Semihypergroups obtained by merging of 0-semigroups with groups

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Abstract

We consider the class of 0-semigroups $(H,\star)$ that are obtained by adding a zero element to a group $(G,\cdot)$ so that for all $x,y\in G$ it holds $x\star y\neq 0 \quad\Rightarrow\quad x\star y = xy$. These semigroups are called 0-extensions of $(G,\cdot)$. We introduce a merging operation that constructs a 0-semihypergroup from a 0-extension of $(G,\cdot)$ by a suitable superposition of the product tables. We characterize a class of 0-simple semihypergroups that are merging of a 0-extension of an elementary Abelian 2-group. Moreover, we prove that in the finite case all such 0-semihypergroups can be obtained from a special construction where $(H,\star)$ is nilpotent.

Keywords: 0-semigroups, simple semigroups, fully semihypergroups.

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1 Introduction

In our preceding paper [7] we discovered a family of 0-semihypergroups having the following property, among others: for any 0-semihypergroup $(H, \circ)$ the hyperproduct $\circ$ is obtained by a superposition of the product tables of a 0-semigroup $(H, \star)$ and a left zero semigroup $(H_+, \cdot)$ [13], where $H_+ = H - \{0\}$. Those 0-semihypergroups originated from a study on semihypergroups having the cardinality of all hyperproducts not greater than 2 and the fundamental relation β non-transitive. In [7] we obtained a complete description of the isomorphism classes of that family; if $|H_+| = n$ then the number of these isomorphism classes is the $(n+1)$-th term of the sequence A000070 [14]. The aim of the present work is to analyze algebraic and combinatorial properties of pairs made by a 0-semigroup $(H, \star)$ and a 0-group $(H, \cdot)$ such that the operation $\circ$ defined as $x \circ y = \{x \star y, xy\}$ is associative.

After introducing some basic definitions and notations to be used throughout the paper, in Section 2 we consider 0-semigroups $(H, \star)$ that are obtained by adding a zero element to a group $(H_+, \cdot)$. These semigroups are called 0-extensions of $(H_+, \cdot)$. Moreover, we introduce the merging operation, which constructs a 0-semihypergroup by a suitable superposition of the product tables of $(H, \star)$ and $(H_+, \cdot)$. In Section 3 we consider a class of 0-semihypergroups $(H, \circ)$ that are characterized by being the merging of a 0-extension of an elementary Abelian 2-group. That class is denoted by $\mathfrak{G}_0$.

In Section 4, we prove that every $(H, \circ) \in \mathfrak{G}_0$ belongs to one of two subclasses, denoted by $\mathfrak{G}_{0,d}$ and $\mathfrak{G}_{0,s}$, according to whether $1 \circ 1 = \{0, 1\}$ or $1 \circ 1 = \{1\}$, where 1 is the identity of the group $(H_+, \cdot)$. We denote by $\mathfrak{G}^*_{0,d}$ and $\mathfrak{G}^*_{0,s}$ the subclasses of semihypergroups $(H, \circ)$ in $\mathfrak{G}_{0,d}$ and $\mathfrak{G}_{0,s}$, respectively, such that $|1 \circ x| = |x \circ 1| = |x \circ x| = 2$, for all $x \notin \{0, 1\}$. There is a bijection between the semihypergroups in $\mathfrak{G}^*_{0,d}$ and those in $\mathfrak{G}^*_{0,s}$. We show that every $\mathfrak{G}_0$-semihypergroup can be obtained from a semihypergroup in $\mathfrak{G}^*_{0,d}$ or $\mathfrak{G}^*_{0,s}$ by two special constructions described in Propositions 4.3 and 4.5. In Section 5, we study the class $\mathfrak{G}_0(n)$ of finite $\mathfrak{G}_0$-semihypergroups of size n. In that case the semihypergroups $(H, \circ)$ in $\mathfrak{G}^*_{0,d}$ are merging of a nilpotent semigroup $(H, \star)$ with an elementary Abelian 2-group. In Proposition 5.1 we show a tight bound on the nilpotency rank of the semigroup $(H, \star)$. Finally, in Section 6, with the help of symbolic computation software, we determine the number of isomorphism classes in $\mathfrak{G}_0(5)$ and $\mathfrak{G}_0(9)$. To that goal, we use the results found in Section 4 and 5. We obtain 41 semihypergroups in $\mathfrak{G}_0(5)$ and 7272 in $\mathfrak{G}_0(9)$.

1.1 Basic definitions and results

Throughout this paper we use just a few basic concepts and definitions that belong to common terminology in semigroup and semihypergroup theory, see [2, 3, 13].

A semigroup $(S, \cdot)$ is said to be nilpotent if there exists $r \in \mathbb{N}$ such that $|S^r| = 1$. The minimum positive integer r such that $|S^r| = 1$ is called nilpotency rank or degree of $(S, \cdot)$.

A semigroup with a zero element 0 is called 0-*semigroup*.

A *right zero semigroup* is a semigroup $(S, \cdot)$ such that $xy = y$, for all $x, y \in S$. *Left zero semigroups* are defined in an analogous way.

A group $(G, \cdot)$ in which every element has order less or equal to two is called elementary Abelian 2-group.

If $(G, \cdot)$ is a group and $0 \notin G$ the set $G \cup \{0\}$ is a 0-semigroup respect the product $\star$ defined as follows:

$$0 \star 0 = x \star 0 = 0 \star x = 0, \quad x \star y = xy, \quad \text{for all } x, y \in G.$$

The semigroup $(G \cup \{0\}, \star)$ is called 0-*group*.

Let H be a non-empty set, a hyperoperation $\circ$ on H is a map from $H \times H$ to $P^*(H)$, where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H. If A, B are non-empty subsets of H then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ and $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ for all $x \in H$.

A *semihypergroup* is a non-empty set H endowed with an associative hyperproduct $\circ$, that is $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

If $(H, \circ)$ is a semihypergroup, an element $0 \in H$ such that $x \circ 0 = \{0\}$ (resp., $0 \circ x = \{0\}$) for all $x \in H$ is called *right zero scalar element* (resp., *left zero scalar element*) of $(H, \circ)$. If 0 is both right and left zero scalar element, then it is called *zero scalar* or *absorbing element*, and $(H, \circ)$ is said to be a 0-*semihypergroup*.

A *simple semihypergroup* is a semihypergroup $(H, \circ)$ such that $H \circ x \circ H = H$, for all $x \in H$. A semihypergroup $(H, \circ)$ with a zero scalar element 0 is called *zero-simple* if $H \circ x \circ H = H$, for all $x \in H - \{0\}$ [9].

Given a semihypergroup $(H, \circ)$, the relation β^* of H is the transitive closure of the relation $\beta = \cup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation in H and, for every integer $n > 1$, β_n is defined recursively as follows:

$$x \beta_n y \iff \exists (z_1, \ldots, z_n) \in H^n \ : \ \{x, y\} \subseteq z_1 \circ z_2 \circ \ldots \circ z_n.$$

The relations β, β^* are called *fundamental relations* on H [1, 10, 11, 15]. The interested reader can find all relevant definitions, many properties and applications of fundamental relations, even in more abstract contexts, also in [4, 5, 6, 12].

2 0-extensions and mergings

In this section we introduce a class of 0-semigroups that are obtained by adding a zero element to a group. Furthermore, we provide a construction of 0-semihypergroups by a suitable superposition of the product tables of one such 0-semigroup with the associated group. Here and in the following we indicate with 1 the identity of G and we use the notation I_n as a shorthand for the set $\{0, 1, \ldots, n\}$.

2.1 0-extensions

Definition 2.1. Let $(H, \star)$ and $(G, \cdot)$ be respectively a 0-semigroup and a group. We say that $(H, \star)$ is a 0-*extension of* $(G, \cdot)$ if the following conditions are verified:

1. $0 \notin G$ and $H = G \cup \{0\}$;
2. For all $x, y \in G$ it holds $x \star y \neq 0 \ \Rightarrow \ x \star y = xy$.

Hereafter, we give some examples of 0-extensions.

Example 2.1. Every 0-group is 0-extension of a group.

Example 2.2. Consider the following operations defined on I_2:

$\star_1$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

$\star_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	0	0

$\star_3$	0	1	2
0	0	0	0
1	0	1	0
2	0	2	0

$\star_4$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

$\star_5$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

$\star_6$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

Then, $(I_2, \star_1), \ldots, (I_2, \star_6)$ are 0-extensions of the group $\mathbb{Z}_2$. Moreover, they are pairwise not isomorphic and none of them is isomorphic to the 0-group obtained from $\mathbb{Z}_2$.

Example 2.3. Consider the following operations defined on I_4:

$\star$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	0	4	0
3	0	3	4	0	0
4	0	4	0	0	0

$*$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	1	4	0
3	0	0	0	1	0
4	0	0	0	0	0

Then, $(I_4, \star)$ and $(I_4, *)$ are 0-extensions of the elementary Abelian 2-group of size 4.

Example 2.4. Let $(G, \cdot)$ be a group and let $\{H_i\}_{i \in I}$ be a family of subgroups of G, with $|I| \geq 2$. Moreover, let $\{\{x_{i1}, x_{i2}\}\}_{i \in I}$ be a family of subsets of G which verifies the following conditions for every $i, j \in I$:

1. $\{x_{i1}, x_{i2}\} \subseteq H_i$;
2. $\{x_{i1}, x_{i2}, x_{i1}x_{i2}\} \cap H_j = \varnothing$ if $i \neq j$.

Considering an element $0 \notin G$, we define the following operation $\star$ on the set $H = G \cup \{0\}$:

$$a \star b = \begin{cases} ab & \text{if } \{a, b\} = \{x_{i1}, x_{i2}\} \text{ for some } i \in I \\ 0 & \text{else.} \end{cases}$$

The set H with the operation $\star$ is a 0-extension of the group $(G, \cdot)$.

Now we show a special construction of 0-extensions, that will be largely used in the next section.

Proposition 2.1. *Let $(G, \cdot)$ be an elementary Abelian 2-group and let $(H, \star)$ be a 0-extension of $(G, \cdot)$ such that*

$$x \star y = z \quad \Longrightarrow \quad x \star z = z \star x = y \star z = z \star y = 0 \tag{1}$$

for all (x, y, z) *of distinct elements in* $H - \{0, 1\}$*. On the set* H *we define the following product: If* $(a, b) \neq (1, 1)$ *then*

$$a \otimes b = \begin{cases} 0 & \text{if } a = b \neq 1 \\ 0 & \text{if either } a = 1 \text{ or } b = 1 \\ a \star b & \text{otherwise.} \end{cases} \tag{2}$$

Moreover, $1 \otimes 1$ *can be defined as* 0 *or* 1*, indifferently. Then the product* $\otimes$ *defined in* (2) *is associative and* $(H, \otimes)$ *is a* 0*-extension of the group* $(G, \cdot)$*.*

Proof. Clearly, for every $x, y, z \in H$ such that $\{x, y, z\} \cap \{0, 1\} \neq \varnothing$, we have $(x \otimes y) \otimes z = x \otimes (y \otimes z) = 0$. On the other hand, if $\{x, y, z\} \cap \{0, 1\} = \varnothing$ then we obtain:

- If $x = y = z$ we have $(x \otimes y) \otimes z = x \otimes (y \otimes z) = 0$.
- If $x = y \neq z$, we have $(x \otimes x) \otimes z = 0$ and $x \otimes z = x \star z$. Clearly, if $x \star z = 0$ then $x \otimes (x \otimes z) = 0$. If $x \star z \neq 0$ then $x \star z = xz$ with $xz \neq 1$ since $x \neq z$ and $(G, \cdot)$ is an elementary Abelian 2-group. Hence x, z, xz are three distinct elements in $H - \{0, 1\}$. So, by (1) and (2), we have $x \star (xz) = 0$ and $x \otimes (x \otimes z) = x \otimes (x \star z) = x \otimes (xz) = x \star (xz) = 0$.
- If $x \neq y = z$, as in the preceding case, we obtain that $(x \otimes y) \otimes y = x \otimes (y \otimes y) = 0$.
- If $x \neq y$ and $y \neq z$, we can distinguish three cases:

  1) $x \otimes y = z$, 2) $y \otimes z = x$, 3) $x \otimes y \neq z$ and $y \otimes z \neq x$.

  In the case 1), we have $(x \otimes y) \otimes z = 0$ and $z = x \otimes y = x \star y = xy$. Therefore $x \neq z$ otherwise $y = 1$. Thus, the elements x, y, z are pairwise distinct and, by (1), $x \star y = z \;\Rightarrow\; y \star z = 0$. In consequence $y \otimes z = 0$ and $(x \otimes y) \otimes z = x \otimes (y \otimes z) = 0$.

  The case 2) is similar to the case 1). Finally, in the case 3), we have $x \star y \neq 1$ and $y \star z \neq 1$ otherwise $1 = x \star y = xy$ or $1 = y \star z = yz$ and we have the contradiction $x = y$ or $y = z$ since $(G, \cdot)$ is an elementary Abelian 2-group. Hence $x \otimes y = x \star y \neq z$, $y \otimes z = y \star z \neq x$ and consequently $(x \otimes y) \otimes z = (x \star y) \star z = x \star (y \star z) = x \otimes (y \otimes z)$.

Then $(H, \otimes)$ is a 0-semigroup. From (2) we have that $x \otimes y \neq 0 \;\Rightarrow\; x \otimes y = x \star y \neq 0 \;\Rightarrow\; x \otimes y = xy$. Hence, $(H, \otimes)$ is a 0-extension of $(G, \cdot)$. □

2.2 The merging operation

In this subsection we introduce a construction of a 0-semihypergroup from a 0-extension $(H, \star)$ of a group $(G, \cdot)$. The 0-semihypergroup obtained in that way will be called the *merging* of $(H, \star)$ and $(G, \cdot)$.

Let $(H, \star)$ be a 0-extension of a group $(G, \cdot)$. We define on the set H the following hyperoperation $\circ$: For every $x, y \in G$ let

$$0 \circ 0 = 0 \circ x = x \circ 0 = \{0\}, \quad x \circ y = \{x \star y, xy\}. \tag{3}$$

From (3), we trivially deduce that for every $x, y \in G$ we have

$$x \circ y = \begin{cases} \{xy\} & \text{if } x \star y \neq 0; \\ \{0, xy\} & \text{if } x \star y = 0. \end{cases} \tag{4}$$

We can prove the following result:

Proposition 2.2. *The set H equipped with the hyperproduct defined in* (3) *is a* 0*-semihypergroup such that $\prod_{i=1}^{n} z_i \in z_1 \circ \ldots \circ z_n$ and $|z_1 \circ \ldots \circ z_n| > 1 \Rightarrow z_1 \circ \ldots \circ z_n = \{0, \prod_{i=1}^{n} z_i\}$, for every $z_1, z_2, \ldots, z_n \in G$ and $n \geq 2$.*

Proof. Firstly we prove that the hyperoperation $\circ$ is associative. Let $x, y, z \in H$. If $0 \in \{x, y, z\}$ then $(x \circ y) \circ z = x \circ (y \circ z) = \{0\}$. Therefore, we suppose that $x, y, z \in G$. By (4), we have that $xy \in x \circ y \subseteq \{0, xy\}$ and $xyz \in (xy) \circ z \subseteq \{0, xyz\}$. Hence,

$$xyz \in (x \circ y) \circ z \subseteq \{0, xy\} \circ z = 0 \circ z \cup (xy) \circ z = \{0, xyz\}$$

and we deduce that $xyz \in (x \circ y) \circ z \subseteq \{0, xyz\}$. Analogously we can prove that $xyz \in x \circ (y \circ z) \subseteq \{0, xyz\}$.

Now, we suppose $|(x \circ y) \circ z| = 1$. By (3), we have $\{x \star y \star z, xyz\} \subseteq (x \circ y) \circ z = \{xyz\}$ and so $x \star y \star z = xyz$. Since $xyz \neq 0$, we deduce that $0 \notin \{x \star y \star z, x \star y, y \star z\}$. In consequence we obtain $y \star z = yz$ and $xyz = x \star (y \star z) = x \star (yz) = x(y \star z)$. Thus,

$$x \circ (y \circ z) = \{x \star y \star z, x \star (yz), x(y \star z), xyz\} = \{xyz\} = (x \circ y) \circ z.$$

Analogously, if $|x \circ (y \circ z)| = 1$ then $(x \circ y) \circ z = x \circ (y \circ z) = \{xyz\}$.

Now, suppose that $|(x \circ y) \circ z| = 2$. It follows that $|x \circ (y \circ z)| = 2$ and so $(x \circ y) \circ z = x \circ (y \circ z) = \{0, xyz\}$. Thus $(H, \circ)$ is a 0-semihypergroup.

To prove the second part of the claim, let $n \geq 2$ and $z_1, z_2, \ldots, z_n \in G$. If $n = 2$ then the claim is true by (4). Proceeding by induction, suppose the claim is true for $n - 1 \geq 2$. Clearly, we have that $\prod_{i=1}^{n-1} z_i \in z_1 \circ \ldots \circ z_{n-1} \subseteq \{0, \prod_{i=1}^{n-1} z_i\}$ and $(\prod_{i=1}^{n-1} z_i) \circ z_n \subseteq \{0, \prod_{i=1}^{n} z_i\}$. Hence, we obtain

$$\begin{aligned} z_1 \circ \ldots \circ z_n &= (z_1 \circ \ldots \circ z_{n-1}) \circ z_n \\ &\subseteq \{0, \prod_{i=1}^{n-1} z_i\} \circ z_n = \{0\} \cup (\prod_{i=1}^{n-1} z_i) \circ z_n \subseteq \{0, \prod_{i=1}^{n} z_i\}. \end{aligned}$$

Therefore, $|z_1 \circ \ldots \circ z_n| > 1 \;\Rightarrow\; z_1 \circ \ldots \circ z_n = \{0, \prod_{i=1}^{n} z_i\}$. □

Definition 2.2. We say that the 0-semihypergroup $(H, \circ)$ in Proposition 2.2 is the *merging of* $(H, \star)$ *with* $(G, \cdot)$.

Example 2.5. Consider the following hyperproducts defined on I_2:

$\circ_1$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0, 1

$\circ_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	0, 2	0, 1

$\circ_3$	0	1	2
0	0	0	0
1	0	1	0, 2
2	0	2	0, 1

$\circ_4$	0	1	2
0	0	0	0
1	0	1	0, 2
2	0	0, 2	0, 1

$\circ_5$	0	1	2
0	0	0	0
1	0	0, 1	0, 2
2	0	0, 2	1

$\circ_6$	0	1	2
0	0	0	0
1	0	0, 1	0, 2
2	0	0, 2	0, 1

The 0-semihypergroups $(I_2, \circ_1), \ldots, (I_2, \circ_6)$ are the merging of the 0-semigroups $(I_2, \star_1), \ldots, (I_2, \star_6)$ in Example 2.2 with $\mathbb{Z}_2$. These semihypergroups will be used in the next section to introduce a new class of 0-semihypergroups.

Remark 2.1. Let $(H, \star)$ be a 0-extension of an Abelian 2-group $(G, \cdot)$ that verifies the condition (1). By Proposition 2.1, the semigroup $(H, \otimes)$ is also a 0-extension of $(G, \cdot)$, where $\otimes$ is the product defined from $\star$ as in (2). Moreover, let $(H, \circ)$ be the merging of $(H, \star)$ with $(G, \cdot)$ and let $(H, \bullet)$ be the merging of $(H, \otimes)$ with $(G, \cdot)$. By Definition 2.2, the hyperproduct $\bullet$ fulfills

$$\begin{cases} \{1\} \subseteq 1 \bullet 1 \subseteq \{0, 1\} \\ x \bullet x = \{0, 1\} \\ 0 \bullet 0 = 0 \bullet 1 = 1 \bullet 0 = 0 \bullet x = x \bullet 0 = \{0\} \\ 1 \bullet x = x \bullet 1 = \{0, x\} \\ x \bullet y = \{x \otimes y, xy\} = \{x \star y, xy\} \end{cases} \tag{5}$$

for all $x, y \in H - \{0, 1\}$ and $x \neq y$. In particular, the hyperproducts $\circ$ and $\bullet$ may only differ in the values assumed on the pairs $(1, x)$, $(x, 1)$ and (x, x), for all $x \in G$. In the remaining cases we have $x \circ y = x \bullet y$.

3 The class of $\mathfrak{G}_0$-semihypergroups

The six semihypergroups in Example 2.5 belong to the list of fourteen 0-semihypergroups of size 3 where the relation β is not transitive [9, Thm. 5.6]. In particular, they belong to the family of the fully zero-simple semihypergroups [7]. We remember that a 0-semihypergroup $(H, \circ)$ is called *fully zero-simple* if it fulfills the following conditions:

1. All subsemihypergroups of $(H, \circ)$ (H itself included) are zero-simple;
2. the relation β in $(H, \circ)$ and its restrictions β_K to any subsemihypergroup $K \subset H$ of size ≥ 3 are not transitive.

Since the relation β is transitive in all semihypergroups of size ≤ 2, it follows that every fully zero-simple semihypergroup has size ≥ 3. In [7] the authors study the fully zero-simple semihypergrous satisfying the condition $\{y\} \subseteq x \circ y \subseteq \{0, y\}$, for all $x, y \in H - \{0\}$. In this case, $H - \{0\}$ is a right zero semigroup [13] and $(H, \circ)$ can be regarded as the merging of a 0-semigroup with a right zero semigroup. Moreover, apart of isomorphisms, the fully zero-simple semihypergroups of size n that verify such condition are exactly $\sum_{k=0}^{n} p(k)$, where $p(k)$ denotes the number of non-increasing partitions of k.

In this section we study the fully zero-simple semihypergroups $(H, \circ)$ with an element $1 \neq 0$ such that, for all $x \in H - \{0, 1\}$, the sets $\{0, 1, x\}$ are subsemihypergroups isomorphic to one of semihypergroups in Example 2.5. Moreover, we show that these semihypergroups can be obtained as merging of a 0-semigroup with an elementary Abelian 2-group. Firstly, we borrow the following result from Theorem 5.6 in [9].

Theorem 3.1. *The semihypergroups in Example 2.5 are all and only the fully zero-simple semihypergroups of size* 3 *that are merging of a* 0*-extension of* $\mathbb{Z}_2$ *with* $\mathbb{Z}_2$*, apart of isomorphisms.*

Now we prove the following result:

Proposition 3.1. *Let* $(H, \star)$ *be a* 0*-extension of the group* $(G, \cdot)$*, with* $|G| \geq 2$*, and let* $(H, \circ)$ *be the merging of* $(H, \star)$ *with* $(G, \cdot)$*. If for all* $x \in H -$

$\{0,1\}$ the set $\{0,1,x\}$ is a subsemihypergroup of $(H,\circ)$ isomorphic to one of semihypergroups $(I_2,\circ_1),\ldots,(I_2,\circ_6)$ in Example 2.5 then $(H,\circ)$ is a fully 0-simple semihypergroup. Moreover, $(G,\cdot)$ is an elementary Abelian 2-group.

Proof. First of all we prove that if $K \subseteq H$ is a subsemihypergroup of $(H,\circ)$ then K is zero-simple. We suppose that $|K| \geq 2$ since the thesis is trivial if $|K| = 1$. By hypothesis, if there exists $x \in K - \{0,1\}$ then the set $\{0,1,x\}$ is a subsemihypergroup of $(H,\circ)$ isomorphic to one semihypergroups in Example 2.5. Hence we have $1 \in x \circ x \subseteq K$ and $0 \in \{1,x\} \circ \{1,x\} \subseteq K \circ K \subseteq K$. Thus $K = \{0,1\}$ or $|K| \geq 3$ with $\{0,1\} \subset K$. In both cases K is zero-simple since $x \in 1 \circ x \circ 1$ and $K \circ x \circ K = K$, for all $x \in K - \{0\}$.

Now, we prove that if $|K| \geq 3$ then β_K is not transitive. If $|K| = 3$, there exists $x \in H - \{0,1\}$ such that $K = \{0,1,x\}$ and K is isomorphic to one of the semihypergroups in Example 2.5. By Theorem 3.1, the relation β_K is not transitive. If $|K| \geq 4$ then there exist $x,y \in K - \{0,1\}$ with $x \neq y$. The sets $\{0,1,x\}$ and $\{0,1,y\}$ are subsemihypergroups isomorphic to one of the semihypergroups in Example 2.5. Hence, there exist two hyperproducts P and Q of elements in $\{1,x\}$ and $\{1,y\}$, respectively, such that $\{0,x\} = P$ and $\{0,y\} = Q$. Therefore, $(x,0) \in \beta_K$ and $(0,y) \in \beta_K$. If by absurd $(x,y) \in \beta_K$ then there exists a hyperproduct R of elements in K such that $\{x,y\} \subseteq R$, which is impossible by Proposition 2.2. Thus β_K is not transitive and $(H,\circ)$ is a fully zero-simple semihypergroup. Finally, since $1 \in x \circ x \subseteq \{0,1\}$, by (3) and (4) we have $xx = 1$ for all $x \in G$, hence $(G,\cdot)$ is an elementary Abelian 2-group. □

Let $\mathfrak{F}_0$ be the class of fully zero-simple semihypergroups. We use 0 and H_+ to denote the zero scalar element of a semihypergroup $(H,\circ) \in \mathfrak{F}_0$ and the set $H - \{0\}$, respectively.

Definition 3.1. Let $\mathfrak{G}_0$ be the subclass of semihypergroups in $\mathfrak{F}_0$ with an element $1 \in H_+$ such that for all $x \in H_+$ the set $\{0,1,x\}$ is a subsemihypergroup of $(H,\circ)$ isomorphic to one of semihypergroups in Example 2.5. A semihypergroup $(H,\circ) \in \mathfrak{G}_0$ is called $\mathfrak{G}_0$-*semihypergroup*. Moreover, the family of semihypergroups $\{\{0,1,x\}\}_{x \in H-\{0,1\}}$ is the *spectrum* of $(H,\circ)$.

For reader's convenience, we collect in the following lemma some preliminary results from [7].

Lemma 3.1. *If $(H,\circ) \in \mathfrak{F}_0$ then we have:*

1. *If K is a subsemihypergroup of H such that $0 \notin K$ then $|K| = 1$. Moreover, if $|K| \geq 2$ then the zero element of K is 0;*

2. *for every sequence $z_1, \ldots, z_n$ in H_+ we have $z_1 \circ \cdots \circ z_n \neq \{0\}$;*

3. *the set H_+ endowed with the hyperproduct $a \diamond b = (a \circ b) \cap H_+$ is a simple semihypergroup.*

Consider the following definition:

Definition 3.2. Let $(H, \circ) \in \mathfrak{F}_0$. The semihypergroup $(H_+, \diamond)$ defined in Lemma 3.1(3) is the *residual semihypergroup* of $(H, \circ)$. If $(H_+, \diamond)$ is a group then it is said the *residual group* of $(H, \circ)$.

Lemma 3.2. *Let $(H, \circ) \in \mathfrak{G}_0$. Then we have:*

1. *$|x \circ y| \leq 2$; and $|x \circ y| = 2 \Rightarrow 0 \in x \circ y$, for all $x, y \in H_+$;*

2. *the residual semihypergroup of $(H, \circ)$ is an elementary Abelian 2-group.*

Proof. 1. By considering the hyperproduct tables in Example 2.5, for all $x, y \in H_+$ we have $x \circ (x \circ y) = (x \circ x) \circ y \subseteq \{0, 1\} \circ y = \{0, y\}$. Hence, by Lemma 3.1(2), we deduce that $y \in x \circ a$ for some $a \in x \circ y$. Therefore we have $x \circ y \subseteq x \circ (x \circ a) = (x \circ x) \circ a \subseteq \{0, 1\} \circ a = \{0, a\}$. Hence $|x \circ y| \leq 2$ and $|x \circ y| = 2 \Rightarrow 0 \in x \circ y$.

2. By the previous point and Lemma 3.1(3), $(H_+, \diamond)$ is a simple semigroup. Moreover, since $(H, \circ) \in \mathfrak{G}_0$, by looking at the tables in Example 2.5, for all $x \in H_+$ we have $x \diamond x = \{1\}$ and $x \diamond 1 = 1 \diamond x = \{x\}$. Hence $(H_+, \diamond)$ is an elementary Abelian 2-group. □

By the preceding lemma, if $(H, \circ) \in \mathfrak{G}_0$ then $x \diamond y$ is a nonzero singleton, for all $x, y \in H_+$. By identifying a singleton with the element itself, we can define on the set H the following operation:

$$x \star y = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0; \\ 0 & \text{if } |x \circ y| = 2 \text{ and } x, y \in H_+; \\ x \diamond y & \text{if } |x \circ y| = 1 \text{ and } x, y \in H_+. \end{cases} \tag{6}$$

Note that $x \circ y = \{x \star y,\ x \diamond y\}$. Moreover, we have the following result:

Proposition 3.2. *If $(H, \circ) \in \mathfrak{G}_0$ then $(H, \star)$ defined in (6) is a semigroup that is a 0-extension of the residual group $(H_+, \diamond)$.*

Proof. Let $x, y, z \in H$. We have to prove that $(x \star y) \star z = x \star (y \star z)$. If $0 \in \{x, y, z\}$ the thesis is obvious. Therefore suppose that $x, y, z \in H_+$.

If $0 \notin x \circ y \circ z$ then $x \circ y = x \diamond y$, $(x \diamond y) \circ z = x \diamond y \diamond z$, $y \circ z = y \diamond z$ and $x \circ (y \diamond z) = x \diamond y \diamond z$. Hence we have that $(x \star y) \star z = (x \diamond y) \star z = x \diamond y \diamond z = x \star (y \diamond z) = x \star (y \star z)$.

Now, suppose that $0 \in x \circ y \circ z$. There are four options:

1. $0 \notin x \circ y$ and $0 \notin y \circ z$.

   We have $x \circ y = \{x \diamond y\}$ and $y \circ z = \{y \diamond z\}$. Hence $x \star y = x \diamond y$ and $y \star z = y \diamond z$. Moreover, since $0 \in x \circ y \circ z$, we have $0 \in (x \diamond y) \circ z$ and $0 \in x \circ (y \diamond z)$. Consequently, we obtain $|(x \diamond y) \circ z| = |x \circ (y \diamond z)| = 2$ and $(x \star y) \star z = (x \diamond y) \star z = 0 = x \star (y \diamond z) = x \star (y \star z)$.

2. $0 \in x \circ y$ and $0 \notin y \circ z$.

   We have $x \circ y = \{0, x \diamond y\}$ and $y \circ z = y \diamond z$. Therefore $(x \star y) \star z = 0 \star z = 0$ and $0 \in x \circ (y \circ z) = x \circ (y \diamond z)$. Consequently $x \circ (y \diamond z) = \{0, x \diamond y \diamond z\}$ and $x \star (y \star z) = x \star (y \diamond z) = 0$. Hence, $(x \star y) \star z = x \star (y \star z) = 0$.

3. $0 \notin x \circ y$ and $0 \in y \circ z$.

   The argument is analogous to the preceding one.

4. $0 \in x \circ y$ and $0 \in y \circ z$.

   We have $x \circ y = \{0, x \diamond y\}$ and $y \circ z = \{0, y \diamond z\}$, hence $(x \star y) \star z = 0 \star z = 0 = x \star 0 = x \star (y \star z)$.

Finally, $(H, \star)$ is a 0-extension of the residual group $(H_+, \diamond)$ since we have $x \star y = x \diamond y \iff x \star y \neq 0$, for all $x, y \in H_+$. $\square$

Definition 3.3. The 0-semigroup $(H, \star)$ in Proposition 3.2 is the semigroup *associated to* $(H, \circ)$.

Theorem 3.2. *Let $(H, \circ)$ be a 0-semihypergroup of size ≥ 3. The following conditions are equivalent:*

1. *$(H, \circ) \in \mathfrak{G}_0$;*

2. *there exists an elementary Abelian 2-group $(H_+, \cdot)$, with identity 1, and a 0-semigroup $(H, \star)$, which is a 0-extension of $(H_+, \cdot)$, such that $(H, \circ)$ is the merging of $(H, \star)$ with $(H_+, \cdot)$. Moreover, the set $\{0, 1, x\}$ is a subsemihypergroup of $(H, \circ)$ isomorphic to one of semihypergroups in Example 2.5, for all $x \in H - \{0, 1\}$.*

Proof. 1. $\Rightarrow$ 2. The claim follows from Lemma 3.2 and Proposition 3.2. In fact, $(H, \circ)$ is the merging of the associated semigroup $(H, \star)$ with the residual group $(H_+, \diamond)$.

2. $\Rightarrow$ 1. By hypothesis and Proposition 2.2, $(H, \circ)$ is a 0-semihypergroup such that $xy \in x \circ y$ and $|x \circ y| = 2 \;\Rightarrow\; x \circ y = \{0, xy\}$, for all $x, y \in H_+$. Since $(H_+, \cdot)$ is an elementary Abelian 2-group, we have

$$\{0,1,x\} \circ \{0,1,x\} = \{0\} \cup 1 \circ 1 \cup 1 \circ x \cup x \circ 1 \cup x \circ x = \{0,1,x\},$$

for all $x \in H - \{0,1\}$. Hence the set $X = \{0,1,x\}$ is a subsemihypergroup of $(H, \circ)$, for all $x \in H - \{0,1\}$. By hypothesis $(X, \circ)$ is isomorphic to one of semihypergroups in Example 2.5, for all $x \in H - \{0,1\}$. Finally, by Proposition 3.1 and Definition 3.1, we conclude that $(H, \circ) \in \mathfrak{G}_0$. □

3.1 A special property of $\mathfrak{G}_0$-semihypergroups

In this paragraph we prove a special property of the $\mathfrak{G}_0$-semihypergroups that will turn out to be useful in next sections. We premise a lemma which is valid for all $(H, \circ) \in \mathfrak{F}_0$. Hereafter, if A is a non-empty subset of H then we define $R_A = \{x \in H \mid x \circ A \subseteq A\}$ and $L_A = \{x \in H \mid A \circ x \subseteq A\}$.

Lemma 3.3. *Let $(H, \circ) \in \mathfrak{F}_0$ and let A be a non-empty subset of H. If the set R_A (resp., L_A) is non-empty then it is a subsemihypergroup of $(H, \circ)$. Moreover, $0 \notin A \Rightarrow |R_A| = 1$ (resp., $|L_A| = 1$).*

Proof. Let $x_1, x_2 \in R_A$. For every $z \in x_1 \circ x_2$, we have that $z \circ A \subseteq (x_1 \circ x_2) \circ A = x_1 \circ (x_2 \circ A) \subseteq x_1 \circ A \subseteq A$, hence $x_1 \circ x_2 \subseteq R_A$ and R_A is a subsemihypergroup of $(H, \circ)$, and analogously for L_A. The last part of the claim follows from Lemma 3.1(1), since $0 \notin A \Rightarrow 0 \notin R_A$. □

From the preceding proposition, we deduce the following results:

Proposition 3.3. *Let $(H, \circ) \in \mathfrak{G}_0$ and let x, y, z be distinct elements in $H - \{0,1\}$ such that $x \circ y = \{z\}$. Then we have*

1. *$x \circ z = \{0, y\}$ and $z \circ y = \{0, x\}$;*
2. *$z \circ x = \{0, y\}$ and $y \circ z = \{0, x\}$;*
3. *$z \circ z = \{0, 1\}$;*
4. *$|x \circ 1| = |1 \circ y|$;*

5. $|y \circ 1| = |z \circ 1|$ *and* $|1 \circ x| = |1 \circ z|$*.*

Proof. 1. Let $(H_+, \cdot)$ be the residual group of $(H, \circ)$. Since $x \circ y = \{z\}$, in $(H_+, \cdot)$ we obtain $xy = z$, $y = xz$ and $x = zy$. Therefore in $(H, \circ)$ we have $y \in x \circ z$ and $x \in z \circ y$. Now, by absurd, suppose that $x \circ z = \{y\}$. Letting $A = \{y, z\}$ we obtain $0 \notin R_A$ and $x \in R_A$. Hence, by Lemma 3.3, we deduce that $|R_A| = 1$ and $R_A = \{x\}$, which is impossible because we would have $x \circ x = R_A \circ R_A = R_A = \{x\}$ in $(H, \circ)$ and $xx = 1$ in the residual group. Thus $x \circ z = \{0, y\}$. Analogously we can prove that $z \circ y = \{0, x\}$, by using L_A in place of R_A.

2. From item 1., if we suppose that $z \circ x = \{y\}$ we obtain the contradiction $\{z\} = x \circ y = x \circ (z \circ x) = (x \circ z) \circ x = \{0, y\} \circ x \supseteq \{0\}$. Then $z \circ x = \{0, y\}$. Analogously, if $y \circ z = \{x\}$, we obtain $\{z\} = x \circ y = (y \circ z) \circ y = y \circ (z \circ y) = y \circ \{0, x\} \supseteq \{0\}$, an absurdity. Therefore $y \circ z = \{0, x\}$.

3. We have $z \circ z = (x \circ y) \circ z = x \circ (y \circ z) = x \circ \{0, x\} = \{0, 1\}$.

4. If, by absurd, we suppose that $|x \circ 1| = 2$ and $|1 \circ y| = 1$ then we have $x \circ 1 = \{0, x\}$, $1 \circ y = \{y\}$, hence the following contradiction: $\{z\} = x \circ y = x \circ (1 \circ y) = (x \circ 1) \circ y = \{0, x\} \circ y = \{0, z\}$. The case $|x \circ 1| = 1$ and $|1 \circ y| = 2$ is disproved analogously.

5. We can reason as in the case 4., by using the hyperproducts $z \circ 1 = x \circ y \circ 1$ and $1 \circ z = 1 \circ x \circ y$. □

Corollary 3.1. *Let* $(H, \circ) \in \mathfrak{G}_0$*. The associated* 0*-semigroup* $(H, \star)$ *satisfies the condition* (1)*, that is* $x \star y = z \ \Rightarrow \ x \star z = z \star x = y \star z = z \star y = 0$*, for all distinct elements* $x, y, z \in H - \{0, 1\}$*.*

Proof. By Proposition 3.2 $(H, \star)$ is a 0-extension of the residual group $(H_+, \diamond)$. If x, y, z are distinct elements in $H - \{0, 1\}$ such that $x \star y = z$ then $x \diamond y = z$ and $x \circ y = \{z\}$. From points 1. and 2. of Proposition 3.3, we have $|x \circ z| = |z \circ x| = |z \circ y| = |y \circ z| = 2$ and the claim follows. □

In conclusion, by Theorem 3.2 every $(H, \circ) \in \mathfrak{G}_0$ can be obtained as the merging of a 0-semigroup $(H, \star)$ with a elementary Abelian 2-group $(H_+, \cdot)$. By Corollary 3.1, $(H, \star)$ fulfills the hypotheses of Proposition 2.1. Consequently, we can also define of H the product $\otimes$ in (2); the resulting semigroup $(H, \otimes)$ is a 0-extension of $(H_+, \cdot)$, possibly different from $(H, \star)$. As pointed out in Remark 2.1, the merging of $(H, \otimes)$ with $(H_+, \cdot)$ is the $\mathfrak{G}_0$-semihypergroup $(H, \bullet)$ defined by (5).

4 Principal semihypergroups in $\mathfrak{G}_0$

In what follows, we denote by $\mathfrak{G}_0(n)$ the subclass of $\mathfrak{G}_0$-semihypergroups with size n. Since H_+ is the support of an Abelian 2-group it must hold $n = 2^r + 1$ for some integer r.

We note that if $(H, \circ) \in \mathfrak{G}_0$ and there exists $x \in H$ such that $\{0, 1, x\}$ is a semihypergroup isomorphic to $(I_2, \circ_5)$ or $(I_2, \circ_6)$ in Example 2.5 then all other subsemihypergroups in the spectrum of $(H, \circ)$ are isomorphic to $(I_2, \circ_5)$ or $(I_2, \circ_6)$, because in that case $1 \circ 1 = \{0, 1\}$, otherwise we would have the contradiction $\{0, 1\} = 1 \circ 1 = \{1\}$. This fact divides $\mathfrak{G}_0$ into two disjoint subclasses, that of the semihypergroups whose spectrum contains only semihypergroups isomorphic to $(I_2, \circ_5)$ or $(I_2, \circ_6)$, and that of the semihypergroups whose spectrum contains only semihypergroups isomorphic to $(I_2, \circ_i)$ for $i = 1, \ldots, 4$. We denote these two subclasses by $\mathfrak{G}_{0,d}$ and $\mathfrak{G}_{0,s}$, according to whether $1 \circ 1$ is a doublet or a singleton, respectively. In particular $\mathfrak{G}_{0,d}(n)$ and $\mathfrak{G}_{0,s}(n)$ are the subclasses of semihypergroups of size n in $\mathfrak{G}_{0,d}$ and $\mathfrak{G}_{0,s}$ respectively. Clearly, by Theorem 3.1 we have $\mathfrak{G}_{0,d}(3) = \{(I_2, \circ_5), (I_2, \circ_6)\}$ and $\mathfrak{G}_{0,s}(3) = \{(I_2, \circ_i), i = 1, \ldots, 4\}$, up to isomorphisms.

A simple construction of semihypergroups in $\mathfrak{G}_{0,d}$ or $\mathfrak{G}_{0,s}$ of arbitrary size is obtained as follows.

Example 4.1. Let $(G, \cdot)$ be an elementary Abelian 2-group and let 1 be the identity of G. Let $H = G \cup \{0\}$ where $0 \notin G$ and let K_x be the set $\{0, 1, x\}$ for all $x \in G - \{1\}$. If we equip every set K_x with a hyperoperation $\circ_x$ such that $(K_x, \circ_x)$ is isomorphic to one of the semihypergroups in $\mathfrak{G}_{0,d}(3)$ (resp., $\mathfrak{G}_{0,s}(3)$) then we can define in H the following hyperproduct:

$$a \circ b = \begin{cases} a \circ_x b & \text{if } a, b \in K_x \text{ for some } x; \\ \{0, ab\} & \text{otherwise.} \end{cases}$$

It is easy to prove that $(H, \circ) \in \mathfrak{G}_{0,d}$ (resp., $\mathfrak{G}_{0,s}$).

In this section we will prove some results about semihypergroups $(H, \circ) \in \mathfrak{G}_0$ having the following property:

$$|1 \circ x| = |x \circ 1| = |x \circ x| = 2, \qquad \text{for all } x \notin \{0, 1\}. \tag{7}$$

Note that if $(H, \circ)$ fulfills (7) and belongs to $\mathfrak{G}_{0,d}$ then all semihypergroups in its spectrum are isomorphic to $(I_2, \circ_6)$ in Example 2.5, while if

$(H,\circ) \in \mathfrak{G}_{0,s}$ then its spectrum consists of semihypergroups isomorphic to $(I_2, \circ_4)$. The subclasses of $\mathfrak{G}_{0,s}$ and $\mathfrak{G}_{0,d}$ that contain them are denoted respectively with $\mathfrak{G}^*_{0,s}$ and $\mathfrak{G}^*_{0,d}$. These semihypergroups play an important role in what follows, because any $\mathfrak{G}_0$-semihypergroup can be obtained from a $\mathfrak{G}_0$-semihypergroup fulfilling (7) by means of a particular construction, as we will prove subsequently.

We observe that there is a bijection between semihypergroups in $\mathfrak{G}^*_{0,d}$ and those in $\mathfrak{G}^*_{0,s}$, as claimed hereafter.

Proposition 4.1. *For every $(H,\circ) \in \mathfrak{G}^*_{0,d}$ there exists $(H,\bullet) \in \mathfrak{G}^*_{0,s}$ such that $x \circ y = x \bullet y$ for all pairs $(x,y) \neq (1,1)$. The converse is also true.*

In the following proposition, to any $\mathfrak{G}_0$-semihypergroup we associate a special semihypergroup fulfilling condition (7).

Proposition 4.2. *Let $(H,\circ) \in \mathfrak{G}_0$. Define on H the following hyperproduct:*

$$\begin{cases} 1 \bullet x = x \bullet 1 = \{0, x\} & \text{if } x \in H - \{0,1\} \\ x \bullet x = \{0,1\} & \text{if } x \in H - \{0,1\} \\ x \bullet y = x \circ y & \text{otherwise.} \end{cases} \tag{8}$$

Then $(H,\bullet)$ belongs to $\mathfrak{G}^*_{0,s}$ or $\mathfrak{G}^*_{0,d}$ depending on whether $(H,\circ) \in \mathfrak{G}_{0,s}$ or $(H,\circ) \in \mathfrak{G}_{0,d}$.

Proof. By Theorem 3.2, $(H,\circ)$ is the merging of a 0-semigroup $(H,\star)$ with an elementary Abelian 2-group $(H_+,\cdot)$. From Corollary 3.1, the semigroup $(H,\star)$ satisfies the condition (1) and so, by Remark 2.1, $(H,\bullet)$ is a 0-semihypergroup. Now, observe that for every $x \in H - \{0,1\}$ the set $\{0,1,x\}$ is a subsemihypergroup of $(H,\bullet)$ which is isomorphic to $(I_2,\circ_4)$ or $(I_2,\circ_6)$ of Example 2.5, depending on whether $1 \bullet 1 = \{1\}$ or $1 \bullet 1 = \{0,1\}$, respectively. Therefore, again by Theorem 3.2, $(H,\bullet)$ belongs to $\mathfrak{G}^*_{0,s}$ or $\mathfrak{G}^*_{0,d}$. □

Definition 4.1. The 0-semihypergroups which belong to $\mathfrak{G}^*_{0,s}$ or $\mathfrak{G}^*_{0,d}$ are called *principal semihypergroups*. In particular the 0-semihypergroup $(H,\bullet)$ in Proposition 4.2 is the principal semihypergroup *corresponding to* $(H,\circ)$.

Example 4.2. The following $\mathfrak{G}_0$-semihypergroup $(H,\bullet)$ is a principal semihypergroup of the class $\mathfrak{G}^*_{0,s}$:

$\bullet$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0, 2	0, 3	0, 4
2	0	0, 2	0, 1	4	0, 3
3	0	0, 3	0, 4	0, 1	0, 2
4	0	0, 4	0, 3	0, 2	0, 1

This semihypergroup is the principal semihypergroup corresponding e.g., to the following semihypergroups:

$\circ_1$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	0, 1	4	0, 3
3	0	3	0, 4	0, 1	0, 2
4	0	4	0, 3	0, 2	0, 1

$\circ_2$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	0, 1	4	0, 3
3	0	0, 3	0, 4	0, 1	0, 2
4	0	0, 4	0, 3	0, 2	0, 1

Next, we will show how it is possible to generate $\mathfrak{G}_{0,d}$ or $\mathfrak{G}_{0,s}$ from $\mathfrak{G}^*_{0,d}$ or $\mathfrak{G}^*_{0,s}$, respectively.

Proposition 4.3. *Let $(H, \bullet) \in \mathfrak{G}^*_{0,d}$, let $(H_+, \cdot)$ be its residual group, and let $(H, \star)$ be its associated 0-semigroup. Furthermore, let $*$ be any product on H fulfilling the following conditions:*

$$\begin{cases} x * x \in \{0, 1\} & \text{if } x \in H - (H \star H \cup \{1\}) \\ x * y = x \star y & \text{otherwise.} \end{cases} \tag{9}$$

Then $(H, *)$ is a 0-extension of $(H_+, \cdot)$. Moreover, the merging of $(H, *)$ with $(H_+, \cdot)$ belongs to $\mathfrak{G}_{0,d}$.

Proof. Firstly, we prove that $*$ is associative. By hypotheses, for all $x \in H$ we have

$$0 \star x = x \star 0 = 1 \star x = x \star 1 = x \star x = 0. \tag{10}$$

If $a, b, c \in H$ are such that $\{a, b, c\} \cap \{0, 1\} \neq \varnothing$ or $a = b = c$, it is easy to verify that $(a * b) * c = a * (b * c)$. Therefore, we have to consider only the following cases, with $a, b, c \notin \{0, 1\}$:

1. $a = b \neq c$.

   In $(H_+, \cdot)$ we have $ac \neq a$ and we can distinguish two cases: $a \star c = 0$ and $a \star c = ac$. In the first case, we have $a * (a * c) = a * (a \star c) = a * 0 = 0$. In the second case, by Corollary 3.1, we obtain $a \star (ac) = 0$ and $a * (a * c) = a * (a \star c) = a * (ac) = a \star (ac) = 0$. Hence $(a * a) * c \subseteq \{0, 1\} * c = 0 = a * (a * c)$ and so $(a * a) * c = a * (a * c) = 0$.

2. $a \neq b = c$.

   The proof is similar to that one of the preceding point.

3. $a = c \neq b$.

   If $a * b \neq 0$ then we have $a * b = a \star b = ab$, with a, b, ab distinct elements in $H - \{0, 1\}$. Therefore, by Corollary 3.1, $ab * a = ab \star a = 0$. Consequently, we obtain $(a * b) * a = 0$. Analogously, if $b * a \neq 0$ we obtain $a * (b * a) = 0$, thus $(a * b) * a = a * (b * a) = 0$.

4. a, b, c mutually distinct and $ab = c$.

   We have $a * b = a \star b \in \{0, c\}$ and $b * c = b \star c \in \{0, a\}$. If $a * b = a \star b = c$ then $c \in (H \star H) - \{0\}$ and $c * c = c \star c = 0$. Moreover, from Corollary 3.1, we have $b * c = b \star c = 0$. Hence $(a * b) * c = c * c = 0 = a * 0 = a * (b * c)$. If $b * c = b \star c = a$, we obtain $a \in (H \star H) - \{0\}$ and $a * a = a * b = 0$ as before. Hence, we have $(a * b) * c = 0 * c = 0 = a * a = a * (b * c)$. Finally, if $a * b = 0 = b * c$, it is clearly that $(a * b) * c = a * (b * c) = 0$.

5. a, b, c mutually distinct and $ab \neq c$.

   In $(H_+, \cdot)$ we have $bc \neq a$ and for definition $(a * b) * c = a * (b * c) = a \star b \star c$.

Therefore $(H, *)$ is a 0-semigroup. By definition $(H, *)$ is a 0-extension of the residual group $(H_+, \cdot)$ of $(H, \bullet)$. Now, let $(H, \circ)$ be the merging of $(H, *)$ with $(H_+, \cdot)$. By Proposition 2.2, $(H, \circ)$ is a 0-semihypergroup. By (10), for all $x \in H$, the set $\{0, 1, x\}$ is a subsemihypergroup of $(H, \circ)$ isomorphic to $(I_2, \circ_5)$ or $(I_2, \circ_6)$ in Example 2.5. Finally, by Theorem 3.2, $(H, \circ) \in \mathfrak{G}_{0,d}$. □

The forthcoming result guarantees that all semihypergroups in $\mathfrak{G}_{0,d}$ can be obtained by means of the construction outlined in the preceding proposition.

Proposition 4.4. *Let $(H, \bullet)$ be the principal semihypergroup corresponding to $(H, \circ) \in \mathfrak{G}_{0,d}$ and let $(H_+, \cdot)$ be its residual group. If $(H, *)$ and $(H, \star)$ are the 0-semigroups associated respectively to $(H, \circ)$ and $(H, \bullet)$ then the operations $*$ and $\star$ fulfil conditions (9).*

Proof. By definition (6) and (8), for any $x, y \in H$ we have $x * y = x \star y$ if $x \neq y$ or $x \in \{0,1\}$ or $y \in \{0,1\}$. Moreover, if $x \in H \star H - \{0,1\}$ then there exist $a, b \notin \{0,1\}$ such that $a \neq x \neq b$ and $x = a \star b = ab$. Hence $a \circ b = a \bullet b = \{x\}$ and $x \circ x = \{0,1\}$ by Proposition 3.3(3). Since $x \bullet x = \{0,1\}$, we obtain $x * x = 0 = x \star x$. In all remaining cases we have $x \notin H \star H \cup \{1\}$ and also $x * x \in \{0,1\}$. □

The next proposition is analogous to Proposition 4.3 concerning $\mathfrak{G}_{0,s}$ instead of $\mathfrak{G}_{0,d}$. Since the proof essentially follows the trail of the one of Proposition 4.3, we provide the proof of only one case concerning the associativity of the hyperproduct, which is specific to $\mathfrak{G}_{0,s}$-semihypergroups.

Proposition 4.5. *Let $(H, \bullet) \in \mathfrak{G}^*_{0,s}$, let $(H_+, \cdot)$ be its residual group, and let $(H, \star)$ be its associated 0-semigroup. Moreover, let $*$ be any product on H fulfilling the following conditions:*

$$\begin{cases} \{1 * x, x * 1\} \subseteq \{0, x\} & \text{if } x \notin \{0,1\} \\ x * y = x \star y & \text{otherwise,} \end{cases} \tag{11}$$

provided that, if $(x, y) \in T = \{(a, b) \in [H - \{0,1\}]^2 \mid a \star b \neq 0\}$ then

$$\begin{cases} x * 1 = 0 \iff 1 * y = 0 \\ 1 * x = 0 \iff 1 * (xy) = 0 \\ y * 1 = 0 \iff (xy) * 1 = 0. \end{cases} \tag{12}$$

Then $(H, *)$ is a 0-extension of $(H_+, \cdot)$. Moreover the merging of $(H, *)$ with $(H_+, \cdot)$ belongs to $\mathfrak{G}_{0,s}$.

Proof. The proof of the associativity of $*$ proceeds analogously to that of Proposition 4.3, by considering that here $1 \star 1 = 1$ and the other identities in (10) are still valid. Hereafter we detail the case $a = 1$ and $b, c \neq 1$ where the condition (12) is employed.

If $b * c = 0$ then $1 * (b * c) = 1 * 0 = 0 = (1 * b) * c$ because $1 * b \in \{0, b\}$.

If $b * c \neq 0$ then $b \neq c$, $b * c = b \star c = bc \neq 1$ and $(b, c) \in T$. Therefore, by (12), if $1 * b = 0$ then $1 * (bc) = 0$ and $(1 * b) * c = 0 * c = 0 = 1 * (bc) = 1 * (b * c)$, while if $1 * b = b$ then $1 * (bc) = bc$ and $(1 * b) * c = b * c = bc = 1 * (bc) = 1 * (b * c)$. □

Analogously to Proposition 4.4 we have the following result concerning $\mathfrak{G}_{0,s}$. We refrain from including a complete proof, which requires long but straightforward arguments.

Proposition 4.6. *Let $(H, \bullet)$ be the principal semihypergroup corresponding to $(H, \circ) \in \mathfrak{G}_{0,s}$. If $(H, *)$ and $(H, \star)$ are the 0-semigroups associated respectively to $(H, \circ)$ and $(H, \bullet)$ then the operations $*$ and $\star$ fulfil conditions* (11) *and* (12).

5 Nilpotency of associated semigroups

In this section we consider the 0-semigroups associated to a finite semihypergroup $(H, \circ) \in \mathfrak{G}^*_{0,d}$. By Corollary 3.1, the associated 0-semigroup $(H, \star)$ fulfills conditions (1), (2), and $1 \star 1 = 0$. Moreover, it is also easy to prove that for all $x, y \in H$ we have

1) $x \star y \star x = 0$,

2) if $x = x \star y$ or $x = y \star x$ then $x = 0$.

We observe that if H is finite of size n then $(H, \star)$ is nilpotent. Indeed, $x \star x = 0$ for all $x \in H$. Moreover, if $x_1, x_2, \dots, x_{n+1}$ are elements in H, the elements

$$x_1,\ x_1 \star x_2,\ \dots,\ x_1 \star x_2 \star \dots \star x_{n+1}$$

are not distinct. Hence, there exist two integers l, m such that $l < m \leq n+1$ and $x_1 \star x_2 \star \cdots \star x_l = x_1 \star x_2 \star \cdots \star x_m$. Consequently,

$$\begin{aligned} x_1 \star x_2 \star \dots \star x_l &= x_1 \star x_2 \star \dots \star x_l \star (x_{l+1} \star \dots \star x_m)^2 \\ &= x_1 \star x_2 \star \dots \star x_l \star 0 = 0, \end{aligned}$$

hence $(H, \star)$ is nilpotent.

Theorem 5.1. *Let H be a finite set of size ≥ 3 and $\circ$ a hyperproduct on H. The following conditions are equivalent:*

1. *$(H, \circ) \in \mathfrak{G}^*_{0,d}$;*
2. *there exists a nilpotent semigroup $(H, \star)$ such that*
   - *a) $(H, \star)$ is a 0-extension of an elementary Abelian 2-group $(H_+, \cdot)$;*
   - *b) $x \star 1 = 1 \star x = x \star x = 0$ for all $x \in H_+$;*
   - *c) $(H, \circ)$ is merging of $(H, \star)$ with $(H_+, \cdot)$.*

Proof.

1) $\Rightarrow$ 2) Immediate consequence of Theorem 3.2 and the fact that $(H,\circ)$ belongs to $\in \mathfrak{G}^*_{0,d}$.

2) $\Rightarrow$ 1) By c) and Proposition 2.2, $(H,\circ)$ is a 0-semihypergroup. By b), for all $x \in H_+$, the set $\{0,1,x\}$ is a subsemihypergroup of $(H,\circ)$ isomorphic to $(I_2,\circ_6)$ in Example 2.5. Finally, by a), c) and Theorem 3.2, we have $(H,\circ) \in \mathfrak{G}^*_{0,d}$. □

In the following proposition we show a tight bound on the nilpotency rank of $(H,\star)$.

Proposition 5.1. *Let* $(H,\circ) \in \mathfrak{G}^*_{0,d}(n)$ *and let* ν *be the nilpotency rank of its associated semigroup* $(H,\star)$*. Then* $\binom{\nu}{2} \leq n-2$*.*

Proof. Let $q = \nu - 1$. By hypothesis, there exist q elements $a_1, a_2, \dots, a_q$ in $H - \{0,1\}$ such that $a_1 \star a_2 \star \cdots \star a_q \neq 0$. Necessarily, these elements are pairwise distinct. We arrange the proof in four steps.

α) For all $i \in \{1,2,\dots,q-1\}$ and for all integers k such that $0 < k \leq q-i$ we have $a_i \star a_{i+1} \star \cdots \star a_{i+k} \notin \{0,1\}$. Indeed, if $a_i \star a_{i+1} \star \cdots \star a_{i+k} = 0$ then we obtain the contradiction $a_1 \star a_2 \star \cdots \star a_q = 0$. If $a_i \star a_{i+1} \star \cdots \star a_{i+k} = 1$ then in the residual group of $(H,\circ)$ we have $a_i \cdot a_{i+1} \cdots a_{i+k} = 1$ and $a_i = a_{i+1} \cdots a_{i+k} = a_{i+1} \star \cdots \star a_{i+k}$. It follows the contradiction $1 = a_i \star a_{i+1} \star \cdots \star a_{i+k} = a_i \star a_i = 0$.

β) For all $i \in \{1,2,\dots,q-1\}$ and integers r,s such that $0 \leq r < s \leq q-i$ we have $a_i \star a_{i+1} \star \cdots \star a_{i+r} \neq a_i \star a_{i+1} \star \cdots \star a_{i+s}$. Indeed, if $a_i \star a_{i+1} \star \cdots \star a_{i+r} = a_i \star a_{i+1} \star \cdots \star a_{i+s}$ then

$$\begin{aligned} a_i \star a_{i+1} \star \cdots \star a_{i+s} &= a_i \star \cdots \star a_{i+r} \star a_{i+r+1} \star \cdots \star a_{i+s} \\ &= a_i \star \cdots \star \underbrace{a_{i+s} \star a_{i+r+1} \cdots \star a_{i+s}}_{=0} \\ &= a_i \star \cdots \star a_{i+s-1} \star 0 = 0, \end{aligned}$$

which is impossible for α).

γ) $a_i \star a_{i+1} \star \cdots \star a_{i+r} \neq a_j \star a_{j+1} \star \cdots \star a_{j+s}$, for all $i \in \{1,2,\dots,q-1\}$, $i < j$, $0 \leq r \leq q-i$ and $0 \leq s \leq q-j$. Indeed, if $a_i \star \cdots \star a_{i+r} = a_j \star \cdots \star a_{j+s}$

for $i < j$ then

$$\begin{aligned} a_i \star \cdots \star a_{j-1} \star a_j \star \cdots \star a_{j+s} &= \underbrace{a_i \star \cdots \star a_{j-1} \star a_i}_{=0} \star \cdots \star a_{i+r} \\ &= 0 \star a_{i+1} \star \cdots \star a_{i+r} = 0, \end{aligned}$$

which is impossible for α).

δ) By the preceding points, all the following elements in $H - \{0, 1\}$ are pairwise distinct:

$$a_1, a_2 \ldots, a_q \text{ and } a_i \star a_{i+1} \star \cdots \star a_j, \quad \text{for all } 1 \leq i < j \leq q.$$

Therefore $q + (q-1) + (q-2) + \ldots + 2 + 1 = \binom{q+1}{2} \leq |H| - 2$ and we obtain the proof. ☐

Remark 5.1. We can exploit a construction found in [8] in order to determine the nilpotent semigroups of rank 3 which are 0-extensions of a group $(G, \cdot)$ of size ≥ 2. Fix an element $0 \notin G$ and a non-empty set $A \subseteq G$ and let $B = (G - A) \cup \{0\}$. On the set $S = G \cup \{0\}$ consider a product $\otimes$ fulfilling the following condition: for every $x, y \in S$

$$x \otimes y = \begin{cases} xy \text{ or } 0 & \text{if } x, y \in A \text{ and } xy \in B; \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

For all $x, y, z \in S$, we have that $(x \otimes y) \otimes z = x \otimes (y \otimes z) = 0$, and so $(S, \otimes)$ is a nilpotent 0-semigroup of rank 2 or 3. Clearly $(S, \otimes)$ is a 0-extension of $(G, \cdot)$. We note that $(S, \otimes)$ is a nilpotent semigroup of rank 2 if and only if $x \otimes y = 0$ for all $x, y \in A$, and in particular if A is a subsemigroup of $(G, \cdot)$.

Conversely, if $(S, \otimes)$ is a nilpotent semigroup of rank 3 which is also a 0-extension of a group $(G, \cdot)$ then $S \otimes S \subset S$. Thus, putting $B = S \otimes S$, the set $A = G - B$ is a non-empty subset of G and $0 \notin A$. Moreover, for all $x, y \in S$ we have:

- if $x, y \in A$ then $x \otimes y \in B$ and $(x \otimes y \neq 0 \ \Rightarrow\ x \otimes y = xy)$;
- if $x \notin A$ or $y \notin A$ then $\{x, y\} \cap B \neq \varnothing$ and $x \otimes y = 0$.

Therefore the operation $\otimes$ fulfills the condition (13). The result is true even if the nilpotency rank of $(S, \otimes)$ is 2: in that case $A = G$ and $B = \{0\}$.

The Theorem 5.1 and the construction described in the remark above characterizes the semigroups having nilpotency rank 3 associated to a semihypergroup in $\mathfrak{G}^*_{0,d}$.

Theorem 5.2. *Let $(H, \circ)$ be a finite semihypergroup in $\mathfrak{G}^*_{0,d}$ and let $(H, \star)$ and $(H_+, \cdot)$ be its associated 0-semigroup and residual group, respectively. The nilpotency rank of $(H, \star)$ is 3 if and only if $\{0\} \neq H \star H \subset H$ and*

$$x \star y = \begin{cases} xy \text{ or } 0 & \text{if } x, y \in H - (H \star H) \text{ and } xy \in H \star H; \\ 0 & \text{otherwise.} \end{cases} \qquad (14)$$

6 Computation of isomorphism classes in $\mathfrak{G}_0(5)$ and $\mathfrak{G}_0(9)$

In this section we present two results obtained with the help of symbolic computation software. We determine the number of semihypergroups in $\mathfrak{G}_0(5)$ and $\mathfrak{G}_0(9)$, apart of isomorphisms. To these goals, first we find the semihypergroups in $\mathfrak{G}^*_{0,s}(t)$ and $\mathfrak{G}^*_{0,d}(t)$ with $t \in \{5, 9\}$ and after, using Propositions 4.3 and 4.5, we find all the elements in $\mathfrak{G}_0(5)$ and $\mathfrak{G}_0(9)$, up to isomorphisms. Clearly, by Proposition 4.1, it is sufficient to determine the semihypergroups in $\mathfrak{G}^*_{0,d}(t)$ because those in $\mathfrak{G}^*_{0,s}(t)$ differ only in the hyperproduct $1 \bullet 1 = \{1\}$. By Theorem 5.1, the characterization of finite semihypergroups in $\mathfrak{G}^*_{0,d}$ is based on the determination of all nilpotent semigroups $(H, \star)$ that are 0-extensions of an Abelian 2-group. Furthermore, by Corollary 3.1, $(H, \star)$ must also fulfil the condition (1).

6.1 Semihypergroups in $\mathfrak{G}_0(5)$

The support of semihypergroups in $\mathfrak{G}^*_{0,d}(5)$ is $H = \{0, 1, 2, 3, 4\}$, and the residual group $(H_+, \cdot)$ is $\mathbb{Z}_2^2$, which is represented by the following table:

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

By Theorem 5.1, every semihypergroup $(H, \bullet) \in \mathfrak{G}^*_{0,d}(5)$ is the merging of $(H, \star)$ with $(H_+, \cdot)$, where $(H, \star)$ verifies the following conditions, for all distinct elements $x, y, z \in H_+$:

$$\begin{cases} 1 \star x = x \star 1 = x \star x = 0; \\ x \star y = z \ \Rightarrow \ x \star z = z \star x = y \star z = z \star y = 0. \end{cases} \tag{15}$$

By Proposition 5.1, the nilpotency rank of $(H, \star)$ is 2 or 3. Obviously, if the rank is 2 we have $x \star y = 0$, for all $x, y \in H$. If the nilpotency rank is 3, then there exists a product $x \star y \neq 0$, with $x, y \in \{2, 3, 4\}$. By Theorem 5.2 and (15), it is not restrictive to suppose that $B = H \star H = \{0, 2\}$ and $A = \{1, 3, 4\}$. In this case, at least one of the products $3 \star 4$ and $4 \star 3$ differs from 0, and so we have $3 \star 2 = 2 \star 3 = 4 \star 2 = 2 \star 4 = 0$. Apart of isomorphisms, we have two nilpotent semigroups with rank 3 which are 0-extension of $(H_+, \cdot)$. Their product tables are the following:

$\star_1$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	2
4	0	0	0	0	0

$\star_2$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	2
4	0	0	0	2	0

By considering also the (trivial) nilpotent semigroup of rank 2, we obtain in $\mathfrak{G}^*_{0,d}(5)$ three semi-hypergroups which are merging of the preceding nilpotent semigroups with $(H_+, \cdot)$, whose products are the following:

$\bullet_0$	0	1	2	3	4
0	0	0	0	0	0
1	0	0, 1	0, 2	0, 3	0, 4
2	0	0, 2	0, 1	0, 4	0, 3
3	0	0, 3	0, 4	0, 1	0, 2
4	0	0, 4	0, 3	0, 2	0, 1

$\bullet_1$	0	1	2	3	4
0	0	0	0	0	0
1	0	0, 1	0, 2	0, 3	0, 4
2	0	0, 2	0, 1	0, 4	0, 3
3	0	0, 3	0, 4	0, 1	2
4	0	0, 4	0, 3	0, 2	0, 1

$\bullet_2$	0	1	2	3	4
0	0	0	0	0	0
1	0	0, 1	0, 2	0, 3	0, 4
2	0	0, 2	0, 1	0, 4	0, 3
3	0	0, 3	0, 4	0, 1	2
4	0	0, 4	0, 3	2	0, 1

Using the construction shown in Proposition 4.3, an exhaustive search of all 0-semigroups that extend $(H_+, \cdot)$ yields 11 semihypergroup in $\mathfrak{G}_{0,d}(5)$. In particular, the number of semihypergroups which are obtained from $(H, \bullet_0)$, $(H, \bullet_1)$ and $(H, \bullet_2)$ is 4, 3, and 4, respectively.

The semihypergroups in $\mathfrak{G}^*_{0,s}(5)$ can be determined from those in $\mathfrak{G}^*_{0,d}(5)$, by applying Proposition 4.1. We denote such semi-hypergroups with $(H, \bullet_3)$, $(H, \bullet_4)$ and $(H, \bullet_5)$. Analogously to the previous case, by using Proposition 4.5 we can derive 30 semihypergroup in $\mathfrak{G}_{0,s}$. Those obtained from $(H, \bullet_3)$, $(H, \bullet_4)$ and $(H, \bullet_5)$ are 20, 8, and 2, respectively.

In conclusion, we have the following result:

Theorem 6.1. *Up to isomorphisms there exist* 11, 30 *and* 41 *semihypergroups in* $\mathfrak{G}_{0,d}(5)$, $\mathfrak{G}_{0,s}(5)$ *and* $\mathfrak{G}_0(5)$, *respectively.*

6.2 Semihypergroups in $\mathfrak{G}_0(9)$

As in the preceding case, first of all we outline the arguments which allow us to determine, up to isomorphisms, the number of semihypergroups in $\mathfrak{G}^*_{0,d}(9)$ and $\mathfrak{G}^*_{0,s}(9)$. The support of semihypergroups in $\mathfrak{G}^*_{0,d}(9)$ is $H = \{0, 1, 2, \ldots, 8\}$, and the residual group $(H_+, \cdot)$ is $\mathbb{Z}_2^3$, which is represented by the following table:

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

(16)

By Theorem 5.1, every semihypergroup $(H, \bullet) \in \mathfrak{G}^*_{0,d}(9)$ is the merging of the associated nilpotent semigroup $(H, \star)$ with $(H_+, \cdot)$, where $(H, \star)$ verifies the conditions (15). Moreover, for Proposition 5.1, the nilpotency rank of $(H, \star)$ can be 2 or 3 or 4.

If the nilpotency rank of $(H, \star)$ is 4 then there will be at least a nonzero product of three distinct elements in $H - \{0, 1\}$. It is not restrictive to suppose that $2 \star 3 \star 5 \neq 0$ and so $2 \star 3 = 4$, $4 \star 5 = 8$, $3 \star 5 = 7$ and $2 \star 7 = 8$. Consequently, by (15), we deduce that $4 \star 2 = 2 \star 4 = 3 \star 4 = 4 \star 3 = 0$, $8 \star 4 = 4 \star 8 = 8 \star 5 = 5 \star 8 = 0$, $7 \star 3 = 3 \star 7 = 7 \star 5 = 5 \star 7 = 0$ and $8 \star 2 = 2 \star 8 = 8 \star 7 = 7 \star 8 = 0$. Therefore we arrive at the following partial product table:

$\star$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	4	0			8	0
3	0	0		0	0	7		0	
4	0	0	0	0	0	8			0
5	0	0				0		0	0
6	0	0					0		
7	0	0		0		0		0	0
8	0	0	0		0	0		0	0

where the empty cells can assume the value 0 or the corresponding value in (16). Extensive computations performed with the help of a symbolic computation software produced 13 nilpotent semigroups of rank 4, apart of isomorphisms.

Now we determine the nilpotent semigroups of rank three which are 0-extension of (16). From Theorem 5.2 and (15) we obtain that $H \star H$ can be equal to one of the following sets, up to isomorphisms:

$$B_1 = \{0, 2\},\ B_2 = \{0, 2, 3\},\ B_3 = \{0, 2, 3, 4\},\ B_4 = \{0, 2, 3, 5\}$$

For instance, in case B_3 the operation $\star$ must verify the conditions

$$\begin{aligned} 2 &\in \{5 \star 6, 6 \star 5, 7 \star 8, 8 \star 7\} \subseteq \{0, 2\} \\ 3 &\in \{5 \star 7, 7 \star 5, 6 \star 8, 8 \star 6\} \subseteq \{0, 3\} \\ 4 &\in \{5 \star 8, 8 \star 5, 6 \star 7, 7 \star 6\} \subseteq \{0, 4\} \end{aligned}$$

and all other products must be 0. Symbolic computations yield $n_{B_3} = 176$ nilpotent semigroups of rank 3. Analogously, using the sets B_1, B_2 and B_4

we obtain the following numbers: $n_{B_1} = 10$, $n_{B_2} = 36$ and $n_{B_4} = 10$. By considering also the nilpotent semigroup of rank 2, we get a total of 246 nilpotent semigroups. This number is equal to the number of semihypergroups in $\mathfrak{G}^*_{0,d}(9)$ and, by Proposition 4.1, also the number of semihypergroups in $\mathfrak{G}^*_{0,s}(9)$, apart of isomorphisms. We collect in the following table the results obtained for all possible cases arising from Propositions 4.3 and 4.5:

rank	2	3				4
		n_{B_1}	n_{B_2}	n_{B_3}	n_{B_4}	
$\mathfrak{G}^*_{0,d}(9)$, $\mathfrak{G}^*_{0,s}(9)$	1	10	36	176	10	13
$\mathfrak{G}_{0,d}(9)$	10	265	990	2495	112	124
$\mathfrak{G}_{0,s}(9)$	264	1014	1204	566	110	118

In conclusion, we have the following statement.

Theorem 6.2. *Up to isomorphisms, there exist* 3996*,* 3276 *and* 7272 *semihypergroups in* $\mathfrak{G}_{0,d}(9)$*,* $\mathfrak{G}_{0,s}(9)$ *and* $\mathfrak{G}_0(9)$*, respectively.*

7 Conclusions

In our preceding papers [5, 6, 9] we faced the study of simple semihypergroups where the fundamental relation β is not transitive, in all subsemihypergroups of size ≥ 3. These semihypergroups, which we called *fully simple*, own a right (or left) zero scalar element and all their hyperproducts have size ≤ 2. In finite case, the number of isomorphism classes of fully simple semihypergroups of size $n \geq 3$ is n-th term of the sequence A000712. Motivated by these results, in [7] we considered a class of simple semihypergroups having an absorbing element 0 and the relation β not transitive in every subsemihypergroup of size ≥ 3. These semihypergroups, which we call *fully 0-simple*, differ substantially from fully simple semihypergroups since their hyperproducts can have size greater than two. An example is the following:

$\circ$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	$0,2$	$0,5$	$0,2$	$0,2,6$	$0,1,3,4$	$0,5$
2	0	$0,5$	$0,1,3,4$	$0,5$	$0,5$	$0,2,6$	$0,1,3,4$
3	0	$0,2$	$0,5$	$0,2$	$0,2,6$	$0,1,3,4$	$0,5$
4	0	$0,2,6$	$0,5$	$0,2,6$	$0,2,6$	$0,1,3,4$	$0,5$
5	0	$0,1,3,4$	$0,2,6$	$0,1,3,4$	$0,1,3,4$	$0,5$	$0,2,6$
6	0	$0,5$	$0,1,3,4$	$0,5$	$0,5$	$0,2,6$	$0,1,3,4$

Still in [7] we analyzed the subclass of fully 0-simple semihypergroups having all hyperproducts of size ≤ 2. The hyperproduct tables of those semihypergroups can be regarded as the superposition of the product table of a 0-semigroup and that of either a (right or left) zero semigroup or an elementary Abelian 2-group. In particular, we considered the class $\mathfrak{R}_0$ of fully 0-simple semihypergroups such that for all pairs (x, y) of distinct nonzero elements the subset $\{0, x, y\}$ is a subsemihypergroup whose hyperproduct table contains the product table of a right zero semigroup. Finally, we proved that the number of isomorphism classes of semihypergroups in $\mathfrak{R}_0$ having size n is the n-th term of the sequence A000070.

In the present paper, we deepen the understanding of that superposition of product tables, which we call *merging*, see Definition 2.2. Moreover, we define and study the class $\mathfrak{G}_0$ of fully 0-simple semihypergroups with a particular element $1 \neq 0$ such that, for all $x \notin \{0, 1\}$, the subset $\{0, 1, x\}$ is isomorphic to one of the semihypergroups listed in Example 2.5. Those semihypergroups of size 3 are the only fully 0-simple semihypergroups which can be obtained by a merging with the group $\mathbb{Z}_2$. In particular, we prove that all semihypergroups in $\mathfrak{G}_0$ are obtained as a merging with an elementary Abelian 2-group. Apart of isomorphisms, there are 41 $\mathfrak{G}_0$-semihypergroups of size 5 and 7272 of size 9.

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