, \ldots, z_n$ of elements in H_+ we have $\prod_{i=1}^{n} z_i \neq \{0\}$;*
5. *the set H_+ equipped with hyperproduct $a \star b = (a \circ b) \cap H_+$, for all $a, b \in H_+$, is a semihypergroup.*

By points 2. and 4. of Lemma 1.1 we deduce the following result:

Corollary 1.1. *Let S be a subsemihypergroup of $H \in \mathfrak{F}_0$, then we have:*

1. *if $0 \notin S$ then there exists $a \in H_+$ such that $S = \{a\}$ and $a \circ a = \{a\}$;*
2. *if $|S| = 2$ then there exists $a \in H_+$ such that $S = \{a, 0\}$ and $\{a\} \subseteq a \circ a \subseteq \{0, a\}$.*

2 Properties of semihypergroup $(H_+, \star)$

From point 5. of Lemma 1.1, we know that the set of nonzero element H_+ of a fully 0-simple semihypergroup $(H, \circ)$ is a simple semihypergroup equipped with hyperoperation $a \star b = (a \circ b) \cap H_+$, for all $a, b \in H_+$. This semihypergroup is called *residual semihypergroup* of $(H, \circ)$.

In this section we show some properties of the quotient semigroups H/β^* and $H_+/\beta^*_{H_+}$. We prove that the quotient semigroup H/β^* is trivial, and we find that the semigroup $H_+/\beta^*_{H_+}$ is a periodic completely simple semigroup. Moreover, as a consequence of this result, we prove that the class of fully zero-simple semihypergroups $(H, \circ)$ such that the set $S = \{0, x, y\}$ is a subsemihypergroup, for all pairs (x, y) of distinct elements in H_+, consists of three subclasses, namely $\mathfrak{R}_0$, $\mathfrak{L}_0$ and $\mathfrak{G}_0(3)$. The semihypergroups $(H, \circ)$ in $\mathfrak{R}_0$ satisfy the condition $\{y\} \subseteq x \circ y \subseteq \{0, y\}$, for all $x, y \in H$. The semihypergroups in $\mathfrak{L}_0$ are such that $\{x\} \subseteq x \circ y \subseteq \{0, x\}$, and finally the class $\mathfrak{G}_0(3)$ consists of fully zero-simple semihypergroups of size 3 such that $(H_+, \star)$ is isomorphic to group $\mathbb{Z}_2$.

Theorem 2.1. *Let $(H, \circ) \in \mathfrak{F}_0$. For all $x \in H$, we have $(x, 0) \in \beta$. Moreover H/β^* is trivial.*

Proof. Let $[0]_H = \{a \in H \mid (a, 0) \in \beta\}$. Clearly we have $0 \in [0]_H$ and, by points 3. and 4. of Lemma 1.1, we obtain $[0]_H \neq \{0\}$. Now, let $a \in [0]_H - \{0\}$. Since H is zero-simple, we have $H \circ a \circ H = H$. Hence, for all $x \in H$, there exist $y, z \in H$ such that $x \in y \circ a \circ z$. Moreover, since $(a, 0) \in \beta$, there exists a hyperproduct P of elements in H_+ such that $\{0, a\} \subseteq P$ and we deduce

$$\{x, 0\} \subseteq y \circ a \circ z \cup \{0\} = y \circ a \circ z \cup y \circ 0 \circ z = y \circ \{a, 0\} \circ z \subseteq y \circ P \circ z.$$

Hence we have $(x, 0) \in \beta$, for every $x \in H$. Obviously, $|H/\beta^*| = 1$. □

Now we premise an easy lemma:

Lemma 2.1. *Let A, B be two non-empty subsets of $(H, \circ) \in \mathfrak{F}_0$ different from the singleton $\{0\}$. We have:*

1. $(A - \{0\}) \star (B - \{0\}) = A \circ B - \{0\}$.

2. *If $(A, \circ)$ is a subsemihypergroup of $(H, \circ)$ then $(A - \{0\}, \star)$ is a subsemihypergroup of $(H_+, \star)$.*

3. *If $0 \in A$ and $(A - \{0\}, \star)$ is a subsemihypergroup of $(H_+, \star)$ then $(A, \circ)$ is a subsemihypergroup of $(H, \circ)$.*

4. *If $A_+ = A - \{0\}$ and $(\widehat{A}, \circ), (\widehat{A}_+, \star)$ are the subsemihypergroups of $(H, \circ)$ and $(H_+, \star)$ generated from A and A_+ respectively, then $\widehat{A}_+ = \widehat{A} - \{0\}$.*

Proof. 1. If $x \in (A - \{0\}) \star (B - \{0\})$, there exist $a \in A - \{0\}$ and $b \in B - \{0\}$ such that $x \in a \star b$. Obviously $x \neq 0$ because $a \star b = a \circ b - \{0\}$, hence $x \in A \circ B - \{0\}$. Conversely, if $x \in A \circ B - \{0\}$ then there exist $a \in A$ and $b \in B$ such that $x \in a \circ b$. If $a = 0$ or $b = 0$, we have $x = 0$, against the hypothesis. Therefore $a, b, x \in H_+$ and $x \in a \circ b - \{0\} = a \star b \subseteq (A - \{0\}) \star (B - \{0\})$.

Points 2. and 3. are immediate consequences of the previous point.

4) It follows from item (1). In fact, if we put $\star(A_+)^n = \underbrace{A_+ \star \ldots \star A_+}_{\text{n times}}$ and $\circ A^n = \underbrace{A \circ \ldots \circ A}_{\text{n times}}$ then we have $\star(A_+)^n = \circ A^n - \{0\}$ and so

$$\widehat{A}_+ = \bigcup_{n \geq 1} \star(A_+)^n = \bigcup_{n \geq 1} (\circ A^n - \{0\}) = \widehat{A} - \{0\}. \quad \square$$

Remark 2.1. We note that if $(H, \circ) \in \mathfrak{F}_0$ then for every $x \in H - \{0\}$ such that $x \circ x \neq \{x\}$, the cyclic semihypergroup $\widehat{x} = \bigcup_{n \geq 1} \circ x^n$ generated by x has size ≥ 2, by point 2. of Lemma 1.1. Therefore it is a zero-simple subsemihypergroup of $(H, \circ)$, with 0 as zero scalar element. Moreover $\widehat{x} \circ \widehat{x} = \widehat{x}$ and $\widehat{x} \circ x \circ \widehat{x} = \widehat{x}$.

Before the next result we recall some notions that are typical of the semigroup theory. Let $E(S)$ be the set of idempotent elements in a semigroup S. If $E(S) \neq \emptyset$ then it can be partially ordered in a natural way, by means of the relation defined by the following rule: $e \leq f \;\Leftrightarrow\; ef = fe = e$.
If S is a semigroup with zero, the zero element of S is the unique minimum idempotent. The idempotents which are minimal in the poset $(E(S) - \{0\}, \leq)$ of non-zero idempotents are called *primitive*. Thus a primitive idempotent f has the property that for all $e \in E(S) - \{0\}$ we have $e \leq f \;\Leftrightarrow\; e = f$.
If S is a semigroup without zero and $E(S) \neq \emptyset$, an idempotent f is primitive if and only if it is minimal, that is, $e \leq f \;\Leftrightarrow\; e = f$.
If a simple semigroup S owns at least a primitive element then it is said to be *completely simple* and every idempotent of S is minimal [19].

Proposition 2.1. *Let* $(H_+, \star)$ *the residual semihypergroup of* $(H, \circ) \in \mathfrak{F}_0$ *and* $[0, 0]_H = \{(a, b) \in H \times H \mid a = 0$ *or* $b = 0\}$. *Then we have:*

1. $\beta_{H_+} = \beta - [0, 0]_H$;
2. $H_+/\beta^*_{H_+}$ *is a periodic semigroup;*
3. $H_+/\beta^*_{H_+}$ *is a completely simple semigroup.*

Proof.

1. It follows immediately from the fact that $\star\prod_{i=1}^{n} z_i = \circ\prod_{i=1}^{n} z_i - \{0\}$, for every $z_1, z_2, ..., z_n \in H_+$.

2. For every $x \in H_+$, we have that $x \circ x = \{x\}$ or the cyclic subsemihypergroup $\widehat{x}$ of $(H, \circ)$ has size ≥ 2. In both cases, by Remark 2.1, there exists an integer $n \geq 2$ such that $x \in \circ x^n$. Therefore we have $x \in \star x^n$. If $\varphi : H_+ \to H_+/\beta^*_{H_+}$ is the canonical projection, we obtain that $\varphi(x) = \otimes(\varphi(x))^n$. Thus every element in $H_+/\beta^*_{H_+}$ has finite period, hence $H_+/\beta^*_{H_+}$ is periodic.

3. We begin to observe that the quotient semigroup H_+/β^*_{H+} is simple. In fact, for every $x \in H_+$ we have:

$$\begin{aligned} H_+/\beta^*_{H_+} = \varphi(H_+) &= \varphi(H_+ \star x \star H_+) \\ &= \varphi(H_+) \otimes \varphi(x) \otimes \varphi(H_+) \\ &= H_+/\beta^*_{H_+} \otimes \varphi(x) \otimes H_+/\beta^*_{H_+}. \end{aligned}$$

Moreover, for item (2) and Proposition 1.2.3 in [19], the set $E(H_+/\beta^*_{H_+})$ of idempotent elements in $H_+/\beta^*_{H_+}$ is non-empty. If $H_+/\beta^*_{H_+}$ owns a zero element, obviously $|H_+/\beta^*_{H_+}| = 1$ and $H_+/\beta^*_{H_+}$ is completely simple. Therefore, we suppose that $H_+/\beta^*_{H_+}$ does not own a zero element and let a, b two distinct elements in H_+ such that $\overline{a} = \varphi(a), \overline{b} = \varphi(b)$ are idempotents and $\overline{a} \leq \overline{b}$. By definition, we have that $\overline{a} \otimes \overline{b} = \overline{b} \otimes \overline{a} = \overline{a}$. Moreover, the sets $h = \beta^*_{H_+}(a)$ and $k = \beta^*_{H_+}(b)$ are subsemihypergroups of H_+, since $\overline{a}, \overline{b}$ are idempotents in $H_+/\beta^*_{H_+}$. Now, for every $x \in h$ and $y \in k$, we obtain:

$$\varphi(x \star y) = \varphi(x) \otimes \varphi(y) = \overline{a} \otimes \overline{b} = \overline{a} = \varphi(a);$$

$$\varphi(y \star x) = \varphi(y) \otimes \varphi(x) = \overline{b} \otimes \overline{a} = \overline{a} = \varphi(a).$$

Therefore $h \star k \cup k \star h \subseteq h$ and

$$(h \cup k) \star (h \cup k) = h \star h \cup h \star k \cup k \star h \cup k \star k \subseteq h \cup k,$$

hence $h \cup k$ is a subsemihypergroup of H_+. For item (3) of Lemma 2.1, $h \cup k \cup \{0\}$ is a subsemihypergroup of H of size ≥ 3, because

$a \neq b$. Moreover, since $(H, \circ) \in \mathfrak{F}_0$, also $(h \cup k \cup \{0\}, \circ) \in \mathfrak{F}_0$, and in consequence:

$$\begin{aligned} h \cup k \cup \{0\} &= (h \cup k \cup \{0\}) \circ h \circ (h \cup k \cup \{0\}) \\ &= h \circ h \circ h \cup h \circ h \circ k \cup k \circ h \circ h \cup k \circ h \circ k \cup \{0\} \\ &\subseteq h \cup \{0\}. \end{aligned}$$

Thus, we have that $h \cup k \subseteq h$ since $0 \notin h \cup k$. Therefore $\beta^*_{H_+}(b) = k \subseteq h = \beta^*_{H_+}(a)$ and $\overline{a} = \overline{b}$, that is $H_+/\beta^*_{H_+}$ is completely simple. □

The following result is a consequence of the previous proposition.

Corollary 2.1. *Let $(H_+, \star)$ be the residual semihypergroup of $(H, \circ) \in \mathfrak{F}_0$. If $(H_+, \star)$ is a group then it is a torsion group.*

Proof. By hypothesis the residual semihypergroup $(H_+, \star)$ is a group, hence $\beta^*_{H_+}$ is the identity relation and $H_+/\beta^*_{H_+}$ is isomorphic to $(H_+, \star)$. Finally, from Proposition 2.1 (2), $(H_+, \star)$ is a torsion group. □

Remark 2.2. Let $(G, \cdot)$ be a group of size ≥ 2 and $0 \notin G$. In $H = G \cup \{0\}$ we can define the following hyperproduct:

$$a \circ b = \begin{cases} \{0\} & \text{if } a = 0 \text{ or } b = 0 \\ \{0, ab\} & \text{else.} \end{cases}$$

$(H, \circ)$ is a zero-semihypergroup such that the relation β^*_K is not transitive for all subsemihypergroups $K \subseteq H$ of size $|K| > 2$. Clearly we have $H_+ = G$, the relation $\beta^*_{H_+}$ is the identity relation and $H_+/\beta^*_{H_+} \cong G$. By Corollary 2.1, if G is not a torsion group then $(H, \circ)$ is not a fully zero-simple semihypergroup. We note that if $x \in G$ is an element of infinite period then the set

$$S = \{x^n \mid n \in \mathbb{N} - \{0\}\}$$

is a subsemigroup of $(G, \cdot)$ and $K = S \cup \{0\}$ is a subsemihypergroup of $(H, \circ)$. Moreover $(K, \circ)$ is not zero-simple, indeed we have $K \circ x \circ K \neq K$ since $x \notin K \circ x \circ K$. Hence the first axiom of fully zero-simple simihypergroup is not verified.

If $(G, \cdot)$ is a torsion group then all subsemigroups of $(G, \cdot)$ are subgroups. Moreover, $K \subseteq H$ is a subsemihypergroup of $(H, \circ)$ of size ≥ 2 if and only if there exists a subgroup k of $(G, \cdot)$ such that $K = k \cup \{0\}$. Clearly we have $K \circ x \circ K = K$, for all $x \in K$. Thus in this case $(H, \circ)$ is a fully zero-simple semihypergroup.

Remark 2.3. Let $\mathfrak{F}_{0,p}$ be the class of fully zero-simple semihypergroups $(H, \circ)$ such that the set $S = \{0, x, y\}$ is a subsemihypergroups, for all pair (x, y) of distinct elements in H_+. For definition of fully zero-simple semihypergroup we have $(S, \circ) \in \mathfrak{F}_{0,p}$. If we suppose that there exists a hyperproduct P of elements in S_+ such that $\{x, y\} \subseteq P$ then $x \circ x \cup x \circ y \cup y \circ x \cup y \circ y \subseteq P \circ P$. By Lemma 1.1 (3), we have $S = P \circ P$ and the relation β_S is transitive, that is impossible since $(S, \circ) \in \mathfrak{F}_{0,p}$. In consequence, by Lemma 1.1 (4), if $x, y \in H_+$ and $|x \circ y| = 2$ then $x \circ y \in \{\{0, x\}, \{0, y\}\}$. Hence the residual semihypergroup $(H_+, \star)$ is a semigroup and $(S_+, \star)$ is a subsemigroup of size 2 isomorphic to $S_+/\beta^*_{S_+}$ because $\beta^*_{S_+}$ is the identity relation. Moreover, by Proposition 2.1 (3), $(S_+, \star)$ is a completely simple semigroup of size 2. Thus $(S_+, \star)$ is isomorphic to one of the following semigroups:

$\mathbb{R}_2$:

	1	2
1	1	2
2	1	2

$\mathbb{L}_2$:

	1	2
1	1	1
2	2	2

$\mathbb{Z}_2$:

	1	2
1	1	2
2	2	1

Now, we show that if $H \geq 4$ then there are no subsets $\{x, y, z\}$ of distinct elements in H_+ such that the semihypergroups $S' = \{0, x, y\}$ and $S'' = \{0, x, z\}$ verify the following cases:

a.) $S'_+ \cong \mathbb{R}_2$ and $S''_+ \cong \mathbb{Z}_2$;

b.) $S'_+ \cong \mathbb{L}_2$ and $S''_+ \cong \mathbb{Z}_2$;

c.) $S'_+ \cong \mathbb{Z}_2 \cong S''_+$;

d.) $S'_+ \cong \mathbb{R}_2$ and $S''_+ \cong \mathbb{L}_2$;

The case *a.*) is impossible. Indeed, if $S'_+ = \{x, y\} \cong \mathbb{R}_2$ and $S''_+ = \{x, z\} \cong \mathbb{Z}_2$ then the subset $T = \{0, y, z\}$ is not a subsemihypergroup of $(H, \circ)$ since $x \in z \circ z \subseteq T \circ T$.

The cases *b.*) anc *c.*) are similar to previous case.

Finally, in the last case, we obtain the following partial table

	x	y	z
x	x	y	x
y	x	y	
z	z		z

Since $x \star (z \star y) = (x \star z) \star y = \{y\}$ then $z \star y = \{y\}$ and so we have the contradiction $\{x\} = y \star x = (z \star y) \star x = z \star (y \star x) = z \star x = \{z\}$.

An immediate consequence of the considerations made in the previous remark is the following result

Theorem 2.2. *Let $\mathfrak{F}_{0,p}$ be the class of fully zero-simple semihypergroups $(H, \circ)$ such that the set $S = \{0, x, y\}$ is a subsemihypergroup, for all subsets $\{x, y\}$ of distinct elements in H_+. Moreover, let $T = \{0, 1, 2\}$ a subset of H. Then we have*

1. *$(H_+, \star)$ is a semigroup;*
2. *for all x, y of distinct elements in H_+, $(\{x, y\}, \star)$ is a completely simple semigroup of size 2;*
3. *all subsemihypergroups of size 2 in $(H_+, \star)$ are isomorphic to only one of three semigroups $\mathbb{R}_2$, $\mathbb{L}_2$, $\mathbb{Z}_2$;*
4. *if $(T_+, \star)$ is isomorphic to $\mathbb{R}_2$ then $(H_+, \star)$ is a right zero-semigroup and $\{y\} \subseteq x \circ y \subseteq \{0, y\}$, for all $x, y \in H$;*
5. *if $(T_+, \star)$ is isomorphic to $\mathbb{L}_2$ then $(H_+, \star)$ is a left zero-semigroup and $\{x\} \subseteq x \circ y \subseteq \{0, x\}$, for all $x, y \in H$;*
6. *if $(T_+, \star)$ is isomorphic to $\mathbb{Z}_2$ then $|H| = 3$.*

As a consequence of points 4., 5. and 6. of previous theorem we have that the class $\mathfrak{F}_{0,p}$ consists of three subclasses, namely $\mathfrak{R}_0$, $\mathfrak{L}_0$ and $\mathfrak{G}_0(3)$. The semihypergroups $(H, \circ)$ in $\mathfrak{R}_0$ (resp. $\mathfrak{L}_0$) satisfy the condition $\{y\} \subseteq x \circ y \subseteq \{0, y\}$ (resp. $\{x\} \subseteq x \circ y \subseteq \{0, x\}$), for all $x, y \in H$. In finite case, in paper [11] the number of semihypergroups in $\mathfrak{R}_0$ (resp. $\mathfrak{L}_0$) has been calculated, up to isomorphisms. If $n \geq 2$ and $|H| = n + 1$ then such number is the $(n + 1)$-th term of sequence $A000070$ [24], namely $\sum_{k=0}^{n} p(k)$, where $p(k)$ denotes the number of nonincreasing partitions of integer k. Moreover, the semihypergroups in class $\mathfrak{G}_0(3)$ are the six semihypergroups listed below, apart of isomorphisms:

$\circ_1$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0, 1

$\circ_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	0, 2	0, 1

$\circ_3$	0	1	2
0	0	0	0
1	0	1	0, 2
2	0	2	0, 1

$\circ_4$	0	1	2
0	0	0	0
1	0	1	0, 2
2	0	0, 2	0, 1

$\circ_5$	0	1	2
0	0	0	0
1	0	0, 1	0, 2
2	0	0, 2	1

$\circ_6$	0	1	2
0	0	0	0
1	0	0, 1	0, 2
2	0	0, 2	0, 1

Obviously, the six semihypergroups listed above belong to class $\mathfrak{G}_0$ of fully zero-simple semihypergroups $(H, \circ)$ whose the residual semihypergroup $(H_+, \star)$ is an Abelian 2-group.

In the next theorem if $(H, \circ) \in \mathfrak{F}_0$ and $x \in H_+$ then we denote by $\widehat{x}_+$ the cyclic subsemihypergroup of $(H_+, \star)$ generated by the element x. Moreover we put $\star x^n = \underbrace{x \star \ldots \star x}_{\text{n times}}$.

Theorem 2.3. *Let $(H, \circ) \in \mathfrak{F}_0$, then the following conditions are equivalent*

1. *the residual semihypergroup $(H_+, \star)$ is a torsion group;*
2. *$\bigcap_{x \in H_+} \widehat{x}_+ \neq \emptyset$ and $\widehat{x}_+$ is a finite subgroup of $(H_+, \star)$, for all $x \in H_+$.*

Proof. The implication 1. $\Rightarrow$ 2. is obvious. We prove that 2. $\Rightarrow$ 1. If $y \in H_+$ and ε is the identity of $\widehat{y}_+$ then $\bigcap_{x \in H_+} \widehat{x}_+ \subseteq \widehat{\varepsilon}_+ = \{\varepsilon\}$. Hence $\bigcap_{x \in H_+} \widehat{x}_+ = \{\varepsilon\}$ and so $\varepsilon \in \widehat{x}_+$, for all $x \in H_+$. Since ε is an idempotent element, we have that ε is the identity of group $\widehat{x}_+$, for all $x \in H_+$. Consequently ε is also identity of $(H_+, \star)$ and every element $x \in H_+$ has inverse because $\widehat{x}_+$ is a group. Now, we show that $|a \star b| = 1$, for all $a, b \in H_+$. Clearly the thesis is true if $\varepsilon \in \{a, b\}$. Therefore we suppose that $a, b \in H_+ - \{\varepsilon\}$ and let $\{x, y\} \subseteq a \star b$. Since $\widehat{a}_+$ and $\widehat{b}_+$ are finite subgroups of $(H_+, \star)$, there exist $n, m \in \mathbb{N} - \{0, 1\}$ such that $\star a^n = \{\varepsilon\} = \star b^m$. We have $\star a^{nm} = \star b^{nm} = \{\varepsilon\}$ and $\star a^{nm-1} \star x \cup \star a^{nm-1} \star y = \star a^{nm-1} \star \{x, y\} \subseteq \star a^{nm-1} \star (a \star b) = (\star a^{nm-1} \star a) \star b = \star a^{nm} \star b = \varepsilon \star b = \{b\}$. Hence $\star a^{nm-1} \star x = \star a^{nm-1} \star y = \{b\}$ and we have $\{y\} = \varepsilon \star y = \star a^{nm} \star y = (a \star (\star a^{nm-1})) \star y = a \star ((\star a^{nm-1}) \star y) = a \star ((\star a^{nm-1}) \star x) = (a \star (\star a^{nm-1})) \star x = \star a^{nm} \star x = \varepsilon \star x = \{x\}$. Thus $|a \star b| = 1$ and $(H_+, \star)$ is a torsion group. □

Notice that the semihypergroups $(H, \circ) \in \mathfrak{R}_0$ (resp. $\mathfrak{L}_0$) satisfy the condition that $\widehat{x}_+$ is a subgroup of $(H_+, \star)$, for all $x \in H_+$. In this case we have $\bigcap_{x \in H_+} \widehat{x}_+ = \emptyset$ because $\widehat{x}_+ = \{x\}$, for all $x \in H_+$, and $(H_+, \star)$ is a right zero semigroup (resp. a left zero semigroup).

As a consequence of the previous theorem we obtain the following

Corollary 2.2. *Let* $(H, \circ) \in \mathfrak{F}_0$*, then the following conditions are equivalent*

1. *the residual semihypergroup* $(H_+, \star)$ *is an Abelian 2-group;*
2. *there exist an element* $1 \in H_+$ *such that* $\{1, x\}$ *is a subgroup of* $(H_+, \star)$ *for all* $x \in H_+$.

The class of semihypergroups that verify one of the equivalent conditions of the previous corollary is indicated with $\mathfrak{G}_0$. In [13], the authors show that the hyperproduct $\circ$ of semihypergroups in $\mathfrak{G}_0$ can be regarded as superposition of the product table of a zero-semigroup and an elementary Abelian 2-group. In particular, in case of size 5 or 9 the number of semihypergroups in $\mathfrak{G}_0$ are 41 or 7272 respectively.

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