QUOTIENTS OF LOCALLY MINIMAL GROUPS

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ABSTRACT. A topological group G is called *locally minimal* if there exists a neighbourhood V of the identity of G such that whenever H is a Hausdorff group and $f : G \to H$ is a contuinuous isomomorphism such that $f(V)$ is a neighbourhood of 1 in H, then f is open. This paper is focused on the study of quotients of locally minimal groups.

A surprizing connection between locally q-minimality and divisibility is found, by showning that a dense subgroup of $\mathbb{R}^n$ is locally q-minimal if and only if it is divisible. This provides examples showing that a topological group with a dense locally q-minimal subgroup need not be locally q-minimal. We also propose a weaker notion, namely local q^*-minimality and show that a dense subgroup H of a Hausdorff group G is locally q^*-minimal if and only if G is locally q^*-minimal and H is locally t-dense in G.

CONTENTS

1. INTRODUCTION

Call a Hausdorff topological group G *minimal*, if G admits no properly coarser Hausdorff group topology. Obviously, G is minimal precisely when G satisfies the open mapping theorem with respect to continuous isomotphisms with domain G. Compact groups are minimal, the first examples of non-compact minimal groups were found by Doïchinov [20] and Stephenson [31] and the research in this field, inspired by G. Choquet, was quite intensive for almost five decades (see the [4, 7, 12, 13, 15, 16, 19, 21, 25, 27, 28, 29, 32, 33], as well as the surveys or monographs [6, 8, 9, 10, 14]).

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As noticed by Stephenson, locally compact groups need not be minimal (actually, a compact abelian group is minimal precisely when it is compact). This motivated Morris and Pestov [26] (see also [3]) to introduce the notion of locally minimal groups: a Hausdorff topological group (G, τ) is *locally minimal with respect to a neighbourhood* V of the identity of G if every Hausdorff group topology $\sigma \leq \tau$ and such that V is a σ-neighbourhood of the identity, coincides with τ (sometimes, for simpllcitiy, we simply speak of local minimality without mentioning an explicit neighbourhood V). Further detail on locally minimal groups can be found in [11, 1, 2]. The relevant permanence of local minimality related to the passage to closed or dense subgroup was largely studied in these papers. Nevertheless, perhaps the most relevant (from the point of view of the open mapping theorem) permanence property was not discussed so far, namely the preservation of local minimality under taking quotients. The aim of this paper is to fill this gap by a careful study of the stability of the class of locally minimal groups under taking quotients.

Minimality fails to be preserved under taking quotients. This is why the smaller class of *totally minimal groups*, namely the minimal groups that are minimal along with all their Hausdorff quotients, was introduced in [12] (somewhat later also in [33], under the name *q-minimal groups*). Equivalently, a topological group G is totally minimal if every surjective continuous homomorphism of G onto a Hausdorff topological group is open. Inspired by the latter formulation (and the definition of local minimality, depending on a fixed neighbourhood V of the identity), the locally q-minimal groups were introduced in [11] as follows:

Definition 1.1. [11] *A topological group G is called* locally q-minimal with respect to a neighbourhood *V of the identity of G if every contuinuous surjective homomorphism $f : G \to H$ onto a Hausdorff group H such that $f(V)$ is a neighbourhood of 1 in H, is open.*

Often we say briefly *G is locally q-minimal* if there exists such a neighbourhood V. Obviously, every discrete group G is locally q-minimal (with respect to the neighbourhood $\{e_G\}$).

The point of view adopted in the first of the above two equivalent formulations of total minimality (*every Hausdorff quotient of the group is minimal*) provides an obvious alternative way to obtain a "local" version of total minimalily as follows:

Definition 1.2. *A topological group G is called* locally t-minimal *if each Hausdorff quotient group of G is a locally minimal.*

This property was given and used in [34] under the term local q-minimality. We prefer to dedicate a different term (namely, local t-minimality), since one of the aims of this paper is to show that these two notions differ substantially. It is clear that a topological group G is locally q-minimal with respect to a neighbourhood V of the identity of G iff G/N is locally minimal with respect to VN/N for each closed normal subgroup N of G [1]. Hence, a locally q-minimal group is obviously locally t-minimal, but the converse is not true (see Examples 4.3 and 4.8).

The third generalization of total minimality (the local q^*-minimality) is closely related to another relevant property of total minimality that we recall first.

The following notion was proposed in [12] and independently (but somewhat later) also in [33]: a subgroup H is a topological group G is called *totally dense* if $H \cap N$ is dense in N for every closed normal subgroup N of G. This notion was used to provide the following crucial criterion for total minimality of dense subgroups:

Theorem 1.3. [12] *A dense subgroup H of a topological group G is totally minimal iff G is totally minimal and H is totally dense in G.*

[1] in particular, G/N is minimal whenever $VN = G$, we shall see that this become a too strong restraint to impose on G.

Definition 1.4. *A Hausdorff topological group G is called* locally q^*-minimal with respect to a neighbourhood V *of the identity of G if every continuous surjective homomorphism $f : G \to H$ onto a Hausdorff group H such that $f(V)$ is a neighbourhood of the identity in H and* $\ker f \subset V$, *is open.*

Often we say briefly G is *locally q^*-minimal* if there exists such a neighbourhood V. Obviously, local q-minimality implies local q^*-minimality, but the converse implication may fail even for subgroups of $\mathbb{R}$ (see Examples 4.3 and 4.8). We provide an example showing that this property does not coincide with (actually, does not imply) local t-minimality (Example 5.9).

It has been an open problem for some time to find a criterion for local q-minimality of dense subgroups in this line. Our second main result shows that such a criterion for local q-minimality cannot be available, as a group containing a dense locally q-minimal subgroup need not be locally q-minimal itself (Example 4.8). The weaker notion of local q^*-minimality we propose (see Definition 1.4) allows for such a criterion for local q^*-minimality of dense subgroups (see Theorem 5.4). In order to obtain this characterization one needs the notion of *local t-density* (see Definition 5.2).

Notation and terminology. We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive natural numbers and primes, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, and by $\mathbb{T}$ the unit circle group which is identified with $\mathbb{R}/\mathbb{Z}$. The cardinality of the continuum 2^ω will be also denoted by $\mathfrak{c}$. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$. For a prime p the symbol $\mathbb{Z}(p^\infty)$ stands for the quasicyclic p-group and $\mathbb{Z}_p$ stands for the p-adic integers.

The subgroup generated by a subset X of a group G is denoted by $\langle X\rangle$, and $\langle x\rangle$ is the cyclic subgroup of G generated by an element $x \in G$. The abbreviation $K \leq G$ is used to denote a subgroup K of G.

Throughout this note all topological groups are assumed to be Hausdorff, unless otherwise stated explcitely. We denote by $\mathcal{V}_\tau(1)$ (or simply by $\mathcal{V}(1)$) the filter of neighbourhoods of 1 in a topological group (G, τ).

For a topological group G we denote by $\widetilde{G}$ the Raĭkov compeltion of G. We recall here some compactness-like conditions on a topological group G. A group G is *precompact* (some authors prefer "totally bounded") if $\widetilde{G}$ is compact. The centre $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$ of G is a closed subgroup of G.

The *torsion part* $t(G)$ of an Abelian group G is the set $\{g \in G : ng = 0 \text{ for some } n \in \mathbb{N}\}$. Clearly, $t(G)$ is a subgroup of G. For a prime p, the *p-primary component* G_p of G is the subgroup of G that consists of all $x \in G$ satisfying $p^n x = 0$ for some positive integer n. The group G is *divisible* if $nG = G$ for every $n \in \mathbb{N}$. [2]

All unexplained topological terms can be found in [23]. For background on Abelian groups, see [24] and [30].

2. Background on locally minimal groups

Fact 2.1. [1, Proposition 2.5] If G is a locally minimal topological group and H is a closed central subgroup of G, then H is locally minimal. More precisely, if local minimality of G is witnessed by V, then H is locally minimal with respect to $V_1 \cap H$ for any neighbourhood V_1 of the identity of G such that $V_1^2 \subset V$.

[2]maybe can be omitted:
For a prime p and element x of a topological group G, p is said to be *topologically p-torsion*, if $p^n x \to 0$. A subgroup H of G is called p-monothetic, if H has a dense cyclic subgroup generated by a topologically p-torsion element of H.

Corollary 2.2. *Let G be a locally minimal abelian group. Then there exists a neighbourhood U of the identity in G such that each closed subgroup N of G contained in U is minimal (so, precompact).*

Proof. Assume that G is locally minimal with respect to U^2, where U is a neighbourhood of the identity of G. Let N be a closed subgroup of G contained in U. According to Remark 2.1, N is locally minimal with respect to $U \cap N = N$, hence, N is minimal. For the last assertion recall that according by the celebrated Prodanov-Stoyanov Theorem, every minimal abelian group is precompact. □

In order to formulate the local minimality criterion from [2], we need to recall first the following notion:

Definition 2.3. [2] *Let H be subgroup of a topological group G. We say that H is* locally essential *in G if there exists a neighborhood V of 0 in G such that $H \setminus \{0\}$ meets each nontrivial closed normal subgroup N of G which is contained in V.*

When necessary, we shall say H is locally essential with respect to V to indicate that V witnesses local essentiality. Note that if V witnesses local essentiality, then any smaller neighborhood of zero does too.

Definition 2.4. *A topological group G is said to have* no small subgroups *(or shortly, to be an* NSS group*), if G has a neighbourhood of the identity element that contains no non-trivial subgroups.*

Remark 2.5. *Obviously, every subgroup of an NSS group is locally essential.*

The following criterion for local minimality was established in [2]:

Fact 2.6. [Criterion for local minimality] *Let H be a dense subgroup of a topological group G. Then H is locally minimal iff G is locally minimal and H is locally essential in G.*

Remark 2.7. *The proof of Fact 2.6 in* [2, Theorem 3.5] *shows more. Namely, for a dense subgroup H of G:*

(1) When H is locally minimal and if W is a closed neighbourhood of 1 in G such that $W \cap H$ witnesses local minimality of H, then each neighbourhood W_1 of the identity in G satisfying $W_1^2 \subset W$ witnesses local essentiality of H in G and W witnesses local minimality of G.

(2) When G is locally minimal and if the neighbourhood V of the identity in G witnesses both local minimality of G and local essentiality of H in G, then for every neighbourhood V_1 of the identity in G with $V_1^2 \subset V$ the neighbourhood $V_1 \cap H$ witnesses local minimality of H.

[3] According to [14, §4.1], for a prime number p, an element x of a topological group G is called *quasi-p-torsion* iff $\langle x \rangle$ is either a cyclic p-group or topological isomorphic to $(\mathbb{Z}, \tau_p)$. Following [14, p.145], call a subgroup H of G p-monothetic, if G has a dense cyclic subgroup generated by a quasi-p-torsion element of G. If G is complete, then the closed p-monothetic subgroups of G are cyclic p-groups or topologically isomorphic to $\mathbb{Z}_p$. According to a useful local criterion for minimality in [14, Theorem 4.3.7], a precompact abelian group G is minimal if and only if G non-trivially meets every non-trivial closed p-monothetic subgroup of G for every prime p. Using this criterion for minimality, along with Corollary 2.2, we obtain now a useful local (w.r.t. the primes p) version of Fact 2.6:

[3]This definition, along with the corollary that follows are maybe not needed. Then they should be deleted (and kept for the second paper).

Corollary 2.8. *Let G be a complete locally minimal abelian group. A dense subgroup H of G is locally minimal if and only if there exists a neighbourhood U of 0 in G, such that for every prime p the subgroup H non-trivially meets every non-trivial closed p-monothetic subgroup of G, i.e., every subgroup $N \cong \mathbb{Z}_p$ of G contained in U non-trivially meets G and H contains every cyclic group of order p that is entirely contained in U.*

Proof. The necessity follows from Fact 2.6. Now assume that neighbourhood U of 0 with the above properties exist in G. According to Fact 2.6, it is enough to check that H is locally essential in G w.r.t. U. Shrinking, if necessary, U we can assume that every closed subgroup of G contained in U is compact, by Corollary 2.2. Pick an arbitrary closed subgroup $N \neq \{0\}$ of G contained in U. By [14, Theorem 4.1.7], N contains a non-trivial closed monothetic subgroup N_1. Now our hypothesis implies that $H \cap N_1 \neq \{0\}$, so $H \cap N \neq \{0\}$ as well. ☐

Fact 2.9. The following useful facts will be used in the sequel:

(a) ([11, Lemma 2.8]) Locally compact groups and totally minimal groups are locally q-minimal, hence locally t-minimal.

(b) local t-minimality and local q-minimality are both preserved under taking quotients. More precisely, a topological group G is locally q-minimal with respect to some neighbourhood V of 1 if and only if for every closed normal subgroup N of G the quotient group G/N is locally minimal with respect to the neighbourhood $(VN)/N$ of 1 in G/N.

Fact 2.10. [11, 34] If a group G has an open locally minimal subgroup, then G itself is locally minimal.

In item (b) we see some examples of locally minimal groups without open locally minimal subgroups.

Example 2.11. (a) *According to* [11, 34, 26] *every subgroup G of a Lie group L is locally minimal. To see that G is also locally t-minimal, assume that G is dense in L (obviously, this is not a restrictive assumption). Let G/N be a quotient of G w.r.t. a closed normal subgroup N of G. Then G/N is isomorphic to a (dense) subgroup of the quotient $L/cl_L(N)$ of L, hence G/N is locally minimal.*

On the other hand, G is also locally q^*-minimal, witnessed by any neighbourhood of e_G containing no non-trivial subgroups.

(b) *Consider the subgroup $G = \mathbb{Z}(p^\infty)$ of $\mathbb{T}$. Then G is locally t-minimal by (a), but H has no proper open subgroups and H itself is not minimal. Thus H has no open minimal subgroup. Analogous argument shows that any dense embedding of $\mathbb{Z}$ in $\mathbb{T}$ induces on $\mathbb{Z}$ a locally q-minimal topology on $\mathbb{Z}$ without open minimal subgroups.*

For a symmetric subset U of a group $(G, +)$ with $0 \in U$, and $n \in \mathbb{N}$ let

$$(1/n)U := \{x \in G : kx \in U \text{ for all } k \in \{1, 2, \dots, n\}\}.$$

The following group analog of a normed space was introduced by Enflo [22]. A Hausdorff topological group (G, τ) is said to be *uniformly free from small subgroups* (UFSS for short) if for some neighbourhood U of 0, the sets $(1/n)U$ form a neighborhood basis at 0 for τ. The class of UFSS groups is stable under taking subgroups, completions, local isomorphisms and has the three space property (and so stability under finite direct product). Finally, UFSS group is both NSS and locally minimal [1, Proposition 3.12].

The property UFSS is not stable under taking quotients, nevertheless this cannot exclude a priori that (subgroups of) UFSS groups are locally q-minimal. In Remark 4.4 we show that $\mathbb{R}$ has subgroups that are not locally q-minimal, so subgroups of Banach spaces (in particular, UFSS groups) need not be locally q-minimal.

3. Some general properties of the local q-, q^*- and t-minimality

3.1. **Invariance under taking closed or open subgroups.** Now we show that a closed central subgroup of a locally q-minimal (resp., locally q^*-minimal, locally t-minimal) group is locally q-minimal (resp., locally q^*-minimal, locally t-minimal).

Proposition 3.1. *Let H be a closed central subgroup of a topological group G.*

(a) if G is locally q-minimal then also H is locally q-minimal.

(b) if G is locally q^*-minimal then also H is locally q^*-minimal.

(c) if G is locally t-minimal then also H is locally t-minimal.

Proof. (a) Suppose that G is locally q-minimal with respect to U, a neighbourhood of the identity of G. Let H be a closed central subgroup of G. Take a neighbourhood V of the identity such that $V^2 \subset U$, we are going to prove that $V \cap H$ witness local q-minimality of H.

Let N be a closed subgroup of H, then it is a closed normal subgroup of G as well. Denote by π the natural quotient mapping of G onto G/N. By our assumption, G/N is locally minimal with respect to $\pi(U)$. Since $\pi(V)$ is a closed subgroup of G/N and $\pi(V)^2 = \pi(V^2) \subset \pi(U)$, we apply Remark 2.1 and get that $\pi(H)$ is locally minimal with respect to $\pi(V) \cap \pi(H)$, so to $\pi(V \cap H) \subseteq \pi(V) \cap \pi(H)$ as well.

The proofs of (b) and (c) are similar (do it!). □

The next proposition shows that the implication of the above proposition can be inverted in case the closed subgroup is actually open (see also Remark 3.3 below):

Proposition 3.2. *A Hausdorff topological group with an open locally q-minimal (resp., q^*-minimal, t-minimal) subgroup is locally q-minimal (resp., q^*-minimal, t-minimal).*

Proof. We first prove the case of local q-minimality, the case of local q^*-minimality is similar. Let H be a locally q-minimal group witnessed by $U \in \mathcal{V}_H(0)$ and suppose that H is an open subgroup of the (G, τ). Then U is a neighborhood of 0 in G. Assume that $f : G \to G_1$ is a surjective homomorphism such that $f(U)$ is a neighborhood of 0 in G_1. Let $f \restriction_H$ be the restriction of f to H, considered as a surjective homomorphism of H onto $f(H)$. Since $U \subseteq H$, we have that $f(U) \subseteq f(H)$, then $f(U)$ is a neighbourhoodof 0 in $f(H)$ and $f(H)$ is open in G_1(because it contains a neighbourhood$f(U)$ of the identity of G_1). By the U-local minimality of H, one readily gets that $f \restriction_H: H \to f(H)$ is open. Since that H is open in G and $f(H)$ is open in G_1, $f : G \to G_1$ is also open.

Now suppose that the open subgroup H of G is locally t-minimal. Let N be an arbitrary closed normal subgroup of G, it suffices to show that G/N is locally minimal. Obviously, $M = H \cap N$ is a closed normal subgroup of H. By local t-minimality of H, H/M is locally minimal.

Let $\pi : G \to G/N$ and $\xi : H \to H/M$ be the canonical maps. They are continuous and open; moreover, there exists a continuous isomorphisms $j : H/M \to \pi(H)$ with $\pi \restriction_H = j \circ \xi$. Pick an open neighbourhood U of the identity in H, such that $\xi(U)$ witnesses local minimality of H/M. Since H is open in G, this yields that U is an open neighbourhood of the identity in G as well. Then $\pi(U)$ is an open neighbourhood of the identity in G/N contained in $\pi(H)$. To the continuous isomorphism $j : H/M \to \pi(H)$ we can apply the local minimality of H/M (with respect to $\xi(U)$) to conclude that j is a topological isomorphism, as $j(\xi(U)) = \pi(U)$ is open in $\pi(H)$. Hence, the open subgroup $\pi(H)$, being topologically isomorphic to H/M, is locally minimal. Hence, G/N is locally minimal as well. □

Remark 3.3. *Let H be a closed central subgroup of a topological group G. We do not know if any of the three implication above can be inverted in general. We show below that this is true in the abelian case.*

Definition 3.4. [17, Definition 3.1] *A subgroup H of a group G is called:*

(i) Hausdorff embedded *in G if for every Hausdorff group topology τ on H there exists a Hausdorff group topology τ' on G such that $\tau = \tau' \restriction_H$ (and in this case we say that τ' extends τ);*

(ii) super-normal *(in G) if $G = c_G(H)H$, i.e., for every $x \in G$ there exists $y \in H$ such that $x^{-1}hx^{-1} = y^{-1}hy$ for every $h \in H$.*

Super-normal subgroups are Hausdorff embedded, hence central subgroups, as well as direct summands are Hausdorff embedded [17]. As far as extension of a *fixed* Hausdorff group topology is concerned, one has the following:

Fact 3.5. *Let H be a normal subgroup of a group G. A group topology λ on H, can be extended to a topology λ^* on G in a standard way, by declaring the family $\mathcal{V}_{(H,\lambda)}(e)$ to forms a local base at e of a λ^*. Then λ^* is a group topology if and only if all automorphism of H, obtained by restriction to G of conjugations by elements of G are continuous* [17, Theorem 3.4]*. Clearly, then $\lambda^* \restriction_H = \lambda$ and H is λ^*-open.*

Theorem 3.6. *Let H be an open subgroup of a topological group G.*

(a) If H is Hausdorff embedded and G is locally minimal, then so is H.

(b) If H is super-normal and G is locally q^*-minimal, then so is H.

Proof. Let τ be the topology of G.

(a) Let $V \in \mathcal{V}_G(1)$ witness local minimality of (G, τ). Since H is open, we can assume slog that $V \subseteq H$, so $V \in \mathcal{V}_H(1)$. To show that V witnesses local q^*-minimality of $(H, \tau \restriction_H)$ pick a Hausdorff group topology $\sigma \leq \tau \restriction_H$ with $V \in \sigma$. Since H is a Hausdorff embedded subgroup of G, the standard extension σ^* of σ is a Hausdorff group topology on G such that $\sigma^* \restriction_H = \sigma \leq \tau \restriction_H$ and H is σ^*-open in G. Since $V \in \sigma$, we deduce that $V \in \sigma^*$ as well. Since H is τ-open, we deduce that $\sigma^* \leq \tau$. Now the local minimality of (G, τ) implies that the identity $(G, \tau) \to (G, \sigma^*)$ is open. Hence, $\sigma^* = \tau$ and consequently $\sigma = \sigma^* \restriction_H = \tau \restriction_H$.

(b) Let us note that the (stronger) assumption that H is super-normal implies that every normal subgroup of H is normal in G as well. Let $V \in \mathcal{V}_G(1)$ witness local q^*-minimality of (G, τ). Since H is open, we can assume slog that $V \subseteq H$, so $V \in \mathcal{V}_H(1)$. To show that V witnesses local q^*-minimality of $(H, \tau \restriction_H)$ pick a normal closed subgroup N of H, it will be a normal subgroup in G. Let $f : H \to H/N$ and $h : G \to G/N$ be the quotient maps and let σ be a Hausdorff group topology on H/N such that $f : (H, \tau \restriction_H) \to (H/N, \sigma)$ is continuous and $f((V) \in \sigma$. Let $j : H/N \to G/H$ be the obvious idenitification of H/N with a subgroup of the abstract group G/N. As H is open in G, $j(H/N)$ will be open in $(G/N, \bar{\tau})$, where $\bar{\tau}$ denotes the quotient topology of G/N. Since $j(H)$ is $\bar{\tau}$-open, we deduce that $\sigma^* \leq \bar{\tau}$, so the local q^*-minimality of (G, τ) implies that $h : (G, \tau) \to (G/N, \sigma^*)$ is open. Since H is τ-open, this yields that $f = h \restriction_H$ is open as well. □

Corollary 3.7. *Let H be an open subgroup of a topological abelian group G. Then H is locally (q^*-)minimal iff G is locally (q^*-)minimal.*

For a topological group G denote by $o(G)$ the intersection of all open subgroups of G. This is a closed normal subgroup of G containing the connected component $c(G)$ of G. The subgroup $o(G)$ coincides with the whole G precisely when G has no proper open subgroup. If G is locally precompact, then this occurs precisely when its locally compact completion K is connected.

For locally precompact G with locally compact completion K, $o(G)$ is a proper open subgroup of G precisely when G intersects c(K) into a dense subgroup and c(K) is an open subgroup of K.

Assume that we have a group G such that o(G) is open in G. Then such a G is locally q-minimal precisely when $o(G)$ is locally q-minimal.

3.2. **The 3-space property.** In case the open subgroup H as in Proposition 3.2 is also normal, one can formulate the above results in a way to connect to the 3-space problem: if a group G has a normal subgroup H such that G/H is discrete (hence, locally q-minimal), then G is locally q-minimal (resp., locally q^*-minimal, locally t-minimal) whenever the subgroup H is locally q-minimal (resp., locally q^*-minimal, locally t-minimal). We shall see in Example 4.9 that the counterpart of this property, when H is supposed to be discrete and G/H locally q-minimal, fails.

In the sequel, for a group X a subgroup $G \leq X$ and a topology τ on G the symbol τ/G stand for the quotient topology on X/G with respect to τ.

Lemma 3.8. *Let X be a group and $G \subset X$ be a subgroup. Let τ, σ be group topologies such that $\sigma \subset \tau$, $\sigma \restriction_G = \tau \restriction_G$, and $\sigma/G = \tau/G$. Then $\sigma = \tau$.*

Lemma 3.9. *Let (G, τ) be a Hausdorff topological group and K a closed normal subgroup of G. Suppose that for some neighbourhood U of the identity of G the group K is locally minimal with respect to $U \cap K$ and UK/K witness local minimality of G/K. Then for any Hausdorff group topology σ which is coarser than τ, if U is a σ-neighbourhood of the identity and K is σ-closed, then $\sigma = \tau$.*

Proof. To conclude that $\sigma = \tau$ we intend to apply Lemma 3.8. To this end we use local minimality of $(K, \tau \restriction_K)$ w.r.t. $U \cap K$ and the fact that $U \cap K \in \sigma \restriction_K \leq \tau \restriction_K$. This gives $\sigma \restriction_K = \tau \restriction_K$. Now consider $\sigma/H \leq \tau/H$. Since K is σ-closed, it is Hausdorff. Moreover, as $U \in \sigma$, we deduce that $UK/K \in \sigma/K$. By the local minimality of G/K w.r.t. UK/K we deduce that $\sigma/H = \tau/H$. Hence, we get $\sigma = \tau$ applying a well-known fact (Merzon lemma, see [19, Lemma 1]).

□

Now we show that the 3-space property is available both for local minimality and for local q^*-minimality under appropriate natural conditions. Recall that a group is called *totally complete* if every Hausdorff quotient group is complete.

Theorem 3.10. *If H is a totally complete subgroup of G such that both H and G/H are locally (q^*-)minimal, then G is also locally (q^*-)minimal.*

Proof. First we prove the version with local minimality and in this case it is enough the have H simply complete.

Suppose that U is a neighbourhood of the identity of G such that K is locally minimal with respect to $U \cap K$ and UK/K witness local minimality of G/K. We are going to prove that G is U-locally minimal, by using apply Lemma 3.9. Denote by τ the original topology on G and σ another Hausdorff group topology on G such that $\sigma \subset \tau$ and U is a σ-neighbourhood of the identity. Clearly $U \cap K$ is a neighbourhood of the identity of $K \leq (G, \sigma)$, hence the local minimality of K implies that $\sigma \restriction_K = \tau \restriction_K$. As $(K, \tau \restriction_K) = (K, \sigma \restriction_K)$ is complete, it is σ-closed in (G, σ). Hence, we can apply Lemma 3.9 to complete the this part of the proof.

Now we pass to the proof of the stronger property of local q^*-minimality. Without loss of generality, we can choose a neighbourhood U of 1 in G such that $U^2 \cap H$ witnesses local q^*-minimality of H and UH/H witnesses local q^*-minimality of G/H. We will show that G is locally q^*-minimal with respect to U. Take a closed normal subgroup N of G contained in U, it suffices to prove that G/N is locally minimal with respect to $\pi(U)$, where $\pi : G \to G/N$ is the quotient homomorphism. By local q^*-minimality of H, $\pi \restriction_H$ is open, i.e. $\pi(H) \cong H/(H \cap N)$.

Therefore, $\pi(H)$ is locally minimal with respect to $\pi(U^2 \cap H) \supset \pi(UN \cap H) = \pi(U) \cap \pi(H)$. Since H is totally complete, $\pi(H)$ is complete. Hence, $\pi(H)$ is a closed normal subgroup of $\pi(G) = G/N$. Note that $\pi(G)/\pi(H) \cong G/NH \cong (G/H)/(NH/H)$. Since $N \subset U$, $NH/H \subset UH/H$. By local q^*-minimality of G/H, $\pi(G)/\pi(H)$ is locally minimal with respect to $\pi(U)\pi(H)/\pi(H)$. So, according to Theorem **??**, $\pi(G)$ is locally minimal with respect to $\pi(U)$. □

4. Local q-minimality vs divisibility of groups

For an abelian group G, we define the subgroups

$$\nu(G) = \bigcap_{n \in \mathbb{N}} nG \quad \text{and} \quad \pi(G) = \bigcap_{p} pG.$$

Obviously,

$$d(G) \subseteq \nu(G) \subseteq \pi(G),$$

where $d(G)$ is the maximum divisible subgroup of G. Moreover, $\nu(G) = d(G)$ when G is torsion-free. On the other hand, $G = \pi(G)$ implies that $G = d(G)$ is divisible.

We will give a example to show that a locally t-minimal group need not to be locally q-minimal. In order to produce the examples we need the following lemma:

Lemma 4.1. *If a dense subgroup G of $\mathbb{R}$ is not divisible, then for any $\varepsilon > 0$, there exists an element g in G such that $|g| < \varepsilon$ and $g \notin pG$ for some prime p.*

Proof. Fix an $\varepsilon > 0$. As $U = (-\varepsilon, \varepsilon) \cap G$ is dense in $(-\varepsilon, \varepsilon)$, it suffices to show that the set

$$X := \bigcup_{p} G \setminus pG = G \setminus \bigcap_{p} pG = G \setminus \pi(G)$$

is dense in G (hence, in U as well). Our hypothesis $G \neq d(G)$ yields $G \neq \pi(G)$, hence there exists $g \in G \setminus \pi(G)$. Obviously, the coset $Y = g + \pi(G)$ is contained in X.

Consider now two cases. If $\pi(G)$ is dense in G, then obviously Y will be dense in G, hence $X \supseteq Y$ will be dense as well.

If $\pi(G)$ is not dense in G, then it is not dense in $\mathbb{R}$ either, so $\pi(G)$ is cyclic. Therefore, $\pi(G)$ is a closed set with empty interior of G, hence $X = G \setminus \pi(G)$ is dense in G. □

Proposition 4.2. *A dense subgroup of $\mathbb{R}$ endowed with the usual topology is locally q-minimal iff it is divisible.*

Proof. First, assume that G is locally q-minimal with respect to $U = W \cap G$, where W is a connected neighbourhood of 0 in $\mathbb{R}$. We can assume without loss of generality that $W = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Assume that G is not divisible. Then there exists $g \in U$ such that

$$g \notin pG \tag{1}$$

for some prime p, by Lemma 4.1. Let $N = \langle g \rangle$. Then $N + W = \mathbb{R}$. Moreover, the local q-minimality of G with respect to U yields that G/N is locally minimal with respect to

$$(U + N)/N = ((W \cap G) + N)/N = G/N,$$

hence G/N is minimal. Since G/N is dense in $\mathbb{R}/N$, which is topological isomorphic to $\mathbb{T}$, G/N must contain the socle of $\mathbb{R}/N$. In particular, G/N must contain an element $\bar{x} = x + N$ ($x \in G$) of order p. Then $px \in N$, while $x \notin N$. So, there exists a positive integer n such that $px \in ng$ and n is co-prime with p, i.e. $(n, p) = 1$. Hence, there exists an integer u such that $un \equiv 1 (mod\ p)$. Then $ung = upx \in pG$ and $(un - 1)g \in pG$ which implies that $g \in pG$, in contradiction to (1). A contradiction.

Conversely, suppose that $G \leq \mathbb{R}$ is divisible. We claim that G is locally q-minimal with respect to $U := (-1,1) \cap G$. Let N be a closed subgroup of G, we may assume that $\{0\} \neq N \neq G$. While, since each nontrivial closed proper subgroup of $\mathbb{R}$ is cyclic, N is a subgroup of an infinite cyclic group, so, N itself is an infinite cyclic group. We identify G/N with a subgroup of $\mathbb{T}$, then the divisibility of G and $N \cong \mathbb{Z}$ imply that G/N contains the torsion part of $\mathbb{T}$. Hence G/N is minimal, this implies that G/N is locally minimal with respect to $(U+N)/N$. □

Now we are in position to produce an example of a locally t-minimal group that is not locally q-minimal.

Example 4.3. *Let p be a prime and let G be the subgroup of $\mathbb{R}$ defined as follows*

$$\{r \in \mathbb{R} : p^n r \in \mathbb{Z} \text{ for some } n \in \mathbb{N}\}.$$

Since G is a dense non-divisible subgroup of $\mathbb{R}$, G is not locally q-minimal, by Proposition 4.2. On the other hand, G is locally t-minimal, by Example 2.11 (a).

Remark 4.4. *Proposition 4.2 also produces an example which shows that a topological group with a dense locally q-minimal subgroup need not to be locally q-minimal.*

In fact, let $G = \langle a \rangle + \mathbb{Q}$ be the subgroup of $\mathbb{R}$, where a is an irrational number. Then $\mathbb{Q}$ is dense in G and locally q-minimal, while G is not locally q-minimal since G is not divisible. Note that G is also an example that a locally t-minimal group fails to be locally q-minimal.

This example shows that a subgroup of a Banach space need not be locally q-minimal. Since G is a UFSS group (as a subgroup of $\mathbb{R}$), this example also means that there exists an UFSS group which is not locally q-minimal.

Proposition 4.2 gives a necessary and sufficient condition for a dense subgroup of $\mathbb{R}$ to be locally q-minimal. A natural question is to consider the high-dimensional situation, i.e. for $n > 1$, what about the dense locally q-minimal subgroup of $\mathbb{R}^n$? The following theorem gives a positive answer to this question.

Theorem 4.5. *Let n be a positive integer. A dense subgroup of $\mathbb{R}^n$ is locally q-minimal iff it is divisible.*

Proof. Necessity. Suppose that G is a dense subgroup of $\mathbb{R}^n$ that is not divisible.

We first note that G has no proper open subgroup. Indeed, if H is an open subgroup of G, then $H = O \cap G$, where O is an open subset of $\mathbb{R}^n$. Hence, $\overline{O} = \overline{H}$ is a subgroup of $\mathbb{R}^n$ that contains the open set O. So $\overline{O}$ is an open subgroup and hence a closed subgroup of $\mathbb{R}^n$, by the connectedness of $\mathbb{R}^n$ we have $\overline{O} = \mathbb{R}^n$. This implies that H is dense in $\mathbb{R}^n$, so dense in G. As H is also closed in G, H must coincide with G.

Assume that G is locally q-minimal with respect to $V = W \cap G$, where W is an open neighbourhood of 0 in $\mathbb{R}^n$. Take a convex open neighbourhood U of $\mathbb{R}^n$ such that

$$\underbrace{U + U + ... + U}_{n \text{ times}} \subset W.$$

Since $\pi(G) := \bigcap_{p \in \mathbb{P}} pG \neq G$, $\pi(G)$ is not an open subgroup of G. So $U \cap G \not\subseteq \pi(G)$. Then there exists a prime p with $U \cap G \not\subseteq pG$. Pick an element $g_1 \in (U \cap G) \setminus pG$. Since $\overline{G \cap U} = \overline{U}$ has a non-empty interior in $\mathbb{R}^n$, we can choose $g_2, g_3, ..., g_n \in G \cap U$ such that $\{g_1, g_2, ..., g_n\}$ forms a basis of the vector space $\mathbb{R}^n$. Let $N_i = \langle g_i \rangle \subset L_i$ and denote by L_i the linear hull of g_i for $i = 1, 2, ..., n$. Then $N_i = \langle g_i \rangle \subset L_i$ and $\mathbb{R}^n$ can be identified with $\prod_{i=1}^n L_i$ and each point in $\mathbb{R}^n$ can be represented as $a_1 g_1 + a_2 g_2 + ... + a_n g_n$ uniquely, where a_i is a real number for each

$i = 1, 2, ..., n$. The group $N := \prod_{i=1}^n N_i$ is a closed subgroup of both $\mathbb{R}^n$ and G. Moreover,

$$\mathbb{R}^n/N = \prod_{i=1}^n T_i, \quad \text{where} \quad T_i = L_i/N_i \cong \mathbb{T} \quad \text{for each } i.$$

We claim that that $W + N = \mathbb{R}^n$. It suffices to show that if $x = \sum_{i=1}^n a_i g_i$ with $a_i \in [0, 1)$ for each i, then $x \in W$. Since U is convex and $g_i \in U$, we have that $a_i g_i \in U$, for $i = 1, ..., n$. Hence

$$x = a_1 g_1 + a_2 g_2 + ... + a_n g_n \in \underbrace{U + U + ... + U}_{n \text{ times}} \subset W.$$

We now prove that $(W \cap G) + N = G$. To show the equation we only need to prove that $G \subset (W \cap G) + N$. Fix $g \in G$, then there exits $x \in W$ and $y \in N$ such that $g = x + y$ since $W + N = \mathbb{R}^n$. So, $x \in G + N = G$, i.e. $x \in W \cap G$. Hence, $g = x + y \in (W \cap G) + N$. Since G is locally q-minimal with respect to V, G/N is locally minimal with respect to $(V + N)/N = ((W \cap G) + N)/N = G/N$, hence, it is minimal. A similar argument with that in Proposition 4.2 shows that G/N does not contain the cyclic subgroup of order p of T_1. So G/N is not essential in $\prod_{i=1}^n L_i$, a contradiction.

Sufficiency. Let G be a divisible dense subgroup of $\mathbb{R}^n$ and U a bounded neighbourhood of 0 in $\mathbb{R}^n$. Take a neighbourhood U_1 of 0 in $\mathbb{R}^n$ such that $U_1 + U_1 \subset U$. Let $V = U_1 \cap G$. We will see that G is locally q-minimal with respect to V. Let N be a closed subgroup of G contained in V. Consider the linear hull of N, clearly it is topologically linearly isomorphic to $\mathbb{R}^m$ for some positive integer $m \le n$.

If $m = n$, then N contains a subset $P = \{x_1, x_2, ..., x_n\}$ that is a basis of $\mathbb{R}^n$, i.e. each element in $\mathbb{R}^n$ can be represented as the form $a_1 x_1 + a_2 x_2 + ... + a_n x_n$ uniquely, where $a_1, a_2, ..., a_n \in \mathbb{R}$. This implies that $\mathbb{R}^n$ can be identified with $\prod_{i=1}^n L_i$, where $L_i = \mathbb{R}x_i$. Put $N' = \langle P \rangle$, then N' is a discrete (hence closed) subgroup of both G and $\mathbb{R}^n$. Note that $\mathbb{R}^n/N'$ is naturally topologically isomorphic to $\prod_{i=1}^n T_i$, where $T_i = L_i/\langle x_i \rangle \cong \mathbb{T}$ for each i. Since G is divisible, G/N' contains the torsion part of $\mathbb{R}^n/N'$. So G/N' is totally minimal. Then $G/N \cong (G/N')/(N/N')$ is minimal, so locally minimal with respect to $(V + N)/N$.

Now we consider the case $m < n$. Similarly, we can choose a subset $P = \{x_1, x_2, ..., x_m\}$ of N such that P is a basis of X, where X is the linear hull of N. Since G is dense in $\mathbb{R}^n$, the linear hull of G is exactly $\mathbb{R}^n$, so we can choose $Q = \{y_1, y_2, ..., y_{n-m}\} \subset G$ such that $P \cup Q$ is a basis of $\mathbb{R}^n$. Let Y be the linear hull of Q, then $\mathbb{R}^n$ can be identified with $X \times Y$. Since G is divisible, the subgroup $P' = \{q_1 x_1 + q_2 x_2 + ... + q_m x_m : q_i \in \mathbb{Q}\}$ of X is contained in G. Clearly P' is dense in X, so $G_1 := G \cap X$ is dense in X. Similarly, $G_2 := G \cap Y$ is dense in Y. Further, both G_1 and G_2 are divisible since G, X and Y are divisible and $\mathbb{R}^n$ is torsion-free. Let $N' = \langle P \rangle$, then $N' \subset N \subset G \cap X = G_1$. A similar argument with the case $m = n$ shows that G_1/N' is totally minimal, so G_1/N is minimal. Hence, G_1/N is essential in X/N. Let π be the natural projection of $X \times Y$ onto Y. Then $W = \pi(U)$ is bounded since U is bounded. Clearly, $U \subset X \times W$.

We claim that $G_1/N \times G_2$ is locally essential with respect to $X/N \times W$ in $X/N \times Y$. Indeed, Y is NSS with respect to W, so any closed subgroup K of $X/N \times Y$ contained in $X/N \times Y$ is also contained in $X/N \times \{0\}$. Then the essentiality of G_1/N in X/N implies that K intersects $G_1/N \times \{0\}$ non-trivially. Since $G_1 \times G_2 \subset G$, G/N is also locally essential with respect to $X/N \times W$. Note that $X/N \times W$ also witnesses local minimality of $X/N \times Y$ (since X/N is compact as a quotient group of X/N' and $Y \cong \mathbb{R}^{n-m}$). Further, by the choice of V, we obtain that

$$(V+N)/N+(V+N)/N=(V+V+N)/N\subset(U+N)/N\subset(X\times W+N)/N=X/N\times W.$$

According to (1) of Remark 2.7, G/N is locally minimal with respect to $(V+N)/N$. □

At this point we can apply all this observation to describe which (not necessarily DENSE) subgroups G of $\mathbb{R}^n$ are locally q-minimal.

Indeed, assume that G is a subgroup of R^n. Then its closure K in $\mathbb{R}^n$ is locally compact and isomorphic to $\mathbb{R}^m\times\mathbb{Z}^k$ (let us identify for simplicity $K=\mathbb{R}^m\times\mathbb{Z}^k$), so $o(G)=G\cap K$. Now $o(G)$ is dense in $\mathbb{R}^m$, so $o(G)$ is locally minimal if and only if o(G) is divisible. This gives the following corollary of Thm. 5.5 that reinforces the theorem itself:

Corollary 4.6. *If G is a subgroup of $\mathbb{R}^n$, then G is locally q-minimal iff $o(G)$ is divisible.*

Unfortunately, one cannot go too far with this since even $\mathbb{T}$ has plenty of dense divisible subgroups that are not locally q-minimal. probably this will easily give dense divisible subgroups of the compact group $Q^\wedge$ that are not locally q-minimal. (Actually, one can see with "plain eye" dense subgroups of $Q^\wedge$ that are divisible and not even q^*-minimal, i.e., not totally locally dense).

In the next proposition we characterise the infinite locally q-minimal and non-totally minimal subgroups of $\mathbb{T}$ as those having finite torsion part (recall that an infinite subgroup $\mathbb{T}$ is totally minimal precisely when it contains $t(\mathbb{T})=\mathbb{Q}/\mathbb{Z}$).

In the next proof we put for brevity $V_n=(-1/4n,1/4n)+\mathbb{Z}$ in $\mathbb{T}$ and note that V_1 witnesseses the NSS property of $\mathbb{T}$.

Proposition 4.7. *A dense subgroup G of $\mathbb{T}$ is locally q-minimal iff it satisfies one of the following two conditions:*

(1) G is totally minimal;

(2) The torsion part T of G is finite.

Proof. We split the proof of the sufficiency into two cases according to (1) and (2).

Case 1. G is totally minimal. Then G is locally q-minimal with respect to G.

Case 2. The torsion part T of G is finite. Let $n:=|T|$. We claim that G is locally q-minimal with respect to $U=V_n\cap G$. Let N be a proper closed subgroup of G. Then N is finite cyclic, let $m=|N|$, so $m|n$. The map f of $\mathbb{T}/N$ onto $\mathbb{T}$ defined by $f(x+N)=mx$, for any $x\in\mathbb{T}$, is an topological isomorphism (it is well defined, as $x+N=y+N$ yields $x-y\in N$, so $mx=my$). So we can identify G/N with the dense subgroup mG of $\mathbb{T}$, then $UN/N\subset mV_n=(-m/4n,m/4n)+\mathbb{Z}\subset(-1/4,1/4)$

Since $(-1/4,1/4)$ witnesses local essentiality of G/N in $\mathbb{T}$, according to Remark 2.7 (2), G/N is locally minimal with respect to UN/N.

Necessity. Assume that G satisfies neither of the conditions (1) and (2), and G is locally q-minimal with respect to a connected neighbourhood U of 1. In particular, T is infinite, hence dense. Therefore, we can choose $a\in G$ of finite order such that $\langle a\rangle U=G$. By the negation of case 2, there exists p such that G does not contain $\mathbb{Z}(p^\infty)$, so $G\cap\mathbb{Z}(p^\infty)=\mathbb{Z}(p^n)$ for some $n\in\mathbb{N}$. We can choose the torsion element $a\in G$ with the additional property $\langle a\rangle\cap\mathbb{Z}(p^\infty)=G\cap\mathbb{Z}(p^\infty)=\mathbb{Z}(p^n)$. Then $G/\langle a\rangle$ has no non-trivial p-torsion elements, while its completion $\widetilde{G/\langle a\rangle}$ is isomorphic to $\mathbb{T}$. Hence, $G/\langle a\rangle$ is not minimal, i.e., not $(\langle a\rangle U)/\langle a\rangle$-locally minimal. This proves that G is not locally minimal w.r.t. U. □

Example 4.8. *Let G be a quasi-cyclic subgroup of $\mathbb{T}$ with the usual compact topology. Proposition 4.7 shows that G is not locally q-minimal. According to Example 2.11(a), every Hausdorff quotient group of G is locally minimal. This example shows that a locally t-minimal group, even it is precompact and divisible, needs not to be locally q-minimal.*

The next example shows that local q-minimality has not a three-space-property of such form.

Example 4.9. *Let H be the discrete subgroup $\langle\sqrt{2}\rangle$ of $\mathbb{R}$. Let $G = \mathbb{Q} \oplus H < \mathbb{R}$, then H is an totally complete and locally q-minimal subgroup of G. Further, G/H is topologically isomorphic to a torsion-free dense subgroup of $\mathbb{T}$, so G/H is also locally q-minimal. While, G is not locally q-minimal since it is not divisible.*

5. Local q^*-minimality criterion

5.1. **Local t-density and the local q^*-minimality criterion.** As shown in [12] (see also [8, Theorem 2.6]), a dense subgroup H of G is totally minimal iff G is totally minimal and H is totally dense in G. This theorem is called *criterion of total minimality.* The criterion of total minimality implies that a topological group containing a dense total minimal subgroup must by total minimal on its own account. Remark 4.4 implies that a similar criterion for local q-minimality cannot be available.

We will give a criterion for local q^*-minimality.

It is easy to check that the Hausdorff group G is locally q^*-minimal iff there exists a neighbourhood V of the identity such that G/N is $\pi(V)$-locally minimal for each closed normal subgroup N of H contained in V, where π is the natural quotient mapping of G onto G/N. Hence, a locally q-minimal group is locally q^*-minimal.

Lemma 5.1. *If a topological group G is locally q-minimal (resp. locally q^*-minimal) with respect to U^2, then G/N is locally minimal with respect to $\overline{UN/N}$ for any closed normal subgroup N (resp. for any closed normal subgroup $N \subset U$) of G.*

Proof. We prove the case of local q-minimality, the other is similar.

Since G is locally q-minimal with respect to U^2, G/N is locally minimal with respect to $U^2N/N = (UN/N)^2 \supset \overline{UN/N}$. The last conclusion is from the openness of the quotient mapping of G onto G/N. □

Definition 5.2. *A dense subgroup H is a topological group G is called* locally t-dense *if there exists a neighbourhood $V \in \mathcal{V}_G(1)$ such that $H \cap N$ is dense in N for every closed normal subgroup N of G contained in V.*

In an NSS group every dense subgroup is obviously locally t-dense.

The next proposition shows that every locally t-dense subgroup H of K is actually totally dense and consequently coincides with K.

Proposition 5.3. *Let K be a compact torsion abelian group. Then every locally t-dense subgroup H of K coincides with K.*

Proof. Indeed, let U be the neighbourhood of 0 witnessing the local t-density of H. Since K has a local base of open subgroups, we can assume slog that U is an open subgroup of K. Moreover, as $K = \prod_{i\in I} C_i$ is a topological product of finite cyclic groups C_i, we can assume (by further shrinking U), that U is a direct summand of K, i.e., $K = F \times U$, where F is a finite group. Now $H_1 = H \cap U$ is dense in U and the local t-density of H w.r.t. U means that H_1 is totally dense in U. Since U_1 is torsion, total density of H_1 implies $H_1 = U$. This proves that H contains U. Consequently, H itself is open and consequently also closed. Therefore, $H = K$. □

Compactness plays a relevant role in this proposition. Indeed, the torsion group $\mathbb{Q}/\mathbb{Z}$ has plenty of proper dense subgroups and they are all locally t-dense as $\mathbb{Q}/\mathbb{Z}$ is NSS.

Theorem 5.4. *A dense subgroup H of a Hausdorff group G is locally q^*-minimal iff G is locally q^*-minimal and H is locally t-dense in G.*

Proof. First we assume that H is locally q^*-minimal with respect to a neighbourhood V^2 of the identity e in H, where $V = W \cap H$ and W is a closed neighbourhood of the identity in G. Take a neighbourhood U of e in G such that $U^2 \subset W$. We prove that U witnesses both local q^*-minimality of G and local t-density of H in G.

Let N be a closed normal subgroup of G contained in U and N' the closure of $N \cap H$. Then N' is normal in G. Denote by ψ the quotient mapping of G onto G/N and by π the quotient mapping of G onto G/N'. Then we can identify G/N with the quotient group of G/N' with respect to the closed normal subgroup N/N' of G/N'. Let p be the above quotient mapping of G/N' onto G/N. Clearly, $\psi = p \circ \pi$. Since $H \cap N' = H \cap N$ is dense in N', the quotient mapping π remains open when restricted to H, hence we identify $H/(H \cap N)$ with the dense subgroup $\pi(H)$ of G/N'. By the local q^*-minimality assumption of H and Lemma 5.1, $\pi(H)$ is $\overline{\pi(V)} \cap \pi(H)$-locally minimal, where $\overline{\pi(V)}$ is the closure of $\pi(V)$ in G/N'. Clearly, $W \subset \overline{V}$ yields $\pi(W) \subset \pi(\overline{V}) \subset \overline{\pi(V)}$. According to Remark 2.7 (1), G/N' is locally minimal with respect to $\overline{\pi(V)}$, so locally minimal with respect to $\pi(W)$ and $\pi(U)$. Moreover, the inclusion $\pi(U)^2 = \pi(U^2) \subset \pi(W)$ implies that $\pi(H)$ is locally essential in G/N' with respect to $\pi(U)$. Therefore, the proofs of both the local q^*-minimality and local t-density will be complete if we show that $N = N'$(i.e., p is a topological isomorphism). Denote by K the kernel N/N' of p. We aim to show that

$$K \cap \pi(H) = \{e_q\}. \tag{2}$$

Take $h \in H$ such that $\pi(h) \in K \cap \pi(H) = \pi(N) \cap \pi(H)$, then

$$h \in NN' \cap H = N \cap H \subset N' = \ker \pi,$$

hence $\pi(h) = \{e_q\}$, where $\{e_q\}$ is the identity of G/N'. Since $K = \pi(N) \subset \pi(U)$, this proves (2). Hence, by the $\pi(U)$-local essentiality of $\pi(H)$ in G/N', K is trivial, which implies that $N = N'$.

Now we assume that G is locally q^*-minimal with respect to a neighbourhood U of the identity and H is locally t-dense in G with respect to U. Take neighbourhoods W, W' of the identity in G such that $W^2 \subset U$, $W'^2 \subset W$, and let $V = W' \cap H$. We are going to prove that H is locally q^*-minimal with respect to V.

Let N be a closed normal subgroup of H such that $N \subset V$.Denote by $\overline{N}$ the closure of N in G. Then the natural quotient mapping of H onto H/N can be extended to the quotient mapping $\pi : G \to G/\overline{N}$ when we identify H/N with the dense subgroup $\pi(H)$ of $G/\overline{N}$. The assumption that $N \subset V$ implies that $\overline{N} \subset \overline{V} = \overline{W'} \subset W \subset U$. Then $G/\overline{N}$ is locally minimal with respect to $\pi(U)$, hence, with respect to $\pi(W)$, by our assumption. We claim that $\pi(H)$ is locally essential in $G/\overline{N}$ with respect to $\pi(W)$. Take a closed normal subgroup $K \subset \pi(W)$ of $\pi(H)$ such that $K \cap \pi(H) = \{e_q\}$, where e_q is the identity of $G/\overline{N}$. Then $\pi^{-1}(K)$ is a closed normal subgroup of G and

$$\pi^{-1}(K) \subset W\overline{N} \subset WW \subset U.$$

So, $\pi^{-1}(K) \cap H$ is dense in $\pi^{-1}(K)$. Therefore $\{e_q\} = K \cap \pi(H)$ is dense in K, which implies that $K = \{e_q\}$. Hence we finished the proof of the local essentiality. Moreover, it is clear that $\pi(W')^2 = \pi(W'^2) \subset \pi(W)$. Again, Remark 2.7 (2) shows that $\pi(H)$ is locally minimal with respect to $\pi(W') \cap \pi(H)$, so to $\pi(W' \cap H) = \pi(V)$. □

We give now a result about totally dense subgroups.

Proposition 5.5. *A totally dense subgroup H of G is locally q-minimal iff G is locally q-minimal.*

Proof. We first assume that U is a neighbourhood of the identity of G such that H is locally q-minimal with respect to V^2, where V denotes $U \cap H$. Let W be a neighbourhood of the identity in G such that $W^2 \subset U$, we are going to prove that G is locally q-minimal with respect to W. Let N be a closed normal subgroup of G and $N' = H \cap N$. By the total density of H in G, we can identity H/N' with the dense subgroup HN/N of G/N. Denote by π the natural quotient mapping of G on to G/N.

Since H is locally q-minimal with respect to V^2, Lemma 5.1 implies that $\pi(H)$ is locally minimal with respect to $\overline{\pi(V)} \cap \pi(H)$, where $\overline{\pi(V)}$ is the closure of $\pi(V)$ in G/N. According to Remark 2.7 (1), G/N is locally minimal with respect to $\overline{\pi(V)}$, so it suffices to prove that $\pi(W) \subset \overline{\pi(V)}$. Since $\pi(H)$ is dense in G/N, $\pi(W) \subset \overline{\pi(W) \cap \pi(H)}$. Therefore, it is enough to check that $\pi(W) \cap \pi(H) \subset \pi(V) \subset \overline{\pi(V)}$. Since, $N = \overline{N'}$ and $N' \subset H$, we have the following chain of inclusions:

$$\pi(W)\cap\pi(H) = \pi(WN\cap H) \subset \pi(WWN'\cap H) = \pi((W^2\cap H)N') \subset \pi((W^2\cap H)N) = \pi(W^2\cap H) \subset \pi(V).$$

Conversely, assume that G is locally q-minimal with respect to U. Choose a neighbourhood W of the identity in G such that $W^2 \subset U$, let $V = W \cap H$. We claim that V witness local q-minimality of H. Let N' be a closed normal subgroup of H and N the closure of N', then N is a closed normal subgroup of G. Denote by π the natural quotient mapping of G onto G/N, clearly $\pi(H) = H/N'$ is dense in G/N. Moreover, since H is totally dense in G, $\pi(H)$ is also totally dense, hence locally essential with respect to any neighbourhood of the identity, in G/N. By the U-local q-minimality assumption of G we know that G/N is locally minimal with respect to $\pi(U)$. Since $\pi(V) \subset \pi(W) \cap \pi(H)$, according to Remark 2.7 (2), it suffices to prove that $\pi(W)^2 \subset \pi(U)$. This obviously follows from the choice of W. □

Since local minimality and local q^*-minimality coincide on NSS-groups and since every subgroup of a Lie group is locally minimal, we conclude (as in Example 2.11 (a)) that every subgroup of a Lie group is locally q^*-minimal.

5.2. **Applications of the local q^*-minimality criterion.** The next example was given in [11, Example 2.10] to show the difference between local minimality and local q-minimality. We will see that the group in this example is not even locally q^*-minimal either. Moreover, it can be used to produce a divisible locally minimal abelian group that is not locally q^*-minimal. (According to [7, Proposition 2.1], all divisible minimal abelian groups are totally minimal, this motivates the question of whether all divisible locally minimal abelian groups are locally q*-minimal.)

Example 5.6. Let $c = (a_p)_{p\in\mathbb{P}}$ be a topological generator of the compact monothetic group $K = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$.

(a) Consider the subgroups $N = \prod_{p\in\mathbb{P}} p\mathbb{Z}_p$ and $G = \langle c\rangle + N$ of K. Then G is dense in K and minimal, hence, locally minimal ([11, [Example 2.10]). Let us see that G is not locally q^*-minimal. Indeed, if G were locally q^*-minimal, then G would be locally t-dense with respect to some neighbourhood U of the identity in K. One can choose $p \in \mathbb{P}$ such that $\mathbb{Z}_p \subset U$. Theorem 5.4 implies that $G \cap \mathbb{Z}_p$ is dense in $\mathbb{Z}_p$. While, $G \cap \mathbb{Z}_p = p\mathbb{Z}_p$, a contradiction.

(b) Consider the topological group (G, τ) introduced in (a). Let H be divisible hull of G and let τ^* be the topology (standard extension of τ) on H defined in Fact **??**. By Proposition

3.1 (b), local q^*-minimality is stable under taking open subgroups in abelian groups. Hence, we deduce that (H, τ^*) is not locally q^*-minimal, as G is not locally q^*-minimal, by item (a). However, (H, λ) is locally minimal since it contains the open locally minimal subgroup G (see [1, Proposition 2.4]).

Since any locally q-minimal group is locally q^*-minimal, we get the following corollary immediately:

Corollary 5.7. *If H is a dense locally q-minimal subgroup of a Hausdorff topological group G, then H is locally t-dense in G.*

Notice that the converse of Corollary 5.7 is not true, Example 4.8 provides a counter example. Another corollary is obtained by making use of Example 5.3.

Corollary 5.8. *A locally q^*-minimal precompact torsion abelian group is compact.*

Remark 4.4 and Example 4.8 together show that we can not obtain a criterion of local q-minimality for dense subgroups (at last in the same format as the criterions for (total) minimality).

We saw that local t-minimality and local q^*-minimality are both strictly weaker than local q-minimality. (By Proposition 4.2 and Theorem 4.5, any dense non-divisible subgroup G of $\mathbb{R}$ is not locally q-minimal, while it is both locally t-minimal and locally q^*-minimal.)

The following example shows that a locally q^*-minimal abelian group needs not to be locally t-minimal, i.e., has a non-locally minimal quotient. This shows that local q^*-minimality, unlike local q-minimality and local t-minimality, is not preserved by taking quotients.

Example 5.9. The Hilbert space ℓ^2 considered as a topological abelian group ℓ^2 is UFSS (see [1, Example 3.14]). Hence, every subgroup of ℓ^2 is also UFSS (see [1, Lemma 3.12(b)]), so locally q^*-minimal. Let $\{e_n : n \in \mathbb{N}\}$ be the canonical basis of ℓ^2. Take a prime p, let P be the dense subgroup of the $\mathbb{R}$ generated by $\{\frac{1}{p^n}, n \in \mathbb{N}\}$. Following [1, Example 3.14], $H = \overline{\langle\{\frac{1}{p^n}e_n : n \in \mathbb{N}\}\rangle}$ and Consider the group $G = \{(x_n) \in \ell^2 : x_n \in P\} = P^{\mathbb{N}} \cap \ell^2$. We prove that for the closed subgroup $N := H \cap G$ of G the quotient G/N is not locally minimal.

(a) We prove first that G is dense in ℓ^2. Indeed, fix $y = (y_n) \in \ell^2$ and $\varepsilon > 0$, by $\overline{P} = \mathbb{R}$, we can choose $x_n \in P$ such that $|x_n - y_n| < \frac{\varepsilon}{2^n}$ for each $n \in \mathbb{N}$. Since

$$\sqrt{\sum_{n\in\mathbb{N}}(x_n - y_n)^2} < \sqrt{\sum_{n\in N}\left(\frac{\varepsilon}{2^n}\right)^2} \leq \frac{\varepsilon}{\sqrt{3}} < \varepsilon,$$

we deduce that $z := (x_n - y_n) \in \ell^2$, so $x = (x_n) = z + y \in \ell^2$, and hence, $x \in G$. The former inequality also implies that $||x - y|| < \varepsilon$, thus G is dense in ℓ^2.

(b) Denote by π the natural projection of ℓ^2 onto ℓ^2/H. As N is dense in H, the subgroup $\pi(G)$ of ℓ^2/N is naturally topologically isomorphic to G/N, according to [14, Lemma 4.3.2].

(c) We now show that $\pi(G)$ is not locally minimal. Indeed, if $\pi(G)$ were locally minimal, there must exits $\varepsilon > 0$ such that each closed subgroup of $\pi(G)$ contained in $\pi(\varepsilon B)$ is minimal, by Corollary 2.2, where B is the unit ball in ℓ^2.

An argument similar to that in [1, Example 3.14] shows that there exists a positive integer k_0 such that $\pi(S) \subset \pi(\varepsilon B)$, where S is the linear hull of the set $\{e_k : k > k_0\}$. Let k be an integer such that $k > k_0$, then $\pi(Pe_k) \subset \pi(\mathbb{R}e_k) \subset \pi(S) \subset \pi(\varepsilon B)$. We note that $\pi(\mathbb{R}e_k)$ is closed in $\pi(\ell^2)$ since $\pi(\mathbb{R}e_k)$ is topologically isomorphic to $\mathbb{T}$. Moreover, $\ker \pi \restriction_{\mathbb{R}e_k} = \langle \frac{1}{p^k}e_k\rangle \subset G$. So $\pi(Pe_k) = \pi(G) \cap \pi(\mathbb{R}e_k)$, hence, it is a closed subgroup of $\pi(G)$. This implies that $\pi(Pe_k)$ is minimal, by Corollary 2.2. Since $\pi(Pe_k)$ is topologically isomorphic to the p-torsion part of $\mathbb{T}$, so $\pi(Pe_k)$ is not essential in its completion, a contradiction.

6. Local q^*-minimality combined with other compactness properties

A topological group G is said to be *sequentially complete* is every Cauchy sequence in G is convergent (equivalently, when G is sequentially closed in its Raĭmov completion). Clearly, complete groups are sequentially complete, so the latter is a rather weak compactness-like property. Now we shall combine it with pseudocompactness.

Theorem 6.1. *Every sequentially complete locally q^*-minimal pseudocompact abelian group is compact.*

Proof. Assume that G is a sequentially complete locally q^*-minimal pseudocompact abelian group. Since pseudocompact groups are precompact, the completion K of G is compact. By Theorem 5.4, G is locally t-dense in K and let U be a neighbourhood of 0 in K witnessing that. By the structure theory of compact groups, one can find a closed subgroup N of K contained in U such that K/N is metrizable (actually, one can have it even a Lie group). Then the subgroup N of K is a G_δ-set. By Comfort-Ross' criterion for pseudocompactness, we deduce that G is G_δ-dense in K. In particular, the subgroup $G_1 = N \cap G$ of G is G_δ-dense in N. On the other hand, G_1 is closed in G, hence G_1 is sequentially complete. Next we note that G_1 is totally dense in its completion N, by the local t-density of G w.r.t. U. Hence, G_1 is totally minimal and sequentially complete, hence compact, according to [9, Theorem 3.4]. This proves that $G_1 = N$, so $N \leq G$. Since G is G_δ-dense and N is a G_δ-subgroup, we deduce that $K = G + N$, hence $G = K$. □

We are not aware whether "pseudocompact" can be replaced by"precompact" in Theorem 6.1 (see Question 7.6).

One cannot omit "pseudocompact" in the above theorem, since there are plenty of complete non-compact UFSS groups (e.g., the Hilbert space ℓ^2) which are, of course, locally q^*-minimal. On the other hand, since countably compact groups are both sequentially complete and pseudocompact, we obtain:

Corollary 6.2. *Every countably compact locally q^*-minimal abelian group is compact.*

Theorem 6.3. *For a compact abelian group K TFAE:*
(a) K has no proper pseudocompact totally dense subgroup;
(b) K has no proper pseudocompact locally t-dense subgroup;
(c) there exists a torsion closed G_δ-subgroup of K.
(b**) K has no proper dense pseudocompact and locally q^*-minimal subgroup;*

Proof. The equivalence of (a) and (c) is contained in [15] and the implication (b) → (a) is trivial. To prove (c) → (b) assume that N is a torsion closed G_δ-subgroup of K. Assume that H is a pseudocompact locally t-dense subgroup of K. Since dense pseudocompact subgroups are G_δ-dense by Comfort-Ross criterion for pseudocompactness of dense subgroups of compact groups, the subgroup $H_1 := H \cap N$ is G_δ-dense in N. Moreover, the closed subgroup H_1 of H is locally q^*-minimal (by 3.1).

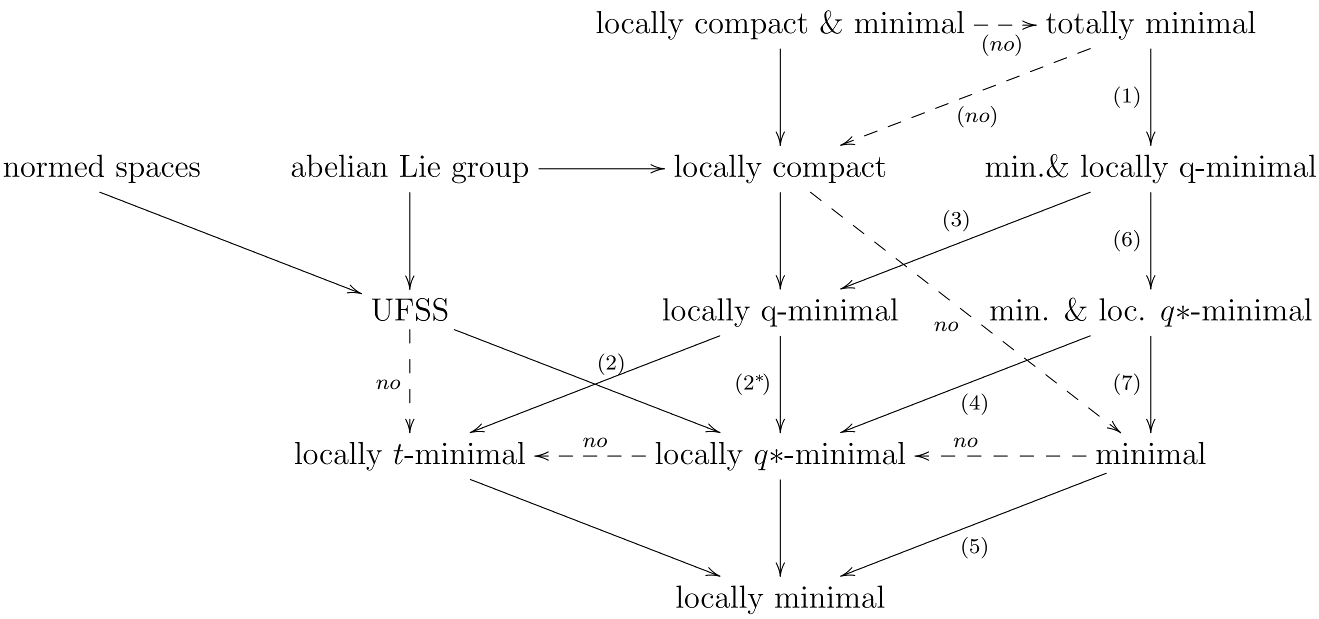
The equivalence of (b) and (b*) follows from Theorem 5.4. □

7. Open problems

We have three types of notions to describe the local minimality of quotient groups——locally t-minimal groups, locally q-minimal groups and locally q^*-minimal groups. We do not know whether local t-minimality imply local q^*-minimality.

Question 7.1. *Does local t-minimality imply local q^*-minimality ?*

The following diagram shows the implication we have proved or disproved:



The non-implication "totally minimal $\not\to$ locally compact" yields also a non-implication "minimal & locally q-minimal$\not\to$ locally compact".

The non-implication "UFSS$\not\to$ locally t-minimal" follows from Example 5.9.

The non-reversibility of the implications (2) and (2*) follows from Example 2.11.

The non-reversibility of the implication (7) witness by the the group subgroup $G = K[2] + \bigoplus_\omega \mathbb{Z}(4)$ of the group $K = \mathbb{Z}(4)^\omega$,

The non-reversibility of the implication (6) is witness by the socle G of $\mathbb{T}$ group ,

The non-reversibility of the "parallel" implications (3), (4) and (5) follows from the fact that locally compact abelian groups are not minimal.

The following example show that the implication (1) in the above diagram cannot be inverted.

Example 7.2. *Let p be a prime and $K = \mathbb{Z}_p \times \mathbb{Z}(p^2)$. Let $\xi \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$ be independent with $1 \in \mathbb{Z}_p$ and let c be the generator of the group $\mathbb{Z}(p^2)$. The subgroup $G = \langle(\xi, c)\rangle + \mathbb{Z} \times \langle pc\rangle$ of K is dense,essential and contains an open totally minimal (hence, locally q-minimal) subgroup. Hence, G is minimal and locally q-minimal, but G is not totally minimal as G is not totally dense in K.*

The subgroup G of the upper triangular linear group $T_2 + (\mathbb{R})$, consisting of all matrices with $a_{21} = 0$ and $a_{22} = 1$, is minimal and locally compact, hence locally q-minimal, but not totally minimal.

According to [7], minimal abelian groups that are also divisible, are totally minimal. A similar phenomenon can be observed in Proposition 4.2 and Theorem 4.5, where we prove the counterpart of this property for local q-minimality for the dense subgroups of $\mathbb{R}$ and $\mathbb{R}^n$. Moreover, we see that the implication can be inverted, namely local q-minimality for these subgroups implies divisibility. (Such a phenomenon is not present for arbitrary minimal abelian groups, More precisely, the compact Pontryagin dual $K = \mathbb{Q}^\wedge$ of the discrete group $\mathbb{Q}$ is divisible and has dense totally minimal subgroups that are not divisible. Same applies to $\mathbb{T}$, it has dense totally minimal subgroups that are not divisible.) On the other hand, Proposition 4.2 provides plenty of examples of non-locally q-minimal subgroups of $\mathbb{T}$. Among them there are many divisible locally minimal groups:

Question 7.3. *Can we give a necessary and sufficient condition for a dense subgroup G of $\mathbb{T}^2$ (or $\mathbb{T}^n$) to be locally q-minimal ?*

Question 7.4. *Does local t-minimality have the three space property, e.g., if G has a totally complete normal subgroup K such that G/K is total locally minimal, does G have the same property?*

Example 7.2 leaves open the following:

Question 7.5. *If G has an h-complete*[4] *(in particular, compact*[5] *) normal subgroup K such that G/K is localy q-locally minimal, does G have the same property?*

If $K \leq G$ is a totally complete subgroup of G that is locally t-minimal along with G/K, then also G is locally t-minimal. Assume that local t-minimality of G/K is witnessed by $\pi(U)$ for some neighbourhood of e_G such that $K \cap U$ witnesses local t-minimality of K. Let N be a closed normal subgroup of G. To prove that G/N is locally minimal $q : G \to G/N$ is a continuous homomorphism such that $q(U)$ is open in G/N. Consider the quotient map $h : G/K \to G/KN$ and assume that G/KN carries the quotient topology of G/N. Then $l : G/N \to G/NK$ takes the neighbourhood $q(U)$ of $e_{G/N}$ to a neighbourhoodof e in G/KN. Since $l(q(U)) = h(\pi(U))$, this allows as to claim that h is open since G/K is locally q-minimal w.r.t. $\pi(U)$. yet it is not clear if this can prove that q is open ?

Question 7.6. *Can "pseudocompact" be replaced by"precompact" in Theorem 6.1.*

Call a property P of topological spaces *contageuous*, if whenever X is a dense subspace of a space Y, then $X \in P$ implies $Y \in P$. When we speak of topological groups, we shall consider dense subgroups, of course. Here is a list of contagious properties:
(a) connectedness;
(b) pseudocompactness;
(c) minimality;
(d) total minimality;
(e) local minimality;
(f) local q^*-minimality;
(g) commutativity.
We showed that local q-minimality is not a contagious property.

Question 7.7. *Is total local minimalty a contagious property ?*

Our criterions for (total) minimality, of local (q^*-) minimality are designed for contagious properties, that's why we cannot produce such a criterion for local q-minimality. This circumstance determines our major interest in locally q^*-minimal groups (rather than locally q-minimal ones).

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[4] i.e., all continuous homomorphic images of H are complete.

[5] I believe this must be true when H is finite, although I have no proof at hand.

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