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CICLO XXX

Nonlinear differential equations having non-sign-definite weights

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I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original. The present dissertation is my own work and contains nothing which is the outcome of work done under the supervision of Professor Fabio Zanolin in order to achieve the title of Research Doctorate at the University of Udine.

Udine, Academic Year 2016-2017

To my mom and dad
my loving *fixed points*.

Publications

- E. Sovrano and F. Zanolin. “Remarks on Dirichlet problems with sublinear growth at infinity.” In: *Rend. Istit. Mat. Univ. Trieste* 47 (2015), pp. 267-305. Referred as [SZ15].
- E. Sovrano and F. Zanolin. “The Ambrosetti-Prodi periodic problem: Different routes to complex dynamics.” In: *Dynamic Systems and Applications* (2017). Referred as [SZ17d].
- E. Sovrano and F. Zanolin. “Indefinite weight nonlinear problems with Neumann boundary conditions.” In: *J. Math. Anal. Appl.* 452.1 (2017), pp. 126-147. Referred as [SZ17c].
- E. Sovrano. “How to get complex dynamics? A note on a topological approach.” (submitted, 2016). Referred as [Sovbm].
- E. Sovrano. “A negative answer to a conjecture arising in the study of selection-migration models in population genetics.” In: *J. Math. Biol.* (to appear, 2018). Referred as [Sov17].
- E. Sovrano. “Ambrosetti-Prodi type result to a Neumann problem via a topological approach.” In: *Discrete Contin. Dyn. Syst. Ser. S* (2018). Referred as [Sov18].
- G. Feltrin and E. Sovrano, “Three positive solutions to an indefinite Neumann problem: a shooting method?.” In: *Nonlinear Analysis* (to appear, 2018). Referred as [FS18].
- E. Sovrano and F. Zanolin, “Ambrosetti-Prodi periodic problem under local coercivity conditions.” In: *Adv. Nonlinear Stud.* (to appear, 2018). Referred as [SZ17b].
- E. Sovrano and F. Zanolin, “A periodic problem for first order differential equations with locally coercive nonlinearities.” In: *Rend. Istit. Mat. Univ. Trieste* (2017). Referred as [SZ17a].

Abstract

In the present PhD thesis we deal with the study of the existence, multiplicity and complex behaviors of solutions for some classes of boundary value problems associated with second order nonlinear ordinary differential equations of the form

$$u'' + f(u)u' + g(t,u) = s, \quad t \in \mathcal{I}, \tag{$\star$}$$

or

$$u'' + g(t,u) = 0, \quad t \in \mathcal{I}, \tag{$\star\star$}$$

where $\mathcal{I}$ is a bounded interval, $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, $s \in \mathbb{R}$ and $g\colon \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ is a perturbation term characterizing the problems.

The results carried out in this dissertation are mainly based on dynamical and topological approaches. The issues we address have arisen in the field of partial differential equations. For this reason, we do not treat only the case of ordinary differential equations, but also we take advantage of some results achieved in the one dimensional setting to give applications to nonlinear boundary value problems associated with partial differential equations.

In the first part of the thesis, we are interested on a problem suggested by Antonio Ambrosetti in "Observations on global inversion theorems" (2011). In more detail, we deal with a periodic boundary value problem associated with ($\star$) where the perturbation term is given by $g(t,u) := a(t)\phi(u) - p(t)$. We assume that a, $p \in L^\infty(\mathcal{I})$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $\lim_{|\xi|\to\infty} \phi(\xi) = +\infty$. In this context, if the weight term $a(t)$ is such that $a(t) \geq 0$ for a.e. $t \in \mathcal{I}$ and $\int_{\mathcal{I}} a(t)\,dt > 0$, we generalize the result of multiplicity of solutions given by C. Fabry, J. Mawhin and M.N. Nakashama in "A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations" (1986). We extend this kind of improvement also to more general nonlinear terms under local coercivity conditions. In this framework, we also treat in the same spirit Neumann problems associated with second order ordinary differential equations and periodic problems associated with first order ones.

Furthermore, we face the classical case of a periodic Ambrosetti-Prodi problem with a weight term $a(t)$ which is constant and positive. Here, considering in ($\star\star$) a nonlinearity $g(t,u) := \phi(u) - h(t)$, we provide several conditions on the nonlinearity and the perturbative term that ensure the presence of complex behaviors for the solutions of the associated T-periodic problem. We also compare these outcomes with the result of stability carried out by Rafael Ortega in "Stability of a periodic problem of Ambrosetti-Prodi type" (1990). The case with damping term is discussed as well.

In the second part of this work, we solve a conjecture by Yuan Lou and Thomas Nagylaki stated in "A semilinear parabolic system for migration and selection in population genetics" (2002). The problem refers to the number of positive solutions for Neumann boundary value problems associated with ($\star\star$) when the perturbation term is given by $g(t,u) := \lambda w(t)\psi(u)$ with $\lambda > 0$, $w \in L^\infty(\mathcal{I})$ a sign-changing weight term such that $\int_{\mathcal{I}} w(t)\,dt < 0$ and $\psi\colon [0,1] \to [0,\infty[$ a non-concave continuous function satisfying $\psi(0) = 0 = \psi(1)$ and such that the map $\xi \mapsto \psi(\xi)/\xi$ is monotone decreasing.

In addition to this outcome, other new results of multiplicity of positive solutions are presented as well, for both Neumann or Dirichlet boundary value problems, by means of a particular choice of indefinite weight terms $w(t)$ and different positive nonlinear terms $\psi(u)$ defined on the interval $[0,1]$ or on the positive real semi-axis $[0,+\infty[$.

du'' + a(t) f(u) = 0
a⁺ > 0 [0, τ]
a⁻ > 0 [τ, T]
f(s) =
0
S
s₀
∫₀¹ f
= ∫₀¹ f
∫₀¹ f
f'(s₀) > 0
a(t) = a⁺ − a⁻
λ μ

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Contents

Introduction

> “ ... *nonlinear systems are surely the rule, not the exception, not only in research, but also in the everyday world.* ”
>
> Robert M. May, *Simple mathematical models with very complicated dynamics*, Nature (1976).

It is generally recognized that phenomena occurring in our world have nonlinear features so that the field of Nonlinear Analysis becomes essential in the study of several kinds of problems in mathematics, biology, mechanics, and so on ...

The research carried out through the PhD program did explorations in this exciting field. In fact, we have examined qualitative aspects for classes of nonlinear ordinary differential equations (ODEs) and some related issues for nonlinear partial differential equations (PDEs). In the present dissertation, we devote our attention to the study of existence, multiplicity and complex behaviors of solutions for a selection of boundary value problems (BVPs). More precisely, we consider concrete problems associated with differential equations characterized by “jumping nonlinearities” or, alternatively, by nonlinear terms with “indefinite weights”, for which it has been a case of love at first sight. Among them, there are two challenging questions that were raised in the early 2000’s in the context of *Ambrosetti-Prodi problems* and *indefinite weight problems*, respectively.

We now introduce the reader to the issues tackled, highlighting two questions, with the purpose to figure out the class of problems considered. Afterwards, we will point out the main contributions achieved and we will briefly discuss the content of the thesis.

Motivations

One of the goals of the present manuscript is to discuss and give answers on two previously open problems coming from the field of PDEs. These problems firstly appeared in the following works:

¶1. “Observations on global inversion theorems” by A. Ambrosetti [Amb11];

¶2. “A semilinear parabolic system for migration and selection in population genetics” by Y. Lou and L. Nagylaki [LN02].

These works were motivating new perspectives of investigations on this field of research. With special emphasis to the papers just reported, we are going to summarize as follows the main questions to which we have sought solutions.

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Part ¶1. The first question addressed regards the *periodic case* of the so-called *Ambrosetti-Prodi problem.* This problem is part of a classical topic in Nonlinear Analysis which involved, at the beginning, a Dirichlet problem of the following type

$$\begin{cases} \Delta u + \phi(u) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, is a bounded domain with boundary of class $C^{2,\alpha}$, Δ denotes the Laplace operator, $h \in C^{0,\alpha}(\overline{\Omega})$ for $\alpha \in]0,1[$ and $\phi \in C^2(\mathbb{R})$ a strictly convex function such that $\phi(0) = 0$ and $0 < \lim_{\xi\to-\infty} \phi'(\xi) < \lambda_1^{\mathcal{D}}(-\Delta;\Omega) < \lim_{\xi\to+\infty} \phi'(\xi) < \lambda_2^{\mathcal{D}}(-\Delta;\Omega)$ with $\lambda_1^{\mathcal{D}}(-\Delta;\Omega)$, $\lambda_2^{\mathcal{D}}(-\Delta;\Omega)$ the first two eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$ (see [AP72]). A precise description of the set of the solutions for this Dirichlet BVP was carried out in the groundbreaking work [AP72] and it provided the classical alternative of *zero, one or two solutions*, depending on the position of the function h with respect to a suitable manifold of codimension 1 in the space $C^{0,\alpha}(\overline{\Omega})$. The main feature which characterizes this problem is the interference of the derivative of the nonlinear term ϕ with the spectrum of the linear operator. Nowadays, it is well known that this interference has a strongly influence on the number of solutions for the problem. Beyond the Dirichlet BVP just mentioned, such kinds of PDEs have stimulated further investigations that consider different boundary conditions. In particular, addressing the periodic case, an open question is pointed out in [Amb11] and, in our context, it states:

> To study the periodic BVP for the second order ODE given by
>
> $$u'' + \phi(u) = h(x)$$
>
> where h is T-periodic, for some $T > 0$, and ϕ satisfies
>
> $$-\infty < \lim_{\xi\to-\infty} \frac{\phi(\xi)}{\xi} < \lambda_1 < \lim_{\xi\to+\infty} \frac{\phi(\xi)}{\xi} < \lambda_2$$
>
> for λ_1, λ_2 the first two eigenvalues associated with the differential operator $-u''$ with T-periodic boundary conditions [Amb11, p. 13].

In the mathematical literature, a great deal of work has already done for Ambrosetti-Prodi problems under periodic boundary conditions, concerning existence, multiplicity and stability of periodic solutions (e.g. [FMN86; NO03; Ort90; PMM92]).

Despite all this, there are still new directions to undertake in research. One of these could be the possibility to improve some assumptions, which are standard in this subject starting from the Eighties, up to guarantee the same (weak) alternative of zero, one or two solutions [FMN86], typical for periodic Ambrosetti-Prodi problems. Another one could be querying about the observation of complex behaviors due to high multiplicity results of periodic solutions starting from [PMM92].

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Part ¶2. The second question attacked take place in the context of population genetics and deals with PDEs of reaction-diffusion type. At this juncture, the interest is in the following class of parabolic PDEs with no flux on the boundary:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda w(x)\psi(u) & \text{in } \Omega\times]0,+\infty[, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\times]0,+\infty[, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded open connected set with boundary of class C^2, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$, λ is a positive real parameter, $w \in L^\infty(\Omega)$ a sign changing function and $\psi : [0,1] \to \mathbb{R}$ is a function of class C^2 such that

$$\psi(0) = 0 = \psi(1),\ \psi(\xi) > 0 \text{ for all } \xi \in]0,1[\text{ and } \psi'(1) < 0 < \psi'(0).$$

Since the function w is allowed to change its sign, we enter in the classical topic of *indefinite weight problems*.

Under the additional hypothesis of concavity for the function ψ, it is known that for λ sufficiently large and $\int_\Omega w(x)\,dx < 0$ this problem admits a unique nontrivial equilibrium, i.e. positive stationary nontrivial solution (see [Hen81]). Close to this achievement, there are results of multiplicity of nontrivial equilibria when ψ is not concave and the map $\xi \mapsto \psi(\xi)/\xi$ is not monotone decreasing (see [LNS10]).

Taking advantage of the above preface, we are ready to introduce the open problem pointed out in [LN02] which involves the study of the number of equilibria for indefinite problems when the concavity assumption of ψ is weakened to the decreasing monotonicity of the map $\xi \mapsto \psi(\xi)/\xi$. This way, in our framework, the question reads as follows.

> Conjecture. Suppose that $w(x) > 0$ on a set of positive measure in Ω and $\int_\Omega w(x)\,dx < 0$ then, if the map $\xi \mapsto \psi(\xi)/\xi$ is monotone decreasing in $]0,1[$, the considered parabolic problem has at most one nontrivial equilibrium $0 < u(t,x) < 1$ for every $x \in \overline{\Omega}$, which, if it exists, is globally asymptotically stable [LNN13, p. 4364].

This conjecture can be interpreted as win-win, in the sense that either a positive or a negative answer would be a great advance in this field leading to further directions of work. Accordingly, giving a reply to this open problem, it motivates in turn the interest in addressing new issues on BVPs associated with differential equations characterized by the presence of an indefinite weight function.

Main contributions

All the results we are going to present are collected in the papers written during the PhD program [FS18; Sovbm; Sov17; Sov18; SZ15; SZ17b; SZ17c; SZ17d; SZ17a].

First of all, from Part ¶1 and Part ¶2, we notice how some of the addressed issues have been born on the broad and active field of PDEs. Nevertheless, to give answers, we will almost restrict ourselves to treat the corresponding cases in an ODE environment. This way, we will take advantage of the one-dimensional setting to face the problems we are interested in, from several points of view, which are mainly based on dynamical or topological approaches. For this reason, our attention in this dissertation will be principally directed to second order nonlinear ODEs of the form

$$u'' + f(u)u' + g(t,u) = s \quad \text{with } t \in \mathcal{I}, \tag{$\star$}$$

or

$$u'' + g(t,u) = 0 \quad \text{with } t \in \mathcal{I}, \tag{$\star\star$}$$

where $\mathcal{I}$ is a bounded open interval, $s \in \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $g\colon \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ is a perturbation term which gathers the features of the considered problems and it will be discussed step by step in the following two parts.

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Contributions Part ¶1. Investigations in the context of periodic Ambrosetti-Prodi problems are primarily carried out by considering a T-periodic BVP associated with an ODE of the form in ($\star$). In this setting, thanks to the work of [FMN86], equation ($\star$) has no T-periodic solutions, at least one T-periodic solution or at least two T-periodic solutions according to $s < s_0$, $s = s_0$ or $s > s_0$, provided that

$$\lim_{|u| \to +\infty} g(t,u) = +\infty, \quad \text{uniformly in } t.$$

A first achievement in this thesis is the weakening of the above condition of coercivity in order to still guarantee a weak alternative of Ambrosetti-Prodi type for the T-periodic solutions (cf. [SZ17b]). As a consequence of this improvement, there is the possibility to treat cases in which the coercivity condition $g(t,u) \to +\infty$ holds only locally. To illustrate the point, let us look at the following result.

Corollary. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{|u|\to+\infty} \phi(u) = +\infty. \tag{Hϕ}$$

Let a, $p\colon \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic functions with

$$a(t) \geq 0 \text{ for all } t \in [0,T] \text{ and } a \not\equiv 0. \tag{Ha}$$

Then, there exists $s_0 \in \mathbb{R}$ such that equation

$$u'' + f(u)u' + a(t)\phi(u) = s + p(t),$$

has no T-periodic solutions, at least one T-periodic solution or at least two T-periodic solutions according to $s < s_0$, $s = s_0$ or $s > s_0$.

In the previous result, we have considered an ODE involving a nonlinearity

$$g(t,u) := a(t)\phi(u) - p(t),$$

that does not suit to be treat in a classical handling, since the uniform requirement is not satisfied. In this manner we highlight how we considerably relax the requirement assumed in [FMN86].

Finally, our main goal is to explore the number of T-periodic solutions for the parameter dependent equation ($\star$) in a Carathéodory setting under local coercivity conditions on $g(t,u)$. In [FMN86], the uniformity condition is essential to construct upper and lower solutions for equation ($\star$) (see [DCH06]). This construction is then used to proceed within the topological degree theory in function spaces presented in [GM77]. Indeed, the scheme proposed in [FMN86], produce bounded sets in the Banach space of continuous and T-periodic functions where the topological degree is different form zero, that lead to the existence of T-periodic solutions, depending on the parameter s. Without uniformity on the limits at infinity the search of a lower solution becomes a tricky and delicate question. In [SZ17b], the novelty in our approach is to provide some lower bounds for the solutions, by using a Villari's type condition (see [Vil66]). In this manner, we offset the apparently loss of a lower solution and go ahead with the same approach of [FMN86]. As a result, in this periodic framework, we recover the alternative of Ambrosetti-Prodi with more general assumptions in comparison to the ones treated up to now.

We also extend this kind of results to the case of periodic problems associated with first order ODEs of the form $u' + g(t,u) = s$. In more detail, we generalize in [SZ17a] previous works appeared in this area (cf. [Maw87b; Maw87a; Maw87c; Nka89]) including the case of a locally coercive nonlinearity g. As a first example, we consider the generalized Riccati differential equation, and so the result performed reads as follows.

Corollary. *Let a, b, $c\colon \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic functions and let $\alpha > 1$. Assume that $a(t)$ satisfies* (Ha). *Then, there exists $s_0 \in \mathbb{R}$ such that equation*

$$u' + a(t)|u|^\alpha + b(t)u + c(t) = s,$$

has no solutions, at least one solution or at least two solutions according to $s < s_0$, $s = s_0$ or $s > s_0$.

As a second example, we achieve an analogous result to the one provided for the second order case.

Corollary. *Let* $\phi\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (Hϕ). *Let* $a, p\colon \mathbb{R} \to \mathbb{R}$ *be continuous and* T*-periodic functions. Assume that* $a(t)$ *satisfies* (Ha). *Then, there exists* $s_0 \in \mathbb{R}$ *such that equation*

$$u' + a(t)\phi(u) = s + p(t),$$

has no solutions, at least one solution or at least two solutions according to $s < s_0$, $s = s_0$ *or* $s > s_0$.

We also notice that the previous corollaries are actually proved assuming the more general Carathéodory context and so we obtain a generalization of the results in [Nka89] (cf. [SZ17a, Corollary 4.1 and Corollary 4.2]).

At this point the well-known connection between periodic problems and problems under Neumann boundary conditions leads to the natural question whether we are allowed to weak the uniform condition also in the Neumann context. In this respect a second contribution in this thesis involves the case of Neumann BVPs and gives an affirmative answer to the previous question (cf. [Sov18; SZ17b]). As an application, here we report a result of Ambrosetti-Prodi type for a Neumann problem associated to ($\star$) with no damping term for a locally coercive nonlinear term.

Corollary. *Let* $\phi\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (Hϕ). *Let* $a, p\colon [0,T] \to \mathbb{R}$ *be continuous functions. Assume that* $a(t)$ *satisfies* (Ha). *Then, there exists* $s_0 \in \mathbb{R}$ *such that problem*

$$\begin{cases} u'' + a(t)\phi(u) = s + p(t), \\ u'(0) = u'(T) = 0, \end{cases}$$

has no T*-periodic solutions, at least one* T*-periodic solution or at least two* T*-periodic solutions according to* $s < s_0$, $s = s_0$ *or* $s > s_0$.

To demonstrate the previous result of multiplicity we follow two different arguments. One is still within the framework of topological degree by means of simple modifications of the approach just presented (cf. [SZ17b]). Another one exploits the shooting argument which has the advantage of being more elementary nevertheless it requires the uniqueness of the solutions for the associated initial value problems and their continuability (cf. [Sov18]).

Further analyses are carried out by considering a T-periodic BVP associated with a second order nonlinear ODE of the form ($\star\star$), or associated with the analogous equation having damping term $u'' + cu' + g(t, u) = 0$, where $c > 0$. A third achievement in this thesis is the discussion of chaotic dynamics when the nonlinearity is given by

$$g(t, u) := \phi(u) - h(t)$$

with $h(t)$ a T-periodic forcing term and ϕ a function containing the principal features about the crossing of the first eigenvalue $\lambda_1 = 0$. Namely, we consider a convex function $\phi \in C^2$ such that $\phi(0) = 0$, $\phi(\xi) > 0$ for all $\xi \neq 0$ and satisfying (Hϕ). The intent is to show the existence of many periodic solutions (harmonic and subharmonic) for

$$u'' + \phi(u) = h(t),$$

as well as, "chaos" for the Poincaré map Φ (or for its iterate) associated to the previous equation (cf. [SZ17d]).

A first case study is when the ODE considered may be treated as a small perturbation of the associated autonomous system, e.g. $h(t) = \varepsilon \sin(\omega t)$ with $\omega > 0$ arbitrary and $\varepsilon > 0$ sufficiently small. This way, by exploiting a Melnikov type approach [GH83], we detect complex behaviors for a suitable iterate of the map Φ defined on the interval $[0, T]$.

A second case study is when the perturbation $h(t)$ is not necessary small. In this case, we take into account two different methods to address the issue. Firstly, we base our approach on the works [Ged+02; KMO96] settled in Conley index theory. In this vein, assuming that $h(t) = \varepsilon h_0(\omega t)$ with h_0 an arbitrary T-periodic function and both $\omega > 0$ and $\varepsilon > 0$ sufficiently small, we detect complex behaviors for a suitable iterate of the map Φ defined on the interval $[0, T/\omega]$. Secondly, given $0 < \tau < T$, we assume $h(t) := k_1 \mathbb{1}_{[0,\tau[}(t) + k_2 \mathbb{1}_{[\tau,T]}(t)$, where $0 \leq k_1 < k_2$ and $\mathbb{1}_A$ denotes the indicator function of the set A. In other words, we treat the case of "switched systems" that is a matter of general interest in the field of control theory [Bac14]. Consequently, we consider a topological argument called "stretching along the paths" method [MPZ09] and, assuming that τ and $T - \tau$ are sufficiently large, we detect complex behaviors for the map Φ defined on the interval $[0, T]$. This way, we deduce that the chaotic region found out require $\lim_{\xi \to +\infty} \phi(\xi)/\xi > \lambda_2$, where λ_2 is the first positive eigenvalue of $-u''$ with T-periodic boundary conditions. This fact is not surprising looking at stability results achieved in this framework in [Ort89; Ort90].

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Contributions Part ¶2. Investigations on the topic of indefinite weight problems are carried out mainly with respect to BVPs associated with $(\star\star)$, where $g(t, u) := w(t)\psi(u)$, involving different boundary conditions, for instance Dirichlet or Neumann ones. This way the indefinite problems we face became

$$(\mathscr{IP}) \qquad \begin{cases} u'' + w(t)\psi(u) = 0, \\ \mathfrak{BC}(u, u') = 0_{\mathbb{R}^2}, \end{cases}$$

with $\mathfrak{BC}(u, u') = (u(\omega_1), u(\omega_2))$ or $\mathfrak{BC}(u, u') = (u'(\omega_1), u'(\omega_2))$, $\omega_1 < \omega_2$. In this context, the weight term $w\colon [\omega_1, \omega_2] \to \mathbb{R}$ is a sign-changing function and the nonlinearity ψ satisfies one of the following two conditions:

- (ψ_1) $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ continuous and such that $\psi(0) = 0$, $\psi(\xi) > 0$ $\forall\, \xi \in \mathbb{R}^+_0$ and $\lim_{\xi \to +\infty} \psi(\xi)/\xi = 0$, i.e. ψ is sublinear at ∞,
- (ψ_2) $\psi\colon [0, 1] \to \mathbb{R}^+$ continuous and such that $\psi(0) = 0 = \psi(1)$ and $\psi(\xi) > 0$ $\forall\, \xi \in]0, 1[$.

Our main goal is to discuss the number of positive solutions for problem $(\mathscr{IP})$ with respect to the features of the nonlinear term ψ or of the indefinite weight term w. This way further achievements in the present thesis regard multiplicity results of positive solutions for indefinite weight problems of the form $(\mathscr{IP})$ when ψ satisfies either (ψ_1) or (ψ_2).

First of all, let us start by considering Neumann boundary conditions and functions ψ satisfying (ψ_2). Here, we give two counterexamples to a conjecture by Lou and Nagylaki in population genetics [LN02]. More specifically, we construct two explicit non concave functions $\psi(\xi)$ which satisfy the decreasing monotonicity of $\xi \mapsto \psi(\xi)/\xi$ so that for a certain weight function w the migration-selection model can have at least three positive steady states. We point out one of our counterexamples as follows.

Proposition. *Let* $\psi : [0, 1] \to \mathbb{R}$ *be such that* $\psi(\xi) := \xi(1 - \xi)(1 - 3\xi + 3\xi^2)$. *Assume* $w : [\omega_1, \omega_2] \to \mathbb{R}$ *be defined as* $w(t) := -\mathbb{1}_{[\omega_1,0[}(t) + \mathbb{1}_{[0,\omega_2]}(t)$ *with* $\omega_1 = -0.21$ *and* $\omega_2 = 0.2$. *Then, for* $\lambda = 45$ *the Neumann problem associated with* $u'' + \lambda w(x)\psi(u) = 0$ *on* $[\omega_1, \omega_2]$ *has at least* 3 *solutions such that* $0 < u(t) < 1$ *for all* $t \in [\omega_1, \omega_2]$.

It is worth noting that the concavity of ψ is a sufficient condition to ensure the uniqueness of a nontrivial positive solutions for indefinite problems $(\mathscr{IP})$ when ψ satisfies either (ψ_1) or (ψ_2). Accordingly, aimed by the works of [BH90; BO86], we face also the question whether the result of uniqueness under the monotonicity request on $\xi \mapsto \psi(\xi)/\xi$ is still true for Dirichlet problems of the form $(\mathscr{IP})$ when ψ satisfies (ψ_1) and we give a negative answer as well.

Proposition. *Let* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be such that* $\psi(\xi) := 10\xi e^{-3\xi^2} + \xi/(|\xi| + 1)$. *Assume* $w : [\omega_1, \omega_2] \to \mathbb{R}$ *be defined as* $w(x) := (1 - |x|)^5 \cos\left(\frac{9\pi}{2}|x|^{1.2}\right)$ *with* $\omega_1 = -1$, $\omega_2 = 1$. *Then, for* $\lambda = 80$ *the Dirichlet problem associated with* $u'' + \lambda w(x)\psi(u) = 0$ *on* $[\omega_1, \omega_2]$ *has at least* 5 *solutions such that* $0 < u(t)$ *for all* $t \in]\omega_1, \omega_2[$.

As a next direction of research we focus on the features of the weight term w, considering as starting point of investigation the works by López-Gómez [LG97; LG00]. So that, let us typify the weight term as follows. We suppose that there exist σ, τ with $\omega_1 < \sigma < \tau < \omega_2$ such that

$$\begin{aligned} w(t) &\geq 0, \quad w \not\equiv 0, \quad \text{for a.e. } t \in [\omega_1, \sigma], \\ w(t) &\leq 0, \quad w \not\equiv 0, \quad \text{for a.e. } t \in [\sigma, \tau], \\ w(t) &\geq 0, \quad w \not\equiv 0, \quad \text{for a.e. } t \in [\tau, \omega_2], \end{aligned}$$

and, given two real positive parameters λ and μ, we set

$$\tilde{w}_{\lambda,\mu}(t) := \lambda w^+(t) - \mu w^-(t),$$

with $w^+(t)$ and $w^-(t)$ denoting the positive and the negative part of the function $w(t)$, respectively. Hence the shape of the graph of w is characterized by a finite sequence of positive and negative humps. This description is exploited in the literature to study BVPs with nonlinearities ψ satisfying (ψ_1) which are superlinear at zero, i.e. $\lim_{\xi\to 0^+} \psi(\xi)/\xi = 0$. As a results one can find high multiplicity of positive solutions for the rispectively problem $(\mathscr{IP})$, depending on the number of humps of the weight term [BFZ16].

To our knowledge, this framework is completely new for the case of superlinear nonlinearities at zero satisfying (ψ_2). In this manner, at least for the ODE case, we refine a result already available in literature [LNS10].

Theorem. *Let $\psi\colon [0,1] \to \mathbb{R}^+$ be a locally Lipschitz continuous function satisfying (ψ_2) and such that $\lim_{\xi\to 0^+} \psi(\xi)/\xi = 0$. Let $w \in L^1(\omega_1, \omega_2)$. Then, there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ there exists $\mu^*(\lambda) > 0$ such that for every $\mu > \mu^*(\lambda)$ the Neumann problem associated with $u'' + \tilde{w}_{\lambda,\mu}(x)\psi(u) = 0$ on $[\omega_1, \omega_2]$ has at least 3 solutions such that $0 < u(t) < 1$ for all $t \in [\omega_1, \omega_2]$.*

As a last direction of work we take into account a nonlinear function ψ satisfying (ψ_1) and we replace the sublinear growth condition at ∞ by a more general one as follows

(ψ_3) $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ continuous and such that $\psi(0) = 0$, $\psi(\xi) > 0$ $\forall \xi \in \mathbb{R}^+_0$ and $\liminf_{\xi\to+\infty} \int_0^\xi \psi(s)\, ds/\xi^2 = 0$.

The interest in nonlinearities with growth condition on the potential at ∞ is mainly aimed by the classical work of [Ham30]. In a context of an oscillatory potential, results of high multiplicity of positive solutions for indefinite weight problems of the form $(\mathscr{IP})$ under Dirichlet boundary conditions follow from [OZ96; OO06; MZ93]. Instead, the Neumann problem is not completely explored and we provide a result of high multiplicity for the ordinary case, that reads as follows.

Theorem. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a locally Lipschitz continuous function satisfying (ψ_3) with $\limsup_{\xi\to+\infty} 2\int_0^\xi \psi(s)\, ds/\xi^2 > 0$ and $\xi \mapsto \psi(\xi)/\xi$ upper bounded in a right neighborhood of 0. Let $w : [\omega_1, \omega_2] \to \mathbb{R}$ be a bounded piecewise continuous function satisfying $w(t) \geq 0$, $w \not\equiv 0$, for a.e. $t \in [\omega_1, \sigma]$, and $w(t) \leq 0$, $w \not\equiv 0$, for a.e. $t \in [\sigma, \omega_2]$. Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda, r, k) > 0$ such that for each $\mu > \mu^*$ the Neumann problem associated with $u'' + \tilde{w}_{\lambda,\mu}(x)\psi(u) = 0$ has at least k solutions which are nonincreasing on $[\omega_1, \omega_2]$ and satisfy $0 < u(t) \leq r$, for each $t \in [\sigma, \omega_2]$. Moreover, if $\limsup_{\xi\to+\infty} 2\int_0^\xi \psi(s)\, ds/\xi^2 = +\infty$ the result holds with $\lambda^* = 0$.*

Remark. At first glance, the problems faced in the previous two parts could seem not related. Instead, we stress the fact that most of the applications, introduced in the present PhD thesis, involve nonlinearities which are interlinked by the presence of *non-sign-definite weight functions*. For instance, the results aimed by Part ¶1 apply to ODEs with a weight term $a(\cdot)$ such that $a(t) \geq 0$ with $a \not\equiv 0$. On the other hand, the achievements motivated by Part ¶2 deal with BVPs characterized by a weight term $w(\cdot)$ which change its sign. Therefore, the treatment of non-sign-definite weight problems will be the main character of all the manuscript. $\lhd$

Outline

Keeping the classes of problems presented above in mind, we divide the dissertation into two parts. The first part consists of four chapters and is devoted to study the problem of Ambrosetti-Prodi. The second part includes five chapters and is concerned on indefinite weight BVPs. Let us now quickly describe the content of each chapter separately.

❄

Chapter 1 contains a systematic overview on the literature of Ambrosetti-Prodi problems.

Chapter 2 is dedicated to Ambrosetti-Prodi problems under periodic boundary conditions. In Section 2.1 we improve standard assumptions in this subject, starting from the Eighties, up to consider new local coercivity conditions, in order to still guarantee the Ambrosetti-Prodi alternative for periodic solutions (cf. [SZ17b]). In Section 2.2 we highlight how such kind of problems could be a valuable source of complex behaviors, so that we present a comparison of different level of “chaos” detected by means of several approaches available in literature (cf. [SZ17d]).

Chapter 3 treats the ordinary case of Ambrosetti-Prodi problems under Neumann boundary conditions with local coercivity conditions (cf. [Sov18; SZ17b]).

Chapter 4 collects possible perspectives of investigations that have been arisen through the study of this topics.

❄❄

Chapter 5 introduce on the indefinite weight BVPs faced in this dissertation presenting a survey on the state of the art.

Chapter 6 concerns the discussion of positive solutions of Dirichlet problems for indefinite PDEs having a positive nonlinearity, defined on the positive real line, with liner-sublinear growth or satisfying more general growth conditions on his primitive (cf. [SZ15]).

Chapter 7 deals with indefinite Neumann BVPs associated to ODEs with a positive nonlinearity, defined on the positive real line, whose primitive presents oscillations at infinity. We propose a multiplicity result of positive solutions that we extend also to the case of PDEs in radially symmetric domains (cf. [SZ17c]).

Chapter 8 focuses on indefinite Neumann problems associated to ODEs with positive nonlinearities, defined on the interval $[0,1]$, that arise in the field of population genetics. In Section 8.1 the negative answer to a conjecture proposed in [LN02] showing multiplicity of positive solutions (cf. [Sov17]). In Section 8.2 we pursue the study of these kind of problems providing a new multiplicity result of positive solutions which highlight the possibility that the number of positive solutions could be related with the features of the weight term.

Chapter 9 collects open questions that, as far as we know, are still unresolved.

Appendix A and **Appendix B** complete this thesis with several mathematical notions and tools that have been used to prove the collected results.

Notation & terminology

Let us introduce some standard notation. $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$, $\mathbb{R}^+ := [0, +\infty[$ is the set of non-negative real numbers and $\mathbb{R}^+_0 :=]0, +\infty[$ is the set of positive real numbers. We denote the restriction of a given function f on a subset A of its domain by $f|_A$. By $\mathbb{1}_A$ we mean the indicator function of a set A.

Given $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, by $C^k(X, Y)$ we indicate the space of continuous maps from X to Y with continuous $k-th$ derivative. Given $0 < \alpha \leq 1$, we denote by $C^{0,\alpha}(X)$ the space of Hölder continuous functions with exponent α in X. Given $p \geq 1$, we indicate by $L^p(X, \mathbb{R}^m)$ the L^p-space and by $W^{k,p}(X, \mathbb{R}^m)$ the Sobolev space with $H^1(X) = W^{1,2}(X, \mathbb{R}^m)$. By C^k_T, L^p_T and $W^{k,p}_T$ we mean the space of maps f defined on $\mathbb{R}$ which are T-periodic and such that $f|_{[0,T]}$ belongs to $C^k([0,T])$, $L^p([0,T])$ and $W^{k,p}([0,T])$, respectively. $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^N$.

Given an interval $\mathcal{J} \subseteq \mathbb{R}$, we say that a map $f\colon \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions if for a.e. $t \in \mathcal{J}$ the function $f(t,\cdot)$ is continuous and for every $\xi \in \mathbb{R}$ the function $f(\cdot,\xi)$ is measurable. If, moreover, for each $r > 0$ there exists $\ell \in L^1(\mathcal{J},\mathbb{R}^+)$ such that $|f(t,\xi)| \leq \ell(t)$ for a.e. $t \in \mathcal{J}$ and for every $|\xi| \leq r$, we say that f satisfies L^1-Carathéodory conditions.

We denote by Δ the Laplace operator in a given domain $\Omega \subseteq \mathbb{R}^N$ and $\mathbf{n}$ the outward unit normal vector on $\partial\Omega$.

We denote the positive and the negative real parts of a given function $f \in L^1(\mathcal{J})$ by $f^+ := (f + |f|)/2$ and $f^- := (-f + |f|)/2$, respectively.

Part I

Ambrosetti-Prodi problems

1. AP problems: brief overview

The roots of the problem of Antonio Ambrosetti and Giovanni Prodi comes from 1972 with the seminal paper [AP72] that can be considered as a milestone in the mathematical literature. It is a classical problem in the theory of nonlinear differential equations and it has influenced the research in the field of Nonlinear Analysis up to the present days. In this chapter we shall present a historical viewpoint of this problem and we shall introduce both notations and definitions concerning the first part of the present dissertation.

The focus in [AP72] was the study of the inversion of functions with singularities in Banach spaces, which led to a different line of work able to face new elliptic BVPs. Indeed, in [AP72; AP93], Ambrosetti and Prodi dealt with the Dirichlet problem

$$(\mathscr{P}_{\text{AP72}}) \qquad \begin{cases} \Delta u + \phi(u) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, is a bounded domain with boundary of class $C^{2,\alpha}$ and $h \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in]0,1[$. An application of their approach to problem $(\mathscr{P}_{\text{AP72}})$ with an asymmetric nonlinearity ϕ whose derivative crosses the first eigenvalue of the associated linear problem lead to the following result of existence and multiplicity of solutions.

Theorem 1.1 (Ambrosetti and Prodi, 1972)**.** *Let $\phi \in C^2(\mathbb{R})$ a strictly convex function such that $\phi(0) = 0$ and*

$$0 < \lim_{\xi \to -\infty} \phi'(\xi) < \lambda_1^{\mathcal{D}}(-\Delta; \Omega) < \lim_{\xi \to +\infty} \phi'(\xi) < \lambda_2^{\mathcal{D}}(-\Delta; \Omega) \qquad (\text{Hp}_{\text{AP72}})$$

with $\lambda_1^{\mathcal{D}}(-\Delta; \Omega)$, $\lambda_2^{\mathcal{D}}(-\Delta; \Omega)$ the first two eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$. Then, there exists a C^1 manifold $\mathcal{M}$ of codimension 1 which separates $C^{0,\alpha}(\overline{\Omega})$ into two disjoint open regions $\mathcal{A}_0$ and $\mathcal{A}_2$ such that $C^{0,\alpha}(\overline{\Omega}) = \mathcal{A}_0 \cup \mathcal{M} \cup \mathcal{A}_2$ and the following alternative holds:

1° *problem $(\mathscr{P}_{\text{AP72}})$ has* zero solutions *if $h \in \mathcal{A}_0$;*

2° *problem $(\mathscr{P}_{\text{AP72}})$ has* exactly one solution *if $h \in \mathcal{M}$;*

3° *problem $(\mathscr{P}_{\text{AP72}})$ has* exactly two solutions *if $h \in \mathcal{A}_2$.*

The previous result received much attention by the mathematical community and since then problems with these kinds of nonlinearities are called "*Ambrosetti-Prodi problems*" (briefly written here as AP problems). Our main intent is now to outline chronologically how the classical assumption ($\mathrm{Hp}_{\mathrm{AP72}}$) have changed, in order to still guarantee the result of multiplicity which characterizes this topic.

In this respect, the same statement of Theorem 1.1 was obtained by Manes and Micheletti in [MM73], by requiring that

$$-\infty \leq \lim_{\xi\to-\infty} \phi'(\xi) < \lambda_1^{\mathcal{D}}(-\Delta;\Omega) < \lim_{\xi\to+\infty} \phi'(\xi) < \lambda_2^{\mathcal{D}}(-\Delta;\Omega), \qquad (\mathrm{Hp}_{\mathrm{MM73}})$$

instead of the assumption in ($\mathrm{Hp}_{\mathrm{AP72}}$). From condition ($\mathrm{Hp}_{\mathrm{MM73}}$), we can thus observe that the positivity of $\lim_{\xi\to-\infty}\phi'(\xi)$ is not necessary. On the contrary, the main assumption is that the derivative of the nonlinearity has to cross the first eigenvalue when u goes from $-\infty$ to $+\infty$ (from which the name of "asymmetric crossing nonlinearity" or "jumping nonlinearities", see [Fuč75; Fuč76]).

Another pioneeristic work in that context was done by Berger and Podolak [BP74], where the previous abstract description of the solution set was proposed in a different formulation by splitting the term h as

$$h = su_1 + p$$

with u_1 the normalized positive eigenfunction associated with $\lambda_1^{\mathcal{D}}(-\Delta;\Omega)$, $s\in\mathbb{R}$ and p the orthogonal component of h, namely $\int_\Omega u_1(x)p(x)\,dx = 0$. This way, they proved the existence of a unique value $s_0 = s_0(p)\in\mathbb{R}$ such that, if $s<s_0$ then $h\in\mathcal{A}_0$, if $s=s_0$ then $h\in\mathcal{M}$ and if $s>s_0$ then $h\in\mathcal{A}_2$.

The next important contribution to the study of problem ($\mathscr{P}_{\mathrm{AP72}}$) comes form the work [KW75] by Kazdan and Warner, who exploited the technique of upper and lower solutions, to obtain an existence result generalizing the assumptions on the nonlinear term. Indeed, they considered a problem of the following form

$$(\mathscr{P}_{\mathrm{KW75}}) \qquad \begin{cases} \Delta u + \Upsilon(x,u) = su_1(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function Υ is sufficiently smooth and satisfies

$$-\infty \leq \limsup_{\xi\to-\infty} \frac{\Upsilon(x,\xi)}{\xi} < \lambda_1^{\mathcal{D}}(-\Delta;\Omega) < \liminf_{\xi\to+\infty} \frac{\Upsilon(x,\xi)}{\xi} \leq +\infty, \qquad (\mathrm{Hp}_{\mathrm{KW75}})$$

uniformly in x. Notice that the assumptions in ($\mathrm{Hp}_{\mathrm{KW75}}$) are weaker than the ones in ($\mathrm{Hp}_{\mathrm{MM73}}$). Moreover, in [KW75], it was proved the existence of $s_0\in\mathbb{R}$ such that problem ($\mathscr{P}_{\mathrm{KW75}}$) has zero solutions if $s<s_0$ and at least one solution if $s>s_0$, provided that ($\mathrm{Hp}_{\mathrm{KW75}}$) holds.

The multiplicity result typical of AP problems was then obtained by many other authors, combining this latter tool with the degree theory or the fixed point index theory (see [AH79; BL81; Dan78] and the survey [Fig80] for a complete list of references). After these classical results, several outcomes were obtained, for example, by exploring the set of the solutions for problems with nonlinearities whose derivates cross higher eigenvalues than the first one or by changing the boundary conditions.

In the first case, wondering that nonlinearity jumps over $\lambda_1^{\mathcal{D}}(-\Delta;\Omega)$, further investigations led to results of higher multiplicity of solutions (see Lazer and McKenna [LM81], Solimini [Sol85]). Moreover, such kind of questions about resonance and non-resonance for jumping nonlinearities can be also interpreted in the light of the interaction with the so-called Dancer-Fučik spectrum starting with the works of Dancer [Dan76a; Dan76b] and Fučik [Fuč76] (see [Maw07] for a detailed presentation of this topic). In the second case, issues concerning periodic boundary conditions or Neumann boundary conditions were addressed as well (see [FMN86; FS17; Maw87a; Maw06; Ort89; Ort90]).

Looking at the periodic problem, for a fixed period $T>0$ and the differential operator $-u''$ (or $-u''-cu'$), it follows that zero is the first eigenvalue of the associated linear problem with periodic boundary conditions and the function constantly 1 is the corresponding eigenfunction. In this manner the splitting proposed by Berger and Podolak becomes now

$$h = s + p,$$

with p a T-periodic function with mean value zero in a period. Notice also that, in the periodic setting, condition ($\mathrm{Hp}_{\mathrm{MM73}}$) implies

$$\lim_{|u|\to+\infty} \phi(u) = +\infty. \tag{Hϕ}$$

Avoiding the convexity assumption and dealing with the periodic problem associated with the Liénard equation

$$u'' + f(u)u' + g(t,u) = s, \tag{$\mathscr{LE}_s$}$$

a relevant contribution in this direction is contained in the work of Fabry, Mawhin and Nkashama [FMN86].

Theorem 1.2 (Fabry, Mawhin and Nkashama, 1986)**.** *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, g is T-periodic in t and satisfies hypothesis*

$$\lim_{|u|\to+\infty} g(t,u) = +\infty, \text{ uniformly in } t. \tag{$\mathrm{Hp}_{\mathrm{FMN86}}$}$$

Then, there exists $s_0 \in \mathbb{R}$ such that

1° *for $s < s_0$, equation ($\mathscr{LE}_s$) has no T-periodic solutions;*

2° *for $s = s_0$, equation ($\mathscr{LE}_s$) has at least one T-periodic solution;*

3° *for $s > s_0$, equation ($\mathscr{LE}_s$) has at least two T-periodic solutions.*

In this periodic environment, the settings considered in Theorem 1.2 are more general with respect to the classical ones introduced by Ambrosetti and Prodi. As a result, we lose the sharp alternative of Ambrosetti-Prodi type, in favor of a weaker one, due to minimal conditions assumed on g. An immediate consequence of Theorem 1.2 is the following result, which involves a generalized Liénard equation with a weighted restoring term.

Corollary 1.3. *Let f, $\phi\colon \mathbb{R}\to\mathbb{R}$ be continuous functions. Let a, $p\colon \mathbb{R}\to\mathbb{R}$ be continuous and T-periodic functions with*

$$\min_{t\in[0,T]} a(t) > 0.$$

Assume (Hϕ)*. Then, for equation*

$$u'' + f(u)u' + a(t)\phi(u) = s + p(t), \tag{$\mathscr{WLE}_s$}$$

the following result holds. There exists $s_0 \in \mathbb{R}$ such that

1° *for $s < s_0$, equation ($\mathscr{WLE}_s$) has no T-periodic solutions;*

2° *for $s = s_0$, equation ($\mathscr{WLE}_s$) has at least one T-periodic solution;*

3° *for $s > s_0$, equation ($\mathscr{WLE}_s$) has at least two T-periodic solutions.*

Still in the periodic case, an Ambrosetti-Prodi type result can be recovered by replacing condition ($\mathrm{Hp}_{\mathrm{MM73}}$) with an analogous one that take into account the periodic setting of the problem. This was done by Ortega in [Ort89; Ort90] where sharp results about the stability of the T-periodic solutions were obtained as well. In more detail, the case of equation ($\mathscr{LE}_s$) with a constant damping term was studied in [Ort89; Ort90]. In particular, for a periodic problem associated with

$$u'' + cu' + g(t,u) = s \tag{$\mathscr{E}_{\mathrm{O89}}$}$$

where $c > 0$, we can state the following result.

Theorem 1.4 (Ortega, 1990)**.** *Let $g\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function, T-periodic in t and strictly convex in u:*

$$(g_u(t,v)-g_u(t,w))(v-w)>0\,,\quad \text{if } v\neq w,\quad t\in\mathbb{R}.$$

Assume $(\mathrm{Hp}_{\mathrm{FMN86}})$ *and*

$$\lim_{\xi\to+\infty} g_u(t,\xi)\leq\left(\frac{\pi}{T}\right)^2+\left(\frac{c}{2}\right)^2,\quad t\in\mathbb{R}. \qquad (\mathrm{Hp}_{\mathrm{Ort89}})$$

Then, there exists $s_0\in\mathbb{R}$ such that

1° *for $s<s_0$, every solution of equation $(\mathscr{E}_{\mathrm{O89}})$ is unbounded;*

2° *for $s=s_0$, equation $(\mathscr{E}_{\mathrm{O89}})$ has a unique T-periodic solution, which is not asymptotically stable;*

3° *for $s>s_0$, equation $(\mathscr{E}_{\mathrm{O89}})$ has exactly two T-periodic solutions, one asymptotically stable and another unstable.*

Furthermore, studying T-periodic solutions of equation

$$u''+cu'+\phi(u)=h(t), \qquad (\mathscr{E}_{\mathrm{O90}})$$

with $c>0$, condition $(\mathrm{Hp}_{\mathrm{Ort89}})$ can be improved and an analogous version of Theorem 1.1 for an AP periodic problem reads as follows.

Theorem 1.5 (Ortega, 1990)**.** *Let $h\colon\mathbb{R}\to\mathbb{R}$ be a continuous and T-periodic function. Let $\phi\in C^2(\mathbb{R})$ a strictly convex function such that*

$$-\infty\leq\lim_{\xi\to-\infty}\phi'(\xi)<0<\lim_{\xi\to+\infty}\phi'(\xi)\leq\Gamma_1:=\left(\frac{2\pi}{T}\right)^2+\left(\frac{c}{2}\right)^2, \qquad (\mathrm{Hp}_{\mathrm{Ort90}})$$

Then, in the space C^0_T of T-periodic and continuous solutions, there exists a C^1 manifold $\mathcal{M}$ of codimension 1 which separates C^0_T into two disjoint open regions $\mathcal{A}_0$ and $\mathcal{A}_2$ such that $C^0_T=\mathcal{A}_0\cup\mathcal{M}\cup\mathcal{A}_2$ and the following alternatives hold:

1° *equation $(\mathscr{E}_{\mathrm{O90}})$ has zero T-periodic solutions if $h\in\mathcal{A}_0$;*

2° *equation $(\mathscr{E}_{\mathrm{O90}})$ has exactly one T-periodic solution if $h\in\mathcal{M}$;*

3° *equation $(\mathscr{E}_{\mathrm{O90}})$ has exactly two T-periodic solutions if $h\in\mathcal{A}_2$.*

In $(\mathrm{Hp}_{\mathrm{Ort90}})$ the constant Γ_1 plays the role of the second eigenvalue in self-adjoint problems (which is exactly the first positive one). This way we notice, reflected in $(\mathrm{Hp}_{\mathrm{Ort90}})$, the same condition of crossing eigenvalues inherent in AP problems.

At this point a natural state of progress in the study of periodic AP problems was to investigate on the relation between the number of periodic solutions with the number of eigenvalues crossed by the derivative of the nonlinearity $\phi'(\xi)$ as ξ tends to $+\infty$. A contribution in this direction arises from the work [PMM92] by Del Pino, Manásevich and Murua which generalizes the results in [LM90]. This way, the study of problems with asymmetric nonlinearities has stimulated a great deal of works on the investigation of the existence and the multiplicity of periodic and subharmonic solutions (see [BZ13; FG10; NO03; Ort96; Reb97; RZ96; Wan00; ZZ05] and the references therein). These researches find also motivation from both issues on the periodic Dancer-Fučik spectrum and topics related to the Lazer-McKenna suspension bridge models [LM90]. In this respect, due to the stability results achieved in [Ort89; Ort90], one could wonder what can happen to the behavior of the T-periodic solutions when $\lim_{\xi\to+\infty}\phi'(\xi)$ in $(\mathrm{Hp}_{\mathrm{Ort90}})$ skips away from Γ_1. In this case, high multiplicity results of T-periodic solutions are expected in accord to the literature already recalled. Nevertheless, we highlight that the possibility to recover "complex behaviors", as far as we know, has not yet been discussed in detail.

Furthermore, it is significant to consider AP problems under Neumann boundary conditions in view of the strict relation between periodic and Neumann problems. Indeed, solving the Neumann problem on an interval of length T, one can provide solutions also for the $2T$-periodic problem associated with ODEs presenting suitable symmetries in the variable t. In particular, the Neumann problem on the interval $[0,T]$ can be viewed as a subproblem of the periodic problem on the interval $[0,2T]$, since one can find a $2T$-periodic solution starting from a solution to the Neumann problem on the interval $[0,T]$ via a even reflection and a periodic extension. So that, for completeness, we recall here the work by Mawhin [Maw87a] in the case of second order differential equations with Neumann boundary conditions:

$$(\mathscr{N}_{\mathrm{Ma87}}) \qquad \begin{cases} u'' + g(t,u) = s, \\ u'(0) = u'(\mathrm{T}) = 0. \end{cases}$$

For problem $(\mathscr{N}_{\mathrm{Ma87}})$ a result of Ambrosetti-Prodi type is the following.

Theorem 1.6 (Mawhin, 1987)**.** *Let $g\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying* $(\mathrm{Hp}_{\mathrm{FMN86}})$*. Then, there exists $s_0\in\mathbb{R}$ such that*

1° *for $s<s_0$, problem $(\mathscr{N}_{\mathrm{Ma87}})$ has no solutions;*

2° *for $s=s_0$, problem $(\mathscr{N}_{\mathrm{Ma87}})$ at least one solution;*

3° *for $s>s_0$, problem $(\mathscr{N}_{\mathrm{Ma87}})$ at least two solutions.*

At last, in parallel to periodic problems associated with second order ODEs, we give a look at such a kind of periodic problems associated with first order equations. Namely, we consider a problem of the form

$$(\mathscr{P}_{\mathrm{Ma87}}) \qquad \begin{cases} u' + g(t,u) = s, \\ u(0) = u(T) = 0. \end{cases}$$

Results involving an Ambrosetti-Prodi alternative for such a kind of first order periodic ODEs date back from the works [Maw87b; MS86; Vid87]. Moreover, a version of Theorem 1.2 is given in [Maw87b] and it reads as follows.

Theorem 1.7 (Mawhin, 1987)**.** *Suppose $g\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous and T-periodic in t. Assume* $(\mathrm{Hp}_{\mathrm{FMN86}})$*. Then, there exists $s_0\in\mathbb{R}$ such that*

1° *for $s<s_0$, problem $(\mathscr{P}_{\mathrm{Ma87}})$ has no T-periodic solutions;*

2° *for $s=s_0$, problem $(\mathscr{P}_{\mathrm{Ma87}})$ has at least one T-periodic solution;*

3° *for $s>s_0$, problem $(\mathscr{P}_{\mathrm{Ma87}})$ has at least two T-periodic solutions.*

Notwithstanding these original contributions, the periodic AP problem is still a subject that deserves to be studied further, as observed by Ambrosetti in his note concerning "some global inversion theorems with applications to semilinear elliptic equations", (see [Amb11, p. 13]). Up to the present days, as far as we know, the uniform condition firstly assumed in [FMN86] was then considered also by all other authors interested in this topic. Looking at new perspectives concerning periodic AP problems, as well as AP problems under Neumann boundary conditions, from Theorem 1.2, Theorem 1.6 and Theorem 1.7, the natural question arises is whether the uniform coercivity condition in $(\mathrm{Hp}_{\mathrm{FMN86}})$ can be weakened. In particular, with respect to Corollary 1.3, one could wonder where the result still holds when $a(t)\geq 0$ and it vanishes somewhere. In other words, a novel interest is in the study of periodic problems associated with $(\mathscr{WLE}_s)$ when the weight term $a(t)$ is *non-sign definite.*

2. Periodic AP problems

The present chapter, which is based on [SZ17b; SZ17d], is concerned in periodic AP problems

$$(\mathscr{P}) \qquad \begin{cases} u'' + \phi(u) = h(t), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}$$

whose comprehensive picture is given in Chapter 1. In particular, we recall the peculiar hypotheses involving the eigenvalues crossing:

$$(\mathrm{H}\phi_1) \qquad -\infty \leq \lim_{\xi\to-\infty} \frac{\phi(\xi)}{\xi} < \lambda_1 < \lim_{\xi\to+\infty} \frac{\phi(\xi)}{\xi} < \lambda_2,$$

where $\lambda_1 = 0$ and $\lambda_2 = (2\pi/T)^2$ are the first two eigenvalues associated with the differential operator $-u''$ subject to T-periodic boundary conditions. Condition $(\mathrm{H}\phi_1)$ is included, as special case, in the following one

$$(\mathrm{H}\phi_2) \qquad \lim_{|u|\to+\infty} \phi(u) = +\infty.$$

As pointed out in the Introduction, we are going to treat these kinds of problems from two point of views which take into account $(\mathrm{H}\phi_1)$ or $(\mathrm{H}\phi_2)$, respectively.

On the one hand, in Section 2.1, we will face periodic AP problems in the classical framework of Fabry, Mawhin and Nakashama [FMN86]. Recalling the splitting proposed by Berger and Podolak, namely

$$h(t) = s + p(t)$$

where s is a real parameter and p is a T-periodic function such that $\int_0^T p(t)\,dt = 0$, we will face periodic problems of more general type than the ones in $(\mathscr{P})$ which are associated with

$$u'' + f(u)u' + g(t,u) = s.$$

This way the periodic AP problem becomes a sub-case of the previous ones. Therefore, we will study the set of the T-periodic solutions for equations with locally coercive nonlinearities g, by improving the classical assumption

$$(\mathrm{H}g_1) \qquad \lim_{|u|\to+\infty} g(t,u) = +\infty, \quad \text{uniformly in } t,$$

that reflects (Hϕ_2). In more detail, we will obtain in Theorem 2.1.9 the same (weak) alternative of Ambrosetti-Prodi type achieved in [FMN86] under global coercive conditions and in Corollary 2.1.10 an application to a weighted Liénard equation with non sign-definite weight term.

On the other hand, in Section 2.2, we will study the existence of infinitely many periodic solutions along with the detection of complex behaviors for problem ($\mathscr{P}$), by varying the conditions on the perturbative term $h(t)$ in accord with the approaches considered. In particular by means of Theorem 2.2.12, we will show an example of "chaotic dynamics" for problem ($\mathscr{P}$) for $\lambda_3 < \lim_{\xi\to+\infty} \phi(\xi)/\xi < \lambda_4$, where λ_3 and λ_4 are the third and the fourth eigenvalue of the associated linear problem, respectively. This conclusion will be at the end compared with the stability results for problem ($\mathscr{P}$) carried by Ortega in [Ort90].

2.1 Generalization of the result by Fabry, Mawhin & Nkashama

In this section we study the periodic BVP associated with the Liénard differential equation given by

$$(\mathscr{L}\mathscr{E}_s) \qquad u'' + f(u)u' + g(t,u) = s,$$

where we tacitly assume in the sequel that s is a real parameter, $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $g\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions.

In order to discuss the number of T-periodic solutions for the parameter dependent equation ($\mathscr{L}\mathscr{E}_s$), we will collect in Section 2.1.1 some basic facts by exploiting coincidence degree theory, which is a powerful tool developed by Jean Mawhin in [GM77; Maw79; Maw93]. All the notations and results in coincidence degree theory needed for our discussion are collected in Appendix A. Furthermore, we will introduce conditions of Villari's type [Vil66] that are useful to provide lower bounds for the solutions. These preliminary studies will be then applied in the proofs of our main results in Section 2.1.2, which yield an alternative of Ambrosetti-Prodi type for the periodic solutions of equation ($\mathscr{L}\mathscr{E}_s$).

2.1.1 Preliminary results

First of all we are interested in providing an existence result of periodic solutions for the second order ODE

$$u'' + f(u)u' + \nu(t,u) = 0, \tag{2.1.1}$$

where $\nu\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. In this setting, a T-periodic solution of equation (2.1.1) is meant in the generalized sense, namely a solution u of equation (2.1.1) under periodic boundary conditions

$$u(0) = u(T), \quad u'(0) = u'(T),$$

is an absolutely continuous (AC) function $u : [0,T] \to \mathbb{R}$ such that u' is AC and $u(t)$ satisfies (2.1.1) for a.e. $t \in [0,T]$. Equivalently, we could also extend the map $\nu(\cdot, u)$ on $\mathbb{R}$ by T-periodicity and so consider T-periodic solutions $u : \mathbb{R} \to \mathbb{R}$ such that u' is AC.

At this point our intent is to enter in the framework of Mawhin's coincidence (see Appendix A). For that reason, we define the space

$$X = C^1_T := \{u \in C^1([0,T]) : u(0) = u(T),\, u'(0) = u'(T)\},$$

endowed with the norm

$$\|u\|_X := \|u\|_\infty + \|u'\|_\infty$$

and the space $Z = L^1(0,T)$ with the norm $\|u\|_Z := \|u\|_{L^1}$. We consider also the operator $L : X \supseteq \mathrm{dom}L \to Z$ defined as $Lu := -u''$, with

$$\mathrm{dom}L = W^{2,1}_T := \{u \in X : u' \in AC\}.$$

In accord to [Maw72a], as a natural choice for the projections, we take

$$Qu := \frac{1}{T}\int_0^T u(t)\,dt, \quad \forall\, u \in Z,$$

and

$$Pu := Qu, \quad \forall\, u \in X.$$

Thus, we have $\ker L = \mathrm{Im}P = \mathbb{R}$ and $\mathrm{coker}L = \mathrm{Im}Q = \mathbb{R}$. As linear isomorphism J, we take the identity in $\mathbb{R}$. Notice that, for each $w \in Z$, the vector $v = K_P(Id - Q)w$ is the unique solution of the linear boundary value problem

$$\begin{cases} -v''(t) = w(t) - \frac{1}{T}\int_0^T w(t)\,dt, \\ v(0) = v(T), \quad v'(0) = v'(T), \quad \int_0^T v(t)\,dt = 0. \end{cases}$$

At last, as nonlinear operator N, we take the associated Nemytskii operator, namely

$$(Nu)(t) := f(u(t))u'(t) + \nu(t, u(t)), \quad \forall\, u \in X.$$

By a standard argument, it is possible to verify that the operator N is L-completely continuous and, moreover, the map $\tilde{u}(\cdot)$ is a T-periodic solution of (2.1.1) if and only if $\tilde{u} \in \mathrm{dom}L$ with $L\tilde{u} = N\tilde{u}$. Analogously, solutions to the abstract coincidence equation $Lu = \lambda Nu$, with $0 < \lambda \leq 1$, correspond to T-periodic solutions of

$$u'' + \lambda f(u)u' + \lambda\, \nu(t, u) = 0, \quad 0 < \lambda \leq 1. \tag{2.1.2}$$

In the next two lemmas we provide some a priori bounds for the solutions of the parameter dependent equation (2.1.2).

Lemma 2.1.1. *Let $\nu\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following condition:*

(A_1) $\quad \exists\, \gamma \in L^1([0,T], \mathbb{R}^+):\ \nu(t,u) \geq -\gamma(t),\ \forall\, u \in \mathbb{R}$ *and a.e.* $t \in [0,T]$.

Then, there exists a constant $K_0 = K_0(\gamma)$ such that any T-periodic solution u of (2.1.2) *satisfies* $\max u - \min u \leq K_0$.

Proof. Without loss of generality, we extend the map $\nu(\cdot, u)$ by T-periodicity on $\mathbb{R}$ and we suppose that the solutions satisfy $u(t+T) = u(t)$ for all $t \in \mathbb{R}$. Let t^* be such that $u(t^*) = \max u$. We also define $x(t) := \max u - u(t)$, which satisfies $x' = -u'$ and $x'' = -u''$. From (2.1.2) we have

$$-x''(t) = u''(t) = -\lambda f(u(t))u'(t) - \lambda\nu(t, u(t)) \leq \lambda f(u(t))x'(t) + \gamma(t), \text{ for a.e. } t.$$

Multiplying the previous inequality by $x(t) \geq 0$ and integrating on $[t^*, t^* + T]$, after an integration by parts, we obtain

$$\begin{aligned} \|x'\|_{L^2}^2 = \int_0^T x'(t)^2\,dt = \int_{t^*}^{t^*+T} x'(t)^2\,dt = -\int_{t^*}^{t^*+T} x''(t)x(t)\,dt \\ \leq \lambda \int_{t^*}^{t^*+T} f(u(t))x(t)x'(t)\,dt + \int_{t^*}^{t^*+T} \gamma(t)x(t)\,dt. \end{aligned}$$

Since

$$\int_{t^*}^{t^*+T} f(u(t))x(t)x'(t)\,dt = \int_{t^*}^{t^*+T} f(u(t))u(t)u'(t)\,dt - \max u \int_{t^*}^{t^*+T} f(u(t))u'(t)\,dt = 0,$$

it follows that

$$\|x'\|_{L^2}^2 \leq \|\gamma\|_{L^1}\|x\|_\infty. \tag{2.1.3}$$

Using the fact that $x(t^*) = 0$, for a suitable embedding constant c_1, we have

$$\|x\|_\infty \leq c_1 \|x'\|_{L^2}. \tag{2.1.4}$$

From (2.1.3) and (2.1.4), we obtain $\|x'\|_{L^2} \leq c_1 \|\gamma\|_{L^1}$ and then

$$\|x\|_\infty \leq K_0 := c_1^2 \|\gamma\|_{L^1}.$$

The thesis follows, since $\max u - \min u = \|x\|_\infty$. □

Lemma 2.1.2. *Let $\nu : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Let $[a,b] \subset \mathbb{R}$ and $\rho \in L^1([0,T], \mathbb{R}^+)$ be such that $|\nu(t,u)| \leq \rho(t)$ for all $u \in [a,b]$ and a.e. $t \in [0,T]$. Then, there exists a constant $\kappa = \kappa(a,b,\rho)$ such that any T-periodic solution u of* (2.1.2)*, with $a \leq u(t) \leq b$ for all $t \in [0,T]$ satisfies $\|u'\|_\infty \leq \kappa$.*

Proof. The thesis follows straightforward from [DCH06; FMN86; Maw81, Chapter 1, Proposition 4.7] because the term $f(u)u' + \nu(t,u)$ satisfies a Bernstein-Nagumo condition. Indeed, $|f(u)u' + \nu(t,u)| \leq \gamma(t)\psi(|u'|)$ where $\gamma(t) := (K + \rho(t))$ for some positive constant K (depending on a and b) and $\psi(\xi) := (\xi + 1)$ with $\int_0^\infty d\xi/\psi(\xi) = \infty$. □

To proceed with our discussion it is useful, at this point, to introduce the following definitions.

Definition 2.1.3. We say that $\nu(t,u)$ satisfies the *Villari's condition* at $-\infty$ (respectively, at $+\infty$) if there exists a constant $d_0 > 0$ such that

$$\exists \delta = \pm 1 \, : \, \delta \int_0^T \nu(t, u(t)) \, dt > 0$$

for each $u \in C_T^1$ such that $u(t) \leq -d_0$, $\forall\, t \in [0,T]$ (respectively, $u(t) \geq d_0$, $\forall\, t \in [0,T]$).

We notice that Definition 2.1.3 is adapted here from [Vil66]. Moreover, we refer to [BM07; MM98; MSD16], for more information about these conditions as well as generalizations in different contexts.

Definition 2.1.4. We say that a function $\beta \in W_T^{2,1}$ is a *strict upper solution* for equation (2.1.1) if

$$\beta''(t) + f(\beta(t))\beta'(t) + \nu(t, \beta(t)) < 0, \quad \text{for a.e. } t \in [0,T] \tag{2.1.5}$$

and if u is any T-periodic solution of (2.1.1) with $u \leq \beta$, then $u(t) < \beta(t)$ for all t.

We warn that Definition 2.1.4 is a particular case of the definition of strict upper solution considered in [DCH06]. Furthermore, we stress the fact that if ν is a function which is continuous and T-periodic in t and $\beta \in C_T^2$ satisfies (2.1.5) for all t, then β is strict. Indeed, from [DCH06, Chapter 3, Proposition 1.2], if u is a T-periodic solution of (2.1.1) with $u \leq \beta$ then $u < \beta$.

We are now in position to state our main result in this section, which makes use of Theorem A.1 in Appendix A and provides an existence result of T-periodic solutions for equation (2.1.1).

Theorem 2.1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $\nu : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (A_1) *and the Villari's condition at $-\infty$ with $\delta = 1$. Suppose there exists $\beta \in W_T^{2,1}$ which is a strict upper solution for equation* (2.1.1)*. Then,* (2.1.1) *has at least a T-periodic solution $\tilde{u}$ such that $\tilde{u} < \beta$. Moreover, there exist $R_0 \geq d_0$ and $M_0 > 0$, such that for each $R > R_0$ and $M > M_0$, we have*

$$D_L(L - N, \Omega) = 1$$

for $\Omega = \Omega(R, \beta, M) := \{u \in C_T^1 : -R < u(t) < \beta(t), \ \forall\, t \in [0,T], \ \|u'\|_\infty < M\}$.

Proof. First of all we use the upper solution β to truncate the problem as usual in the theory of upper and lower solutions. Accordingly, we define the truncated function

$$\hat{\nu}(t,u) := \begin{cases} \nu(t,u) & \text{for } u \leq \beta(t), \\ \nu(t,\beta(t)) & \text{for } u \geq \beta(t), \end{cases}$$

and consider the parameter dependent equation

$$u'' + \lambda f(u)u' + \lambda\,\hat{\nu}(t,u) = 0, \quad 0 < \lambda \leq 1. \tag{2.1.6}$$

By condition (A_1), we have $\hat{\nu}(t,u) \geq -\gamma(t)$ for all $u \in \mathbb{R}$ and a.e. $t \in [0,T]$, thus $\hat{\nu}$ satisfies (A_1), too. Hence we are in position to apply Lemma 2.1.1, with $\hat{\nu}$ in place of ν, and so we obtain the existence of a constant K_0 (which depends on γ) such that any T-periodic solution u of (2.1.6) satisfies $\max u - \min u \leq K_0$.

Now we claim that $\max u > -d_1$, for some fixed constant $d_1 \geq d_0$ with $d_1 > \|\beta\|_\infty$. Accordingly, if we suppose by contradiction that $u(t) \leq -d_1$ for all $t \in [0,T]$, then $u(t) < \beta(t)$ for all $t \in [0,T]$ and so $u(t)$ is a T-periodic solution of (2.1.2). Hence, an integration on $[0,T]$ of (2.1.2) (divided by $\lambda > 0$), yields to $\int_0^T \nu(t,x(t))\,dt = 0$, which clearly contradicts Villari's condition at $-\infty$ as $-d_1 \leq -d_0$. Since $u(t) > -d_1$ for some $t \in [0,T]$ and hence $\max u > -d_1$, we immediately obtain that

$$\min u > -R_0\,, \qquad \text{for } R_0 := K_0 + d_1\,.$$

At this point, we claim that there exists $\bar{t} \in [0,T]$ such that $u(\bar{t}) < \beta(\bar{t})$. If, by contradiction, $u(t) \geq \beta(t)$ for all $t \in [0,T]$, then u turns out to be a T-periodic solution of

$$u'' + \lambda f(u)u' + \lambda\,\nu(t,\beta(t)) = 0, \quad 0 < \lambda \leq 1.$$

Hence, an integration on $[0,T]$ of the previous equation (divided by $\lambda > 0$), yields to $\int_0^T \nu(t,\beta(t))\,dt = 0$. However, since β is T-periodic and satisfies (2.1.5), an integration of (2.1.5) on $[0,T]$ gives $\int_0^T \nu(t,\beta(t))\,dt < 0$, which leads to a contradiction. Since $u(t) < \|\beta\|_\infty$ for some $t \in [0,T]$ and hence $\min u < \|\beta\|_\infty$, we immediately obtain that

$$\max u < \|\beta\|_\infty + K_0\,.$$

By Lemma 2.1.2, applied to $\hat{\nu}$ in place of ν, we find a constant m_0 which depends on R_0, $\|\beta\|_\infty + K_0$ and a L^1-function bounding $|\hat{\nu}(t,u)|$ on $[0,T] \times [-R_0, \|\beta\|_\infty + K_0]$, such that $\|u'\|_\infty \leq m_0$.

Writing equation

$$-\,u'' = f(u)u' + \hat{\nu}(t,u) \tag{2.1.7}$$

as a coincidence equation of the form $Lu = \hat{N}u$ in the space C^1_T, from the a priori bounds, we find that the coincidence degree $D_L(L-\hat{N},\mathcal{O})$ is well defined for any open and bounded set $\mathcal{O} \subset C^1_T$ of the form

$$\mathcal{O} := \{u \in C^1_T : -R < u(t) < C,\ \forall\, t \in [0,T],\ \|u'\|_\infty < m\}$$

where $R \geq R_0$, $C \geq \|\beta\|_\infty + K_0$ and $m > m_0$.

Finally, we consider the averaged scalar map

$$\hat{\nu}^{\#} : \mathbb{R} \to \mathbb{R}, \quad \hat{\nu}^{\#}(\xi) := \frac{1}{T}\int_0^T \hat{\nu}(t,\xi)\,dt, \quad \forall\, \xi \in \mathbb{R},$$

and we observe that the following holds

$$-JQ\hat{N}|_{\ker L} = -\hat{\nu}^{\#}.$$

Indeed, $\ker L$ is made by the constant functions which are identified with the real numbers. Moreover, we have

$$\hat{\nu}^{\#}(-R) > 0 > \hat{\nu}^{\#}(C).$$

In fact, the first inequality comes from Villari's condition and the choice $R \geq d_1$, while the second inequality follows from $\int_0^T \nu(t,\beta(t))\,dt < 0$ and the choice $C \geq \|\beta\|_\infty$. Thus, an application of Theorem A.1 guarantees that $D_L(L - \hat{N}, \mathcal{O}) = 1$ and hence equation (2.1.7) has a T-periodic solution $\tilde{u}$ with $-R < \tilde{u}(t) < C$, for all $t \in [0,T]$.

To conclude with the proof we have only to check that $\tilde{u} < \beta$. This fact comes from standard arguments in the theory of strict upper solutions. For sake of completeness, we give a short proof. At this point we know that any T-periodic solution of (2.1.6) is below β, at least for some t. Since $\tilde{u}$ is a solution of (2.1.6) for $\lambda = 1$, we have that there exists t_* such that $\tilde{u}(t_*) < \beta(t_*)$. Suppose by contradiction that there exists a t^* such that $\tilde{u}(t^*) > \beta(t^*)$. By the T-periodicity of $v(t) := \tilde{u}(t) - \beta(t)$, there exists an interval $[t_1,t_2]$ such that $t_1 < t^* < t_2$ with $v(t) > 0$ for all $t \in]t_1,t_2[$ and, moreover, $v(t_1) = v(t_2) = 0$ with $v'(t_1) \geq 0 \geq v'(t_2)$. On the interval $[t_1,t_2]$, we have that $\tilde{u}''(t) + f(\tilde{u}(t))\tilde{u}'(t) + \nu(t,\beta(t)) = 0$. Therefore, recalling (2.1.5), we have

$$v''(t) + f(\tilde{u}(t))\tilde{u}'(t) - f(\beta(t))\beta'(t) > 0, \qquad \text{for a.e. } t \in [t_1,t_2].$$

An integration on $[t_1,t_2]$ gives a contradiction, because

$$\int_{t_1}^{t_2} v''(t)\,dt = v'(t_2) - v'(t_1) \leq 0$$

and, for $F' = f$, we have

$$\int_{t_1}^{t_2} f(\tilde{u}(t))\tilde{u}'(t)\,dt = F(\tilde{u}(t_2)) - F(\tilde{u}(t_1)) = F(\beta(t_2)) - F(\beta(t_1)) = \int_{t_1}^{t_2} f(\beta(t))\beta'(t)\,dt.$$

Hence, we obtain $\tilde{u}(t) \leq \beta(t)$ for all $t \in [0,T]$ and so $\tilde{u}$ is a T-periodic solution of (2.1.1) satisfying $\tilde{u} \leq \beta$. Since β is strict (cf. Definition 2.1.4), we conclude that $\tilde{u}(t) < \beta(t)$ for all $t \in [0,T]$.

Applying Lemma 2.1.2, we can find a positive constant M_0, depending on R_0, $\|\beta\|_\infty$ and a L^1-function bounding $|\nu(t,u)|$ on $[0,T] \times [-R_0, \|\beta\|_\infty]$, such that $\|u'\|_\infty \leq M_0$. The conclusion follows from the excision property of coincidence degree (cf. Appendix A). □

2.1.2 Existence and multiplicity results under local coercivity conditions

Now we are ready to provide a (weak) Ambrosetti-Prodi type alternative for T-periodic solutions of equation $(\mathscr{LE}_s)$. Since we are dealing in a Carathédory setting, we need also to assume the following condition:

(A_2) *for all* $t_0 \in [0,T]$, $u_0 \in \mathbb{R}$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|t - t_0| < \delta$, $|u - u_0| < \delta \Rightarrow |g(t,u) - g(t,u_0)| < \varepsilon$.

Taking into account [DCH06, Chapter 3, Proposition 1.5], we notice that assumption (A_2) contains the regularity conditions needed for g to guarantee that any function β satisfying (2.1.5) is a strict lower solution for (2.1.1).

Moreover, in the sequel, the following working hypotheses will be considered as well:

(Hg_2) $\exists\, \gamma_0 \in L^1([0,T],\mathbb{R}^+):\ g(t,u) \geq -\gamma_0(t),\ \forall\, u \in \mathbb{R}$ *and a.e.* $t \in [0,T]$;

(Hg_3) $\exists\, g_0:\ g(t,0) \leq g_0$ *for a.e.* $t \in [0,T]$;

(Hg_4^-) *for each* σ *there exists* $d_\sigma > 0$ *such that* $\frac{1}{T}\int_0^T g(t,u(t))\,dt > \sigma$ *for each* $u \in C^1_T$ *such that* $u(t) \leq -d_\sigma$ *for all* $t \in [0,T]$;

(Hg_4^+) *for each* σ *there exists* $d_\sigma > 0$ *such that* $\frac{1}{T}\int_0^T g(t,u(t))\,dt > \sigma$ *for each* $u \in C^1_T$ *such that* $u(t) \geq d_\sigma$ *for all* $t \in [0,T]$.

Remark 2.1.6. Let us make some comments on the previous assumptions. If the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic in the variable t, then hypothesis (Hg_3) and condition (A_2) are always satisfied, and moreover the solutions of $(\mathscr{LE}_s)$ become of class C^2 (see [DCH06]). This situation is the one studied in [FMN86], instead, in our framework, we do not require $g(t,u) \to +\infty$ for $|u| \to +\infty$, uniformly in t. In this respect, the uniform condition is replaced by assuming the existence of a lower bound as in $(\mathrm{H}g_2)$ and the Villari's type conditions $(\mathrm{H}g_4^-)$ and $(\mathrm{H}g_4^+)$. $\lhd$

Theorem 2.1.7. *Assume* (A_2), $(\mathrm{H}g_2)$, $(\mathrm{H}g_3)$ *and* $(\mathrm{H}g_4^-)$. *Then, there exists* $s_0 \in \mathbb{R}$ *such that:*

1° *for* $s < s_0$, *equation* $(\mathscr{LE}_s)$ *has no* T*-periodic solutions;*

2° *for* $s > s_0$, *equation* $(\mathscr{LE}_s)$ *has at least one* T*-periodic solution.*

Proof. The proof follows the scheme proposed in [FMN86, Theorem 1] which is adapted from [KW75]. For any given parameter $s \in \mathbb{R}$, we set

$$\nu_s(t,u) := g(t,u) - s, \tag{2.1.8}$$

so that equation $(\mathscr{LE}_s)$ is of the form (2.1.1).

Let us start by fixing a parameter $s_1 > g_0$. In this situation, the constant function $\beta(t) \equiv 0$ is a strict upper solution. Indeed, we have

$$\beta''(t) + f(\beta(t))\beta'(t) + g(t,\beta(t)) - s_1 = g(t,0) - s_1 \leq -(s_1 - g_0) < 0$$

and then condition (A_2) guarantees our claim, according to [DCH06, Section 3, Proposition 1.6]. On the other hand, for $\sigma = s_1$, condition $(\mathrm{H}g_4^-)$ implies the Villari's condition at $-\infty$ with $\delta = 1$. Hence, an application of Theorem 2.1.5 guarantees the existence of at least one T-periodic solution u of $(\mathscr{LE}_s)$ for $s = s_1$ with $u < 0$.

As a second step, we claim that if, for some $\tilde{s} < s_1$ the equation has a T-periodic solution (that we will denote by w), then equation $(\mathscr{LE}_s)$ has a T-periodic solution for each $s \in [\tilde{s}, s_1]$. Clearly, it will be sufficient to prove this assertion for s with $\tilde{s} < s < s_1$. Writing equation $(\mathscr{LE}_s)$ as

$$u'' + f(u)u' + g(t,u) - \tilde{s} - (s - \tilde{s}) = 0,$$

we find that $\beta(t) \equiv w(t)$ is a strict upper solution of $(\mathscr{LE}_s)$. Indeed, we have

$$\beta''(t) + f(\beta(t))\beta'(t) + g(t,\beta(t)) - s = w'' + f(w(t))w'(t) + g(t,w(t)) - s = -(s - \tilde{s}) < 0$$

and then property (A_2) guarantees our claim, according to [DCH06, Section 3, Proposition 1.6]. On the other hand, for $\sigma = s$, condition $(\mathrm{H}g_4^-)$ implies the Villari's condition at $-\infty$ with $\delta = 1$. Again, an application of Theorem 2.1.5 guarantees the existence of at least one T-periodic solution u of $(\mathscr{LE}_s)$ with $u < w$ and the claim is proved.

If u is any T-periodic solution of $(\mathscr{LE}_s)$, then, taking the average of the equation on $[0,T]$, we have $\frac{1}{T}\int_0^T g(t,u(t))\,dt = s$ and, using $(\mathrm{H}g_2)$, we obtain

$$s \geq \alpha_0 := -\frac{1}{T}\int_0^T \gamma_0(t)\,dt. \tag{2.1.9}$$

Hence, if $s < \alpha_0$, equation $(\mathscr{LE}_s)$ has no T-periodic solution.

At this point, we have proved that the set of the parameters s for which equation $(\mathscr{LE}_s)$ has T-periodic solutions is an interval which is bounded from below. Let

$$s_0 := \inf\{s \in \mathbb{R} : (\mathscr{LE}_s) \text{ has at least one } T\text{-periodic solution}\}.$$

By the previous discussion, we know that $\alpha_0 \leq s_0 \leq g_0$ and the thesis follows. $\square$

Remark 2.1.8. Let us make some comments on the parameter s_0 in relation with Theorem 2.1.7. Indeed, at this point no information is given about existence or nonexistence of T-periodic solutions of $(\mathscr{LE}_s)$ when $s = s_0$. Without supplementary conditions, we are not able to determine whether the equation $(\mathscr{L}\mathscr{E}_s)$ has T-periodic solutions. For instance, let us consider the T-periodic problem associated with

$$u'' + \phi(u) = s \text{ with } \phi(u) = 2\alpha\left(\sqrt{1+u^2} - u\right), \text{ for } 0 < \alpha < (\pi/T)^2.$$

The T-periodic solutions of the considered equation are only the constant ones, namely the real solutions of $\phi(u) = s$. In this case, $s_0 = 0$ and no solutions exist for $s = s_0$. Similar examples of equations admitting T-periodic solutions for $s = s_0$, can be provided too. $\lhd$

To state our multiplicity result of T-periodic solutions now we are going to assume both the two Villari's condition at $-\infty$ and $+\infty$.

Theorem 2.1.9. *Assume* (A_2), $(\mathrm{H}g_2)$, $(\mathrm{H}g_3)$, $(\mathrm{H}g_4^-)$ *and* $(\mathrm{H}g_4^+)$. *Then, there exists* $s_0 \in \mathbb{R}$ *such that:*

1° *for* $s < s_0$, *equation* $(\mathscr{L}\mathscr{E}_s)$ *has no* T*-periodic solutions;*

2° *for* $s = s_0$, *equation* $(\mathscr{L}\mathscr{E}_s)$ *has at least one* T*-periodic solution;*

3° *for* $s > s_0$, *equation* $(\mathscr{L}\mathscr{E}_s)$ *has at least two* T*-periodic solutions.*

Proof. Without loss of generality, we can suppose that the map $\sigma \mapsto d_\sigma$ is defined on $[0, +\infty)$ and is monotone non-decreasing. The proof of our result follows the outline in [FMN86, Theorem 2]. As before, using (2.1.8), we write equation $(\mathscr{L}\mathscr{E}_s)$ in the form of (2.1.1). Following the functional-analytic approach introduced in Appendix A, we also denote by N_s the corresponding Nemytskii operator, namely

$$(N_s u)(t) := f(u(t))u'(t) + \nu_s(t, u(t)), \quad \forall\, u \in C^1_T.$$

Let us start by fixing a parameter $s_1 > \max\{0, g_0\}$. We claim that it is verified the following property:

$(\mathfrak{P})$ *there exist a positive constant* $\Lambda = \Lambda(s_1)$ *such that for each* $s \le s_1$ *any solution of* $Lu = \lambda N_s u$, *with* $0 < \lambda \le 1$, *satisfies* $\|u\|_\infty < \Lambda$.

In order to prove property $(\mathfrak{P})$, we observe that, by $(\mathrm{H}g_2)$ and $s \le s_1$, it follows that $\nu_s(t, u) \ge -\gamma_0(t) - s_1$ for a.e. $t \in [0, T]$. In this manner, condition (A_1) of Lemma 2.1.1 holds for $\gamma := \gamma_0(t) - |s_1|$ and there exists a constant $K = K(s_1)$ such that, any possible T-periodic solution of

$$u'' + \lambda f(u)u' + \lambda\nu_s(t, u) = 0, \quad 0 < \lambda \le 1, \tag{2.1.10}$$

satisfies

$$\max u - \min u \le K(s_1).$$

Next, we observe that any possible T-periodic solution of (2.1.10) satisfies

$$\max u > -d_{s_1}.$$

Indeed, if $u(t) \le -d_{s_1}$ for all t, taking the average of the equation on $[0, T]$ (and dividing by $\lambda > 0$), we obtain

$$\begin{aligned} 0 = \frac{1}{T}\int_0^T \nu_s(t, u(t))\, dt &= \frac{1}{T}\int_0^T g(t, u(t))\, dt - s \\ &\ge \frac{1}{T}\int_0^T g(t, u(t))\, dt - s_1 > 0, \end{aligned}$$

as a consequence of $(\mathrm{H}g_4^-)$ and so a contradiction is achieved. Similarly, from $(\mathrm{H}g_4^+)$ it follows that

$$\min u < d_{s_1}.$$

By the above inequalities we conclude that $\|u\|_\infty < \Lambda(s_1) := K(s_1) + d_{s_1}$, proving $(\mathfrak{P})$.

As a next step, we observe that there is no T-periodic solution for equation (2.1.10) for $s < \alpha_0$, where α_0 is the constant introduced in (2.1.9) in Theorem 2.1.7.

Let us fix now a constant $s_2 < s_0$. Let also ρ_g a non-negative L^1-Carathéodory function bounding $|g(t,u)|$ for $|u| \leq \Lambda(s_1)$, so that

$$|\nu_s(t,u)| \leq \rho_g(t) + \max\{s_1, |s_2|\}, \text{ for a.e. } t \in [0,T], \forall\, s \in [s_2, s_1], \forall\, u \in [-\Lambda(s_1), \Lambda(s_1)].$$

An application of Lemma 2.1.2 along with property $(\mathfrak{P})$, leads to the existence of a constant $\eta(s_1, s_2) > 0$ such that, for each $s \in [s_2, s_1]$, any solution of $Lu = \lambda N_s u$ with $0 < \lambda \leq 1$, satisfies $\|u'\|_\infty < \eta(s_1, s_2)$.

Following [FMN86], we define the set

$$\Omega_1 = \Omega_1(R_1, R_2) := \{u \in C^1_T : \|u\|_\infty < R_1,\ \|u'\|_\infty < R_2\},$$

which is open and bounded in C^1_T. Putting $\lambda = 1$ and moving $s \in [s_2, s_1]$ as an homotopic parameter, we obtain that

$$D_L(L - N_{s_1}, \Omega_1) = D_L(L - N_{s_2}, \Omega_1) = 0, \quad \forall\, R_1 \geq \Lambda(s_1), \quad \forall\, R_2 \geq \eta(s_1, s_2).$$

From Theorem 2.1.7 we already know that, for $s = s_1$ there is at least one solution and, if there is a solution for some $\tilde{s} < s_1$, then also for every $s \in [\tilde{s}, s_1]$ a solution exists. We claim now that a second solution exists for $s \in]\tilde{s}, s_1]$.

Let w be a T-periodic solution of $(\mathscr{L}\mathscr{E}_s)$ for $s = \tilde{s} < s_1$. Let now $\tilde{s} < s \leq s_1$. Writing equation $(\mathscr{L}\mathscr{E}_s)$ as

$$u'' + f(u)u' + g(t,u) - \tilde{s} - (s - \tilde{s}) = 0,$$

we have that $\beta(t) \equiv w(t)$ is a strict upper solution of $(\mathscr{L}\mathscr{E}_s)$ (as proved in Theorem 2.1.7). On the other hand, for $\sigma = s$, condition $(\mathrm{H}g_4^-)$ implies the Villari's condition at $-\infty$ with $\delta = 1$. Given any constant $R_1 \geq \Lambda(s_1) + 1$ and by fixing a constant $R_2 \geq \eta(s_1, s_2)$, we have that

$$\Omega := \Omega(R_1, w, R_2) \subseteq \Omega_1 := \Omega_1(R_1, R_2),$$

with

$$D_L(L - N_s, \Omega) = 1, \quad D_L(L - N_s, \Omega_1) = 0.$$

Then, the additivity property of the coincidence degree theory (cf. Appendix A) implies that, besides a solution $w_s^{(1)} \in \Omega$, there exists also a second solution $w_s^{(2)} \in \Omega_1 \setminus \overline{\Omega}$.

As in the proof of Theorem 2.1.7, let us define again

$$s_0 := \inf\{s \in \mathbb{R} : (\mathscr{L}\mathscr{E}_s) \text{ has at least one } T\text{-periodic solution}\}.$$

By the above discussion, we know that $\alpha_0 \leq s_0 \leq g_0$ and, moreover:

for every $s < s_0$, there is no T-periodic solution of $(\mathscr{L}\mathscr{E}_s)$ and for every $s > s_0$, there are at least two T-periodic solutions of $(\mathscr{L}\mathscr{E}_s)$.

To conclude with the proof we have to check that, for $s = s_0$, there is at least one T-periodic solution. We thus follow an argument in accord to [FMN86]. Let $s_2 < s_0 < s_1$ be fixed and let θ_n be a decreasing sequence of parameters with $\theta_n \to s_0$ and $\theta_n \in]s_0, s_1]$ for all n. By the previous estimates, we have, for each n, the existence of at least one (actually two) T-periodic solution w_n of equation

$$u'' + f(u)u' + g(t,u) = \theta_n$$

with

$$\|w_n\|_\infty \leq \Lambda(s_1), \quad \|w_n'\|_\infty \leq \eta(s_1, s_2).$$

An application of the Ascoli-Arzelà theorem, passing to the limit as $n \to \infty$, provides the existence of at least one T-periodic solution of $(\mathscr{L}\mathscr{E}_s)$ for $s = s_0$. This completes the proof. □

2.1.3 Non-sign definite weighted Liénard equation

We present now an application of Theorem 2.1.9 with the aim of treat examples classical in literature. In particular, we consider a generalized Liénard equation with a weighted restoring term of the form

$$(\mathscr{WLE}_s) \qquad u'' + f(u)u' + a(t)\phi(u) = s + p(t).$$

This kind of equations can be viewed also as an example of forced Liénard-Mathieu equations. The interest in the study of these equations has been growing up in recent years (see [Kal17]) and it can be traced back to Minorsky's work [Min53].

Corollary 2.1.10. *Let f, $\phi : \mathbb{R} \to \mathbb{R}$ be continuous functions and suppose that*

$$(\mathrm{H}\phi_2) \qquad \lim_{|u|\to\infty} \phi(u) = +\infty.$$

Let a, $p \in L^\infty[0,T]$ with $a(t) \geq 0$ for a.e. $t \in [0,T]$ and $\int_0^T a(t)\,dt > 0$. Then, there exists $s_0 \in \mathbb{R}$ such that:

1° *for $s < s_0$, equation $(\mathscr{WLE}_s)$ has no T-periodic solutions;*

2° *for $s = s_0$, equation $(\mathscr{WLE}_s)$ has at least one T-periodic solution;*

3° *for $s > s_0$, equation $(\mathscr{WLE}_s)$ has at least two T-periodic solutions.*

Proof. We apply Theorem 2.1.9 for

$$g(t,u) := a(t)\phi(u) - p(t).$$

Let us set $\phi_0 := \min_{\xi\in\mathbb{R}} \phi(\xi)$. For any $d > \max\{\phi_0, 0\}$, we introduce the following constants:

$$\zeta^-(d) := \min\{\phi(u) : u \leq -d\}, \quad \zeta^+(d) := \min\{\phi(u) : u \geq d\}.$$

From $(\mathrm{H}\phi_2)$ it follows that both $\zeta^-(d) \to +\infty$ and $\zeta^+(d) \to +\infty$ for $d \to +\infty$. Let $u \in C^1_T$ be such that $|u(t)| \geq d$ for all $t \in [0,T]$. Clearly, $u(t) \leq -d$, $\forall\, t$ or $u(t) \geq d$, $\forall\, t$. In the former case we have that

$$\begin{aligned}\frac{1}{T}\int_0^T g(t,u(t))\,dt &= \frac{1}{T}\int_0^T a(t)\phi(u(t))\,dt - \frac{1}{T}\int_0^T p(t)\,dt \\ &\geq \frac{\zeta^-(d)}{T}\int_0^T a(t)\,dt - \frac{1}{T}\int_0^T p(t)\,dt.\end{aligned}$$

In the other case, we analogously have

$$\frac{1}{T}\int_0^T g(t,u(t))\,dt \geq \frac{\zeta^+(d)}{T}\int_0^T a(t)\,dt - \frac{1}{T}\int_0^T p(t)\,dt.$$

This way, both the Villari's conditions $(\mathrm{H}g_4^-)$ and $(\mathrm{H}g_4^+)$ are satisfied. Condition $(\mathrm{H}g_2)$ is satisfied by choosing as $\gamma_0(t)$ the positive part of $p(t) - a(t)\phi_0$. Hypothesis $(\mathrm{H}g_3)$ holds for any constant $g_0 \geq \|a\|_\infty \phi(0) + \|p\|_\infty$. At last, we observe that condition (A_2) holds for this special choice of $g(t,u)$ (see [DCH06]). □

As a direct consequence we obtain an improvement of Corollary 1.3, since the weight term $a(t) \geq 0$ vanishes somewhere. So that, the classical coercivity condition in [FMN86] is weakened to a local one, $\lim_{|u|\to+\infty} a(t)\phi(u) - p(t) \to +\infty$ for a.e. $t \in [0,T]$.

2.2 Complex dynamics

In this section we study the periodic BVP associated with equation

$$(\mathscr{E}_1) \qquad u'' + \phi(u) = h(t),$$

or with equation

$$(\mathscr{E}_2) \qquad u'' + cu' + \phi(u) = h(t),$$

where we tacitly assume in the sequel that the friction coefficient $c > 0$, the forcing term $h\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic locally integrable function and the nonlinearity satisfies

(Hϕ_3) $\phi\colon \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitz continuous function with* $\phi(0) = 0$*, which is strictly decreasing on* $]-\infty, 0]$ *and strictly increasing on* $[0, +\infty[$ *and* $\lim_{|u|\to+\infty} \phi(u) = +\infty$.

If c is assumed to be small, equation ($\mathscr{E}_2$) can be viewed as a perturbation of the conservative equation ($\mathscr{E}_1$).

Remark 2.2.1. Let us make some comments on condition (Hϕ_3). The features assumed for the nonlinearity ϕ remember the typical ones which appear in AP problems. Indeed, we notice that any sufficiently smooth strictly convex function satisfying

$$\lim_{\xi\to-\infty} \phi'(\xi) < 0 < \lim_{\xi\to+\infty} \phi'(\xi)$$

verifies (Hϕ_2) and it has a unique point of strict absolute minimum $\xi = \xi_m$. So that, without loss of generality (i.e. possibly replacing $\phi(\xi)$ with $\phi(\xi + \xi_m) - \phi(\xi_m)$), we can suppose to work with a nonlinear function ϕ having a strict absolute minimum at $\xi = 0$ and such that $\phi(0) = 0$. Hence, the nonlinearity considered in this section contains the principal features about the crossing of the first eigenvalue $\lambda_1 = 0$. ◁

We are now going to discuss the existence of infinitely many T-periodic solutions as well as detect "chaotic dynamics" under several conditions on $h(t)$. We will refer to the different concepts of chaos which are presented in Appendix B. In particular, we will be interested in the search of "*Smale's horseshoes*" (cf. Definition B.1) as well as "*topological horseshoes*" (cf. Definition B.2).

A graphical motivation for these investigations is suggested in the phase portrait in Figure 2.1 where it is represented a very complicated behavior for solutions of a second order ODE with periodic coefficients and a nonlinearity satisfying our working conditions. A characteristic displayed by Figure 2.1 is the typical alternation of regions of stability and instability or randomness that is common in Hamiltonian systems (cf. [Mos73, Chapter 3]).

We will adopt a dynamical system approach, for the investigations on both ($\mathscr{E}_1$) and ($\mathscr{E}_2$). In this respect, we will analyze the local flow associated with the corresponding systems in the phase-plane. In particular, dealing with ($\mathscr{E}_1$), we consider the planar Hamiltonian system

$$(\mathscr{S}_1) \qquad \begin{cases} x' = y, \\ y' = -\phi(x) + h(t). \end{cases}$$

As usual, by the local flow determined by ($\mathscr{S}_1$) we mean the map $\Phi_{t_0}^t$ which associates to any initial point $z_0 = (x_0, y_0) \in \mathbb{R}^2$ the point $\zeta(t)$, where $\zeta(\cdot)$ is the solution of ($\mathscr{S}_1$) satisfying the initial condition $\zeta(0) = z_0$ and defined on its maximal interval of existence. In the sequel, when not otherwise specified, we will take $t_0 = 0$ and we will consider the Poincaré operator $\Phi := \Phi_0^T$. The fundamental theory of ODEs guarantees that Φ is a homeomorphism defined on an open set dom$\Phi \subseteq \mathbb{R}^2$. Similar considerations can be done for system

$$(\mathscr{S}_2) \qquad \begin{cases} x' = y, \\ y' = -cy - \phi(x) + h(t). \end{cases}$$

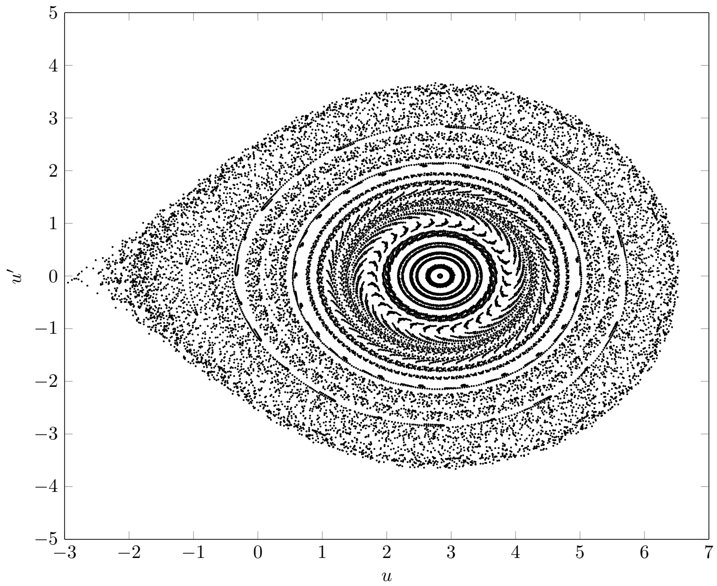


Figure 2.1: Evolution of $u'' + \sqrt{1+u^2} - 1 = 2 + \varepsilon \sin(\omega t)$ in terms of the iterates of the Poincaré map, with $\varepsilon = 0.01$, $\omega = 10$ and varying 500 initial conditions $(u(0), u'(0))$ where $u(0)$ is within the interval $[-4, 6]$ and $u'(0) = 0$.

which is equivalent to $(\mathscr{E}_2)$. Finally, we will describe the complex behavior of T-periodic solutions of both equations $(\mathscr{E}_1)$ and $(\mathscr{E}_2)$, in terms of chaotic dynamics of the discrete dynamical system identified by the Poincaré map associated with system $(\mathscr{S}_1)$ or with system $(\mathscr{S}_2)$, respectively.

In more detail, we will observe that the planar phase-portrait associated with the autonomous equation $u'' + \phi(u) = k$, for $k > 0$, is that of a local center enclosed by a homoclinic trajectory of a hyperbolic saddle point. Owing to this saddle-center geometry, if $(\mathscr{E}_1)$ may be treated as a small perturbation of the associated autonomous system, we will exploit a Melnikov's type approach. On the contrary, when the perturbation is not necessarily small, we will discuss two other different methods. One is coming from the Conley index theory and it is borrowed from [Ged+02; KMO96]. The other one is based on a topological argument called "stretching along the paths method" (SAP method), set out in Appendix B. In any case, we will divide our results into two types, according to the detection of a Smale's horseshoe or a topological one. In view of the different methods used, the conditions for ϕ assumed in $(\mathrm{H}\phi_3)$ represent the minimum equipment of requirements which are common in all the different approaches we are going to discuss and further regularity conditions will be also introduced in the sequel when needed.

2.2.1 Phase-plane analysis

The study of system $(\mathscr{S}_1)$ should become easier after a preliminary qualitative analysis of the autonomous system with a constant forcing term. Roughly speaking, this corresponds to the case in which the time variable is "freezed". Therefore, let us introduce a model problem by means of the autonomous ODE

$$u'' + \phi(u) = k, \tag{2.2.1}$$

with k a real parameter. The phase-plane analysis and geometric considerations give us information about the qualitative behavior of the solutions of (2.2.1) and in turn of $(\mathscr{E}_1)$.

Accordingly, equation (2.2.1) can be written equivalently as a planar system in the phase-plane (x, y):

$$\begin{cases} x' = y, \\ y' = -\phi(x) + k. \end{cases} \tag{2.2.2}$$

First of all, let us find the equilibria of (2.2.2) by solving $\phi(x) = k$. In view of $\min_{\xi\in\mathbb{R}} \phi(\xi) = \phi(0) = 0$, we consider from now on only the case $k \geq 0$.

If $k = 0$, the origin is an unstable equilibrium of the system. In particular, it is the coalescence of a saddle point with a center. It seems interesting to observe that in literature such a geometry appears in the so called Bogdanov-Takens bifurcation (see [GH83]). On the other hand, if $k > 0$, the properties of the function ϕ lead to the existence of exactly two equilibria. Under the assumption (Hϕ_3) made on ϕ, we can define two homeomorphisms

$$\phi_l := \phi|_{]-\infty,0]} :]-\infty, 0] \to [0, +\infty[,$$
$$\phi_r := \phi|_{[0,+\infty[} : [0, +\infty[\to [0, +\infty[,$$

such that ϕ_l is strictly decreasing and ϕ_r is strictly increasing. Therefore, the inverse functions of both ϕ_l and ϕ_r are well defined and we denote them by ϕ_l^{-1} and ϕ_r^{-1}, respectively. By setting

$$x_u = x_u(k) := \phi_l^{-1}(k), \quad x_s = x_s(k) := \phi_r^{-1}(k),$$

we have $x_u < x_s$. The equilibria are the points $(x_u, 0)$ and $(x_s, 0)$ where the first one has got the topological structure of an unstable saddle and the second one is a stable center.

The system (2.2.2) is a hamiltonian system with total energy given by

$$E_k(x, y) := \frac{1}{2}y^2 + F(x) - kx, \tag{2.2.3}$$

where F is defined by

$$F(x) := \int_0^x \phi(\xi)\, d\xi.$$

Notice that $F(\pm\infty) = \pm\infty$.

To describe the associated phase portrait, for each $\rho \in \mathbb{R}$, we define the energy level lines of (2.2.2) as follows

$$\mathcal{L}_\rho := \{(x, y) \in \mathbb{R}^2 : E_k(x, y) = \rho\}.$$

In order to study the geometry of each $\mathcal{L}_\rho$ it is useful to introduce the auxiliary function

$$\Lambda_k(x) := F(x) - kx. \tag{2.2.4}$$

Observe that, for each $k > 0$, the graph of the function Λ_k is that of a N-shaped curve passing through the origin with negative slope.

Proposition 2.2.2. *Let Λ_k be defined as in* (2.2.4) *for $k = 0$. Then, $\Lambda_0(x) = \rho$ has a unique solution for every $\rho \in \mathbb{R}$. In particular, the following hold.*

- *If $\rho = 0$ the solution is $x = 0$.*
- *If $\rho < 0$ we denote it by $x_*(\rho)$ and it is such that $x_*(\rho) < 0$.*
- *If $\rho > 0$ we denote it by $x^*(\rho)$ and it is such that $x^*(\rho) > 0$.*

Proof. From (Hϕ_3) we obtain that Λ_0 is strictly increasing on $\mathbb{R}$ and also $\Lambda_0(0) = 0$. Hence, thanks to the monotonicity of Λ_0, the conclusions follow straightaway. □

Proposition 2.2.3. *Let k be a fixed positive real number and Λ_k defined as in* (2.2.4)*. Then, the following hold.*

- *If $\rho = \Lambda_k(x_u)$, then $\Lambda_k(x) = \rho$ has two solutions. One is x_u and the other one, denoted by $x_h = x_h(k)$, is such that $x_s < x_h$.*
- *If $\rho = \Lambda_k(x_s)$, then $\Lambda_k(x) = \rho$ has two solutions. One is x_s and the other one, denoted by $x_*(\rho)$, is such that $x_*(\rho) < x_s$.*

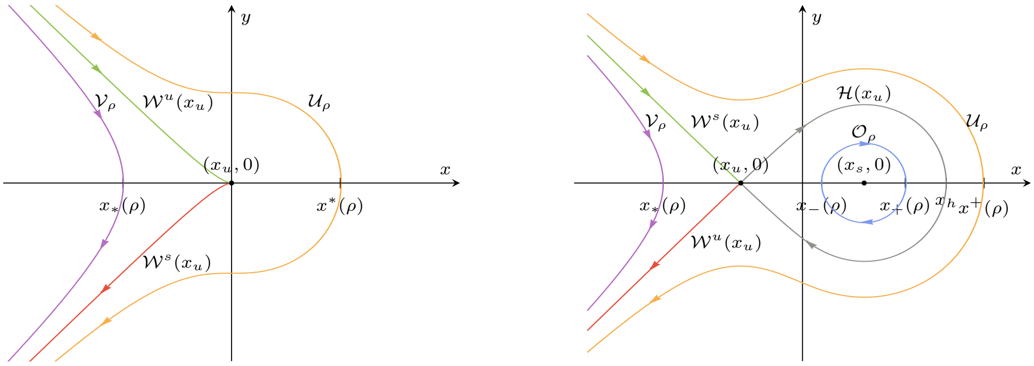


Figure 2.2: Phase portraits of the autonomous system (2.2.2) where the nonlinearity is given by $\phi(u) = \sqrt{1+u^2} - 1$. Right: $k = 0$. Left: $k > 0$. In both cases the geometry of the different energy level lines is pointed out and the arrows show the direction of the flow along the trajectories.

- *If $\rho > \Lambda_k(x_u)$, then $\Lambda_k(x) = \rho$ has a unique solution, denoted by $x^*(\rho)$, and it is such that $x^*(\rho) > x_h$.*
- *If $\Lambda_k(x_s) < \rho < \Lambda_k(x_u)$, then $\Lambda_k(x) = \rho$ has three solutions. These solutions, denoted by $x_*(\rho)$, $x_-(\rho)$ and $x_+(\rho)$, are such that $x_*(\rho) < x_u < x_-(\rho) < x_s < x_+(\rho)$.*
- *If $\rho < \Lambda_k(x_s)$, then $\Lambda_k(x) = \rho$ has a unique solution, denoted by $x_*(\rho)$, and it is such that $x_*(\rho) < x_u$.*

Proof. Assumption (Hϕ_3) leads to $\Lambda_k(0) = 0$. By definition of Λ_k, its derivative is $\Lambda_k'(x) = \phi(x) - k$. Thus, $\lim_{x\to\pm\infty} \Lambda_k'(x) = +\infty$ because of condition $\lim_{|x|\to\infty} \phi(x) = +\infty$. This way, we have $\Lambda_k(\pm\infty) = \pm\infty$. Moreover, Λ_k has exactly two critical points which are the abscissa of the equilibria of system (2.2.2). From (Hϕ_3), we deduce that x_u is a local maximum and x_s is a local minimum. Therefore, it follows that Λ_k is strictly decreasing on $[x_u, x_s]$ and strictly increasing on $]-\infty, x_u]$ and $[x_s, +\infty[$. Since $0 \in]x_u, x_s[$, we have $\Lambda_k(x_u) > 0 > \Lambda_k(x_s)$.

So that, if $\rho = \Lambda_k(x_u)$, then there exists unique $x_h \in]x_s, +\infty[$ such that $\Lambda_k(x_h) = \Lambda_k(x_u)$. Analogously, if $\rho = \Lambda_k(x_s)$ then there exists unique $x_*(\rho) \in]-\infty, x_u[$ such that $\Lambda_k(x_*(\rho)) = \Lambda_k(x_s)$. Instead, for every $\rho \in]\Lambda_k(x_s), \Lambda_k(x_u)[$, there exist $x_*(\rho) \in]-\infty, x_u[$, $x_-(\rho) \in]x_u, x_s[$ and $x_+(\rho) \in]x_s, x_h[$ which are zeros of the equation $\Lambda_k(x) = \rho$. At last, if $\rho \in]\Lambda_k(x_u), +\infty[$, or $\rho \in]-\infty, \Lambda_k(x_s)[$, the equation $\Lambda_k(x) = \rho$ has exactly one solution $x^*(\rho) \in]x_h, +\infty[$, respectively $x_*(\rho) \in]-\infty, x_u[$. □

An application of Proposition 2.2.2 along with Proposition 2.2.3 reveals the geometry of the phase portrait associated with system (2.2.2) for any given $k \geq 0$. Examples of phase portraits which mimic the behavior of the solutions of (2.2.1) are shown in Figure 2.2. Moreover, for all $\rho \in \mathbb{R}$, we can characterize the energy level lines $\mathcal{L}_\rho$ according to their type with respect to the level ρ. Since, the different kinds of energy level lines for case $k > 0$ include the ones for $k = 0$, here we give just a detailed discussion about positive reals k.

For $\rho = \Lambda_k(x_u)$, the saddle like structure is characterized by the union of the unstable equilibrium point with the unstable manifold $W^u(x_u)$, the stable manifold $W^s(x_u)$ and the homoclinic orbit $\mathcal{H}(x_u)$. This way, we have

$$\mathcal{L}_{\Lambda_k(x_u)} = \{(x_u, 0)\} \cup W^u(x_u) \cup W^s(x_u) \cup \mathcal{H}(x_u),$$

where

$$\begin{aligned}
W^u(x_u) &:= \{(x,y) \in \mathbb{R}^2 : x < x_u,\ y < 0,\ E_k(x,y) = \Lambda_k(x_u)\},\\
W^s(x_u) &:= \{(x,y) \in \mathbb{R}^2 : x < x_u,\ y > 0,\ E_k(x,y) = \Lambda_k(x_u)\},\\
\mathcal{H}(x_u) &:= \{(x,y) \in \mathbb{R}^2 : x > x_u,\ E_k(x,y) = \Lambda_k(x_u)\}.
\end{aligned}$$

For $\Lambda_k(x_s) < \rho < \Lambda_k(x_u)$, the energy level line splits as follows

$$\mathcal{L}_{\rho \in]\Lambda_k(x_s),\Lambda_k(x_u)[} = \mathcal{O}_\rho \cup \mathcal{V}_\rho,$$

where

$$\mathcal{O}_\rho := \{(x,y) \in \mathbb{R}^2 : x > x_u,\, E_k(x,y) = \rho\} \tag{2.2.5}$$

is a closed symmetric curve surrounding the center which intersects the x-axis at the points $(x_-(\rho), 0)$ and $(x_+(\rho), 0)$ and it is run in the clockwise sense, on the contrary,

$$\mathcal{V}_\rho := \{(x,y) \in \mathbb{R}^2 : x < x_u,\, E_k(x,y) = \rho\} \tag{2.2.6}$$

is an unbounded symmetric curve which intersects the x-axis at the point $(x_*(\rho), 0)$.

If $\rho = \Lambda_k(x_s)$, then

$$\mathcal{L}_{\Lambda_k(x_s)} = \{(x_s, 0)\} \cup \mathcal{V}_{\Lambda_k(x_s)},$$

where $\{(x_s, 0)\}$ is the stable equilibrium point and $\mathcal{V}_{\Lambda_k(x_s)}$ is defined according to (2.2.6).

For every $\rho < \Lambda_k(x_s)$, $\mathcal{L}_\rho$ is a curve identified by (2.2.6) and so, also in this case, we denote each energy level line with $\mathcal{V}_\rho$.

For every $\rho > \Lambda_k(x_u)$, $\mathcal{L}_\rho$ is an unbounded symmetric curve over the saddle like structure which intersects the x-axis at the point $(x^*(\rho), 0)$ and it is run in the clockwise sense. In this case, the energy level line is

$$\mathcal{U}_\rho := \mathcal{L}_{\rho \in]\Lambda_k(x_u),+\infty[} = \{(x,y) \in \mathbb{R}^2 : E_k(x,y) = \rho\}. \tag{2.2.7}$$

We conclude the phase-plane analysis performing a study, depending on k, of the intersection points between the saddle like structure with the x-axis.

Proposition 2.2.4. *Let $k_1, k_2 \in \mathbb{R}$ such that $0 \le k_1 < k_2$ and $\Lambda_{k_1}, \Lambda_{k_2}$ defined as in (2.2.4), then there exist unique $x_h(k_i)$ for $i \in \{1,2\}$ such that $\Lambda_{k_i}(x_u(k_i)) = \Lambda_{k_i}(x_h(k_i))$ and $x_u(k_2) < x_u(k_1) < x_h(k_1) < x_h(k_2)$.*

Proof. From the growth conditions of ϕ in (Hϕ_3) it follows that

$$x_u(k_2) < x_u(k_1) < x_s(k_1) < x_s(k_2).$$

By the definition of Λ_k, we deduce that

$$\Lambda_{k_1}(x) < \Lambda_{k_2}(x), \quad \forall\, x < 0, \tag{2.2.8}$$

$$\Lambda_{k_1}(x) > \Lambda_{k_2}(x), \quad \forall\, x > 0. \tag{2.2.9}$$

Since $x_u(k_2) < x_u(k_1) < 0$, the condition in (2.2.8) and the fact that Λ_{k_2} is strictly decreasing on $]x_u(k_2), 0]$, imply

$$\Lambda_{k_1}(x_u(k_1)) < \Lambda_{k_2}(x_u(k_1)) < \Lambda_{k_2}(x_u(k_2)). \tag{2.2.10}$$

Thanks to Proposition 2.2.3 there exist exactly two positive real numbers $x_h(k_1)$, $x_h(k_2)$ such that $x_s(k_1) < x_h(k_1)$, $x_s(k_2) < x_h(k_2)$ and

$$\Lambda_{k_i}(x_u(k_i)) = \Lambda_{k_i}(x_h(k_i)), \quad \text{for } i = 1, 2.$$

Using these equalities in (2.2.10) we can get

$$\Lambda_{k_1}(x_h(k_1)) < \Lambda_{k_2}(x_h(k_2)). \tag{2.2.11}$$

Whereas $x_h(k_2) > 0$, then from the condition in (2.2.9) follows

$$\Lambda_{k_2}(x_h(k_1)) < \Lambda_{k_1}(x_h(k_1)). \tag{2.2.12}$$

Combining (2.2.11) and (2.2.12), we obtain $\Lambda_{k_1}(x_h(k_1)) < \Lambda_{k_1}(x_h(k_2))$. Since Λ_{k_1} is strictly increasing on $[x_s(k_1), +\infty[$, we conclude that

$$x_h(k_1) < x_h(k_2),$$

because of $x_s(k_1) < x_h(k_2)$. □

Time mapping formulas. Let us introduce some more notation that will be used throughout this dissertation. Considering (2.2.3) and (2.2.4), the time needed to a solution to move in the phase-plane (x, y) along an orbit path identified by the energy level ρ, from a point (x_1, y_1) to a point (x_2, y_2), is given by

$$\tau(\rho;\, x_1, x_2) := \int_{x_1}^{x_2} \frac{1}{\sqrt{2(\rho - \Lambda_k(s))}}\, ds. \tag{2.2.13}$$

The function $\rho \mapsto \tau(\rho;\, x_1, x_2)$ is called time-map associated with the autonomous equation (2.2.1). The phase-plane analysis has highlighted the presence of a saddle like structure and also mainly two types of orbits. More in detail, there are the periodic orbits, $\mathcal{O}_\rho$, and the non-periodic ones, $\mathcal{V}_\rho$ and $\mathcal{U}_\rho$. With this in mind, we can characterize the time-map formulas in three different kinds.

In the case of the periodic orbits, by (2.2.13) we can evaluate the time elapsed to move along the orbit $\mathcal{O}_\rho$ which is defined as in (2.2.5). In particular, we set the time needed to travel from $(x_-(\rho), 0)$ to a point $(r, 0)$ on $\mathcal{O}_\rho$, with $x_-(\rho) < r \leq x_+(\rho)$, as follows

$$\tau_{\mathcal{O}}(\rho;\, r) := \tau(\rho;\, x_-(\rho), r) = \int_{x_-(\rho)}^{r} \frac{1}{\sqrt{2(\rho - \Lambda_k(s))}}\, ds. \tag{2.2.14}$$

This way, since $\mathcal{O}_\rho$ is a closed symmetric curve, its fundamental period is given by $2\,\tau_{\mathcal{O}}(\rho;\, x_+(\rho))$. With respect to the non-periodic orbits, firstly we consider the unbounded curve $\mathcal{V}_\rho$ defined as in (2.2.6). To evaluate the travel time on $\mathcal{V}_\rho$, let us fix a value r with $r < x_*(\rho) < x_u$. Then, we define two points that belongs to $\mathcal{V}_\rho$: one is $P_\rho^+(r) := \big(r, \sqrt{2(\rho - \Lambda_k(r))}\big)$, in the upper half plane, and the other symmetric one is $P_\rho^-(r) := \big(r, -\sqrt{2(\rho - \Lambda_k(r))}\big)$, in the lower half plane. Therefore, the time needed to move along $\mathcal{V}_\rho$ from $P_\rho^+(r)$ to $(x_*(\rho), 0)$ is

$$\tau(\rho;\, r, x_*(\rho)) = \int_{r}^{x_*(\rho)} \frac{1}{\sqrt{2(\rho - \Lambda_k(s))}}\, ds,$$

which is equal to the time needed to travel from $(x_*(\rho), 0)$ to $P_\rho^-(r)$. It follows that the time elapsed to go from $P_\rho^+(r)$ to $P_\rho^-(r)$ on $\mathcal{V}_\rho$ is

$$\tau_{\mathcal{V}}(\rho;\, r) := 2\int_{r}^{x_*(\rho)} \frac{1}{\sqrt{2(\rho - \Lambda_k(s))}}\, ds. \tag{2.2.15}$$

In a similar way we face the time-map associated with the orbit $\mathcal{U}_\rho$ defined as in (2.2.7). In this case, we fix a value $r < x^*(\rho)$ and so, as before, the time needed to go from $P_\rho^+(r)$ to $P_\rho^-(r)$ along $\mathcal{U}_\rho$ is given by

$$\tau_{\mathcal{U}}(\rho;\, r) := 2\int_{r}^{x^*(\rho)} \frac{1}{\sqrt{2(\rho - \Lambda_k(s))}}\, ds. \tag{2.2.16}$$

2.2.2 Smale's horseshoes

Taking into account the phase-plane analysis performed in Section 2.2.1, we are motivated to exploit the Melnikov's method which is certainly a powerful tool to detect Smale's horseshoes and

> "one of the few analytical methods available for the detection and the study of chaotic motions."[1]

This idea is aimed by the existence of a hyperbolic fixed point for system (2.2.2) which is connected to itself by a homoclinic orbit for $k > 0$, provided that ϕ is sufficiently smooth with $\phi'(x_u) < 0$. In order to have satisfied such a condition for every possible choice of $k > 0$, we assume, along this subsection a more restrictive condition than (Hϕ_3), that is the following one.

[1] Quotation from [GH83, p.186].

(Hϕ_4) *$\phi\colon \mathbb{R} \to \mathbb{R}$ is a strictly convex function of class C^r, for $r \geq 2$, with $\phi(0) = 0$, $\phi(\xi) > 0$ for all $\xi \neq 0$ and $\lim_{|u|\to+\infty} \phi(u) = +\infty$.*

The phase-plane analysis shows the presence of an equilibrium point $A = A_k := (x_s, 0)$, which is a center, and a hyperbolic saddle equilibrium point $B = B_k := (x_u, 0)$ with a homoclinic orbit $\mathcal{H} = \mathcal{H}(x_u)$ enclosing A. This is the classical scheme considered in the Melnikov's theory, where system $(\mathscr{S}_1)$ can be viewed as a perturbation of the autonomous system (2.2.2). In the special case of $(\mathscr{S}_1)$ we can apply this theory by splitting the forcing term $h(t)$ as

$$h(t) = k + \varepsilon h_0(t), \quad k > 0. \tag{2.2.17}$$

Without loss of generality, we can also suppose that $h_0(t)$ changes its sign. In particular, by transferring the mean value of h_0 to the constant k, we can assume

$$\int_0^T h_0(t)\, dt = 0. \tag{2.2.18}$$

Let $q(t) = q_k(t)$ be the solution of equation (2.2.1) such that $q(0) = x_h$ and $q'(0) = 0$, where x_h is the solution of $\Lambda_k(x) = \Lambda_k(x_u)$ depending on k with $x > x_s$ (equivalently, the point $(x_h, 0)$ is the intersection of the homoclinic trajectory $\mathcal{H}$ with the x-axis). The curve $t \mapsto (q(t), q'(t))$ is a particular parametrization of $\mathcal{H}$ and it is unique up to a shift in the time variable. Our choice, which is the standard one in similar situations, is convenient because $q(t)$ is an even function. Moreover, by standard results on hyperbolic saddle points, note that $|q(t) - x_u| + |q'(t)| \to 0$ with exponential decay as $t \to \pm\infty$ (cf. [Hal80, Ch. III.6]). Thus, in particular, the improper integrals $\int_0^{+\infty} (q(t) - x_u)\, dt$ and $\int_0^{+\infty} |q'(t)|\, dt$ are convergent.

Now, the *Melnikov function* associated with system $(\mathscr{S}_1)$ for $h(t)$ as in (2.2.17), is given by

$$\mathfrak{M}(\alpha) := \int_{-\infty}^{+\infty} q'(t) h_0(t + \alpha)\, dt. \tag{2.2.19}$$

Notice that, by the T-periodicity of $h_0(t)$, it turns out that also $\mathfrak{M}(\alpha)$ is a T-periodic function. Moreover, from (2.2.18) we have $\int_0^T \mathfrak{M}(\alpha)\, d\alpha = 0$, so that either $\mathfrak{M} \equiv 0$ or $\mathfrak{M}(\alpha)$ changes its sign.

An application of the Melnikov method to system

$$\begin{cases} x' = y, \\ y' = -\phi(x) + k + \varepsilon h_0(t), \end{cases} \tag{2.2.20}$$

gives the following result (cf. [GH83, Th. 4.5.3] or [Wig03, Th. 28.1.7]).

Theorem 2.2.5. *Assume* (Hϕ_4) *and let $(q(t), q'(t))$ be the homoclinic solution at the saddle point $B = B_k$ for the autonomous system* (2.2.2) *for some $k > 0$. Let also h_0 be a sufficiently smooth, C^r for $r \geq 2$, T-periodic function satisfying* (2.2.18)*. If there exists $\alpha \in [0, T[$ such that $\mathfrak{M}(\alpha) = 0$ and $\mathfrak{M}'(\alpha) \neq 0$, then there is $\varepsilon_0 > 0$ such that for each ε with $0 < |\varepsilon| < \varepsilon_0$ a Smale horseshoe occurs for some iterate of the Poincaré map associated with system* (2.2.20)*.*

The result expressed in Theorem 2.2.5 is robust for small smooth perturbations. More in detail, the presence of a Smale horseshoe is guaranteed also for system

$$\begin{cases} x' = y, \\ y' = -cy - \phi(x) + k + \varepsilon h_0(t), \end{cases}$$

provided that c is sufficiently small, depending on ε. Hence the result applies to equation

$$u'' + cu' + \phi(u) = k + \varepsilon h_0(t)$$

as well. More precisely, if we write the coefficient c as

$$c := \varepsilon c_0\,,$$

the Melnikov function takes the form

$$\mathfrak{M}(\alpha) := \int_{-\infty}^{+\infty} \big(q'(t)h_0(t+\alpha) - c_0 h'(t)^2\big)\,dt$$

and Theorem 2.2.5 applies to system

$$\begin{cases} x' = y, \\ y' = -\varepsilon c_0 y - \phi(x) + k + \varepsilon h_0(t). \end{cases}$$

Usually the test of the existence of a simple zero for the Melnikov function is a hard task, especially if an explicit analytical expression for $q(t)$ is not given. The first important and pioneering applications of this method to some second order nonlinear ODEs, such as the pendulum or the Duffing equation, have taken advantage of the fact that the expression of $q(t)$ was known (see [GH83, p. 191]). On the contrary, when an explicit expression of $q(t)$ is not given, some results can be still produced by exploiting further qualitative information about the homoclinic orbit or even about the forcing term, if they are available. From this point of view, we refer to the work [BF02b] of Battelli and Fečkan since they have evaluated the Melnikov function when $q(t)$ is a rational function of $\exp(t)$. A general result, which does not require any specific assumption on $q(t)$ by involving only a simply verifiable condition on $h_0(t)$, was obtained by Battelli and Palmer in [BP93]. This result applies to system (2.2.20) provided that the period of the forcing term is sufficiently large. For this reason, instead of (2.2.20), it is convenient to consider the system

$$\begin{cases} x' = y, \\ y' = -\phi(x) + k + \varepsilon^2 h_0(\varepsilon t). \end{cases} \tag{2.2.21}$$

In this setting, we can state what follows (cf. [BP93, p. 293, Theorem]).

Theorem 2.2.6. *Assume* (Hϕ_4) *with* $\phi \in C^{r+3}$, *for* $r \geq 5$, *and let* $(q(t), q'(t))$ *be the homoclinic solution at the saddle point* $B = B_k$ *for the autonomous system* (2.2.2) *for some* $k > 0$. *Let also* h_0 *be a sufficiently smooth,* C^{r+3} *for* $r \geq 5$, T*-periodic function satisfying* (2.2.18). *If there exists* $\alpha \in [0, T[$ *such that*

$$h_0'(\alpha) = 0 \neq h_0''(\alpha),$$

then there is $\varepsilon_0 > 0$ *such that for each* ε *with* $0 < |\varepsilon| < \varepsilon_0$ *a Smale horseshoe occurs for some iterate of the Poincaré map associated with system* (2.2.21).

By Theorem 2.2.6, if we suppose that $h_0(t) := \sin(\omega t)$ is the periodic forcing term of period $T := 2\pi/\omega$ for a given $\omega > 0$, then we can state the following result.

Corollary 2.2.7. *Assume* (Hϕ_4) *and let* $(q(t), q'(t))$ *be the homoclinic solution at the saddle point* $B = B_k$ *for the autonomous system* (2.2.2) *for some* $k > 0$. *Then, for any* $\omega > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\omega) > 0$ *such that for each* ε *with* $0 < |\varepsilon| < \varepsilon_0$ *a Smale horseshoe occurs for some iterate of the Poincaré map associated with system*

$$\begin{cases} x' = y, \\ y' = -\phi(x) + k + \varepsilon \sin(\omega t). \end{cases}$$

The same result also holds for the damped system

$$\begin{cases} x' = y, \\ y' = -\varepsilon c_0 y - \phi(x) + k + \varepsilon \sin(\omega t), \end{cases}$$

for c_0 *sufficiently small.*

Proof. For simplicity, we investigate only the frictionless case because with a similar argument one can also derive the result when a small friction term c_0 is present.

Recalling that $q'(t)$ is an odd function, from (2.2.19) we obtain

$$\mathfrak{M}(\alpha) = \int_{-\infty}^{+\infty} q'(t)\sin(\omega t + \omega\alpha)\,dt = -2\omega\cos(\omega\alpha)\eta(\omega),$$

for

$$\eta(\omega) := \int_0^{+\infty} \tilde{q}(t)\cos(\omega t)\,dt, \quad \text{with} \;\; \tilde{q}(t) := q(t) - x_u\,.$$

In this manner, we have reduced the search of a simple zero for $\mathfrak{M}(\alpha)$ to the verification that $\eta(\omega) \neq 0$.

Since $\eta'(\omega) = -\omega\int_0^{+\infty}\tilde{q}(t)\sin(\omega t)\,dt = -\int_0^{+\infty}\tilde{q}(\xi/\omega)\sin(\xi)\,d\xi$, we find that

$$-\eta'(\omega) = \sum_{j=0}^{\infty}(-1)^j\int_{j\pi}^{(j+1)\pi}\tilde{q}\left(\frac{\xi}{\omega}\right)|\sin(\xi)|\,d\xi \;= \sum_{j=0}^{\infty}(-1)^j\Xi_j$$

where we have set

$$\Xi_j := \int_0^{\pi}\tilde{q}\left(\frac{t+j\pi}{\omega}\right)\sin(t)\,dt.$$

By observing that $\tilde{q}(t)$ is positive and decreasing on $[0,+\infty[$, follows that the sequence $(\Xi_j)_j$ is positive, decreasing and $\Xi_j \to 0$ as $j \to +\infty$. The theory of alternating series guarantees that $\sum(-1)^j\Xi_j > 0$ and hence $\eta'(\omega) < 0$ for each $\omega > 0$. Since $\eta(\omega) \to \int_0^{+\infty}\tilde{q}(t)\,dt > 0$ as $\omega \to 0^+$, we conclude that either $\eta(\omega) > 0$ for each $\omega > 0$ or $\eta(\omega)$ vanishes exactly once. On the other hand, by the Riemann-Lebesgue lemma, it follows that $\eta(\omega) \to 0$ as $\omega \to +\infty$. This implies that the second alternative never occurs because η is strictly decreasing. Hence, in view of Theorem 2.2.5 the proof is completed. □

Remark 2.2.8. In the statement of Corollary 2.2.7 no condition on ω, and thus on the period T, is required. This advantage leads limited applicability. Indeed, for a broad family of periodic functions h_0, we should look at Theorem 2.2.6. In this case, however, we warn that the period of the forcing term is modified by the parameter $\varepsilon > 0$. If h_0 is T-periodic, then the forcing term in (2.2.21) has period $T_\varepsilon := T/\varepsilon$ and the second eigenvalue of the corresponding periodic problem becomes $\lambda_2 := (2\pi/T)^2\varepsilon^2$. As a result, for a sufficiently small $\varepsilon > 0$ it follows $\lim_{\xi\to+\infty}\phi'(\xi) > \lambda_2$ and thus the nonlinearity jumps certainly the second eigenvalue. In this manner, we enter in a range of parameters for which several T_ε-periodic solutions exist. Accordingly, we recall that at least the Hamiltonian system $(\mathscr{S}_1)$ has plenty of periodic solution (see [LM87; LM90; Reb97; Wan00; ZZ05]). Hence, it is reasonable to expect to find also chaotic-like solutions for forcing terms which are not necessarily small. This will be discussed in the next section. ◁

2.2.3 Topological horseshoes

From Section 2.2.2 we notice that the Melnikov's theory involves the verification of hypothesis on the simplicity of the zero for the Melnikov function. This task may be very laborious when an explicit analytical expression of the homoclinic solution is not available. Consequently, we discuss two different approaches which are more affordable from applications point of view and require assumption less stringent. Despite all of this, they lead to the detection of a weaker "level of chaos" given by topological horseshoes, instead of a Smale horseshoes. The first one is still within the Melnikov's theory and look at slowly varying hamiltonian dynamical systems. The second one is called stretching along the paths method (SAP method) and look at switched systems.

Slowly varying systems

This subsection concerns the case of periodic forcing terms with a very large period. First of all, we take into account a tool that come from the work by Battelli and Fečkan [BF02a] where they generalized the hypothesis about the existence of a simple zero for the Melnikov function by using topological degree and assuming that $\mathfrak{M}(\alpha)$ changes it sign (cf. [BF02a, Theorem 4.4 and Remark 5.4]).

Theorem 2.2.9. *Assume* (Hϕ_4) *and let* $(q(t), q'(t))$ *be the homoclinic solution at the saddle point* $B = B_k$ *for the autonomous system* (2.2.2) *for some* $k > 0$*. Let also* p_0 *be a sufficiently smooth,* C^r *for* $r \geq 2$*,* T*-periodic function satisfying* (2.2.18)*. If*

$$\mathfrak{M} \not\equiv 0,$$

then there is $\varepsilon_0 > 0$ *such that for each* ε *with* $0 < |\varepsilon| < \varepsilon_0$ *a topological horseshoe occurs for some iterate of the Poincaré map associated with* (2.2.20)*.*

As a possible application of Theorem 2.2.9, we consider in system $(\mathscr{S}_1)$ a forcing term given by $h(t) = k + \varepsilon h_0(\Omega t)$ with $h_0 : \mathbb{R} \to \mathbb{R}$ a T-periodic function of class C^2 and $\Omega > 0$ a fixed constant. Hence, we deal with

$$\begin{cases} x' = y, \\ y' = -\phi(x) + k + \varepsilon h_0(\Omega t), \end{cases} \tag{2.2.22}$$

and prove what follows.

Corollary 2.2.10. *Assume* (Hϕ_4) *and let* $(q(t), q'(t))$ *be the homoclinic solution at the saddle point* $B = B_k$ *for the autonomous system* (2.2.2) *for some* $k > 0$*. Suppose that* p_0 *is not constant. Then, there exists* $\Omega_0 > 0$ *such that for every* Ω *with* $0 < \Omega < \Omega_0$*, there is* $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ *such that for each* ε *with* $0 < |\varepsilon| < \varepsilon_0$ *a topological horseshoe occurs for some iterate of the Poincaré map associated with system* (2.2.22)*. The same result also holds for the damped system*

$$\begin{cases} x' = y, \\ y' = -\varepsilon c_0 y - \phi(x) + k + \varepsilon h_0(\Omega t), \end{cases} \tag{2.2.23}$$

for c_0 *sufficiently small.*

Proof. We prove now the statement for system (2.2.22). The same conclusion holds for system (2.2.23) because the result in Theorem 2.2.9 is stable for small perturbations, since it is based on topological degree theory.

The Melnikov function defined in (2.2.19), associated to (2.2.22), takes here the form $\mathfrak{M}(\alpha) = -\Omega \int_{-\infty}^{+\infty} \tilde{q}(t) h_0'(\Omega t + \Omega\alpha)\, dt$, where $\tilde{q}(t) = q(t) - x_u$. Since h_0 is not constant, there exists s^* such that $h_0'(s^*) > 0$. Then, there exist a constant $\delta^* > 0$ and an interval $[s^*-r^*, s^*+r^*]$ such that $h'(\xi) \geq \delta^*$ for all $\xi \in [s^*-r^*, s^*+r^*]$. Taking $\alpha^* = \alpha^*(\Omega) := s^*/\Omega$, we have that

$$\begin{aligned} -\frac{\mathfrak{M}(\alpha^*)}{\Omega} &\geq \int_{-r^*/\Omega}^{r^*/\Omega} \tilde{q}(t) h_0'(s^* + \Omega t)\, dt - 2\|h_0'\|_\infty \int_{r^*/\Omega}^{+\infty} \tilde{q}(t)\, dt \\ &\geq 2\delta^* \int_0^{r^*/\Omega} \tilde{q}(t)\, dt - 2\|h_0'\|_\infty \int_{r^*/\Omega}^{+\infty} \tilde{q}(t)\, dt. \end{aligned}$$

Since, one can deduce the existence of a constant $\Omega_1 > 0$ such that for each Ω with $0 < \Omega < \Omega_1$ it holds that $\int_0^{r^*/\Omega} \tilde{q}(t)\, dt > (\delta^*)^{-1} \|h_0'\|_\infty \int_{r^*/\Omega}^{+\infty} \tilde{q}(t)\, dt$, then we have $\mathfrak{M}(\alpha^*) < 0$.

Similarly, there exists s_* such that $h_0'(s_*) < 0$. Accordingly, there are a constant $\delta_* > 0$ and an interval $[s_* - r_*, s_* + r_*]$ such that $h'(\xi) \leq -\delta_*$ for all $\xi \in [s_* - r_*, s_* + r_*]$. Taking now $\alpha_* = \alpha_*(\Omega) := s_*/\Omega$, by an argument similar to the previous one, there exists a constant $\Omega_2 > 0$ such that for each Ω with $0 < \Omega < \Omega_2$ we have $\mathfrak{M}(\alpha_*) > 0$. The conclusion now follows from Theorem 2.2.9 by taking $\Omega_0 := \min\{\Omega_1, \Omega_2\}$. □

The assumptions in Corollary 2.2.10 involve an arbitrary non-constant periodic function h_0 of class C^2 with period $T_\Omega := T/\Omega$ such that its displacement from a constant value $k > 0$ is very small and T_Ω is very large.

Avoiding the smallness of the displacement, we consider a topological approach that comes from the work by Gedeon, Kokubu, Mischaikow and Oka [Ged+02] and is based on Conley index theory. The method in [Ged+02] is stable for small perturbations and applies also to systems which are not necessarily periodic in the time variable. Here, we give an application to system

$$\begin{cases} x' = y, \\ y' = -\phi(x) + h(\varepsilon t), \end{cases} \tag{2.2.24}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ is a non-constant periodic function of class C^2 such that $h(t) > 0$ for all $t \in \mathbb{R}$. First we need to introduce a few definitions from [Ged+02]. Writing (2.2.24) as

$$\begin{cases} x' = y, \\ y' = -\phi(x) + h(\theta), \\ \theta' = \varepsilon, \end{cases} \tag{2.2.25}$$

we set, for a moment, θ as a constant parameter and consider the planar autonomous Hamiltonian system

$$\begin{cases} x' = y, \\ y' = -\phi(x) + h(\theta). \end{cases} \tag{2.2.26}$$

Concerning this latter system, for each θ there exist an equilibrium point $A(\theta) := (x_s(\theta), 0)$ which is a center and also a hyperbolic saddle equilibrium point $B(\theta) := (x_u(\theta), 0)$ with a homoclinic orbit enclosing $A(\theta)$. By definition, $\phi(x_u(\theta)) = \phi(x_s(\theta)) = 0$, with $x_u(\theta) < 0 < x_s(\theta)$. We denote also with $\mathscr{A}$ the set of all the points $(A(\theta), \theta)$ which is a curve in $\mathbb{R}^3$. A solution $X(t) := (x(t), y(t), \theta(t))$ of system (2.2.25) is said to oscillate k times over an interval $I = [\theta^-, \theta^+]$ with respect to $\mathscr{A}$, if $k \in \mathbb{N}$ identifies the homotopy class of the closed loop

$$\left(\bigcup_{\theta(t) \in I} X(t) \cup B(\theta(t)) \right) \cup \left(\bigcup_{\theta(t) \in \partial I} \overline{X(t)B(\theta(t))} \right)$$

in the fundamental group of $\mathbb{R}^3 \setminus \mathscr{A}$ (isomorphic to $\mathbb{Z}$). Then, the results in [Ged+02; KMO96], applied to system (2.2.24), give the following conclusion.

Theorem 2.2.11. *Assume* ($\mathrm{H}\phi_4$) *and let also* $h\colon \mathbb{R} \to \mathbb{R}$ *be a non-constant periodic function of class* C^2 *such that* $h(t) > 0$ *for all* $t \in \mathbb{R}$*. Then there exists a choice of infinitely many pairwise disjoint closed intervals* $I_i := [\theta_i^-, \theta_i^+]$ *with*

$$\ldots \theta_{i-1}^- < \theta_{i-1}^+ < \theta_i^- < \theta_i^+ < \theta_{i+1}^- < \theta_{i+1}^+ \ldots, \quad i \in \mathbb{Z},$$

with the following property: for any given positive integer K *there exists* $\bar{\varepsilon} > 0$ *such that for any* ε *with* $0 < \varepsilon < \bar{\varepsilon}$*, there are at least two non-negative integers* m_i' *and* m_i'' *(for* i *odd) and at least* K *non-negative integers* $m_i^1, \ldots m_i^K$ *(for* i *even), such that for each sequence* $(s_i)_{i \in \mathbb{Z}}$ *of integers with* $s_{2i+1} \in \{m_{2i+1}', m_{2i+1}''\}$ *and* $s_{2i} \in \{m_{2i}^1, \ldots, m_{2i}^K\}$*, there is at least one solutions of* (2.2.25) *which oscillates* s_i *times over* I_i*.*

Proof. The thesis follows from [Ged+02, Cor. 1.2]. Therefore, let us briefly check that we enter in the settings of applicability of that result. First of all, we notice that (2.2.24) is a periodically perturbed planar Hamiltonian system of the form $z' = J\nabla H(z, \varepsilon t)$, where J is the 2×2 symplectic matrix. Let us denote by $S(\theta)$ the area of the planar region containing the elliptic equilibrium point $A(\theta)$ and bounded by the homoclinc orbit of (2.2.26) enclosing it. Then, the intervals I_i are chosen so that $S'(\theta_i^-) > 0 > S'(\theta_i^+)$ for i odd and $S'(\theta_i^-) < 0 < S'(\theta_i^+)$ for i even. Since the method in [Ged+02] applies also when the forcing term h is not necessarily periodic, it requires the additional condition that $\theta_{i+1}^- - \theta_i^+$ is uniformly bounded away from zero. However, in our situation, h is a non-constant periodic function. By denoting its fundamental period by T, we can choose the intervals I_i such that $\theta_{i+2}^\pm = \theta_i^\pm + T$ for all $i \in \mathbb{Z}$, without any further hypothesis and this completes the proof. □

Notice that the result achieved in [Ged+02] is stable under small perturbations. That being so, Theorem 2.2.11 applies also to system

$$\begin{cases} x' = y, \\ y' = -\varepsilon^2 c_0 y - \phi(x) + h(\varepsilon t), \end{cases}$$

with $c_0 \in \mathbb{R}$. Moreover, by Theorem 2.2.11, we stress that chaotic dynamics appears also in presence of a periodic perturbation h which is no longer required to be small.

Switched systems

This subsection concerns a step-wise periodic forcing terms that yields switched systems, which is an attractive topic in the field of control theory (see [Bac14]). In this case we take advantage of the SAP method that is presented in the Appendix (see also [MPZ09; PZ04] for the details). By considering switched systems, we are looking for a geometry similar to the one of the "linked twist maps" (see [PZ09; WO04]). More precisely, the configuration of our problem recalls that of the work [PPZ08], where the interplay between an annulus and a strip is considered instead of the usual two annuli.

Let us introduce a periodic piecewise constant forcing term of the form which takes two values as follows

$$h_{k_1,k_2}(t) := \begin{cases} k_1 & \text{for } t \in [0, t_1[, \\ k_2 & \text{for } t \in [t_1, t_1 + t_2[, \end{cases} \tag{2.2.27}$$

with $k_1, k_2 \geq 0$, $k_1 \neq k_2$ and $t_1, t_2 > 0$. We will perform our analysis by assuming

$$0 < k_1 < k_2.$$

Notice that, via minor changes in the argument which follow, one can deal also with the case $k_1 = 0$. We suppose that the fundamental period of $h(t)$ splits as

$$T := t_1 + t_2.$$

In this setting, system $(\mathscr{S}_1)$ is equivalent to the switched system in the phase-plane (x, y) which alternates between two subsystems:

$$(\mathrm{S}_i) \qquad \begin{cases} x' = y, \\ y' = -\phi(x) + k_i, \end{cases}$$

for $i \in \{1, 2\}$. In other words, the solution to $(\mathscr{S}_1)$ which starts from an initial point $z_0 = (x_0, y_0)$ is governed by the subsystem (S_1) for a fixed period of time t_1 and then it is governed by the subsystem (S_2) for another fixed period of time t_2. At this point, the switched system may change back to subsystem (S_1) until the time elapsed is exactly $t_1 + t_2$. As a consequence, the Poincaré map Φ of system $(\mathscr{S}_1)$ can be decomposed as $\Phi = \Phi_2 \circ \Phi_1$, where Φ_i is the Poincaré map of system (S_i) relatively to the time interval $[0, t_i]$, for $i \in \{1, 2\}$.

Theorem 2.2.12. *Assume* (A_1) *and let also* $h \colon \mathbb{R} \to \mathbb{R}$ *be a* T*-periodic stepwise function, such that* $h(t) > 0$ *for all* $t \in \mathbb{R}$. *Then, there exist* τ_1^* *and* τ_2^* *such that a topological horseshoe occurs for the Poincaré map associated with system* $(\mathscr{S}_1)$ *provided that* $t_1 > \tau_1^*$ *and* $t_2 > \tau_2^*$.

Proof. Consider two fixed values k_1, k_2 and let $h(t) = h_{k_1,k_2}(t)$ be defined as in (2.2.27). The idea of the proof is to apply SAP method, namely Theorem B.6. Our task is now to find two oriented topological rectangles $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$ (Definition B.3) where chaotic dynamics take place (in terms of symbolic dynamics on $2 \times m$ symbols). To do this we divide the analysis into the following two steps which collect the stretching properties (Definition B.4).

Step I. *For any path* γ *contained in* $\mathcal{M}$, *connecting the two sides* $\mathcal{M}_l^-$ *and* $\mathcal{M}_r^-$, *there exist two sub-paths* γ_0, γ_1 *such that* $\Phi_1(\gamma_i)$ *is a path contained in* $\mathcal{N}$ *which joins the two sides* $\mathcal{N}_l^-$ *and* $\mathcal{N}_r^-$ *for each* $i \in \{0, 1\}$.

Step II. *For any path γ contained in $\mathcal{N}$, connecting the two sides $\mathcal{N}_l^-$ and $\mathcal{N}_r^-$, there exist $m \geq 2$ sub-paths $\gamma_0, \ldots, \gamma_{m-1}$ such that $\Phi_2(\gamma_i)$ is a path contained in $\mathcal{M}$ which joins the two sides $\mathcal{M}_l^-$ and $\mathcal{M}_r^-$ for each $i \in \{0, \ldots, m-1\}$.*

We start by giving a suitable construction of these topological oriented rectangles. From Section 2.2.1 follows the existence of two homoclinc orbits $\mathcal{H}(x_u(k_1))$ and $\mathcal{H}(x_u(k_2))$, one for system (S_1) and one for (S_2), associated with the energies $\Lambda_{k_1}(x_u(k_1))$ and $\Lambda_{k_2}(x_u(k_2))$, respectively. Moreover, Proposition 2.2.4 leads to

$$x_u(k_2) < x_u(k_1) < x_h(k_1) < x_h(k_2),$$

which is equivalent to said that the region bounded by the homoclinic orbit $\mathcal{H}(x_u(k_2))$ contains the homoclinc orbit $\mathcal{H}(x_u(k_1))$.

Let us fix three main energy levels $A, B, D \in \mathbb{R}$ as follows. Take $A < \Lambda_{k_1}(x_u(k_1))$ such that the solution $a := x_*(\mathrm{A})$ of the equation $\Lambda_{k_1}(x) = A$ belongs to the interval $]x_u(k_2), x_u(k_1)[$. Choose $\Lambda_{k_2}(x_s(k_2)) < D < \Lambda_{k_2}(x_u(k_2))$ in a way that the solutions $d := x_-(D)$ and $x_+(D)$ of the equation $\Lambda_{k_2}(x) = D$ are such that $x_u(k_2) < d < a$ and $x_s(k_2) < x_+(D) < x_h(k_2)$. At last, consider $B > \Phi_{k_1}(x_u(k_1))$ so that the solution $b := x^*(B)$ of $\Lambda_{k_1}(x) = B$ is such that $x_h(k_1) < b < x^+(D)$. This way, one can determine three different energy level lines which are $\mathcal{V}_A, \mathcal{U}_B$ for system (S_1) and $\mathcal{O}_D$ for (S_2), defined as in (2.2.6), (2.2.7) and (2.2.5), respectively. Now, we consider the closed regions

$$\begin{aligned}
\mathcal{S}_A &:= \{(x, y) \in \mathbb{R}^2 : A \leq E_{k_1}(x, y) \leq \Lambda_{k_1}(x_u(k_1)),\ x \leq x_u(k_1)\},\\
\mathcal{S}_B &:= \{(x, y) \in \mathbb{R}^2 : \Lambda_{k_1}(x_u(k_1)) \leq E_{k_1}(x, y) \leq B\},
\end{aligned}$$

and their union

$$\mathcal{S} := \mathcal{S}_A \cup \mathcal{S}_B\,.$$

They are all invariant for the flow associated with system (S_1). The region $\mathcal{S}$ is topologically like a strip with a hole given by the part of the plane enclosed by the homoclinic trajectory $\mathcal{H}(x_u(k_1))$. We also introduce a closed and invariant annular region for system (S_2), given by

$$\mathcal{A} := \{(x, y) \in \mathbb{R}^2 : D \leq E_{k_2}(x, y) \leq \Lambda_{k_2}(x_u(k_2))\}.$$

The intersection of $\mathcal{S}$ with $\mathcal{A}$ determines two disjoint compact sets that we call $\mathcal{M}$ (the one in the upper half-plane) and $\mathcal{N}$ (the other symmetric one in the lower half-plane), that are

$$\mathcal{M} := \mathcal{A} \cap \mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : y > 0\}, \quad \mathcal{N} := \mathcal{A} \cap \mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : y < 0\}.$$

One can easily check that they are topological rectangles. At last, we give an orientation as follows

$$\begin{aligned}
\mathcal{M}_l^- &:= \mathcal{M} \cap \mathcal{V}_A\,, & \mathcal{M}_r^- &:= \mathcal{M} \cap \mathcal{U}_B,\\
\mathcal{N}_l^- &:= \mathcal{N} \cap \mathcal{O}_D\,, & \mathcal{N}_r^- &:= \mathcal{N} \cap \mathcal{H}(x_u(k_2)).
\end{aligned}$$

See Figure 2.3 for a graphical sketch of $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$.

We are now in position to prove Step I. Let us consider system (S_1). Then, thanks to the analysis performed in Section 2.2.1, we known the time needed to move from the point $\big(x_u(k_2), \sqrt{2(A - \Lambda_{k_1}(x_u(k_2)))}\big)$ to the point $\big(x_u(k_2), -\sqrt{2(A - \Lambda_{k_1}(x_u(k_2)))}\big)$ along $\mathcal{V}_A$. This is, in accord with (2.2.15),

$$\tau_{\mathcal{V}_A} := \tau_{\mathcal{V}}(A; x_u(k_2)).$$

From (2.2.16), the displacement, from the point $\big(x_u(k_2), \sqrt{2(B - \Lambda_{k_1}(x_u(k_2)))}\big)$ to the point $\big(x_u(k_2), -\sqrt{2(B - \Lambda_{k_1}(x_u(k_2)))}\big)$ along $\mathcal{U}_B$, requires the following time

$$\tau_{\mathcal{U}_B} := \tau_{\mathcal{U}}(B; x_u(k_2)).$$

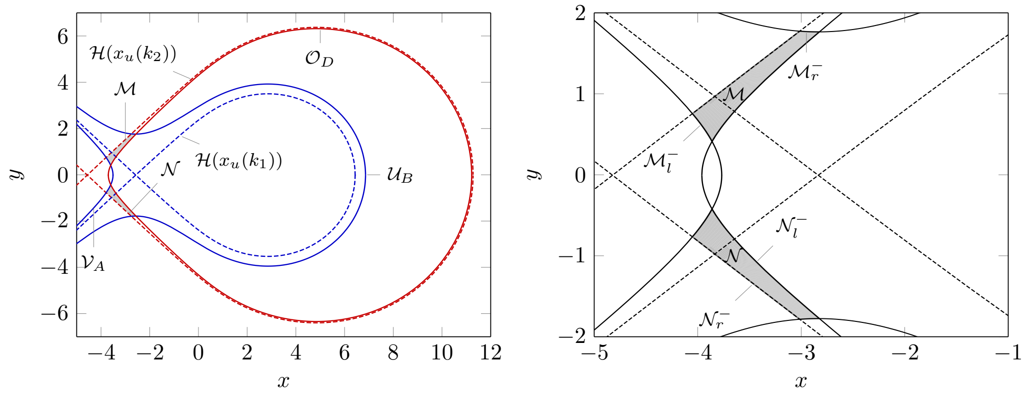


Figure 2.3: Left: Link of an annulus with a "strip with hole". Energy level lines for system (2.2.2) where $\phi(x) = \sqrt{1+x^2} - 1$ are displayed with $k = 2$ (blue) and $k = 4$ (red). Right: Zooming of the topological rectangles with evidence of the boundaries.

As a result of these computations, we fix

$$\tau_1^* := \max\{\tau_{\mathcal{V}_A}, \tau_{\mathcal{U}_B}\}. \tag{2.2.28}$$

Note that each solution of a Cauchy problem with initial conditions taken in $\mathcal{M}$ evolves, through the action of (S_1), inside the invariant region $\mathcal{S}$. More in detail, at any time $t_1 > \tau_{\mathcal{V}_A}$, all the initial points in $\mathcal{M}_l^-$ will be moved, along the level line $\mathcal{V}_A$, to points with $x < x_u(k_2)$ and $y < 0$ by the action of Φ_1. Any solution $u(t)$ of $u'' + f(u) = k_1$ with $(u(0), u'(0)) \in \mathcal{M}_l^-$ starts with $u(0) > x_u(k_2)$ and a positive slope, it is strictly increasing until it reaches its maximum value $u_{\max} = a$ and then it decreases strictly till to the value $u(t_1) < x_u(k_2)$. Moreover, $u'(t)$ is strictly decreasing on the whole interval $[0, t_1]$. Similarly, for $t_1 > \tau_{\mathcal{U}_B}$, all the initial points in $\mathcal{M}_r^-$ will be moved away along the level line $\mathcal{U}_B$. The final points will be such that $x < x_u(k_2)$ and $y < 0$. Analogous considerations can be made for the solution $u(t)$ of $u'' + \phi(u) = k_1$ with $(u(0), u'(0)) \in \mathcal{M}_r^-$ which achieves the maximum value $u_{\max} = b$. In the region $\mathcal{M}$, any path connecting $\mathcal{M}_l^-$ to $\mathcal{M}_r^-$ must intersect the stable manifold $W^s(x_u(k_1))$. Notice that any solution $(x(t), y(t))$ of system (S_1) starting at a point of $W^s(x_u(k_1))$, lies on such a manifold and, therefore, $y(t) > 0$ for all $t \geq 0$.

Let $\gamma : [0, 1] \to \mathcal{M}$ be a continuous path with $\gamma(0) \in \mathcal{M}_l^- \subseteq \mathcal{V}_A$ and $\gamma(1) \in \mathcal{M}_r^- \subseteq \mathcal{U}_B$. First of all, observe that, by the continuity of γ there exists $\bar{s}^\flat, \bar{s}^\sharp \in]0, 1[$ with $\bar{s}^\flat \leq \bar{s}^\sharp$ such that $\gamma(\bar{s}^\flat), \gamma(\bar{s}^\sharp) \in W^s(x_u(k_1))$ and $\gamma(s) \in \mathcal{S}_A$ for all $0 \leq s \leq \bar{s}^\flat$, as well as $\gamma(s) \in \mathcal{S}_B$ for all $\bar{s}^\sharp \leq s \leq 1$. By the choice of τ_1^*, for each $t_1 > \tau_1^*$ it follows that

$$\begin{aligned} &\Phi_1(\gamma(0)),\ \Phi_1(\gamma(1)) \in \{(x, y) : x < x_u(k_2),\ y < 0\}, \\ &\Phi_1(\gamma(\bar{s}^\flat)),\ \Phi_1(\gamma(\bar{s}^\sharp)) \in \{(x, y) : x > x_u(k_2),\ y > 0\}. \end{aligned}$$

Thus, the path γ is folded onto itself in the invariant region $\mathcal{S}$ by the action of system S_1 as shown in Figure 2.4. Now we set

$$s_A'' := \max\{s \in [0, \bar{s}^\flat] : \Phi_1(\gamma(s)) \in \mathcal{N}_l^-\}, \quad s_A' := \max\{s \in [0, s_A''] : \Phi_1(\gamma(s)) \in \mathcal{N}_r^-\}.$$

By definition, $\gamma(s) \in \mathcal{M} \cap \mathcal{S}_A$ and $\Phi_1(\gamma(s)) \in \mathcal{N}$ for all $s \in [s_A', s_A'']$ with $\Phi_1(\gamma(s_A')) \in \mathcal{N}_r^-$ and $\Phi_1(\gamma(s_A'')) \in \mathcal{N}_l^-$. Analogously, we define

$$s_B' := \max\{s \in [\bar{s}^\sharp, s_B''] : \Phi_1(\gamma(s)) \in \mathcal{N}_l^-\}, \quad s_B'' := \max\{s \in [\bar{s}^\sharp, 1] : \Phi_1(\gamma(s)) \in \mathcal{N}_r^-\},$$

and we observe that $\gamma(s) \in \mathcal{M} \cap \mathcal{S}_B$, $\Phi_1(\gamma(s)) \in \mathcal{N}$ for all $s \in [s_B', s_B'']$ with $\Phi_1(\gamma(s_B')) \in \mathcal{N}_l^-$ and $\Phi_1(\gamma(s_B'')) \in \mathcal{N}_r^-$.

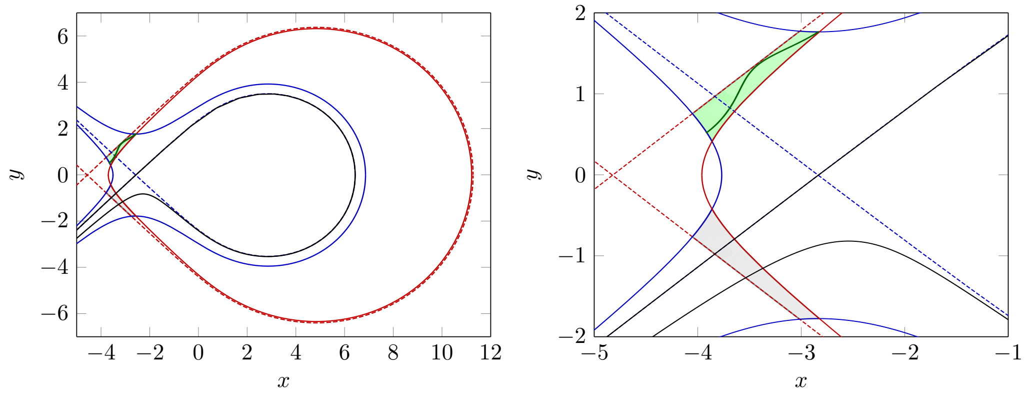


Figure 2.4: Left: Representation of a generic path γ (green) in the topological rectangle $\mathcal{M}$ joining $\mathcal{M}_l^-$ with $\mathcal{M}_r^-$ and its image (black) at time t_1 under the action of the system (S_1) where $\phi(x) = \sqrt{1+x^2} - 1$ and $k = 2$. Right: Zooming of the two crossings between the image of the curve γ with the topological rectangle $\mathcal{N}$.

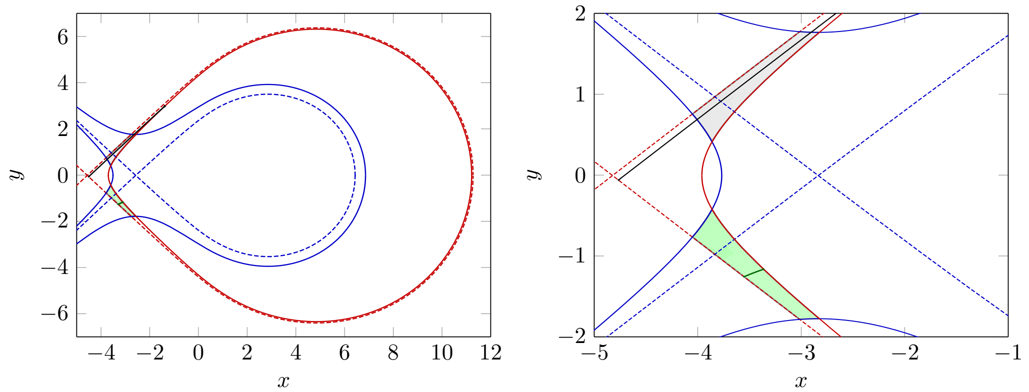


Figure 2.5: Left: Representation of a generic path γ (green) in the topological rectangle $\mathcal{N}$ joining $\mathcal{N}_r^-$ with $\mathcal{N}_l^-$ and its image (black) at time $t_1 + t_2$ under the action of the system (S_2) where $\phi(x) = \sqrt{1+x^2} - 1$ and $k = 4$. Right: Zooming of the crossing between the image of the curve γ with the topological rectangle $\mathcal{M}$.

For any $t_1 > \tau_1^*$ fixed, using an elementary continuity argument, we can determine a (small) open neighborhood $\mathscr{W}$ of $W^s(x_u(k_1)) \cap \mathcal{M}$ such that $y(t) > 0$ for all $t \in [0, t_1]$, whenever $(x(0), y(0)) \in \mathscr{W}$. Thus, finally, if we define

$$\mathcal{K}_{1,0} := \mathcal{M} \cap \mathcal{S}_A \setminus \mathscr{W}, \quad \mathcal{K}_{1,1} := \mathcal{M} \cap \mathcal{S}_B \setminus \mathscr{W},$$

then, in accord with Definition B.4, we have determined two disjoint compact sets such that satisfy the SAP condition with crossing number 2:

$$(\mathcal{K}_{1,0}, \Phi_2) \colon \tilde{\mathcal{M}} \Leftrightarrow\!\!\!\!\rightarrow^2 \tilde{\mathcal{N}}, \quad (\mathcal{K}_{1,1}, \Phi_2) \colon \tilde{\mathcal{M}} \Leftrightarrow\!\!\!\!\rightarrow^2 \tilde{\mathcal{N}}.$$

At last we consider system (S_2) and we prove the stretching property formulated in Step II. Note that each solution of a Cauchy problem with initial conditions taken in $\mathcal{N}$ evolves through the action of (S_2) inside the annular region $\mathcal{A}$ which is invariant for the associated flow. Once the point $(b, 0)$ is fixed as a center for polar coordinates, if the time increases, then all the points of $\mathcal{A} \setminus \{(x_u(k_2), 0)\}$ move along the energy level lines of (S_2) in the clockwise sense. For our purposes, it will be convenient to introduce an angular variable starting from the half-line $L := \{(r, 0) : r < b\}$ and counted positive clockwise from the reference axis L. In this manner all the points of $\mathcal{N}$ are determined by an angle $\vartheta \in]-\pi/2, 0[\pmod{2\pi}$, while those of $\mathcal{M}$ are determined by $\vartheta \in]0, \pi/2[\pmod{2\pi}$. In other words, for our auxiliary polar coordinate system, the region $\mathcal{N}$ (respectively, $\mathcal{M}$) lies in the interior of the fourth quadrant (respectively, first quadrant). Any solution $u(t)$ of $u'' + \phi(u) = k_2$ with $(u(0), u'(0)) \in \mathcal{N}_r^-$ starts with $u(0) > x_u(k_2)$ and a negative slope, it tends as $t \to +\infty$ to the saddle point of (S_2) along the homoclinic orbit, with $u(t)$ decreasing and $u'(t)$ increasing. On the other hand, any solution with $(u(0), u'(0)) \in \mathcal{N}_l^-$ is periodic with period equal to the fundamental period of the orbit $\mathcal{O}_D$, that we denote by

$$\mathcal{T}_{\mathcal{O}_D} := 2\,\tau_{\mathcal{O}}(D;\, x_+(D)),$$

by means of (2.2.14). If we take any path in $\mathcal{N}$ connecting $\mathcal{N}_r^-$ to $\mathcal{N}_l^-$ we have that its image under the action of the flow of (S_2) looks like a spiral curve contained in $\mathcal{A}$ which winds a certain number of times around the center. In order to formally prove this fact and to evaluate the precise number of revolutions, we denote by $\vartheta(t, z)$ the angle at the time $t \geq 0$ associated with the solution $(x(t), y(t))$ of system (S_2) such that $(x(0), y(0)) = z \in \mathcal{N}$. By the previous considerations and the choice of a clockwise orientation, we know that $\frac{d}{dt}\vartheta(t, z) > 0$ for all $z \in \mathcal{N}$. For our next computations we need also to introduce the time needed to go from the point $\big(b, -\sqrt{2(D - \Lambda_{k_2}(b))}\big)$ to the point $\big(b, \sqrt{2(D - \Lambda_{k_2}(b))}\big)$ along $\mathcal{O}_D$, which is given by

$$\tau_{\mathcal{O}_D} := \tau_{\mathcal{O}}(D;\, b),$$

consistently with (2.2.14). Given $m \geq 1$, we fix

$$\tau_2^* := \tau_{\mathcal{O}_D} + (m-1)\mathcal{T}_{\mathcal{O}_D}. \tag{2.2.29}$$

We claim that for each fixed time $t_2 > \tau_2^*$ the SAP property holds for the Poincaré map Φ_2 with crossing number (at least) m. A visualization of this step for $m = 1$ is given in Figure 2.5.

By the previous observations, we have that

$$\begin{aligned} &\vartheta(t_2, z) < 0, \quad \forall\, z \in \mathcal{N}_r^-, \\ &\vartheta(t_2, z) > \frac{\pi}{2} + 2(m-1)\pi, \quad \forall\, z \in \mathcal{N}_l^-. \end{aligned}$$

This allows us to introduce m nonempty subsets $\mathcal{K}_{2,0}, \ldots, \mathcal{K}_{2,m-1}$ of $\mathcal{N}$ which are pairwise disjoint and compact. They are defined by

$$\mathcal{K}_{2,i} := \{z \in \mathcal{N} : \vartheta(t_2, z) \in [2i\pi, (\pi/2) + 2i\pi]\}, \quad \forall i \in \{0, \ldots, m-1\}.$$

Let $\gamma : [0,1] \to \mathcal{N}$ be a continuous path with $\gamma(0) \in \mathcal{N}_r^- \subseteq \mathcal{H}(x_u(k_2))$ and $\gamma(1) \in \mathcal{N}_l^- \subseteq \mathcal{O}_D$. We fix also an index $i \in \{0, \ldots, m-1\}$. First of all, observe that, by the continuity of γ there exists $\bar{s}_i^\flat, \bar{s}_i^\sharp \in\,]0,1[$ with $\bar{s}_i^\flat < \bar{s}_i^\sharp$ such that

$$\vartheta(t_2, \gamma(\bar{s}_i^\flat)) = 2i\pi, \quad \vartheta(t_2, \gamma(\bar{s}_i^\sharp)) = \frac{\pi}{2} + 2i\pi,$$

and

$$2i\pi < \vartheta(t_2, \gamma(s)) < \frac{\pi}{2} + 2i\pi, \quad \forall\, \bar{s}_i^\flat < s < \bar{s}_i^\sharp.$$

For ease of notation, we define as $\mathcal{A}_{\mathrm{I}}$ the intersection of $\mathcal{A}$ with the first quadrant of the auxiliary polar coordinate system and we also consider the following two segments

$$\begin{aligned} &\mathcal{A}_{\mathrm{I}}^x := [x_u(k_2), d] \times \{0\}, \\ &\mathcal{A}_{\mathrm{I}}^y := \{b\} \times \left[\sqrt{2(D - \Lambda_{k_2}(b))}, \sqrt{2(\Lambda_{k_2}(x_u(k_2)) - \Lambda_{k_2}(b))}\right], \end{aligned}$$

which are on the boundary of $\mathcal{A}_{\mathrm{I}}$. By construction, the image $\Phi_2 \circ \gamma|_{[\bar{s}_i^\flat, \bar{s}_i^\sharp]}$ is contained in $\mathcal{A}_{\mathrm{I}}$ and joins $\mathcal{A}_{\mathrm{I}}^x$ to $\mathcal{A}_{\mathrm{I}}^y$. On the other hand, the set $\mathcal{M}$ as well as its sides $\mathcal{M}_l^-$ and $\mathcal{M}_r^-$ separate $\mathcal{A}_{\mathrm{I}}^x$ and $\mathcal{A}_{\mathrm{I}}^y$ inside $\mathcal{A}_{\mathrm{I}}$. An elementary connectivity argument, allows to determine s_i' and s_i'' with $\bar{s}_i^\flat < s_i' < s_i'' < \bar{s}_i^\sharp$ such that $\Phi_2(\gamma(s_i')) \in \mathcal{M}_l^-$, $\Phi_2(\gamma(s_i'')) \in \mathcal{M}_r^-$ and,

$\Phi_2(\gamma(s)) \in \mathcal{M}$, for all $s \in [s_i', s_i'']$. Moreover, $\gamma(s) \in \mathcal{K}_{2,i}$ for all $s \in [s_i', s_i'']$. This way our claim is verified because

$$(\mathcal{K}_{2,i}, \Phi_2) \colon \tilde{\mathcal{N}} \Leftrightarrow\!\!\!\rightarrow \tilde{\mathcal{M}}, \quad \forall\, i \in \{0, \ldots, m-1\}.$$

At the end, from Step I and Step II we can conclude that there exists a topological horseshoe for the Poincaré map $\Phi = \Phi_2 \circ \Phi_1$ with full dynamics on $2 \times m$ symbols. □

Remark 2.2.13. Let us make some comments on the proof of Theorem 2.2.12. Firstly, we notice that this result is stable with respect to small perturbations in the following sense: for any choice of $t_1 > \tau_1^*$ and $t_2 > \tau_2^*$ (so that $T = t_1 + t_2$ is fixed) there exists an $\varepsilon_0 > 0$ such that, for all c with $|c| < \varepsilon_0$ and every forcing term $h(t)$ such that $\int_0^T |h(t) - h_{k_1,k_2}(t)|\, dt < \varepsilon_0$, the conclusion of Theorem 2.2.12 holds for system $(\mathscr{S}_2)$. Hence we can consider also smooth forcing terms $h(t)$ near to $h_{k_1,k_2}(t)$ in the L^1-norm. Secondly, the constants τ_1^* and τ_2^* will be explicitly computed in terms of the forcing term $h(t)$ (cf. (2.2.28) and (2.2.29)). ◁

We conclude, this discussion with an example of chaotic dynamics by considering the nonlinearity $\phi(\xi) = |\xi|$. The essential observation consists in the direct computation of (2.2.28) and (2.2.29), which are not necessarily large, as we are going to see.

Example 2.2.14. Let us consider the second order ODE $u'' + |u| = h_{0,2}(t)$ for $T = t_1 + t_2$, which is equivalent to the differential system

$$\begin{cases} x' = y, \\ y' = -|x| + h_{0,2}(t), \end{cases} \tag{2.2.30}$$

where the forcing term $h_{0,2}(t)$ is defined according to (2.2.27) (one could also consider the case of a smooth nonlinearity sufficiently near to the absolute value).

Our goal is to show the presence of symbolic dynamics on two symbols for system (2.2.30). In this respect, using an argument similar to the one used in Theorem 2.2.12, we consider two regions in the phase-plane defined as follows

$$\mathcal{S} := \{(x, y) \in \mathbb{R}^2 : 0 \le E_0(x, y) \le 8\},$$
$$\mathcal{A} := \{(x, y) \in \mathbb{R}^2 : \rho_\epsilon \le E_2(x, y) \le 2\},$$

where $\rho_\epsilon := (-\epsilon^2 + 4\epsilon)/2$ with $\epsilon > 0$ a sufficiently small fixed real value. The strip region $\mathcal{S}$ is obtained from the equation $u'' + |u| = 0$ by considering the area between the following associated level lines: the unbounded orbit $\mathcal{U}_8$ passing through the point $(4, 0)$ and the line $x = -|y|$ made by the unstable equilibrium point $(x_u(0), 0) = (0, 0)$, the stable manifold $\mathcal{W}^s(0)$ and the unstable one $W^u(0)$. To construct the annular region $\mathcal{A}$, we consider the equation $u'' + |u| = 2$ and from its phase portrait we select the area between the homoclinic orbit $\mathcal{H}(-2)$ at the saddle point $(x_u(2), 0) = (-2, 0)$ and a periodic orbit $\mathcal{O}_{\rho_\epsilon}$ that passes through $(\epsilon, 0)$ which is a point very close to the origin. Dealing with $u'' + |u| = 2$ we can observe that all the periodic orbits enclosing the stable center $(2, 0)$ and contained in the right-half phase-plane are isochronous with period 2π. Now, we set the topological rectangles as follows

$$\mathcal{M} := \mathcal{A} \cap \mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : y > 0\},$$
$$\mathcal{N} := \mathcal{A} \cap \mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : y < 0\},$$

and the orientation is analogous to the one just given in the proof of the previous theorem.

To apply the SAP method we require the following time mapping estimates. First, the time needed to move along $\mathcal{U}_8$ from the point $(2\sqrt{2}, 2\sqrt{2})$ to the point $(2\sqrt{2}, -2\sqrt{2})$, which is

$$\tau_{\mathcal{U}_8} := \tau_{\mathcal{U}}(8; 4) = 2 \int_{2\sqrt{2}}^{4} \frac{ds}{\sqrt{16 - x^2}} = \frac{\pi}{2}.$$

Next, the period $\tau_{\mathcal{O}_{\rho_\epsilon}}$ of the periodic orbit $\mathcal{O}_{\rho_\epsilon}$, which is $\tau_{\mathcal{O}_{\rho_\epsilon}} \sim 2\pi$ when ϵ is chosen small enough. This way, by fixing $t_1 > \pi/2$, the image at time $t = t_1$ of any continuous path contained in $\mathcal{M}$ which connects $\mathcal{M}_l^-$ to $\mathcal{M}_r^-$, is stretched under the action of system (2.2.30) in another continuous path and for it one can find a sub-path entirely contained in $\mathcal{N}$ which connects $\mathcal{N}_l^-$ to $\mathcal{N}_r^-$. Provided that $t_2 > 4\pi$, the previous sub-path is again stretched by system (2.2.30) and at time $t = T = t_1 + t_2$ its image has revolved at least twice around the center $(2, 0)$. From this image, which is a spiral-like curve, we can detect two sub-paths in $\mathcal{M}$ that join the two sides $\mathcal{M}_l^-$ and $\mathcal{M}_r^-$.

In conclusion, if the period of the forcing term $p_{0,2}(t)$ is such that $T > 9\pi/2$, then Theorem B.6 guarantees dynamics on 1×2 symbols for system (2.2.30). In other words, this is the case when a topological horseshoe occurs. Since $\lim_{\xi\to+\infty} \phi(\xi)/\xi = 1$ it follows that the range where complex dynamics take place is between the third and the fourth eigenvalue of the corresponding periodic linear problem. ◁

2.2.4 Comparison

In this section we first of all sum up the result achieved in Section 2.2.2 and Section 2.2.3 for the T-periodic BVP associated with

$$u'' + \phi(u) = h(t)$$

with the intent to compare the different "level of chaos" detected. To do this, we refer to Table 2.1.

Moreover, we make some comment on the results achieved with respect to the classical condition (Hϕ_1), that we recall as follows

$$(\mathrm{H}\phi_1) \qquad -\infty \leq \lim_{\xi\to-\infty} \frac{\phi(\xi)}{\xi} < \lambda_1 = 0 < \lim_{\xi\to+\infty} \frac{\phi(\xi)}{\xi} < \lambda_2 = (2\pi/T)^2.$$

Consequently, as first, we set $h(t) := k + h_0(t)$ and consider

$$u'' + \phi(u) = k + \varepsilon h_0(t), \tag{2.2.31}$$

where $h_0(t) = \sin(\omega t)$ with $\omega > 0$, $k > 0$, ε sufficiently small and period $T = 2\pi/\omega$. For the nonlinearity we assume that $\phi\colon \mathbb{R} \to \mathbb{R}$ is a convex function of class C^2 satisfying $\phi(0) = 0$ and (Hϕ_1). An application of Corollary 2.2.7 leads to the presence of chaos in the sense of the Smale's horseshoe for a suitable iterate of the Poincaré map Φ^N associated with (2.2.31). On the other hand, an application of the abstract theory of Ambrosetti-Prodi leads to the existence of a number $k_0 = k_0(\varepsilon)$ such that equation (2.2.31) has no T-periodic solutions, at least one T-periodic solution or at least two T-periodic solutions according to $k < k_0$, $k = k_0$ or $k > k_0$, respectively. If we assume that $\lim_{\xi\to+\infty} < \omega^2/4 < \lambda_2$, then Theorem 1.4 applied to

$$u'' + \varepsilon c u' + \phi(u) = k + \varepsilon h_0(t), \tag{2.2.32}$$

states that for $k > k_0$ one T-periodic solution is asymptotically stable and the other one unstable. Finally, since Corollary 2.2.7 holds for equation (2.2.32) without any restriction on ω, we obtain the coexistence between chaos zones and regions of stability. This is not in conflict because Melnikov's method ensure the presence of a Smale horseshoe for Φ^N and so it follows also the existence of a large order of subharmonics. Notice that the existence of a great amount of subharmonic solutions has already been obtained for similar Hamiltonian systems in [BZ13; Reb97; RZ96] using the Poincaré-Birkhoff twist theorem (see also [PMM92; LM87; LM90] for previous contributions in this direction). On the contrary, the results in [Ort89; Ort90] prevent the existence of subharmonics of order two for (2.2.32).

At last, we revisit Example 2.2.14 where the forcing term $h(t) := h_{0,2}(t)$ has period $T > 9\pi/2$ and $\phi(u) = |u|$. At this juncture, we observe the presence of chaos in the sense of the topological horseshoe for the Poincaré map Φ associated with (2.2.30). In this case the chaotic zones starts when the derivative of the nonlinearity ϕ crosses also the third eigenvalue, indeed $\lambda_4 > \lim_{\xi\to+\infty} = 1 > \lambda_3 > \lambda_2$ where, $\lambda_j = (j-1)^2(2\pi/T)^2$ for $j \in \mathbb{N}_0$.

Table 2.1: In accordance with different methods, a scheme of results on chaotic dynamics for the periodic model problem $x' = y, \quad y' = -\phi(x) + h(t)$, where the minimal assumptions required which are common to all the approaches are: h T-periodic locally integrable and ϕ locally Lipschitz continuous satisfying $\phi(0) = 0$ and $(\mathrm{H}\phi_3)$. The routes to chaos for the Poincaré map ϕ associated to the system are labeled as A, B, C and D refer to Corollary 2.2.7, Corollary 2.2.10, Theorem 2.2.11 and Theorem 2.2.12, respectively.

Method	**ROUTE A**	**ROUTE B**	**ROUTE C**	**ROUTE D**
Hyphoteses	$\phi \in C^2$, $h(t) = k + \varepsilon h_0(\omega t), \quad k > 0, \quad h_0$ non-constant			$h(t) = k_1 \mathbb{1}_{[0,t_1[}(t) + k_2 \mathbb{1}_{[t_1,t_2[}(t)$ with $t_1 + t_2 = T$ and $0 \le k_1 < k_2$
Further Hyphoteses	$h_0(\omega t) = \sin(\omega t)$ $\omega > 0$ arbitrary $\varepsilon = \varepsilon(\omega) > 0$ small	$h_0(\omega t)$ arbitrary $\omega > 0$ small $\varepsilon = \varepsilon(\omega) > 0$ small	$h_0(\omega t)$ arbitrary $\omega > 0$ small $\lvert\varepsilon\rvert < K$	t_1, t_2 large
Conclusion	Smale horseshoe for an iterate of Φ relatively to $[0, T]$	Topological horseshoe for an iterate of Φ relatively to $[0, T/\omega]$	Symbolic dynamics for Φ relatively to $[0, T/\omega]$	Topological horseshoe for Φ relatively to $[0, T]$

3. Neumann AP problems

The present chapter, coming from [Sov18; SZ17b], is devoted to the study of existence and multiplicity of solutions for Neumann AP problems of the following form

$$(\mathscr{N}_s) \qquad \begin{cases} u'' + g(t,u) = s, \\ u'(0) = u'(T), \end{cases}$$

where s is a real parameter and $g\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

First of all we recall from Chapter 1 that dealing with a continuous function g, under the additional coercivity hypothesis

$$(\mathrm{H}g_1) \qquad \lim_{|u|\to+\infty} g(t,u) = +\infty \quad \text{uniformly in } t$$

introduced in [FMN86; Maw87a], then there exists a number s_0 such that problem $(\mathscr{N}_s)$ has no solutions, at least one solution or at least two solutions according to $s < s_0$, $s = s_0$ or $s > s_0$ (Theorem 1.6). As for the periodic problem in Chapter 2, the goal is here to generalize Theorem 1.6 by weakening the usual condition $(\mathrm{H}g_1)$ considered in the literature [BL81; Maw87a; PP16; Rac93] without requiring any uniformity condition in t. In fact, a typical example that can not be treated within the framework built up in [Maw87a] arises by considering the non-sign definite Neumann problem given by

$$(\mathscr{W}\mathscr{N}_s) \qquad \begin{cases} u'' + a(t)\phi(u) = s + p(t), \\ u'(0) = u'(T), \end{cases}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $(\mathrm{H}\phi_2)$ (i.e. $\lim_{|u|\to+\infty} \phi(u) = +\infty$) and a, $p \in L^\infty(0,T)$ with

$$(\mathrm{H}a_1) \qquad a(t) \geq 0 \text{ for a.e. } t \in [0,T] \text{ with } \int_0^T a(t)\,dt > 0.$$

Indeed, in this case, $g(t,u) = a(t)\phi(u) - p(t)$ does not tend uniformly to infinity as $|u| \to +\infty$ and can even vanishes identically on sets of positive measure. The question now is whether a weak alternative of Ambrosetti-Prodi for the solutions of $(\mathscr{W}\mathscr{N}_s)$ still holds. We first discuss an example to motivate the results in this section.

Example 3.1. Consider the Neumann BVP associated with the parameter dependent equation $u'' + a(t)\phi(u) = s + p(t)$ on $[0,2]$, where $\phi(\xi) = \sqrt{1+\xi^2} - 1$, $p(t) = \sin(t)$ and

$$a(t) = \begin{cases} 0 & \text{for } t \in [0,1[, \\ 1 & \text{for } t \in [1,2]. \end{cases}$$

Notice that $\operatorname{ess\,inf}_{t\in[0,2]} a(t) = 0$ and so the problem does not belong to the setting of the works already quoted. Nevertheless, the multiplicity of solutions is not lost, as suggested in Figure 3.1. In fact, we give numerical evidence of the existence of a number s_* for which the corresponding Neumann problem has at least two solutions. $\lhd$

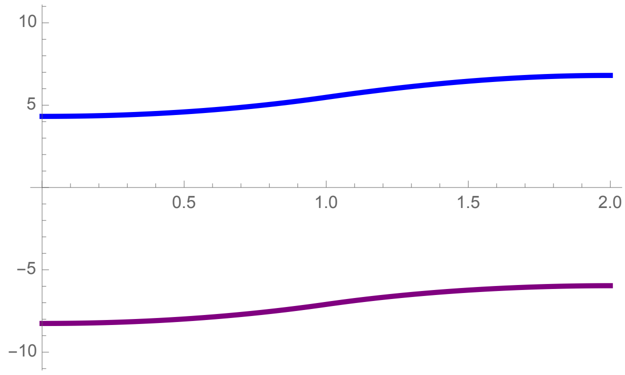


Figure 3.1: Numerical approximation of two solution of the Neumann problem on $[0,2]$ associated with $u''+a(t)\phi(u) = s+p(t)$ with $s = 2$ satisfying the framework of Example 3.1.

In the first part of this chapter we will exploit the "shooting method" in order to provide a multiplicity result of solutions for the non-sign definite Neumann problem $(\mathscr{WN}_s)$ according to Theorem 3.1.1. This kind of approach is handy and well gives the idea of the dynamics of the problem on the phase-plane (u,u'). Nevertheless, apparently does not permit to recover the complete alternative of Ambrosetti-Prodi type.

For this reason, in the second part we will study both problems $(\mathscr{N}_s)$ and $(\mathscr{WN}_s)$ in the same spirit of Section 2.1. In particular, by Theorem 3.2.1 and Corollary 3.2.2, we will give a more detailed description of the set of the solutions for these Neumann BVPs and we will recover the weaker form of the classical scheme zero, one or two solutions.

3.1 Multiplicity result via shooting method

In this section we deal with a nonlinearity $\phi\colon \mathbb{R} \to \mathbb{R}$ of class C^1 which satisfies the following "crossing condition"

$$(\mathrm{H}\phi_5) \qquad -\infty < \lim_{\xi\to-\infty} \phi'(\xi) < 0 < \lim_{\xi\to+\infty} \phi'(\xi) < +\infty,$$

and a weight term $a \in L^\infty(0,T)$ satisfying $(\mathrm{H}a_1)$. In this framework we state and prove the following result of multiplicity of solutions for problem $(\mathscr{WN}_s)$.

Theorem 3.1.1. *Let $p \in L^\infty(0,T)$. Assume that $\phi \in C^1(\mathbb{R})$ is a function which satisfies $(\mathrm{H}\phi_5)$. Moreover, suppose that $a \in L^\infty(0,T)$ is such that conditions in $(\mathrm{H}a_1)$ hold. Then, there exists $s_0 \in \mathbb{R}$ such that the problem $(\mathscr{WN}_s)$ has at least two solutions for all $s > s_0$.*

The proof of Theorem 3.1.1 is performed into two parts. In the first one we present a result of existence and in the second we conclude with a result of multiplicity. In the sequel, without loss of generality we assume that

$$(\mathrm{H}\phi_6) \qquad \phi(0) = 0, \qquad \phi'(\xi) < 0 \quad \forall \xi < 0.$$

In fact, from $(\mathrm{H}\phi_5)$ there exists $r_0 < 0$ such that $\phi'(\xi) < 0$ for each $\xi < r_0$. Therefore, taking $z := u - r_0$, one could also consider the equivalent Neumann problem associated with $z'' + a(t)\tilde{\phi}(z) = s + \tilde{p}(t)$ where $\tilde{p}(t) = p(t) - a(t)\phi(r_0)$ and $\tilde{\phi}(z) := \phi(z + r_0) - \phi(r_0)$ satisfies $(\mathrm{H}\phi_6)$.

Existence result

In this subsection, we present an existence result for $(\mathscr{W}\mathscr{N}_s)$ when the parameter s exceeds some value s_0. Let us consider the truncated Neumann problem associated with the equation

$$u'' + \varphi_s(t,u) = 0, \tag{3.1.1}$$

where

$$\varphi_s(t,\xi) := \begin{cases} a(t)\phi(\xi) - s - p(t) & \xi \leq 0, \\ -s - p(t) & \xi > 0. \end{cases}$$

Notice that (3.1.1) coincides with $(\mathscr{W}\mathscr{N}_s)$ when $u(t) \leq 0$ for all $t \in [0,T]$.

In our framework both uniqueness and global existence for the solutions of the associated Cauchy problems is guaranteed. Thus, let $u(\cdot\,; u_0, u_1)$ be the unique and globally defined on $[0,T]$ solution of the equation (3.1.1) satisfying the initial values

$$u(0) = u_0 \in \mathbb{R}, \quad u'(0) = u_1 \in \mathbb{R}. \tag{3.1.2}$$

We recall that, for every fixed $s \in \mathbb{R}$, the Poincaré map for (3.1.1) on the interval $[0,T]$ is the well defined map

$$\Phi_0^T : \mathbb{R}^2 \to \mathbb{R}^2, \quad (u_0,u_1) \mapsto (u(T), u'(T))$$

where u is the solution of (3.1.1) with the initial values (3.1.2). Moreover, the standard theory of ordinary differential equations guarantees that the Poincaré map is actually a global diffeomorphism of the plane onto itself.

Shooting method. The recipe of the shooting method states that a solution of the Neumann problem associated with equation (3.1.1) can be found by looking for a point $(u_0, 0) \in \mathbb{R}^2$ such that $\Phi_0^T(u_0,0) \in \mathbb{R} \times \{0\}$.

This way our goal is to prove that for any $A > 0$ the Poincaré map associated with (3.1.1) is such that $\Phi_0^T(A,0) \in \mathbb{R}^+ \times \mathbb{R}^+$, while for any $-B \ll 0$ we have that $\Phi_0^T(-B,0) \in \mathbb{R}_0^- \times \mathbb{R}^-$. A continuity argument then leads to the existence of $C \in]-B, A[$ such that $\Phi_0^T(C,0) \in \mathbb{R} \times \{0\}$.

Theorem 3.1.2. *Let $p \in L^\infty(0,T)$. Let $a \in L^\infty(0,T)$ satisfies* $(\mathrm{H}a_1)$. *Assume that $\phi \in C^1(\mathbb{R})$ satisfies* $(\mathrm{H}\phi_5)$ *and* $(\mathrm{H}\phi_6)$. *Then, there exists $s_0 \in \mathbb{R}$ such that for each $s > s_0$ problem $(\mathscr{W}\mathscr{N}_s)$ has at least one solution.*

Before proceeding with the proof of Theorem 3.1.2 we need the following two preliminary lemmas (the prove of the first one follows straightforward by contradiction and so it is omitted).

Lemma 3.1.3. *Let $s > \operatorname{ess\,sup}_{t\in[0,T]} -p(t)$. Then, for any fixed $u_0 > 0$, the solution of* (3.1.1) *with initial values $u(0) = u_0$ and $u'(0) = 0$ is such that $u(t) > 0$ for all $t \in [0,T]$ and $u'(t) > 0$ for all $t \in]0,T]$.*

Lemma 3.1.4. *Let $s \in \mathbb{R}$, $p \in L^\infty(0,T)$. Let $a \in L^\infty(0,T)$ satisfies* $(\mathrm{H}a_1)$. *Assume that $\phi \in C^1(\mathbb{R})$ satisfies* $(\mathrm{H}\phi_5)$ *and* $(\mathrm{H}\phi_6)$. *Then, there exists $r_s < 0$ such that for any fixed $u_0 < r_s$, if u is a solution of* (3.1.1) *with initial values $u(0) = u_0$ and $u'(0) = 0$, then $u(t) < 0$ for each $t \in [0,T]$ and, moreover, $u'(T) < 0$.*

Proof. As long as $u(t)$ is negative, integrating equation (3.1.1) two times with respect to t and taking into account $(\mathrm{H}\phi_6)$, we obtain

$$u(t) = u_0 - \int_0^t \left(\int_0^\xi a(z)\phi(u(z))\,dz \right) d\xi + \frac{st^2}{2} + \int_0^t P(\xi)\,d\xi, \tag{3.1.3}$$

where $P(t) := \int_0^t p(\xi)\,d\xi$. Considering again (H$\phi_6$), from equation (3.1.3), we get

$$u(t) \leq u_0 + \frac{sT^2}{2} + \|P\|_{L^1}.$$

For any $s \in \mathbb{R}$, we define

$$M_s = M(s) := \frac{sT^2}{2} + \|P\|_{L^1}. \tag{3.1.4}$$

Then, from the choice of $u_0 < -M_s$, we have that for each $t \in [0,T]$

$$u(t) \leq u_0 + M_s < 0. \tag{3.1.5}$$

Now we prove $u'(T) < 0$. An integration on $[0,T]$ of equation (3.1.1) leads to the following inequality:

$$u'(T) \leq -\int_0^T a(\xi)\phi(u(\xi))\,d\xi + sT + \|p\|_{L^1}.$$

Recalling (3.1.5) and (Hϕ_6), which implies that the function ϕ is strictly decreasing on $[0,-\infty)$, by using the previous inequality we obtain

$$u'(T) \leq -\phi(u_0 + M_s)\|a\|_{L^1} + sT + \|p\|_{L^1}.$$

From assumption (Hϕ_5) follows $\phi(s) \to +\infty$ as $s \to -\infty$. Therefore, there exists $m_s = m(s) > 0$ such that

$$\phi(u) > \alpha := \frac{sT + \|p\|_{L^1}}{\|a\|_{L^1}}, \quad \forall u < -m_s.$$

Now, choose $r_s = r(s) := -(m_s + M_s)$. Then, $\phi(u_0 + M_s) > \alpha$ for each $u_0 < r_s$. Consequently, taking $u_0 < r_s$, we achieve the thesis, since $u'(T) < 0$ and $u(t) < 0$ for all $t \in [0,T]$. □

Proof of Theorem 3.1.2. Let us take $s_0 := \operatorname{ess\,sup}_{t\in[0,T]} -p(t)$ and divide the proof in two steps.

Step I. *We claim that for every $s > s_0$ there exists $C_1 \in \mathbb{R} \times \{0\}$ such that $\Phi_0^T(C_1, 0) \in \mathbb{R} \times \{0\}$.*

This way, the Neumann problem associated with the truncated equation (3.1.1) has at least a solution for every $s > s_0$.

Let us fix $s > s_0$. We choose a point $(A, 0) \in \mathbb{R}^+ \times \{0\}$ and we denote by u_A the solution of (3.1.1) with initial conditions $u(0) = A$ and $u'(0) = 0$. An application of Lemma 3.1.3 leads to

$$\Phi_0^T(A,0) = \big(u_A(T), u_A'(T)\big) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

On the other hand, thanks to Lemma 3.1.4 there exists a value $r_s < 0$ such that, if we select a point $(-B, 0) \in \mathbb{R}^- \times \{0\}$ with $-B < r_s$ and we denote by u_B the solution of (3.1.1) with initial conditions $u(0) = -B$ and $u'(0) = 0$ then

$$\Phi_0^T(-B,0) = \big(u_B(T), u_B'(T)\big) \in \mathbb{R}^- \times \mathbb{R}^-.$$

At this point, the continuous dependence of the solutions upon the initial data implies that there exists $C_1 \in]-B, A[$ such that the solution u_{C_1} of (3.1.1) with initial conditions $u(0) = C_1$ and $u'(0) = 0$ verifies $\Phi_0^T(C_1, 0) = \big(u_{C_1}(T), u_{C_1}'(T)\big) \in \mathbb{R} \times \{0\}$, and thus the claim is proved. The solution u_{C_1} is in turn a solution of the truncated equation (3.1.1) with Neumann boundary conditions since $u_{C_1}'(T) = 0$.

Step II. *We claim that $u_{C_1}(t) < 0$ for every $t \in [0,T]$.*

This way, the solution of the truncated problem (3.1.1) under Neumann boundary conditions is a solution of $(\mathscr{W}\mathscr{N}_s)$. The proof of the claim is standard from the theory of upper and lower solutions (see [DCH06]), nevertheless, we propose here also an alternative argument. If $u_{C_1}(t) \geq 0$ for all $t \in [0,T]$, then we achieve a contradiction since $\varphi_s(t, u_{C_1}(t)) < 0$ for a.e. $t \in [0,T]$. Now, using a standard maximum principle argument, it is easy to prove that there exists $\delta > 0$ (which depends on the fixed parameter s) such that $u_{C_1}(t) \leq -\delta$ for all $t \in [0,T]$.

At this point, having proved for a fixed $s > s_0$ the existence of a solution u for (3.1.1) satisfying Neumann boundary conditions and such that $u(t) < 0$ for every $t \in [0,T]$, the thesis follows. □

Multiplicity result

In this subsection, we conclude the proof of Theorem 3.1.1. Let us take a fixed value s with $s > s_0 := \operatorname{ess\,sup}_{t\in[0,T]} -p(t)$. By Theorem 3.1.2 there exists at least a solution of $(\mathscr{W}\mathscr{N}_s)$, let us call it $\tilde{u}$.

In order to prove the existence of at least a second solution, we need to introduce the following two preliminary lemmas.

Lemma 3.1.5. *Let $s > s_0$, $p \in L^\infty(0,T)$ and $a \in L^\infty(0,T)$ satisfies* (Ha_1). *Assume that $\phi \in C^1(\mathbb{R})$ satisfies* (Hϕ_5) *and* (Hϕ_6). *Then, there exists $\varepsilon > 0$ such that if u_ε is a solution of $u'' + a(t)\phi(u) = s + p(t)$ with initial values $u(0) = \tilde{u}(0) + \varepsilon$ and $u'(0) = 0$, then $u'_\varepsilon(T) > 0$.*

Proof. Let us take $v_\varepsilon(t) := u_\varepsilon(t) - \tilde{u}(t)$. The Cauchy problem considered here can be equivalently described by the differential equation

$$v''_\varepsilon + a(t)(\phi(v_\varepsilon + \tilde{u}(t)) - \phi(\tilde{u}(t))) = 0 \tag{3.1.6}$$

with initial conditions $v_\varepsilon(0) = \varepsilon$ and $v'_\varepsilon(0) = 0$.

We claim that there exists $\varepsilon > 0$ such that $v'_\varepsilon(T) > 0$. To check this assertion, since $\phi \in C^1(\mathbb{R})$, from equation (3.1.6) we have

$$v''_\varepsilon + a(t)\mathcal{B}_\varepsilon(t)v_\varepsilon = 0 \tag{3.1.7}$$

where

$$\mathcal{B}_\varepsilon(t) := \int_0^1 \phi'(\tilde{u}(t) + \theta v_\varepsilon(x))\, d\theta.$$

Next, by the continuous dependence of the solutions upon the initial data, it follows that $v_\varepsilon(t) \to 0$ uniformly in t as $\varepsilon \to 0^+$. As a consequence, there exists $\varepsilon^* \ll 1$ such that, for each $0 < \varepsilon < \varepsilon^*$ we have $\tilde{u}(t) + \theta v_\varepsilon(t) < 0$ for all $t \in [0,T]$ and for all $\theta \in [0,1]$. This way, recalling (Hϕ_6), we obtain that $\mathcal{B}_\varepsilon(t) < 0$ for each $0 < \varepsilon < \varepsilon^*$.

Thus, if we prove that $v_\varepsilon(t) > 0$ for every $t \in [0,T]$, then the claim is verified. Arguing by contradiction, let us suppose that there exists a first point $t^* \in]0,T]$ such that $v_\varepsilon(t^*) = 0$. Then, from (3.1.7) we deduce that $v_\varepsilon(t) \geq v_\varepsilon(0) = \varepsilon > 0$ for all $t \in [0,t^*]$, a contradiction with respect to $v_\varepsilon(t^*) = 0$. The proof is concluded since $u'_\varepsilon(T) = v'_\varepsilon(T) > 0$. □

Lemma 3.1.6. *Let $s > s_0$, $p \in L^\infty(0,T)$ and $a \in L^\infty(0,T)$ satisfies* (Ha_1). *Assume that $\phi \in C^1(\mathbb{R})$ satisfies* (Hϕ_5) *and* (Hϕ_6). *Then, there exists $R_s > 0$ such that for any fixed $u_0 > R_s$, if u is a solution of $u'' + a(t)\phi(u) = s + p(t)$ with initial values $u(0) = u_0$ and $u'(0) = 0$, then $u'(T) < 0$.*

Proof. From assumptions (Hϕ_6) and (Hϕ_5), it follows that there exists a global minimum $\phi_{\min}$ of ϕ on $\mathbb{R}$ such that

$$\phi_{\min} = \min_{\xi\in[0,+\infty[} \phi(\xi) \leq g(0) = 0.$$

Accordingly, from

$$u'' = -a(t)\phi(u) + s + p(t), \tag{3.1.8}$$

we get the differential inequality

$$u'' \leq -a(t)\phi_{\min} + s + p(t).$$

Now, integrating on $[t_1, t_2] \subseteq [0, T]$, we have

$$u'(t_2) \leq u'(t_1) - \phi_{\min}\|a\|_{L^1} + sT + \|p\|_{L^1}.$$

Then, we fix a constant $K_s = K(s) > 0$ such that

$$K_s > -\phi_{\min}\|a\|_{L^1} + sT + \|p\|_{L^1}.$$

We claim that there exists $t_1 \in [0, T]$ such that $u'(t_1) < -K_s$. From this fact, it immediately follows that $u'(T) < 0$. To check the claim, suppose by contradiction that $u'(t) \geq -K_s$ for every $t \in [0, T]$. It clearly follows that $u(t) \geq u_0 - K_s T$ for every $t \in [0, T]$.

From assumption (Hϕ_5) we deduce that $\phi(\xi) \to +\infty$ as $\xi \to +\infty$, which implies that there exists $k_s = k(s) > 0$ such that

$$\phi(\xi) > \beta := \frac{K_s + sT + \|p\|_{L^1}}{\|a\|_{L^1}}, \quad \forall \xi > k_s.$$

Now, choose $R_s = R(s) := k_s + K_s T > 0$ and take $u_0 > R_s$. In this manner we obtain $u(t) > k_s$ for every $t \in [0, T]$. An integration on $[0, T]$ of (3.1.8) yields to a contradiction, since

$$u'(T) \leq -\inf_{u > k_s} \phi(u)\|a\|_{L^1} + sT + \|p\|_{L^1} < -\beta\|a\|_{L^1} + sT + \|p\|_{L^1} = -K_s.$$

Our claim is thus verified and this completes the proof. □

Remark 3.1.7. We stress the fact that the assumption ϕ of class C^1 is crucial only in the proof of Lemma 3.1.5. For all the other auxiliary lemmas in this section, the condition (Hϕ_2) is enough to achieve the conclusions.

We are now in position to prove our main theorem for this section. Moreover, in Figure 3.2, we illustrate with an example the results reached.

Proof of Theorem 3.1.1. For ease of notation, we still denote by Φ_0^T the Poincaré map associated with the differential equation in $(\mathscr{W}\mathscr{N}_s)$. Let $s > s_0$ be fixed and $\tilde{u}$ the solution to problem $(\mathscr{W}\mathscr{N}_s)$, coming from Theorem 3.1.2. In view of Lemma 3.1.5, there exists a sufficiently small constant ε such that, if we choose a point $(D, 0) \in \mathbb{R} \times \{0\}$ with $D := \tilde{u}(0) + \varepsilon$ and we denote by u_D the solution of $u'' + a(t)\phi(u) = s + p(t)$ with initial conditions $u(0) = D$ and $u'(0) = 0$, then we have $\Phi_0^T(D, 0) = \big(u_D(T), u_D'(T)\big) \in \mathbb{R} \times \mathbb{R}^+$. Clearly, we can take $D < 0$.

On the other hand, in view of Lemma 3.1.6, we can find a sufficiently large constant E depending on s such that, if we choose the point $(E, 0) \in \mathbb{R}^+ \times \{0\}$ and we denote by u_E the solution of $u'' + a(t)\phi(u) = s + p(t)$ with initial conditions $u(0) = E$ and $u'(0) = 0$, then it follows $\Phi_0^T(E, 0) = \big(u_E(T), u_E'(T)\big) \in \mathbb{R} \times \mathbb{R}^-$.

Finally, the existence of a second solution to problem $(\mathscr{W}\mathscr{N}_s)$ follows again by the continuous dependence of the solutions upon the initial data. Indeed, there exists a value $C_2 \in]D, E[$ such that the solution u_{C_2} of $u'' + a(t)\phi(u) = s + p(t)$ with initial conditions $u(0) = C_2$ and $u'(0) = 0$ verifies $\Phi_0^T(C_2, 0) = \big(u_{C_2}(T), u_{C_2}'(T)\big) \in \mathbb{R} \times \{0\}$. The proof is thus completed. □

3.2 Multiplicity result via topological degree

In this section we start by considering problem $(\mathscr{N}_s)$ and we prove an Ambrosetti-Prodi type alternative. More precisely, we propose a generalization of Theorem 1.6 (from [Maw87a]) replacing the coercivity condition (Hg_1) by a local one. We consider again Villari's type conditions introduced in Section 2.1.2. The technique we employ exploits the Mawhin's coincidence degree (see Appendix A) and combines some arguments borrowed by [Maw87a] with other ones newly introduced in Chapter 2.2.

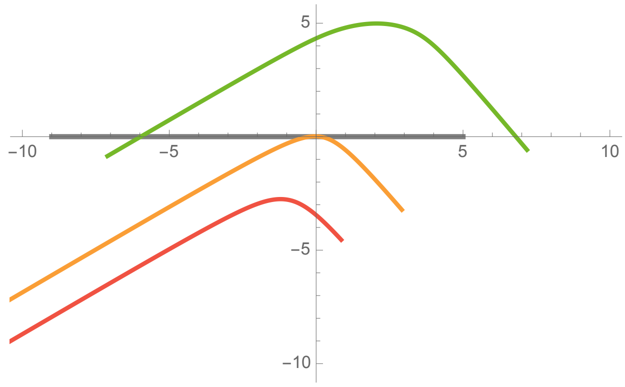


Figure 3.2: In the phase-plane (u, u') application of the shooting method to the Neumann problem on $[0, 2]$ associated with $u'' + a(t)\phi(u) = s + p(t)$ satisfying the framework of Example 3.1. It is displayed the images of the segment $[-9, 5] \subseteq \mathbb{R} \times \{0\}$ (gray) through the action of the Poincaré map varying the parameter s. Considered values: $s = -2$ (red), $s = -0.7$ (yellow) and $s = 2$ (green). Consistently with Theorem 3.1.1, for s sufficiently large, the green line intersects the u-axis two times. Instead, for s sufficiently small no intersection point could be expected (red line).

Theorem 3.2.1. *Let $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (A_2), (G_0) *and* (G_1). *Assume also the Villari's type conditions* $(\mathrm{H}g_4^-)$ *and* $(\mathrm{H}g_4^+)$ *with reference to $x \in C^1_\#$. Then, there exists $s_0 \in \mathbb{R}$ such that:*

1° *for $s < s_0$, problem $(\mathscr{N}_s)$ has no solutions;*

2° *for $s = s_0$, problem $(\mathscr{N}_s)$ has at least one solution;*

3° *for $s > s_0$, problem $(\mathscr{N}_s)$ has at least two solutions.*

Proof. Let us introduce the space

$$X = C^1_\# := \{x \in C^1([0, T]) : x'(0) = x'(T) = 0\}$$

endowed with the norm $\|u\|_X := \|u\|_\infty + \|u'\|_\infty$.

This way we enter in the framework of coincidence degree as done in Section 2.1.1 and so, the proof follows exactly like that of Theorem 2.1.7 and Theorem 2.1.9. □

Notice that the abstract result in Theorem 3.2.1 allows us to treat also $(\mathscr{WN}_s)$. This way we can state the following corollary whose proof is omitted since follows straightway from Theorem 3.2.1.

Corollary 3.2.2. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* $(\mathrm{H}\phi_2)$. *Let $a, p \in L^\infty[0, T]$ with $a(t) \geq 0$ for a.e. $t \in [0, T]$ and $\int_0^T a(t)\, dt > 0$. Then, there exists a number $s_0 \in \mathbb{R}$ such that:*

1° *for $s < s_0$, problem $(\mathscr{WN}_s)$ has no solutions;*

2° *for $s = s_0$, problem $(\mathscr{WN}_s)$ has at least one solution;*

3° *for $s > s_0$, problem $(\mathscr{WN}_s)$ has at least two solutions.*

Finally, we have validate the scheme of zero, one or two solutions for $(\mathscr{WN}_s)$ that we suggested by means of Example 3.1 and Figure 3.2 in the previous section.

4. Further developments from Part I

In Chapter 2 we dealt with the existence of infinitely many periodic solutions for the periodic problem BVP associated with equation

$$u'' + \phi(u) = h(t)$$

where h is a T-periodic forcing term and $\phi\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function with $\phi(0) = 0$, which is strictly decreasing on $]-\infty, 0]$, strictly increasing on $[0, +\infty[$ and $\lim_{|u|\to+\infty} \phi(u) = +\infty$. From the analysis carried out arises a question about the location of the "chaotic region." More precisely, from the prototypical nonlinearity considered in Example 2.2.14 the main observation is that complex behaviors can be detected for a range of parameters such that $\lambda_3 < \lim_{\xi\to+\infty} \phi(\xi)/\xi < \lambda_4$, where $\lambda_j = (j-1)^2(2\pi/T)^2$ denotes the j-th eigenvalue of the associated linear problem. At this point, the question still open is whether we could find "chaos" for such kind of problems when $\lambda_2 < \lim_{\xi\to+\infty} \phi(\xi)/\xi < \lambda_3$?

In Chapter 3 we dealt with the parameter dependent Neumann BVP

$$\begin{cases} u'' + g(t,u) = s, \\ u'(0) = u'(T), \end{cases}$$

where $g\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies a local coercivity condition. In this context we have proved the existence of a number s_0 such that the previous problem has zero, at least one or at least two solutions provided that $s < s_0$, $s = s_0$ or $s > s_0$, respectively (cf. Theorem 3.2.1).

A still open problem concerns the extension of our result to the Neumann problem for an elliptic PDE

$$\begin{cases} \Delta u + g(x,u) = s & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that in [Maw87a] an Ambrosetti-Prodi type alternative is obtained for that BVP if $\lim_{|u|\to+\infty} g(x,u) = +\infty$ uniformly in $\overline{\Omega}$ and $\lim_{u\to+\infty} g(x,u)/u^{\sigma} = 0$ uniformly in $\overline{\Omega}$ with $\sigma = N/(N-2)$ if $N \geq 3$ and σ finite if $N = 2$. The open problem is now to recover the same alternative for the solutions of this problem weakening the uniformity condition in $\overline{\Omega}$ up to work, for example, with nonlinearities $g(x,u) = a(x)\phi(u)$ such that $\lim_{|u|\to+\infty} \phi(u) = +\infty$ and $a(x) \geq 0$ with $\int_{\Omega} a(x)\,dx > 0$ and so also give an extension of Corollary 3.2.2.

Part II

Indefinite weight problems

5. Indefinite weight problems: focused overview

This chapter guides the reader through a selection of BVPs, which are motivated by biological applications, to hopefully justify the abstract formulations in the next chapters. In particular, we take a look to reaction-diffusion equations, which describe, among others, phenomena of population dispersal (see [Bel97; Hen81; LG16; Mur89]). The main aspect that we take into account is the effect of the heterogeneity of a finite habitat on the analysis of persistence/extinction/coexistence of species. In this context, modeling density of a given population $u = u(x,t)$, most common formulations could lead to semilinear parabolic problems of the form

$$(\mathscr{R}\mathscr{D}) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u = w(x)\psi(u) & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0 & \text{in } \partial\Omega, \\ \mathfrak{B}u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where $d > 0$ represents the diffusion rate, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $N \geq 1$, $w\colon \Omega \to \mathbb{R}$ is a weight term, $\psi\colon \mathcal{I} \to \mathbb{R}^+$ is a nonlinear function with $\mathcal{I} = [0,1]$ or $[0,+\infty[$ such that $\psi(0) = 0$ and $\mathfrak{B}$ is the boundary operator, that it could be of Dirichlet type, i.e. $\mathfrak{B}u = u$, or of Neumann type, i.e. $\mathfrak{B}u = \partial u/\partial \mathbf{n}$ (no-flux across the boundary).

About boundary conditions, the Dirichlet one means that the exterior habitat is hostile, instead, the Neumann one means that there exists a inescapable barrier for the population. With respect to the function w we assume that changes its sign, namely we consider the so-called case of an *indefinite weight problems*. In other words, we involve a weight term which is positive, zero or negative in several parts of Ω, this way, one can thought to a “food source” for the population which, in different regions of the habitat, is good (favorable), neutral or worst (unfavorable), respectively.

Dispersal processes are important in the context of population dynamics since describe the distribution of a population and its interaction between the resources in a habitat, up to generate evolutionary selection [BC95]. In this context turns out to be crucial, for the analysis of the dynamics of $(\mathscr{R}\mathscr{D})$, the search of steady states of $(\mathscr{R}\mathscr{D})$ and the study of their stability. It follows that it is essential for the understanding of $(\mathscr{R}\mathscr{D})$ deal with the study of the following nonlinear eigenvalue problem with indefinite weight

$$(\mathscr{I}\mathscr{P}) \qquad \begin{cases} -\Delta u = \lambda w(x)\psi(u) & \text{in } \Omega, \\ \mathfrak{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda := 1/d$ is a positive real parameter. Due to the biological reason, most problems in population dynamics concern the study of existence or nonexistence as well as uniqueness or multiplicity of non-trivial *positive solutions* for $(\mathscr{IP})$, at the varying of λ, under either Dirichlet or Neumann boundary conditions.

The case of an indefinite weight has attracted much attention during the past decades from the pioneering works of Manes and Micheletti [MM73], Hess and Kato [HK80], Brown and Lin [BL80] and López-Gómez [LG96] concerning the properties related to the principal eigenvalue. Assuming several types of boundary conditions or different features for the weight function w along with a wide variety of nonlinear functions ψ, classified according to growth conditions, the research on positive solutions for $(\mathscr{IP})$ has grown up at the end of the Eighties (see, for instance, [AT93; ALG98; BPT87; BPT88; BCDN94; BH90; BO86; BCDN94; BCDN95; Sen83]). Nowadays, it is still a very active area of investigation. Indeed, the recent literature about multiplicity results for positive solutions of indefinite weight problems under Dirichlet or Neumann boundary conditions is really very rich and, in order to cover most of the results achieved with different techniques so far, we quote the following works and the references therein [GRLG00; GHZ03; BGH05; OO06; GG09; FZ15b; Bos11; BG16; BFZ16; LN02; LNN13; LNS10]. Focus on biological applications, from the quoted papers, relevant families of nonlinearities for problem $(\mathscr{IP})$ are the following ones:

Type 1. $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$, $\psi(\xi) > 0\ \forall\, \xi \in \mathbb{R}^+_0$ and $\lim_{\xi\to+\infty} \psi(\xi)/\xi = 0$,

Type 2. $\psi\colon [0,1] \to \mathbb{R}^+$ such that $\psi(0) = 0 = \psi(1)$ and $\psi(\xi) > 0\ \forall\, \xi \in\,]0,1[$.

Remark 5.1. Let us make some comments on the above types of functions. First of all, as long as ψ satisfies the conditions of Type 1, we say that the nonlinearity is *sublinear at infinity.* Secondly, we notice that nonlinearities verifying conditions of Type 2 are very common in the field of population genetics, starting from the pioneering work by W.H. Fleming [Fle75]. Indeed, here $u(x)$ denotes the gene frequency of a population at location $x \in \Omega$ and w is a local selective term. In this case, the dynamics see the existence of a positive trivial solution $u \equiv 1$ and the study of positive solutions is meant avoiding the trivial one. $\lhd$

For both the two nonlinearities of Type 1 and Type 2, the choice of Neumann boundary conditions in the model lead to necessary conditions for the weight w in order to ensure the existence of positive solutions. Indeed, for the Neumann BVPs the existence or nonexistence of positive solutions is influenced by the sign of $\int_\Omega w(x)\,dx$, instead for the Dirichlet ones further assumptions are not needed. On the other hand, the dichotomy between uniqueness or multiplicity of non-trivial positive solutions for problem $(\mathscr{IP})$ is strictly related with conditions of concavity or convexity on ψ. These two conditions are the main features that distinguish the results in this topic. For convenience of the reader, we summarize related work in Table 5.1 and we highlight the lack of knowledge that arises in some cases. It is interesting to note that the question mark regarding Neumann BVP with nonlinearity of Type 2 is exactly the translation in this context of a conjecture appeared in [LN02] (see the Introduction).

At last, by considering the work by Hammerstein [Ham30], we recall other interesting branches of research which are strictly related with Type 1 nonlinearities that consider an oscillatory behavior at infinity for the potential instead of a sublinear growth of ψ. Such features can be provided introducing conditions of the following form

$$\liminf_{\xi\to+\infty} \frac{2\int_0^s \psi(\xi)\,ds}{\xi^2} = 0 < \limsup_{\xi\to+\infty} \frac{2\int_0^s \psi(\xi)\,ds}{\xi^2},$$

as in [OZ96; OO06; MZ93]. In this situation, usually one is interested in results of high multiplicity of positive solutions. In accord, that is already done for indefinite weight problems $(\mathscr{IP})$ under Dirichlet boundary (see for instance [OO06]). Nevertheless, with respect to problems $(\mathscr{IP})$ under Neumann boundary conditions, these kinds of issues turned out to be open.

Table 5.1: Schematic overview of models and results related to problem $(\mathscr{I}\mathscr{P})$ with respect to a nonlinearity ψ of Type 1 or Type 2 and an indefinite weight term w. The nonlinearity and the weight term are assumed to be sufficiently smooth according to the references. Both results of uniqueness and multiplicity of positive solutions are meant for a parameter λ sufficiently large.

Boundary conditions	Hypothesis on w	Hypotheses on ψ	**Type 1**	**Type 2**
$\mathfrak{B}\mathbf{u} = \mathbf{u}$ **Dirichlet**	indefinite	$\psi''(\xi) < 0$ for all $\xi \in \mathbb{R}^+$	for $N \geq 1$, uniqueness [BH90]	for $N \geq 1$, uniqueness [BH90]
		$\psi''(\xi) \nless 0$, $\xi \mapsto \psi(\xi)/\xi$ monotone decreasing for all $\xi \in \mathbb{R}^+$	?	?
		$\lim_{\xi \to 0^+} \psi(\xi)/\xi = 0$	for $N \geq 1$, at least two [Ama76; Lio82; OS99] for $N = 1$ or for $N \geq 1$ in annular domain, high multiplicity [BFZ16]	for $N \geq 1$, at least two [Rab73b]
$\mathfrak{B}\mathbf{u} = \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}$ **Neumann**	indefinite $\int_\Omega w(x)\,dx < 0$	$\psi''(\xi) < 0$ for all $\xi \in \mathbb{R}^+$	for $N \geq 1$, uniqueness [BPT88]	for $N \geq 1$, uniqueness [Hen81]
		$\psi''(\xi) \nless 0$, $\xi \mapsto \psi(\xi)/\xi$ monotone decreasing for all $\xi \in \mathbb{R}^+$	?	?
		$\lim_{\xi \to 0^+} \psi(\xi)/\xi = 0$	for $N = 1$ or for $N \geq 1$ in annular domain, high multiplicity [BFZ16]	for $N \geq 1$, at least two [LNS10]

Note: In the present dissertation, we tackle the question marks.

6. Nonlinearities with linear-sublinear growth

The present chapter, whose content comes from [SZ15], is concerned with Dirichlet problems of the form

$$(\mathscr{I}\mathscr{D}_{\lambda,N}) \qquad \begin{cases} -\Delta u = \lambda w(x)\psi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary, λ is a positive real parameter, the nonlinearity $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying

$$(\mathrm{H}\psi_1) \qquad \psi(0) = 0, \quad \psi(\xi) > 0 \textit{ for every } \xi > 0$$

and, for the weight term, we assume that

$$(\mathrm{H}w_1) \qquad w \in C(\overline{\Omega}) \textit{ is such that there exists } x_0 \in \Omega \textit{ with } w(x_0) > 0$$

as in [HK80] or, otherwise,

$$(\mathrm{H}w_2) \qquad w \in L^\infty(\Omega) \textit{ with } \big|\{x \in \Omega : w(x) > 0\}\big| > 0.$$

In these settings we deal with positive solutions of $(\mathscr{I}\mathscr{D}_{\lambda,N})$, for instance weak, strong or classical solutions (depending on the properties of w, ψ and the domain Ω) such that $u(x) > 0$ for every $x \in \Omega$. We carry out a parallel analysis when the nonlinear term satisfies

$$(\mathrm{H}\psi_2) \qquad \psi_0 := \lim_{\xi \to 0^+} \frac{\psi(\xi)}{\xi} > 0 = \psi_\infty := \lim_{\xi \to +\infty} \frac{\psi(\xi)}{\xi}$$

or the more general condition

$$(\mathrm{H}\psi_3) \qquad \psi_0 > 0 = \liminf_{\xi \to +\infty} \frac{\int_0^\xi \psi(s)\, ds}{\xi^2}.$$

Notice that, if we assume $(\mathrm{H}\psi_2)$, then the nonlinearity ψ becomes of Type 1. This is the case treated in Section 6.1 where we compare two classical results of positive solutions for sublinear elliptic Dirichlet problems, namely the theorem by Brezis and Oswald [BO86] and the one by Brown and Hess [BH90]. Furthermore, by focusing on related results of uniqueness, we will give answer at one of the question mark in Table 5.1. More precisely, we will restrict ourselves to the one-dimensional case and we will provide in Proposition 6.1.6 an example of multiplicity of positive solutions for a function ψ not concave even if the map $\xi \to \psi(\xi)/\xi$ is decreasing on the positive real line.

In Section 6.2 we mainly consider, as underlying assumption, (Hψ_3). As first step, we will show in Proposition 6.2.3 the existence of positive solutions for every λ sufficiently large in the one-dimensional setting with a constant weight, exploiting some time-mapping estimates achieved by Opial [Opi61]. As second step, in the frame of Rabinowitz's global bifurcation theorem, we will provide in Theorem 6.2.8 the existence of a bifurcation branch of positive solution pairs (λ, u) which is unbounded both in the λ and the u components. The conditions we found are, in some sense, optimal (cf. Remark 6.2.15). Our proof is inspired by some arguments developed in [HK80; Coe+12; OO06].

6.1 Remarks on uniqueness and multiplicity of positive solutions

Let us start by considering the following Dirichlet problem

$$\begin{cases} -\Delta u = q(x)\psi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1.1}$$

We can perform a first analysis of (6.1.1) exploiting the main result by Brezis and Oswald [BO86] that leads to the following theorem.

Theorem 6.1.1. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* (Hψ_1) *and let $q \in L^\infty(\Omega) \setminus \{0\}$. If ψ_0 and ψ_∞ are finite and*

$$\lambda_1^{\mathcal{D}}(-\Delta - q(x)\psi_0; \Omega) < 0 < \lambda_1^{\mathcal{D}}(-\Delta - q(x)\psi_\infty; \Omega) \tag{6.1.2}$$

holds, then there exists at least one positive solution u to (6.1.1) *with $u \in C_0^1(\overline{\Omega})$. Moreover, if $q(x) > 0$ for a.e. $x \in \Omega$ and $\xi \mapsto \psi(\xi)/\xi$ is decreasing on $\mathbb{R}_0^+$, then the positive solution is unique and condition* (6.1.2) *is necessary, too.*

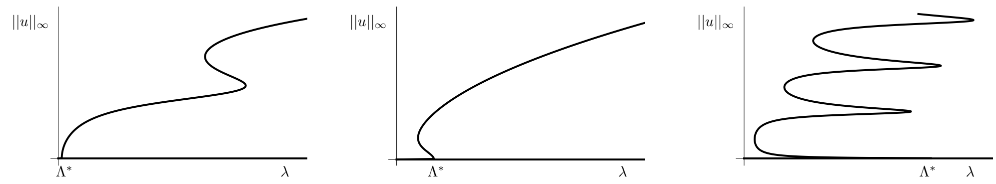
Proof. In order to enter the general setting of [BO86, Theorem 2], we consider

$$f(x,\xi) := q(x)\psi(\xi)$$

and, in such a situation, all the hypotheses are fulfilled. For completeness, we list here the assumptions required in [BO86, Theorem 2]: the map $\xi \mapsto f(x,\xi)$ is continuous on $\mathbb{R}^+$ for a.e. $x \in \Omega$; the map $x \mapsto f(x,\xi)$ belongs to $L^\infty(\Omega)$ for every $\xi \geq 0$; there exists a constant $C > 0$ such that $f(x,\xi) \leq C(\xi + 1)$ for a.e. $x \in \Omega$ and for every $\xi \geq 0$; for each $\delta > 0$ there exists a constant $C_\delta > 0$ such that $f(x,\xi) \geq -C_\delta$ for a.e. $x \in \Omega$ and for every $\xi \in [0,\delta]$ and, moreover,

$$\lambda_1^{\mathcal{D}}\left(-\Delta - \lim_{s\to 0^+} \frac{f(x,\xi)}{\xi}; \Omega\right) < 0 < \lambda_1^{\mathcal{D}}\left(-\Delta - \lim_{s\to +\infty} \frac{f(x,\xi)}{\xi}; \Omega\right).$$

The first two conditions are obviously satisfied. Also, the growth conditions are verified since, $\psi(\xi)/\xi$ is continuous and positive on $\mathbb{R}_0^+$ with finite ψ_0 and ψ_∞ and so, we can find a positive constant $K := \sup_{\xi>0}\{\psi(\xi)/\xi\} < \infty$ such that $|f(x,\xi)| \leq \|q\|_\infty K\xi$, for all $\xi \geq 0$ and for a.e. $x \in \Omega$. The last conditions clearly follows from (6.1.2). At this point, [BO86, Theorem 2] applies and ensures the existence of a non-trivial (weak) nonnegative solution u to problem (6.1.1). By elliptic regularity theory, such a solution belongs to $C_0^1(\overline{\Omega})$ and, moreover, is strictly positive on Ω with negative outward derivative on $\partial\Omega$ (cf. also [BO86, Lemma 1]). About the uniqueness of the positive solution, we just observe that the conditions $q(x) > 0$ on Ω and $\psi(\xi)/\xi$ decreasing on $\mathbb{R}_0^+$, imply that the map $\xi \mapsto f(x,\xi)/\xi$ is decreasing on $\mathbb{R}_0^+$. Therefore, as a result of [BO86, Theorem 1], the conclusion follows. □



(a) $\psi(\xi) = \frac{2\xi+6\sin(\xi)}{10+\xi^2-6\cos(\xi)}$ (b) $\psi(\xi) = \frac{2\xi+12\xi^3}{1+\xi^2+3\xi^4}$ (c) $\psi(\xi) = 10(1-\cos(\xi)) + \frac{\xi}{\sqrt{10(\xi+1)}}$

Figure 6.1: Bifurcation diagrams for one-dimensional Dirichlet (two-point boundary) problems of the form $u'' + \lambda\psi(u) = 0$, $u(0) = 0 = u(\pi)$.

A second analysis of (6.1.1) comes form the work by Brown and Hess [BH90]. A theorem on the existence and uniqueness of classical positive solutions for problem (6.1.1) holds by assuming, among other conditions, that ψ and q are smooth functions and ψ is concave with a sublinear growth at infinity, $\psi_\infty = 0$.

Now, a direct application of Theorem 6.1.1 to the Dirichlet boundary value problem $(\mathscr{ID}_{\lambda,N})$ yields the next results.

Corollary 6.1.2. *Let w satisfy* $(\mathrm{H}w_1)$ *and* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and* $(\mathrm{H}\psi_2)$*. Then, there exists* $\Lambda^* > 0$ *such that problem* $(\mathscr{ID}_{\lambda,N})$ *has a positive solution for each* $\lambda > \Lambda^*$.

Proof. We start by observing that the second inequality in (6.1.2) is trivially satisfied as it refers to the positivity of the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Therefore, we have only to check, for $\lambda > 0$ sufficiently large, the negativity of the first eigenvalue μ_1 of the problem

$$-\Delta u - \lambda\psi_0 w(x)u = \mu u, \quad u|_{\partial\Omega} = 0. \tag{6.1.3}$$

This fact is contained in [Hes82; HK80], here we present the argument of the proof for completeness. To this aim we recall some basic facts from the weighted eigenvalue problem

$$-\Delta u = \nu w(x)u, \quad u|_{\partial\Omega} = 0. \tag{6.1.4}$$

Under assumption $(\mathrm{H}w_1)$ (or, respectively, $(\mathrm{H}w_2)$), according to [BL80; Fig82; HK80; MM73], there exists a sequence of real eigenvalues

$$0 < \nu_1 < \nu_2 \leq \nu_3 \leq \dots$$

to problem (6.1.4), with $\nu_n \to \infty$. Moreover, the principal eigenvalue ν_1 is simple with an associated positive eigenfunction (see, for instance, the work [Fig82, Proposition 1.11 (c) and Theorem 1.13]). In such a situation, we can prove the thesis by taking

$$\Lambda^* := \nu_1/\psi_0\,. \tag{6.1.5}$$

Indeed, let us fix $\lambda > \Lambda^*$ and check that the principal eigenvalue μ_1 of (6.1.3) is negative. Let φ be the corresponding positive eigenfunction, so that φ satisfies

$$-\Delta\varphi(x) - \nu w(x)\varphi(x) = \mu_1\varphi(x) := h(x), \quad \varphi|_{\partial\Omega} = 0, \quad \varphi(x) > 0 \text{ for } x \in \Omega,$$

with $\nu := \lambda\psi_0 > \Lambda^*\psi_0 = \nu_1$. If, by contradiction, $\mu_1 \geq 0$, then $h \geq 0$ and we enter in the setting of [HK80, Proposition 3] which, in turns, implies that $h = 0$ and $\nu = \nu_1$. The last equality clearly contradicts our choice of λ. Hence, $\mu_1 < 0$ and also the first inequality in (6.1.2) is satisfied. By Theorem 6.1.1 we are done. □

Remark 6.1.3. Corollary 6.1.2 is basically a subcase of general results by Brown and Hess (see for instance [BH90, Theorem 3 (ii) and Theorem 4]). Actually, in [BH90] the authors obtain a result of existence and uniqueness of positive classical solutions if and only if

$\lambda > \nu_1/\psi_0 = \Lambda^*$, provided that w and ψ are smooth functions with $\psi''(\xi) < 0$ for all $\xi > 0$. However, in absence of concavity type condition, we cannot guarantee (in general) the uniqueness of the positive solution (see Figure 6.1 (a)) or the fact that positive solutions exist only if $\lambda > \nu_1/\psi_0$ (see Figure 6.1 (b)). Even more complex situations may arise (see Figure 6.1 (c)). $\triangleleft$

Corollary 6.1.4. *Let w satisfy* $(\mathrm{H}w_1)$ *and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and* $(\mathrm{H}\psi_2)$.

- *If $w(x) > 0$ for a.e. $x \in \Omega$ and $\xi \mapsto \psi(\xi)/\xi$ is decreasing on $\mathbb{R}^+_0$ then problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$ has a positive solution if and only if $\lambda > \nu_1/\psi_0$ and such a positive solution is unique* [BO86].
- *If $w(x)$ changes sign and Ω^+ is a set of positive measure and, moreover, $\psi(\xi)$ is smooth on $\mathbb{R}^+_0$ with $\psi''(\xi) < 0$ for all $\xi > 0$, then problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$ has a positive solution if and only if $\lambda > \nu_1/\psi_0$ and such a positive solution is unique* [BH90].

Proof. The first part of the statement follows from Theorem 6.1.1, with the condition $\lambda > \nu_1/\psi_0$ obtained in the same manner as (6.1.5) in the proof of Corollary 6.1.2. The second part of the statement is precisely [BH90, Theorem 4]. $\square$

If we restrict ourselves to the *autonomous case*, i.e. the case of a constant weight $w(x) \equiv 1$, problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$ reduces to the following one

$$\begin{cases} -\Delta u = \lambda\psi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.6}$$

where $\lambda > 0$. As already observed in [BO86, page 56], the next result holds.

Corollary 6.1.5. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and* $(\mathrm{H}\psi_2)$. *Then, problem* (6.1.6) *has a positive solution if*

$$\lambda > \lambda_1^* := \frac{\lambda_1^{\mathcal{D}}(-\Delta;\Omega)}{\psi_0}\,. \tag{6.1.7}$$

Moreover, if the map $\xi \mapsto \psi(\xi)/\xi$ is decreasing on $\mathbb{R}^+_0$ such positive solution is unique and (6.1.7) *is also a necessary condition.*

Notice that if $\psi(\xi)$ is any strictly concave function satisfying $(\mathrm{H}\psi_1)$, then the map $\xi \mapsto \psi(\xi)/\xi$ is decreasing on $\mathbb{R}^+_0$. The converse does not hold, a simple example is given by $\psi(\xi) = \xi/(1+\xi^2)$. In this respect, a natural question is whether the result of uniqueness under the monotonicity request for the map $\xi \mapsto \psi(\xi)/\xi$ is still true also if the weight coefficient is sign-changing. More precisely, when the weight function w is positive, the hypothesis of Brezis-Oswald, concerning the monotonicity of $\xi \mapsto \psi(\xi)/\xi$, is more general than the requirement of Brown-Hess about the concavity of ψ. On the other hand, the monotonicity of $\xi \mapsto \psi(\xi)/\xi$ is not enough to guarantee the uniqueness of positive solutions for an indefinite weight. Here we present an illustrative result in this direction, with the aid of some numerical computations. Furthermore, this example suit well also to deal with the same question in the case of Neumann boundary conditions.

Proposition 6.1.6. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be such that*

$$\psi(\xi) := A\xi e^{-B\xi^2} + \frac{\xi}{|\xi|+1}, \quad A, B > 0.$$

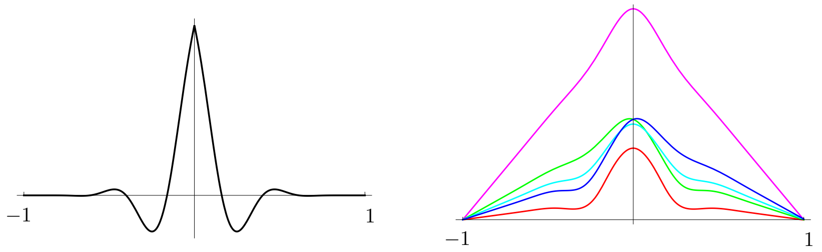
Assume $w\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$w(x) := (1-|x|)^5 \cos\left(\frac{9\pi}{2}|x|^{1.2}\right). \tag{6.1.8}$$

If $A = 10$, $B = 3$, then, for $\lambda = 80$, the following two-point BVP

$$u'' + \lambda w(x)\psi(u) = 0, \quad u(-1) = 0 = u(1), \tag{6.1.9}$$

has at least five solution u such that $u(x) > 0$ for all $x \in\,]-1,1[$.



(a) Graph of the function w as defined in (6.1.8) in the interval $\Omega =]-1,1[$.

(b) Five positive solutions of (6.1.9) for $\lambda = 80$.

Figure 6.2: Multiplicity of positive solutions for the two-point BVP as in Proposition 6.1.6

It is straightforward to check that $\psi_0 = A + 1$, $\psi_\infty = 0$, thus ψ satisfies (Hψ_1) and (Hψ_2). Moreover, the map $\xi \mapsto \psi(\xi)/\xi$ is strictly decreasing on $\mathbb{R}_0^+$; however, the function ψ is not concave. We show now the effect that our indefinite weight, whose graph is in Figure 6.2 (a), has in relation to the number of positive solutions. We observe that multiple positive solutions can be obtained also for different sign-changing weights.

In our case, we give numerical evidence of at least five positive solutions for the Dirichlet problem (6.1.9) on the domain $\Omega =]-1,1[$, via an adaptation of the shooting method already introduced in Section 3.1 at p. 40 to the case of Dirichlet boundary conditions. We start our analysis, for a fixed value of $\lambda = 80$, by shooting solutions from $x = -1$ with initial slope between $r_0 = 0.38$ and $r_1 = 10$. In more detail, for each $r \in [r_0, r_1]$, let $(u(\cdot; -1, 0, r), y(\cdot; -1, 0, r))$ be the solution of

$$\begin{cases} u' = y, \\ y' = -\lambda w(x)\psi(u), \end{cases} \tag{6.1.10}$$

satisfying the initial condition $u(-1) = 0$, $y(-1) = r$. Then, in the phase-plane $(u, y) = (u, u')$, we consider the arc

$$\Gamma := \{(u(1; -1, 0, r), y(1; -1, 0, r)) : r \in [r_0, r_1]\}$$

which is the image of the set $\{0\} \times [r_0, r_1]$ through the Poincaré map associated with (6.1.10). Then, we look for points $p \in \Gamma \cap \{(0, y) : y < 0\}$. The resulting curve Γ is shown in Figure 6.3 (a) where we have also put in evidence the five intersection points.

For each intersection point $p = (0, \rho) \in \Gamma \cap \{(0, y) : y < 0\}$, we then solve the initial value problem

$$u'' + \lambda w(x)\psi(u) = 0, \quad u(-1) = 0, \ u'(-1) = -\rho,$$

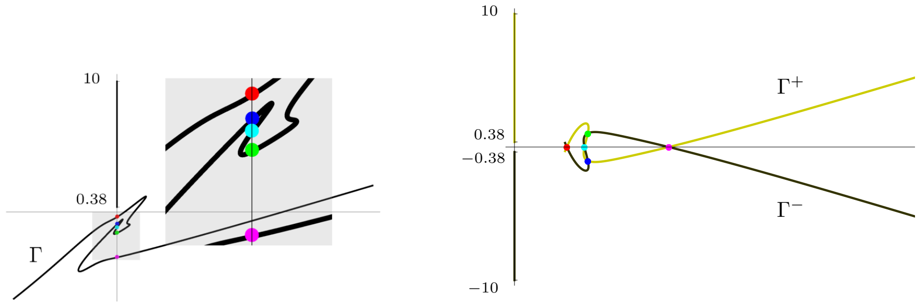
and finally we find a solution of the Dirichlet problem (6.1.9). The symmetry of the weight function, namely $w(-t) = w(t)$, guarantees that the solution $u(\cdot, -\rho)$ is a positive solution of problem (6.1.9) on $]-1,1[$. The corresponding five solutions are represented in Figure 6.2 (b). Notice that, three of these solutions are even functions, while the other two (called u_1 and u_2) are symmetric each other, that is $u_2(-t) = u_1(t)$.

Our example may have some interest also with respect to the result of Gidas, Ni and Nirenber [GNN79] on the symmetry of positive solutions. Notice that [GNN79, Theorem 1] does not applies because the function $[0,1] \ni \xi \mapsto \lambda w(\xi)\psi(u)$ is not decreasing.

Another point of view, in order to distinguish between symmetric and asymmetric solutions, is to consider the intersections points between the curves

$$\Gamma^+ := \{(u(0, r), y(0, r)) : r \in [r_0, r_1]\},$$
$$\Gamma^- := \{(u(0, r), -y(0, r)) : r \in [r_0, r_1]\}.$$

The curve Γ^- can be equivalently described as the locus of the points at the time $t = 0$, shooting back from the negative y-axis with slope $r \in [-r_1, -r_0]$ at the time $t = 1$. In



(a) Shooting of $\{0\} \times [0.38, 10]$ forward over $[-1, 1]$.

(b) Shooting of: $\{0\} \times [0.38, 10]$ forward over $[-1, 0]$; $\{0\} \times [-10, -0.38]$ backward over $[0, 1]$.

Figure 6.3: Phase plane (u, u'): dynamics of the Poincaré map associated with system (6.1.10) in the setting of Proposition 6.1.6.

this way the set of intersection points $p \in \Gamma^+ \cap \{(\alpha, 0) : \alpha > 0\} = \Gamma^- \cap \{(\alpha, 0) : \alpha > 0\}$ is in bijection with the even positive solutions, while the set of intersection points $q \in \Gamma^+ \cap \Gamma^- \setminus \{(\alpha, 0) : \alpha > 0\}$ correspond to the positive solutions symmetric to each other but not even. This point of view is illustrated in Figure 6.3 (b).

6.2 Revisiting the sublinear case

In this section we study $(\mathscr{I}\mathscr{D}_{\lambda,N})$ by assuming $(\mathrm{H}\psi_3)$. First investigations on bifurcation analysis are carried out in the case of ODEs with constant weight terms. The intent is to discuss bifurcation diagrams for positive solutions in term of the analysis of time-mappings.

6.2.1 Time-mapping estimates

Let us consider an open interval $\Omega :=]a, b[$, $a < b$, and reduce problem (6.1.6) to the two-point boundary value problem

$$\begin{cases} u'' + \lambda\psi(u) = 0, \\ u(a) = 0 = u(b), \end{cases} \tag{6.2.1}$$

with $\lambda > 0$. As usual in this case, we indicate by $x = t$ the independent variable.
The set of positive solutions pairs is given by

$$\mathcal{S} = \{(\lambda, u) \in \mathbb{R}^+_0 \times C^1_0([a, b]) : u \text{ is a positive solution of (6.2.1)}\}.$$

Without loss of generality (due to the autonomous nature of system (6.2.1)) we also set $L := b - a$ and observe that problem (6.2.1) is equivalent to

$$\begin{cases} u'' + \lambda\psi(u) = 0, \\ u(-L/2) = 0 = u(L/2). \end{cases}$$

In such a simplified setting, we can provide an interpretation of Corollary 6.1.5 in terms of time-mappings associated to planar autonomous system

$$u' = y, \quad y' = -g(u), \tag{6.2.2}$$

which is equivalent to the scalar equation

$$u'' + \psi(u) = 0. \tag{6.2.3}$$

Since up to now we have assumed $\psi(\xi)$ to be defined only for $\xi \geq 0$, for convenience we take an odd extension of ψ on $\mathbb{R}$ in order to have the solutions $(u(t), y(t))$ of (6.2.2) globally defined in the plane.

System (6.2.2) is conservative with energy

$$E(u,y) := \frac{1}{2}y^2 + P(u),$$

where

$$P(\xi) := \int_0^{\xi} \psi(s)\, ds.$$

Observe that the map $P\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $P(0) = 0$ and is strictly increasing. For every $c > 0$, the solution $(u(t), y(t))$ of (6.2.2) satisfying the initial condition $(u(0), y(0)) = (c, 0)$ is unique, periodic and defined on the whole real line. We denote such a solution with (u_c, y_c) only when we want to stress its dependence on the parameter c.

At this point, as already done in Section 2.2.1 at p. 24, we introduce the time-mapping formula associated with equation (6.2.3). In this framework, it is a continuous function $\tau\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

$$\tau(c) := 2\int_0^c \frac{d\xi}{\sqrt{2(P(c) - P(\xi))}}\,. \qquad (6.2.4)$$

In more detail, $\tau(c)$ is the distance of two consecutive zeros of the solution u of (6.2.3), where $u(t) \geq 0$ for all $t \in \mathbb{R}$ and $||u||_\infty = \max_{t\in\mathbb{R}} u(t) = c$. By a rescaling in the time variable, it is straightforward to check what follows.

Proposition 6.2.1. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and let $R > 0$ be a fixed constant. For each $c > 0$, let us define*

$$v_{c,R}(t) := u_c\Big(\tfrac{\tau(c)}{R}\big(t - \tfrac{a+b}{2}\big)\Big).$$

Then, $v_{c,R}(t)$ is a solution of the equation

$$v'' + \Big(\tfrac{\tau(c)}{R}\Big)^2 \psi(v) = 0$$

with

$$v\big(\tfrac{a+b}{2}\big) = c, \quad v'\big(\tfrac{a+b}{2}\big) = 0$$

and, moreover, the following cases occur:

- $v_{c,R}(t) > 0\ \forall\, t \in [a,b]$ *if and only if* $R > L$,
- $v_{c,R}(t) > 0\ \forall\, t \in\,]a,b[$ *with* $v_{c,R}(a) = 0 = v_{c,R}(b)$ *if and only if* $R = L$,
- $v_{c,R}(t)$ *vanishes in* $]a,b[$ *if and only if* $R < L$.

Considering the second instance in the above proposition, we get immediately what follows.

Proposition 6.2.2. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$. *Then, problem* (6.2.1) *has a positive solution u for some $\lambda > 0$ if and only if*

$$\lambda = \mathfrak{T}(c) := \left(\frac{\tau(c)}{L}\right)^2, \quad \textit{for}\ \ c = ||u_c||_\infty\,.$$

Moreover, the set $\mathcal{S}$ of positive solution pairs is the Cartesian graph of a continuous curve

$$c \mapsto (\mathfrak{T}(c), v_c(\cdot)),$$

where

$$v_c(t) := u_c\Big(\tfrac{\tau(c)}{L}\big(t - \tfrac{a+b}{2}\big)\Big).$$

Proposition 6.2.2 permits to study the global bifurcation branches for positive solutions of problem (6.2.1) by analyzing the behavior of the time-mapping $\tau(\cdot)$. This approach has been already widely exploited by many authors under several different conditions on the nonlinearity (see, for instance, the classical works [Lae70; Sch90; SW81]). The behavior of $\tau(c)$ as $c \to 0^+$ or $c \to +\infty$, as well as other qualitative properties, like monotonicity, has been analyzed by Opial in [Opi61]. In particular, according to [Opi61], if the limits ψ_0 and ψ_∞ exist, then

$$\lim_{c\to 0^+} \tau(c) = \frac{\pi}{\sqrt{\psi_0}} \quad \text{and} \quad \lim_{c\to +\infty} \tau(c) = \frac{\pi}{\sqrt{\psi_\infty}}\,.$$

Moreover, τ is increasing (respectively, decreasing) on $\mathbb{R}^+_0$ provided that $\xi \mapsto \psi(\xi)/\xi$ is decreasing (respectively, increasing) on $\mathbb{R}^+_0$.

If both ψ_0 and ψ_∞ are positive real numbers, then, by Proposition 6.2.2 we can recover a bifurcation result of Ambrosetti and Hess [AH80, Theorem A (iii)]. In fact, in this case, the set $\mathcal{S}$ turns out to be a Cartesian graph joining the bifurcation point $(\pi/L)^2/\psi_0$ from the trivial solution to the bifurcation point $(\pi/L)^2/\psi_\infty$ from infinity.

On the other hand, from (Hψ_2) we obtain

$$\lim_{c\to 0^+} \tau(c) = \frac{\pi}{\sqrt{\psi_0}} \quad \text{and} \quad \lim_{c\to +\infty} \tau(c) = +\infty.$$

Moreover, under the assumptions of Corollary 6.1.5, the map $\mathbb{R}^+_0 \ni c \mapsto \mathfrak{T}(c) \in \mathbb{R}^+_0$ is monotone with

$$\inf \mathfrak{T} = \left(\frac{\pi}{L}\right)^2 / \psi_0 = \lambda_1^* \quad \text{and} \quad \sup \mathfrak{T} = +\infty.$$

From this point of view, one could say that Opial's monotonicity condition for the time-mapping is a dynamical interpretation of the uniqueness condition of Brezis-Oswald.

The inversion of $\mathfrak{T}$ complements Corollary 6.1.5 with a global bifurcation result in the sense that it ensures also the continuity of the map

$$]\lambda_1^*, +\infty) \ni \lambda \mapsto u_\lambda(\cdot),$$

where u_λ is the unique positive solution of (6.1.6) for a given λ (compare with [Hes82; HK80]).

The time-mapping approach based on Proposition 6.2.2 suggests the possibility of improving condition (Hψ_2). More precisely, if we are looking for positive solution pairs (λ, u) of (6.2.1) for all λ in an unbounded interval, we can replace the hypothesis $\psi_\infty = 0$ with appropriate assumptions which yet ensure that $\sup \mathfrak{T} = +\infty$. For example, if we are interested in proving that $\lim_{c\to+\infty} \tau(c) = +\infty$, it will be sufficient to suppose that $P(\xi)/\xi^2 \to 0$ as $\xi \to +\infty$ (cf. [Opi61, Théorème 11]), which is a more general condition than $\psi_\infty = 0$. With this purpose, we introduce the following constants

$$P_\infty := \liminf_{\xi\to+\infty} \frac{2P(\xi)}{\xi^2}, \quad P^\infty := \limsup_{\xi\to+\infty} \frac{2P(\xi)}{\xi^2}\,.$$

By the generalized L'Hôpital's rule, we know that

$$\liminf_{\xi\to+\infty} \frac{\psi(\xi)}{\xi} \le P_\infty \le P^\infty \le \limsup_{\xi\to+\infty} \frac{\psi(\xi)}{\xi}\,.$$

Moreover, using [Opi61, Corollaire 11 and Théorème 16], we find that

$$P_\infty = 0 \Longrightarrow \limsup_{c\to+\infty} \mathfrak{T}(c) = +\infty \Longrightarrow \liminf_{\xi\to+\infty} \frac{\psi(\xi)}{\xi} = 0. \tag{6.2.5}$$

In this setting, we obtain the following result which improves Corollary 6.1.5 in the one-dimensional case.

Proposition 6.2.3. *Let* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and* $(\mathrm{H}\psi_3)$*. Then, the set* $\mathcal{S}$ *of positive solutions pairs* (λ, u) *to problem* (6.2.1) *is a continuous curve which bifurcates from* $(\lambda_1^*, 0)$ *and such that for each* $\lambda > \lambda_1^*$ *there exists at least one positive solution* u *of* (6.2.1) *with* $(\lambda, u) \in \mathcal{S}$*. Furthermore, if* $P^\infty > 0$*, then, for each*

$$\lambda > \eta_* := \lambda_1^{\mathcal{D}}(-\Delta)/P^\infty$$

there is an unbounded set of positive solutions u *of* (6.2.1) *with* $(\lambda, u) \in \mathcal{S}$*.*

Proof. From the first implication in (6.2.5) we know that assumption $(\mathrm{H}\psi_3)$ implies

$$\lim_{c\to 0^+} \mathfrak{T}(c) = \lambda_1^* \quad \text{and} \quad \limsup_{c\to+\infty} \mathfrak{T}(c) = +\infty.$$

Thus, the continuity of the map $\mathfrak{T}$ on $\mathbb{R}_0^+$ implies that the range of $\mathfrak{T}$ contains the interval $]\lambda_1^*, +\infty)$. Then the first part of the claim follows from Proposition 6.2.2. On the other hand, since P is monotone increasing, if we also suppose that $P^\infty > 0$, then necessarily $P(s) \to +\infty$ as $s \to +\infty$. In this manner, we enter in the setting of [Opi61, Corollaire 12] and so we have

$$\liminf_{c\to+\infty} \tau(c) \le \pi/\sqrt{P^\infty}.$$

Hence

$$\liminf_{c\to+\infty} \mathfrak{T}(c) \le \left(\frac{\pi}{L}\right)^2 /P^\infty = \eta_*.$$

We conclude that for each $\lambda \in]\eta_*, +\infty)$ the equation $\mathfrak{T}(c) = \lambda$ has infinitely many solutions. In fact,

$$\liminf_{c\to+\infty} \mathfrak{T}(c) < \lambda < \limsup_{c\to+\infty} \mathfrak{T}(c)$$

and, by the intermediate value theorem, there is a sequence $c_n \to +\infty$ of solutions of the equation $\mathfrak{T}(c) = \lambda$. To each such a solution $c_n > 0$ it corresponds a unique positive solution u_n of (6.2.1) with $||u||_\infty = c_n$. Then also the second part of the claim follows from Proposition 6.2.2. □

The consequence about the existence of infinitely many positive solutions is not related to the condition $\psi_0 > 0$ as it involves only the behavior of the time-mapping at infinity. In particular, infinitely many solutions can occur also when $P_\infty > 0$ as one can see in [FOZ89; MZ93; NZ89; OO06; OZ96]. In this context, the following result can be given for problem (6.2.1) using Opial's estimates, where by convention $1/0^+ = +\infty$ and $1/\infty = 0$.

Proposition 6.2.4. *Let* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and suppose also that*

$$0 \le P_\infty < P^\infty \le +\infty.$$

Then, for each

$$\lambda \in \left] \left(\frac{\pi}{L}\right)^2 /P^\infty, \left(\frac{\pi}{L}\right)^2 /P_\infty \right[$$

there is an unbounded set of positive solutions u *of* (6.2.1)*.*

Proof. We define

$$\eta_* := \left(\frac{\pi}{L}\right)^2 /P^\infty \quad \text{and} \quad \eta^* := \left(\frac{\pi}{L}\right)^2 /P_\infty\,.$$

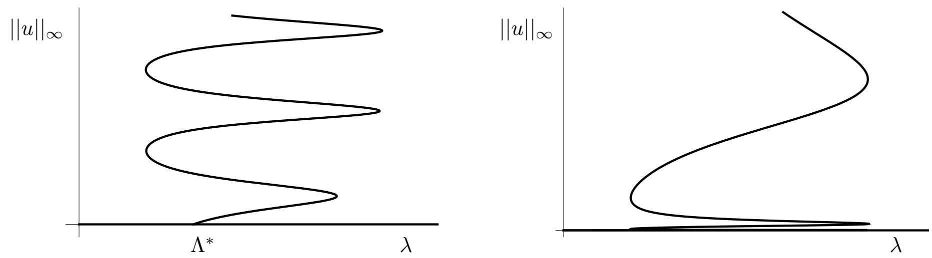
As in the preceding proof, we also note that $P(s) \to +\infty$ as $s \to +\infty$. From [Opi61, Corollaire 12] we find

$$\liminf_{c\to+\infty} \tau(c) \le \pi/\sqrt{P^\infty} < \pi/\sqrt{P_\infty} \le \liminf_{c\to+\infty} \tau(c).$$

Hence

$$\liminf_{c\to+\infty} \mathfrak{T}(c) \le \eta_* < \eta^* \le \limsup_{c\to+\infty} \mathfrak{T}(c).$$

By the intermediate value theorem, for each $\lambda \in]\eta_*, \eta^*[$ there is a sequence $c_n \to +\infty$ of solutions of the equation $\mathfrak{T}(c) = \lambda$ and the proof follows as above. □



(a) $P(\xi) = 2\xi^2 + \xi^2 \cos(2\log(1+\xi))$. (b) $P(\xi) = 2\xi^2 + \xi^2 \cos(2\log(\frac{\xi}{1+\xi}))$.

Figure 6.4: Bifurcation diagrams in logarithmic scale for a two-point boundary problem associated with $u'' + \lambda P'(u) = 0$.

Example 6.2.5. We exhibit a class of nonlinearities consistent with Proposition 6.2.4. Let k, θ, A, B be given constants with $k, A > 0$, $\theta \in [0, 2\pi[$ and

$$|B| < \frac{2A}{\sqrt{k^2+4}}. \tag{6.2.6}$$

Define, for every $\xi \geq 0$,

$$P(\xi) := A\xi^2 + B\xi^2 \cos(k \log(1+\xi) + \theta) \quad \text{and} \quad \psi(\xi) := P'(\xi).$$

Then $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is of class C^∞ and, by (6.2.6), one can easily check that $(\mathrm{H}\psi_1)$ holds. Moreover, $P_\infty = 2(A-B) < 2(A+B) = P^\infty$ and $\psi_0 = 2(A + B\cos\theta) > 0$. $\lhd$

For completeness, let us consider also the case when the conditions at zero and at infinity are interchanged, namely

$$\rho_* := \left(\frac{\pi}{L}\right)^2 / P^0 \quad \text{and} \quad \rho^* := \left(\frac{\pi}{L}\right)^2 / P_0$$

with

$$P_0 := \liminf_{\xi \to 0^+} \frac{2P(\xi)}{\xi^2}, \quad P^0 := \limsup_{\xi \to 0^+} \frac{2P(\xi)}{\xi^2}.$$

Example 6.2.6. We exhibit a class of nonlinearities such that $P_0 < P^0$. Let k, θ, A, B be given constants with $k, A > 0$, $\theta \in [0, 2\pi[$ and B as in (6.2.6). Define, for every $\xi > 0$,

$$P(\xi) := A\xi^2 + B\xi^2 \cos(k \log(\tfrac{\xi}{\xi+1}) + \theta), \quad \psi(\xi) := P'(\xi) \;\; \text{and} \;\; \psi(0) = 0.$$

Then $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is of class C^∞ satisfying $(\mathrm{H}\psi_1)$ and such that $P_0 = 2(A-B) < 2(A+B) = P^0$ and $\psi_\infty = 2(A + B\cos\theta) > 0$. $\lhd$

All the possible combinations of conditions on lower and upper limits for the potential P yield the following result (the proof is omitted since it follows from analogous arguments of that used to prove Proposition 6.2.4).

Proposition 6.2.7. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying $(\mathrm{H}\psi_1)$ and let $\mathcal{S}$ be the set of positive solutions pairs for (6.2.1). Then, the following statements hold.*

- *If $P^0 > P_\infty$, then for each $\lambda \in]\rho_*, \eta^*[$ there exists at least one positive solution u of (6.2.1) with $(\lambda, u) \in \mathcal{S}$.*
- *If $P_0 < P^\infty$, then for each $\lambda \in]\eta_*, \rho^*[$ there exists at least one positive solution u of (6.2.1) with $(\lambda, u) \in \mathcal{S}$.*
- *If $P_0 < P^0$, then for each $\lambda \in]\rho_*, \rho^*[$ there is a sequence of positive solutions u of (6.2.1) which converges uniformly to zero.*

- *If $P_\infty < P^\infty$, then for each $\lambda \in]\eta_*, \eta^*[$ there is a sequence of positive solutions $u_{\lambda,n}$ of* (6.2.1) *with $||u_{\lambda,n}||_\infty \to +\infty$ for $n \to \infty$.*

In Figure 6.4 we represent the bifurcation diagrams for two particular potentials P which belongs to Example 6.2.5 and Example 6.2.6, respectively. We stress that the set of positive solutions pairs is a Cartesian graph bounded in the λ-component and unbounded in the u-component, as expected from Proposition 6.2.4 and Proposition 6.2.7. Moreover, we notice that it is not difficult to combine Example 6.2.5 and Example 6.2.6 in order to produce a class of functions such that both $P^0 > P_0$ and $P^\infty > P_\infty$ are valid.

These analyses allows us to use the time-mapping estimates to show in the next section some possible improvements of Corollary 6.1.2 for the original Dirichlet problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$.

6.2.2 Bifurcation branches

Let us consider a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary of class $C^{1,1}$. Let X be the Banach space $C^1_0(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ with its standard norm, where by $C^1(\overline{\Omega})$ we mean the set of functions $u \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ with the partial derivatives having continuous extension on $\overline{\Omega}$. We denote by

$$\mathcal{P}_X := \{u \in X : u(x) \geq 0, \forall x \in \Omega\}$$

the positive cone in X. For technical reason we suppose also what follows.

(Hw_3) *There exist an open set $\Omega_1 \subset \Omega$ and $\eta > 0$ such that $w(x) \geq \eta$ for a.e. $x \in \Omega_1$.*

Condition (Hw_3) is always satisfied when (Hw_1) holds. Although it is slightly more restrictive than (Hw_2), nevertheless it is a key hypothesis for the study of indefinite problems (see [LG13, Ch. 9]).

Our goal is still the generalization of Proposition 6.2.3 to problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$. In view of the presence of the parameter λ in the differential equation, it seems natural to enter in a bifurcation setting, in order to obtain both the existence of solutions for each λ in a certain range and the existence of a continuum of solution pairs with the desired properties. With this respect, we prove the following result.

Theorem 6.2.8. *Let $w \in L^\infty(\Omega)$ satisfy* (Hw_3) *and $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* (Hψ_1) *and such that ψ_0 is finite. Then, the following conclusions hold:*

i) If $\psi_0 > 0$, there exists an unbounded continuum $\mathcal{C} \subset \mathbb{R}^+_0 \times X$ containing $(\Lambda^*, 0)$ and such that $\mathcal{C} \setminus \{(\Lambda^*, 0)\}$ is made of positive solution pairs (λ, u) to problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$.

ii) If $\psi_0 > 0$ and, moreover, $P_\infty = 0$, then for each $\lambda > \Lambda^*$ there is at least one positive solution u with $(\lambda, u) \in \mathcal{C}$.

iii) If $\psi_0 > 0$ and also $P^\infty > P_\infty = 0$, then there exists M^* such that for each $\lambda > M^*$ there is an unbounded set of positive solutions.

Proof of i). The first part closely follows the schemes proposed in [HK80, Theorem 2] and [Coe+12, Theorem 2.2] which involve the global bifurcation theorem of Rabinowitz [Rab71, Theorem 1.3]. In [HK80] the theory was developed for a continuous weight function, but it can be suitably adapted to cover the case in (Hw_3).

First of all, we extend ψ by oddness to the whole real line (such extension will be still denoted by ψ). We fix a constant $p > N$ and consider the Nemytskii operator F associated to $f(x, u) := w(x)\psi(u)$, namely

$$F : X \to L^\infty(\Omega) \hookrightarrow L^p(\Omega), \quad u(\cdot) \mapsto f(\cdot, u(\cdot)).$$

For each $v \in L^p(\Omega)$ the Dirichlet problem $-\Delta u = v(x)$ in Ω, with $u \in W^{1,p}_0(\Omega)$, has a unique solution in $W^{2,p}(\Omega)$. Since $p > N$, this latter space is compactly embedded in $C^{1,\beta}(\overline{\Omega})$ for $0 \leq \beta < 1 - (N/p)$. Moreover, in this setting, $u \in W^{1,p}_0(\Omega)$ implies that $u \in C(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$.

We denote by

$$\mathfrak{L}^{-1} : L^p(\Omega) \to W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \hookrightarrow C_0^1(\overline{\Omega})$$

the inverse of the Laplacian operator which associates to each $v \in L^p(\Omega)$ the solution $u \in C_0^1(\overline{\Omega})$ of $-\Delta u = v(x)$ in Ω, with $u = 0$ on $\partial\Omega$. This way, problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$ can be settled like a fixed point problem in the space X, as follows

$$u = \lambda K u, \tag{6.2.7}$$

where $K : X \to X$ is the completely continuous operator defined as

$$K(u) := \mathfrak{L}^{-1} F(u).$$

Following [LG13], we define $C^{1,1^-}(\overline{\Omega}) := \bigcap_{0<\theta<1} C^{1,\theta}(\overline{\Omega})$. Observe that a solution of (6.2.7) belongs to $C_0^1(\overline{\Omega}) \cap C^{1,1^-}(\overline{\Omega})$ and is twice classically differentiable almost everywhere in Ω (see [GT83] and [LG13, Theorem 5.8] to justify our assertions on regularity results).

The existence of a finite ψ_0 allows us to express the nonlinearity f as

$$f(x,\xi) = \psi_0 w(x)\xi + w(x)\gamma(\xi), \quad \text{with } \gamma(\xi)/\xi \to 0 \text{ as } \xi \to 0.$$

Therefore K admits a linearization at $u = 0$ of the form $\mathfrak{L}^{-1}A$, where A is the multiplication operator induced by the function $\psi_0 w(\cdot)$. We denote by $\mathcal{W}$ the closure in $\mathbb{R} \times X$ of the set of non-trivial solution pairs (λ, u) of (6.2.7).

Let Λ^* be defined as in (6.1.5). According to Hess-Kato [HK80], the point $(\Lambda^*, 0)$ is a bifurcation point of the nonlinear problem (6.2.7). An application in this setting of Rabinowitz's global bifurcation theorem [Rab71; Rab73a] ensures that the set $\mathcal{W}$ contains a maximal subcontinuum $\mathcal{F}$ such that $\mathcal{F} \ni (\Lambda^*, 0)$ and $\mathcal{F}$ is either unbounded or contains a point $(\hat{\lambda}, 0)$ where $\hat{\lambda}$ is a characteristic value of $\mathfrak{L}^{-1}A$ with $\hat{\lambda} \neq \Lambda^*$. On account of the fact that $(0,0)$ is not a bifurcation point, $\mathcal{F}$ is connected and $\Lambda^* > 0$, we firstly observe that

$$\mathcal{F} \subset \mathbb{R}_0^+ \times X.$$

We are going now to prove that $\mathcal{F}$ contains an unbounded sub-continuum $\mathcal{C}$ starting from $(\Lambda^*, 0)$, satisfying $\mathcal{C} \setminus \big(\mathbb{R} \times \{0\}\big) \subset \mathbb{R}_0^+ \times \mathrm{int}\mathcal{P}_X$, which does not contain any point $(\hat{\lambda}, 0)$ with $\hat{\lambda} \neq \Lambda^*$. To this aim, we will show that

$$\mathcal{F} \setminus \big(\mathbb{R} \times \{0\}\big) \subset \big(\mathbb{R}_0^+ \times \mathrm{int}\mathcal{P}_X\big) \cup \big(\mathbb{R}_0^+ \times -\mathrm{int}\mathcal{P}_X\big) \tag{6.2.8}$$

and

$$\mathcal{F} \cap \big(\mathbb{R} \times \{0\}\big) = \{(\Lambda^*, 0)\}. \tag{6.2.9}$$

In fact, condition (6.2.9) implies that the second alternative of Rabinowitz bifurcation theorem does not occur and therefore $\mathcal{F}$ is unbounded. Then, the continuum we are looking for can be defined as

$$\mathcal{C} := \{(\lambda, |u|) : (\lambda, u) \in \mathcal{F}\} \subset \big(\mathbb{R}_0^+ \times \mathrm{int}\mathcal{P}_X\big) \cup \{(\Lambda^*, 0)\}. \tag{6.2.10}$$

In this manner, assertion *i)* follows because it is obvious that $\mathcal{C}$ is a closed connected unbounded set of solution pairs to (6.2.7) which contains $(\Lambda^*, 0)$ and moreover for each $(\lambda, u) \in \mathcal{C} \setminus \{(\Lambda^*, 0)\}$ we have $\lambda > 0$ and $u > 0$.

Our task is now to check conditions in (6.2.8) and (6.2.9). To do this we divide the proof into some steps.

Step I. *There is a neighborhood U of $(\Lambda^*, 0)$ such that*

$$U \cap \mathcal{F} \subset \big(\mathbb{R}_0^+ \times \mathrm{int}\mathcal{P}_X\big) \cup \big(\mathbb{R}_0^+ \times -\mathrm{int}\mathcal{P}_X\big) \cup \{(\Lambda^*, 0)\}.$$

Indeed, if by contradiction there is no neighborhood U of $(\Lambda^*, 0)$ as above, then one could find a sequence (λ_n, u_n) of solutions to (6.2.7) with $\lambda_n \to \Lambda^*$ and $u_n \notin -\mathrm{int}\mathcal{P}_X \cup \mathrm{int}\mathcal{P}_X$, such that $0 < ||u_n|| \to 0$. Normalizing, we have

$$v_n = \lambda_n \frac{K(||u_n||v_n)}{||u_n||}, \quad \text{where } v_n := \frac{u_n}{||u_n||}.$$

By compactness, we can assume that $v_n \to v$ (up to a subsequence). Moreover,

$$v \notin -\mathrm{int}\mathcal{P}_X \cup \mathrm{int}\mathcal{P}_X\,.$$

Using the linearization of K at zero we obtain

$$v = \Lambda^* \mathfrak{L}^{-1} A v, \qquad \text{with } \;||v|| = 1.$$

This means that v is an eigenfunction corresponding to the positive principal eigenvalue Λ^* and therefore

$$v \in -\mathrm{int}\mathcal{P}_X \cup \mathrm{int}\mathcal{P}_X\,.$$

A contradiction is thus achieved. For what follows, notice that $U \cap \big(\mathcal{F} \setminus \big(\mathbb{R} \times \{0\}\big)\big)$ is nonempty.

Step II. *It holds that*

$$\mathcal{F} \cap \big(\mathbb{R}_0^+ \times \big(-\partial\mathcal{P}_X \cup \partial\mathcal{P}_X\big)\big) = \{(\Lambda^*, 0)\}.$$

Suppose that $(\zeta, u_0) \in \mathcal{F} \cap \big(\mathbb{R}_0^+ \times \big(-\partial\mathcal{P}_X \cup \partial\mathcal{P}_X\big)\big)$. The odd extension of ψ implies that also the operator K is odd. Therefore, when u is a solution of (6.2.7), $-u$ is a solution, too. Accordingly, without loss of generality, we can suppose that $(\zeta, u_0) \in \mathcal{F} \cap \big(\mathbb{R}_0^+ \times \partial\mathcal{P}_X\big)$.

We claim that $u_0 \equiv 0$. If, by contradiction, $u_0 \not\equiv 0$, then u_0 is a non-trivial nonnegative solution to the problem

$$-\Delta u = \zeta w(x)\psi(u), \quad u|_{\partial\Omega} = 0,$$

which is equivalent to

$$-\Delta u + cu = (c + \zeta w(x)\varphi(u))u, \quad u|_{\partial\Omega} = 0,$$

where we have introduced the auxiliary continuous function

$$\varphi(\xi) := \begin{cases} \psi(\xi)/\xi & \text{for } \;\xi \neq 0, \\ \psi_0 & \text{for } \;\xi = 0. \end{cases} \tag{6.2.11}$$

Now, if we take

$$c \geq \zeta ||w||_\infty \sup_{0 \leq s \leq ||u||_\infty} \varphi(s),$$

we obtain that

$$-\Delta u_0(x) + cu_0(x) \geq 0, \quad u_0|_{\partial\Omega} = 0,$$

with $u_0(x) \geq 0$ for all $x \in \Omega$ and $u_0 \not\equiv 0$. By the strong maximum principle $u_0 \in \mathrm{int}\mathcal{P}_X$ follows and this leads to a contradiction.

Since $u_0 \equiv 0$, now we have $(\zeta, 0) \in \mathcal{F} \cap \big(\mathbb{R}_0^+ \times \partial\mathcal{P}_X\big)$. So that, there exists a sequence (λ_n, u_n) of solutions to (6.2.7) with $\lambda_n \to \zeta > 0$ and $u_n \in \mathrm{int}\mathcal{P}_X$ such that $0 < ||u_n|| \to 0$. Normalizing as in Step I and passing up to a subsequence for $v_n := u_n/||u_n||$, we obtain

$$v = \zeta \mathfrak{L}^{-1} A v, \qquad \text{with } \;||v|| = 1 \text{ and } \;v > 0.$$

This means that v is a positive eigenfunction associated with the eigenvalue $\zeta > 0$. Therefore $\zeta = \Lambda^*$, as there is a unique positive eigenvalue having a positive eigenfunction.

Step III. *It holds that*

$$\mathcal{F} \subset \big(\mathbb{R}_0^+ \times \mathrm{int}\mathcal{P}_X\big) \cup \big(\mathbb{R}_0^+ \times -\mathrm{int}\mathcal{P}_X\big) \cup \{(\Lambda^*, 0)\}. \tag{6.2.12}$$

Indeed, let us consider the set

$$\mathcal{F}' := \{(\lambda, u) \in \mathcal{F} : \lambda > 0, \pm u \in \mathrm{int}\mathcal{P}_X\} \cup \{(\Lambda^*, 0)\}.$$

By Step I, the set $\mathcal{F}'$ is open relatively to $\mathcal{F}$. We claim that $\mathcal{F}'$ is closed in $\mathcal{F}$. To do this, we consider a sequence $(\lambda_n, u_n) \to (\zeta, u)$ with $\zeta > 0$ and $u_n \in \mathrm{int}\mathcal{P}_X \cup -\mathrm{int}\mathcal{P}_X$. If

$u \in \text{int}\mathcal{P}_X \cup -\text{int}\mathcal{P}_X$, we are done. Otherwise, if $u \in -\partial\mathcal{P}_X \cup \partial\mathcal{P}_X$, from Step II we have $(\zeta, u) = (\Lambda^*, 0)$. The claim is thus proved. The connectedness of $\mathcal{F}$ implies that $\mathcal{F}' = \mathcal{F}$ and (6.2.12) is verified.

Finally, the proof of *i*) is concluded because (6.2.8) and (6.2.9) directly follow from (6.2.12). □

Proof of ii). Having already built up the continuum $\mathcal{C}$, we will prove that it is unbounded in the λ-component if $P_\infty = 0$. To this aim, we introduce the projection

$$p_1 : \mathbb{R} \times X \to \mathbb{R}, \quad (\lambda, u) \mapsto \lambda$$

and we show that $p_1(\mathcal{C}) \supset [\Lambda^*, +\infty)$.

Suppose, by contradiction, that the inclusion does not hold. So that there exists $\hat{\lambda} > \Lambda^*$ such that $\lambda < \hat{\lambda}$ for each $(\lambda, u) \in \mathcal{C}$.

Let a_1, b_1 be such that $\Omega \subset]a_1, b_1[\times \mathbb{R}^{N-1}$. By hypothesis $P_\infty = 0$ follows that $\tau^\infty = +\infty$, where

$$\tau^\infty := \limsup_{c \to +\infty} \tau(c).$$

Let us fix a constant $R > b_1 - a_1$ and let $d > 0$ be such that

$$\tau(d)^2 > R^2 \hat{\lambda} ||w||_\infty.$$

According to Proposition 6.2.1 the function $v_{d,R}(t)$ is a solution of

$$v'' + \left(\tfrac{\tau(d)}{R}\right)^2 \psi(v) = 0$$

such that $v_{d,R}(t) > 0$ for all $t \in [a_1, b_1]$. Finally, from $v_{d,R}$ we define a function on $\mathbb{R}^N$ as

$$\beta(x) := v_{d,R}(x_1), \quad \forall\, x = (x_1, \ldots, x_N) \in \overline{\Omega}.$$

By construction, for each $\lambda \in]0, \hat{\lambda}[$, the function $\beta(x)$ is an upper solution which is not a solution for problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$. Indeed, there exists a constant $\rho > 0$ such that

$$-\Delta\beta(x) \geq \hat{\lambda} ||w||_\infty \psi(\beta(x)) + \rho, \quad \forall\, x \in \Omega \tag{6.2.13}$$

and, moreover,

$$\inf_{x \in \overline{\Omega}} \beta(x) = \eta > 0. \tag{6.2.14}$$

Now, we claim that

$$u(x) < \beta(x), \forall\, x \in \Omega, \tag{6.2.15}$$

for each positive solution u such that $(\lambda, u) \in \mathcal{C}$. To prove this inequality we follow an argument close to the one in [OO06, Step 4] (for another possible proof, but involving a locally Lipschitz condition, we refer to [Gám97, Theorem 2.2]).

Let us consider the set

$$\mathcal{C}' := \{(\lambda, u) \in \mathcal{C} : u(x) < \beta(x), \forall\, x \in \Omega\},$$

which is nonempty and open relatively to $\mathcal{C}$. In order to prove (6.2.15) we will show that $\mathcal{C}'$ is also closed relatively to $\mathcal{C}$, so we can conclude by the connectedness of $\mathcal{C}$.

Let $U(x) \leq \beta(x)$, for all $x \in \Omega$, be a solution of $(\mathscr{I}\mathscr{D}_{\lambda,N})$ for some λ such that $(\lambda, U) \in \mathcal{C}$. We notice that $U(x) < \beta(x), \forall\, x \in \partial\Omega$. We are going to prove that $U(x) < \beta(x)$, $\forall\, x \in \Omega$. Let us fix $\varepsilon > 0$ such that

$$4\varepsilon\hat{\lambda}||w||_\infty < \rho.$$

By the uniform continuity of $\psi(\xi)$ on the interval $[0, ||\beta||_\infty]$, there exists $\delta > 0$ such that $|\psi(\xi') - \psi(\xi'')| < \varepsilon$ for each $\xi', \xi'' \in [0, ||\beta||_\infty]$ with $|\xi' - \xi''| < \delta$. If there exists a point $x_0 \in \Omega$ such that $U(x_0) = \beta(x_0)$, then we can take a (small) open ball $B(x_0, r) \subset \Omega$ such that $|U(x) - U(x_0)| < \delta$ and $|\beta(x) - \beta(x_0)| < \delta$ for all $x \in B[x_0, r]$. As a consequence, we have

$$|\psi(\beta(x)) - \psi(U(x))| < 2\varepsilon, \quad \forall\, x \in B[x_0, r].$$

A comparison between (6.2.13) and $-\Delta U(x) = \lambda w(x)\psi(U(x))$ for a.e. $x \in B(x_0, r)$ (for $0 < \lambda < \hat{\lambda}$) shows that the function $Q(x) := \beta(x) - U(x)$ satisfies $-\Delta Q(x) \geq \rho/2$ for a.e. $x \in B(x_0, r)$ with $Q \geq 0$ on $\partial B(x_0, r)$ and $Q(x_0) = 0$. This contradicts the strong maximum principle on the ball $B(x_0, r)$ (cf. [HO96, Lemma 3.2 (interior form)]). Therefore we conclude that $\mathcal{C}'$ is closed relatively to $\mathcal{C}$.

Therefore, from (6.2.15) we have that

$$\mathcal{C} \subset]0, \hat{\lambda}[\, \times [0, \beta(\cdot)].$$

Hence, $\mathcal{C}$ is bounded in the product space and this contradicts the alternatives of Rabinowitz's global bifurcation theorem. Assertion *ii*) is thus proved. □

Proof of iii). For the latter assertion, concerning the case $P^\infty > P_\infty = 0$, we rely to [OO06, Theorem 2.2] applied to the problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2.16}$$

where

$$f(x, \xi) := \begin{cases} \lambda w(x)\psi(\xi) & \text{if } \xi \geq 0, \\ 0 & \text{if } \xi < 0. \end{cases}$$

With this respect, we observe that $f(x, \xi) \geq \lambda\eta\psi(\xi)$ for every $\xi \geq 0$ and a.e. $x \in \Omega_1$ and, moreover, $f(x, \xi) \leq h(\xi) := \lambda||w||_\infty\psi(\xi)$ for every $\xi \geq 0$ and a.e. $x \in \Omega$. By our special form of $f(x, \xi)$ (which, in particular, implies $f(x, 0) \equiv 0$), one can see that the assumptions (h_3) and $\psi(\xi) \to +\infty$ as $\xi \to +\infty$ required in [OO06, Theorem 2.2] can be ignored. The condition $P_\infty = 0$ implies $\liminf_{\xi \to +\infty}(\int_0^\xi h(s)\, ds)/\xi^2 = 0$ as in (h_5) of [OO06, Theorem 2.2] and thus the existence of a sequence of upper solutions β_n tending to infinity uniformly in $\overline{\Omega}$ is guaranteed. On the other hand, given $\rho_N = N^N/(N-1)^{(N-1)}$ for $N \geq 2$ otherwise $\rho_1 = 1$ and let $R > 0$ be the radius of the largest ball contained in Ω_1, according to [OO06, Remark 1] if

$$\lambda > M^* := \frac{\rho_N}{\eta P^\infty}\left(\frac{\pi}{2R}\right)^2,$$

then there exists a sequence of lower solutions α_n with $\max(\alpha_n) = \max_{\Omega_1}(\alpha_n)$ tending to infinity. The rest of the proof is similar to [OO06, Theorem 2.2]. It leads to the existence of an unbounded sequence of solutions u_n for (6.2.16) and the strong maximum principle (cf. [HO96, Lemma 3.2 (global form)]) guarantees that $u_n(x) > 0$ for all $x \in \Omega$. □

The construction of an upper solution using conditions on the lower limit at infinity of $P(\xi)/\xi^2$ has been already exploited in [FGZ91; OO06; OZ96].

One could argue that functions satisfying $(\mathrm{H}\psi_3)$ and not $(\mathrm{H}\psi_2)$ seem really artificial. Our opinion is that such kind of functions may look slightly unusual but not too weird. One can easily provide examples of functions in the class $(\mathrm{H}\psi_1)$ which satisfy

$$0 = \liminf_{\xi \to +\infty} \frac{2P(\xi)}{\xi^2} < \limsup_{\xi \to +\infty} \frac{2P(\xi)}{\xi^2}\,.$$

This can be done in different manners. For example, by selecting an increasing sequence of positive reals $(a_n)_n$ such that

$$\lim_{n \to +\infty} n^{-2} a_{2n} = \ell \in]0, +\infty] \quad \text{and} \quad \lim_{n \to +\infty} n^{-2} a_{2n+1} = 0.$$

Then $P(\xi)$ can be constructed as a smooth function satisfying $P(0) = P'(0) = 0 < P''(0)$, $P'(\xi) > 0$ for all $\xi > 0$ and such that its graph interpolates the points (n, a_n). This procedure, even if it permits to define functions satisfying our requests, still may look somehow artificial. For this reason, we show below how to define in an analytical manner suitable maps satisfying $(\mathrm{H}\psi_1)$ and $(\mathrm{H}\psi_3)$ by the use of elementary functions. Such nonlinearities are obtained by a modification of the ones considered in Example 6.2.5.

Example 6.2.9. Let $\rho, \theta, A, k_1, k_2, p, q$, be positive constants, with $\theta \in [0, 2\pi]$, $A \geq e$, and $0 < q < 1 - p < 1$. Define, for every $s \geq 0$,

$$P(\xi) := \rho\xi^2\Big(1 + \cos\big(k_1 \log^p(A + \xi) + \theta\big) + k_2 \log^{-q}(A + \xi)\Big).$$

If

$$k_1 p + k_2 q < 2k_2\,,$$

then $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$, defined as $\psi(\xi) := P'(\xi)$, is a function of class C^∞ satisfying $(\mathrm{H}\psi_1)$. Moreover,

$$P_\infty = 0 < 4\rho = P^\infty \quad \text{and} \quad \psi_0 \in]0, +\infty[.$$

Indeed, to check that $\psi(\xi) > 0$ for all $\xi > 0$, we just observe that

$$P'(\xi) \geq \rho\xi\Big(2k_2 D^{-q} - k_1 p D^{p-1} - k_2 q D^{-q-1}\Big), \quad \text{for } D := \log(A + \xi) > 1.$$

All the other verifications are straightforward. ◁

Under our assumptions, it is natural to ask wether there are further properties of the Rabinowitz's bifurcation continuum $\mathcal{C}$. Indeed, the following result holds.

Proposition 6.2.10. *Let $w \in L^\infty(\Omega)$ and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and such that ψ_0 is finite. If $\psi_0 > 0$, then the continuum $\mathcal{C}$ defined in* (6.2.10) *is unbounded in the u-component.*

Proof. Let $\mathcal{C}$ be the continuum obtained in point *i*) of Theorem 6.2.8. Suppose, by contradiction, that there exists $M > 0$ such that

$$||u|| \leq M \quad \text{for all } (\lambda, u) \in \mathcal{C} \subset \mathbb{R}^+_0 \times X. \tag{6.2.17}$$

This, in turn, implies that $0 < u(x) \leq M$ for all $x \in \Omega$. Then, as a consequence of $(\mathrm{H}\psi_1)$ and $\psi_0 > 0$, we find that $\psi(u(x)) \geq C_M u(x)$ for all $x \in \overline{\Omega}$, for

$$C_M := \inf_{0<s\leq M} \frac{\psi(s)}{s} > 0.$$

In other words, for φ defined as in (6.2.11), we have that $\varphi(u(x)) \geq C_M$ for every $(\lambda, u) \in \mathcal{C}$ and problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$ can be written as

$$\begin{cases} -\Delta u = \lambda w(x)\varphi(u)u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2.18}$$

Now, let $z \in \Omega_1$ and $r > 0$ be such that the open ball $B = B(z, r)$ satisfies $\overline{B} \subset \Omega^+$ and, moreover, let $\rho_1 > 0$ be the first (positive) eigenvalue of the eigenvalue problem with positive weight

$$-\Delta u = \rho w(x)u, \quad u|_{\partial B} = 0.$$

We denote by ν the associated positive eigenfunction with $\max_B \nu(x) = 1$.

We fix a constant $\hat{\lambda} > \rho_1/C_M$ such that there exists a (positive) solution $\hat{u}$ of (6.2.18) with $(\hat{\lambda}, \hat{u}) \in \mathcal{C}$. We know that such a pair always exists because $\mathcal{C}$ is unbounded in the product space and we are assuming (6.2.17). Let $v(x) = \vartheta\nu(x)$ (with $\vartheta > 0$) be the maximal eigenfunction of

$$-\Delta u = \rho_1 w(x)u, \quad u|_{\partial B} = 0$$

such that $v(x) \leq \hat{u}(x)$, $\forall\, x \in B$. By definition, we have $0 = v(x) < \hat{u}(x)$ on ∂B and $v(x_0) = \hat{u}(x_0)$ for some $x_0 \in B$. The function $Q(x) := \hat{u}(x) - v(x)$ satisfies $-\Delta Q(x) > 0$ for a.e. $x \in B$ with $Q(x) > 0$ on ∂B and $\min_B Q(x) = Q(x_0) = 0$, thus contradicting the maximum principle. Therefore, our assertion is proved. □

As a consequence of Theorem 6.2.8 and Proposition 6.2.10, we can say that Proposition 6.2.3 for the one-dimensional case is now extended to any sufficiently regular domain in $\mathbb{R}^N$. In particular, also Corollary 6.1.5 extends as follows (where the constant $\lambda_1^* := \lambda_1^{\mathcal{D};\Omega}(-\Delta)/\psi_0$ is the one defined in (6.1.7)).

Corollary 6.2.11. *Let $w \in L^\infty(\Omega)$ satisfy $(\mathrm{H}w_3)$ and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying $(\mathrm{H}\psi_1)$ and $(\mathrm{H}\psi_3)$. Then, there exists a continuum $\mathcal{C}$ containing $(\lambda_1^*, 0)$ and such that $\mathcal{C} \setminus \{(\lambda_1^*, 0)\}$ is made of positive solution pairs (λ, u) to problem (6.1.6). The continuum $\mathcal{C}$ is unbounded both in the u-component and the λ-component.*

Moreover, if the map $\xi \mapsto \psi(\xi)/\xi$ is decreasing on $\mathbb{R}_0^+$ since the conditions $(\mathrm{H}\psi_2)$ and $(\mathrm{H}\psi_3)$ are equivalent, then the set of positive solution pairs $\mathcal{S}$ coincides with $\mathcal{C} \setminus \{(\lambda_1^*, 0)\}$ and is the graph of a continuous map $]\lambda_1^*, +\infty[\ni \lambda \mapsto u_\lambda \in \mathrm{int}\mathcal{P}_X$.

From the proof of Theorem 6.2.8 it is also clear that a more general version of Theorem 6.2.8 can be given as follows.

Theorem 6.2.12. *Let $w \in L^\infty(\Omega)$ satisfying $(\mathrm{H}w_3)$ and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying $(\mathrm{H}\psi_1)$ and such that ψ_0 is finite. Then, the following conclusions hold:*

- *If $\psi_0 > 0$, there exists an unbounded continuum $\mathcal{C} \subset \mathbb{R}_0^+ \times X$ containing $(\Lambda^*, 0)$ and such that $\mathcal{C} \setminus \{(\Lambda^*, 0)\}$ is made of positive solution pairs (λ, u) to problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$. The continuum $\mathcal{C}$ is always unbounded in the u-component.*
- *If $\psi_0 > 0$ and, moreover, $\tau^\infty = +\infty$, then the continuum $\mathcal{C}$ is also unbounded in the λ-component and, therefore, for each $\lambda > \Lambda^*$ there is at least one positive solution u with $(\lambda, u) \in \mathcal{C}$.*

The method of producing bounds for a PDEs using the ODE $u'' + \psi(u) = 0$ has been also considered in [OO06] and [Kaj09]. Sufficient conditions for the validity of the time-mapping hypothesis have been presented in previous papers (see, for instance [FZ92]).

Theorem 6.2.12 is useful to produce other existence results where explicit hypotheses on $\psi(\xi)$ or $P(\xi)$ at infinity can be employed in order to obtain $\tau^\infty = +\infty$. From [DIZ91], one could require that ψ is such that

$$\liminf_{\xi \to +\infty} \psi(\xi)/\xi = 0 \text{ and } \xi\psi'(\xi) \le M\psi(\xi) \quad \text{for } \xi > d, \tag{6.2.19}$$

for some positive constant M. This hypothesis, according to Omari and Ye [OY92], is said to be a “desultorily sublinear condition”. For the PDE setting, it has been recently used for the Neumann problem in [Sfe12]. Condition (6.2.19) is independent on $P_\infty = 0$ as shown in an example of [DIZ91].

Finally, we notice that the assumption $\liminf_{\xi \to +\infty} \psi(\xi)/\xi = 0$ alone is not enough to guarantee the existence of positive solutions to problem $(\mathscr{I}\mathscr{D}_{\lambda,N})$ for $\lambda \ge \Lambda^* = \nu_1/k$. Indeed, we are able to provide a counterexample at least for a constant weight and in one-dimension case. Namely the following results holds.

Proposition 6.2.13. *Let $\Omega \subset \mathbb{R}$ be a bounded open interval of length $|\Omega| = L$. For each positive constant k, there exists a continuous function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(\mathrm{H}\psi_1)$, with*

$$\psi_0 = k \quad \textit{and} \quad \liminf_{\xi \to +\infty} \psi(\xi)/\xi = 0, \tag{6.2.20}$$

such that there is no positive solution pair for (6.2.1) when $\lambda \ge \lambda_1^* = (\frac{\pi}{L})^2/k$. The function ψ can be defined so that

$$\lim_{\xi \to +\infty} 2P(\xi)/\xi^2 = \limsup_{\xi \to +\infty} \psi(\xi)/\xi = K, \tag{6.2.21}$$

for any prescribed value $K \in]k, +\infty]$.

Proof. Our example is inspired by some analogous considerations in [DIZ93; Njo91], however the proof here is completely different, since we adopt a time-mapping technique.

We discuss in detail the situation when K is a real number. The case $K = +\infty$ it can be treated in the same way with simple modifications.

We start by giving the general structure of the example. Let k, K be two given constants with $0 < k < K$. We consider a strictly increasing continuous function $q_1 : \mathbb{R}^+ \to \mathbb{R}^+_0$ with $q_1(0) = k$ and $\lim_{\xi\to+\infty} q_1(\xi) = K$. Then, let τ_1 be the time-mapping associated to the autonomous scalar equation

$$u'' + \psi_1(u) = 0, \quad \text{for } \ \psi_1(s) := s\, q_1(s).$$

As usual, we set

$$P_1(\xi) := \int_0^\xi \psi_1(s)\, ds.$$

By the properties recalled in Section 6.2.1, we know that $\tau_1 : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is a strictly decreasing function with

$$\lim_{c\to 0^+} \tau_1(c) = \frac{\pi}{\sqrt{k}} \quad \text{and} \quad \lim_{c\to+\infty} \tau_1(c) = \frac{\pi}{\sqrt{K}}\,.$$

Next, we consider a strictly monotone increasing function $\psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi_2(+\infty) = +\infty$ and such that

$$\lim_{\xi\to+\infty} \frac{\psi_2(\xi)}{\xi} = 0.$$

By the properties of ψ_1 and ψ_2 and since $\psi_1(\xi)/\xi \to K > 0$ as $\xi \to +\infty$, there exists a constant $d > 0$ such that

$$0 < \psi_2(\xi) < \psi_1(\xi), \quad \forall\, \xi \geq d.$$

Let $\varepsilon > 0$ be a fixed constant such that

$$3\varepsilon < \frac{\pi}{\sqrt{k}} - \frac{\pi}{\sqrt{K}} \tag{6.2.22}$$

and, subsequently, let us fix a constant $\theta \in]0, 1[$ such that

$$\sqrt{\theta} \geq \Big(\frac{\pi}{\sqrt{K}} + \varepsilon\Big) \Big/ \Big(\frac{\pi}{\sqrt{k}} - \varepsilon\Big). \tag{6.2.23}$$

At this moment, we can determine a constant $d^* \geq d$ such that

$$\psi_1(\xi) > \frac{1}{1-\theta}, \tag{6.2.24a}$$

$$\tau_1(\xi) < \frac{\pi}{\sqrt{K}} + \varepsilon, \tag{6.2.24b}$$

$$\sqrt{8/\psi_2(\xi)} < \varepsilon, \tag{6.2.24c}$$

hold for all $\xi \geq d^*$.

Finally, we take two sequences $(d_n)_n$ and $(r_n)_n$ of positive real numbers with $d_n \nearrow +\infty$ and $r_n \searrow 0^+$ and $d_1 - r_1 > d^* + 2$.
We also define $I_n := [d_n - r_n, d_n + r_n]$. The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ of our example will be defined as

$$\psi(\xi) := \psi_1(\xi) - \varphi(\xi),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with

$$\varphi(\xi) = 0, \ \forall\, \xi \in \mathbb{R}^+ \setminus \big(\bigcup_{n=1}^{\infty} I_n\big);$$

$$\max_{\xi\in I_n} \varphi(\xi) = \varphi(d_n) := \psi_1(d_n) - \psi_2(d_n).$$

If we denote by, $Q(\xi) := \int_0^\xi \varphi(s)\,ds$, we also impose $\lim_{\xi\to+\infty} Q(\xi) \le 1$. We notice that

$$\psi_1(\xi) \ge \psi(\xi) \ge \psi_2(\xi), \quad \forall\, \xi \ge d^* \ge d.$$

Moreover, $\psi(\xi) = \psi_1(\xi)$ for all $0 \le \xi \le d^* + 2$ and $\psi(d_n) = \psi_2(d_n)$. By definition of ψ, we have also that $P(\xi) = P_1(\xi) - Q(\xi)$. Hence (6.2.20) and (6.2.21) follow immediately.
If we denote by τ the time-mapping associated to $u'' + \psi(u) = 0$, from the definition of ψ it is easy to check that

$$\lim_{c\to+\infty} \tau(c) = \lim_{c\to+\infty} \tau_1(c) = \frac{\pi}{\sqrt{K}}\,.$$

However, we want to prove more. Indeed, we claim that

$$\tau(c) < \frac{\pi}{\sqrt{k}} = \frac{\pi}{\sqrt{\psi_0}}\,, \quad \forall\, c > 0. \tag{6.2.25}$$

By construction, we have that $\tau(c) = \tau_1(c) < \pi/\sqrt{k}$, for all $c \in]0, d^* + 2]$. So, we consider now $c > d^* + 2$ and prove that $\tau(c) < \pi/\sqrt{k}$.

In fact, recalling the time-mapping formula given in (6.2.4) and using the fact that $c - 1 > d^*$, we have

$$\begin{aligned}
\tau(c) &= 2\int_0^{c-1} \frac{d\xi}{\sqrt{2(P(c)-P(\xi))}} + 2\int_{c-1}^{c} \frac{d\xi}{\sqrt{2(P(c)-P(\xi))}}\\
&= 2\int_0^{c-1} \frac{d\xi}{\sqrt{2(P_1(c)-P_1(\xi)-(Q(c)-Q(\xi)))}} + 2\int_{c-1}^{c} \frac{d\xi}{\sqrt{2(\int_s^c \psi(s)\,ds)}}\\
&\le 2\int_0^{c-1} \frac{d\xi}{\sqrt{2(P_1(c)-P_1(\xi)-1)}} + 2\int_{c-1}^{c} \frac{d\xi}{\sqrt{2(\int_s^c \psi_2(s)\,ds)}}\\
&\le \underbrace{\frac{2}{\sqrt{\theta}}\int_0^{c-1} \frac{d\xi}{\sqrt{2(P_1(c)-P_1(\xi))}}}_{\substack{P_1(c)-P_1(\xi)-1 \ge \theta(P_1(c)-P_1(\xi)),\\ \text{by condition (6.2.24a)}}} + 2\int_{c-1}^{c} \frac{d\xi}{\sqrt{2(\int_s^c \psi_2(c-1)\,ds)}}\\
&< \frac{2}{\sqrt{\theta}}\int_0^{c} \frac{d\xi}{\sqrt{2(P_1(c)-P_1(\xi))}} + \sqrt{\frac{2}{\psi_2(c-1)}}\int_{c-1}^{c} \frac{d\xi}{\sqrt{c-\xi}}\\
&= \frac{\tau_1(c)}{\sqrt{\theta}} + \sqrt{\frac{8}{\psi_2(c-1)}} < \frac{1}{\sqrt{\theta}}\underbrace{\left(\frac{\pi}{\sqrt{K}} + \varepsilon\right)}_{\text{by (6.2.24b)}} + \underbrace{\varepsilon}_{\text{by (6.2.24c)}}\\
&\le \underbrace{\frac{\pi}{\sqrt{k}} - \varepsilon}_{\text{by (6.2.23)}} + \varepsilon = \frac{\pi}{\sqrt{k}}\,.
\end{aligned}$$

We have thus verified (6.2.25), so that by Proposition 6.2.2 we know that a positive solution to (6.2.1) can exist only for $\lambda < \lambda_1^*$. In other words, with our choice of the function ψ, there is no positive solution pair for problem (6.2.1) when $\lambda \ge \lambda_1^*$. □

Following the instructions given in the proof, it is easy now to provide a concrete function ψ.

Example 6.2.14. As a model example, let us consider the following functions:

$$q_1(\xi) = \begin{cases} k + \frac{2(K-k)}{\pi}\arctan(\xi) & \text{for } K < +\infty,\\ k + \xi\arctan(\xi) & \text{for } K = +\infty,\end{cases}$$

and

$$\psi_2(\xi) = \sqrt{\xi}.$$

The parameters involved in the construction can be explicitly computed once k and K are given. For instance, let us take $k=1$ and $K=25$. In this case, we can choose $d=1$. Next, we fix $\varepsilon=\pi/4$ and $\theta=9/25$, in order to satisfy (6.2.22) and (6.2.23). With such a choice of the constants, simple computations show that $d^*=170$ is more than adequate to have all the three conditions in (6.2.24) fulfilled. At this point, for any positive integer n, we take

$$d_n = 180+n \quad \text{and} \quad r_n = \frac{2^{-n}}{25 d_n}\,.$$

We define the function $\varphi(s)$ as a piecewise linear function, namely

$$\varphi(\xi) = \begin{cases} \psi_1(d_n)-\psi_2(d_n)-\frac{\psi_1(d_n)-\psi_2(d_n)}{r_n}|\xi-d_n| & \text{for } \xi\in I_n, \\ 0 & \text{for } \xi\notin I_n\,. \end{cases}$$

As a last step, we observe that

$$\int_0^{+\infty}\varphi(\xi)\,d\xi = \sum_{n=1}^{\infty} r_n\varphi(d_n) < \sum_{n=1}^{\infty} r_n\psi_1(d_n) < \sum_{n=1}^{\infty} K r_n d_n = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Therefore, all the required conditions are satisfied. ◁

Remark 6.2.15. The function ψ, whose existence is asserted in Proposition 6.2.13, can be more than continuous. Indeed, it can be smooth as we like (it is just a matter of choosing q_1,ψ_2 and φ smooth functions). In particular, in Example 6.2.14 we can easily modify the choice of φ, taking a piecewise polynomial function instead of a piecewise linear function. Hence, when K is finite and ψ is $C^1(\mathbb{R}^+)$, we have $\psi(\xi)/\xi$ bounded but $\sup_{\xi>0}\psi'(\xi)=+\infty$. In this way our example shows that the second condition in (6.2.19) cannot be avoided. ◁

6.2.3 Applications with more general differential operators

Our main results concerning the existence of unbounded connected branches of positive solution pairs (namely Theorem 6.2.8 and Theorem 6.2.12) involve a nonlinear Dirichlet problem for the Laplace differential operator. We briefly sketch how to obtain the same kind of results for problem

$$\begin{cases} \mathfrak{L}u = \lambda w(x)\psi(u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{6.2.26}$$

where $\mathfrak{L}$ is a more general linear differential operator of second order of the form

$$\mathfrak{L} := -\sum_{j,k=1}^{N}\alpha_{jk}(x)D_jD_k + \sum_{j=1}^{N}\alpha_j(x)D_j + \alpha_0(x).$$

To obtain the statement $i)$ in Theorem 6.2.8 for this operator, we suppose that

$$\alpha_{jk}=\alpha_{kj}\in C(\overline{\Omega}) \quad \text{and} \quad \alpha_j,\alpha_0\in L^\infty(\Omega), \quad \text{with } \alpha_0\geq 0.$$

Moreover, we also assume that $\mathfrak{L}$ is strictly elliptic in Ω, indeed there exists a constant $\kappa>0$ such that $\sum_{j,k}\alpha_{jk}(x)\xi_j\xi_k \geq \kappa||\xi||^2$ for all $x\in\Omega$ and $\xi=(\xi_1,\ldots,\xi_N)\in\mathbb{R}^N$. Taking into account these assumptions and following [HK80], we can reproduce the same proof.

In order to obtain the statement $ii)$ in Theorem 6.2.8 we have to prove the existence of an upper solution β satisfying a condition analogous to (6.2.13). To this purpose, we first give the following lemma which is presented in a general form so that it can be applied in principle also in other contexts. We note also that our lemma presents some overlapping with a preceding result by Grossinho and Omari in [GO97, Lemma 2.1].

Lemma 6.2.16. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$ *and such that $\tau^\infty = +\infty$. Let $I := [t_0, t_1]$ and $B, M > 0$ be fixed real constants. Then for every measurable function $b : I \to \mathbb{R}$ with $|b(t)| \leq B$ for a.e. $t \in I$ and for every constant $K > 0$, there exists $k > K$, such that any solution $u(\cdot)$ of the initial value problem*

$$\begin{cases} u'' + b(t)u' + M\psi(u) = 0, \\ u(t_0) = k, \ u'(t_0) = 0, \end{cases} \tag{6.2.27}$$

is such that $u(t) > 0$ for all $t \in I$ and $u'(t) < 0$ for all $t \in \,]t_0, t_1]$.

Proof. Let $u(\cdot) : J \to \mathbb{R}^+$ be a solution of (6.2.27) defined on a right maximal interval of existence contained in I. For a.e. $t \in J$ we have that

$$\frac{d}{dt}\left(u'(t)e^{\mathcal{B}(t)}\right) + Me^{\mathcal{B}(t)}\psi(u(t)) = 0, \tag{6.2.28}$$

where we have set $\mathcal{B}(t) := \int_{t_0}^t b(\xi)\, d\xi$. Integrating on $[t_0, t]$, for $t \in J$ with $t > t_0$, it follows that

$$u'(t) = -M \int_{t_0}^t e^{\int_t^s b(\xi)\, d\xi}\psi(u(s))\, ds$$

holds. This proves that $u'(t) < 0$ for all $t > t_0$ with $t \in J$.

We claim now that $J = I$ and $u(t) > 0$ for all $t \in I$. Suppose, by contradiction, that there exist a function $b : I \to \mathbb{R}$ satisfying $|b(t)| \leq B$ and a first point $t^* \in J$ such that $u(t^*) = 0$. We multiply equation (6.2.28) by $u'(t)e^{\mathcal{B}(t)}$ and so we obtain the relation

$$\frac{1}{2}\frac{d}{dt}\left(u'(t)e^{\mathcal{B}(t)}\right)^2 + Me^{2\mathcal{B}(t)}\frac{d}{dt}P(u(t)) = 0, \quad \forall\, t \in [t_0, t^*]. \tag{6.2.29}$$

Notice that $\frac{d}{dt}P(u(t)) = \psi(u(t))u'(t) < 0$ for all $t \in \,]t_0, t^*[$. Integrating equation (6.2.29) on $[t_0, t] \subset [t_0, t^*[$ and after simple manipulations, we obtain

$$\begin{aligned} |u'(t)|^2 &= 2M \int_{t_0}^t e^{2\int_t^s b(\xi)\, d\xi}\frac{d}{ds}(-P(u(s)))\, ds \\ &\leq 2Me^{2B|I|}(P(u(t_0)) - P(u(t))) = 2Me^{2B|I|}(P(k) - P(u(t))). \end{aligned}$$

Then, recalling that $u'(t) < 0$ on $]t_0, t^*]$, it follows that

$$-u'(t) \leq e^{B|I|}M^{\frac{1}{2}}\sqrt{2(P(k) - P(u(t)))}, \quad \forall\, t \in \,]t_0, t^*[\,.$$

From the previous inequality we have

$$\int_{u(t)}^{u(t_0)=k} \frac{ds}{\sqrt{2(P(k) - P(u(s)))}} \leq e^{B|I|}M^{\frac{1}{2}}(t - t_0), \quad \forall\, t \in \,]t_0, t^*[$$

and then, letting $t \to t^*$, we find

$$\frac{\tau(k)}{2} = \int_0^k \frac{ds}{\sqrt{2(P(k) - P(u(s)))}} \leq e^{B|I|}M^{\frac{1}{2}}(t^* - t_0) \leq M^{\frac{1}{2}}e^{B|I|}|I|.$$

Thus, using the fact that $\limsup_{c \to +\infty} \tau(c) = +\infty$, a contradiction is achieved. As a consequence, we conclude that $J = I$ and, moreover, $u(t) > 0$ for all $t \in I$. □

By Lemma 6.2.16 we give an upper solution β as in the proof of *ii)* in Theorem 6.2.8. Following the same notation, let a_1 and b_1 be such that $\Omega \subset \,]a_1, b_1[\times\mathbb{R}^{N-1}$. We proceed, by introducing the constants:

$$M_0 > \hat{\lambda}||w||_\infty$$

and

$$M := \frac{M_0}{\kappa}, \quad b := \sup_{x \in \Omega}\left|\frac{\alpha_1(x)}{\alpha_{11}(x)}\right| \leq \frac{||\alpha_1||_\infty}{\kappa}.$$

Then, according to Lemma 6.2.16, let $u \in C^2([a_1, b_1])$ be such that

$$\begin{aligned}
&u''(t) - bu'(t) + M\psi(u(t)) = 0, \quad \forall\, t \in [a_1, b_1],\\
&u(t) > 0, \quad \forall\, t \in [a_1, b_1]\\
&u'(t) < 0, \quad \forall\, t \in]a_1, b_1].
\end{aligned}$$

We define

$$\beta(x) := u(x_1), \quad \forall\, x = (x_1, \ldots, x_N) \in \overline{\Omega}.$$

By the positivity of u on $[a_1, b_1]$ we have that (6.2.14) holds for a suitable constant η.

The choice of $\beta(x)$ implies that

$$\begin{aligned}
\mathfrak{L}\beta(x) &= -\sum_{j,k=1}^{N} \alpha_{jk}(x) D_j D_k \beta(x) + \sum_{j=1}^{N} \alpha_j(x) D_j \beta(x) + \alpha_0(x)\beta(x)\\
&= -\alpha_{11}(x)u''(x_1) + \alpha_1(x)u'(x_1) + \alpha_0(x)u(x_1)\\
&\geq \alpha_{11}(x)\Big(- u''(x_1) + \underbrace{\frac{\alpha_1(x)}{\alpha_{11}(x)} u'(x_1)}_{\text{using } u' < 0} \Big)\\
&\geq \alpha_{11}(x)(-u''(x_1) + bu'(x_1))\\
&= \alpha_{11}(x) M\psi(u(x_1)) \geq \kappa M\psi(u(x_1)) = M_0\psi(u(x_1))\\
&= \hat{\lambda}||w||_\infty \psi(u(x_1)) + (M - \hat{\lambda}||w||_\infty)\psi(u(x_1))\\
&\geq \hat{\lambda}||w||_\infty \psi(\beta(x)) + \rho,
\end{aligned}$$

where ρ is a suitable positive constant such that $(M - \hat{\lambda}||w||_\infty)\psi(u(t)) \geq \rho$ for all $t \in [a_1, b_1]$. Thus (6.2.13) is proved for $\mathfrak{L}$ instead of $-\Delta$. The rest of the proof of *ii*) follows in the same manner and we achieve the same conclusion of Theorem 6.2.8 and Theorem 6.2.12 also for problem (6.2.26).

7. Nonlinearities with oscillatory potential at infinity

In the present chapter, which is based on [SZ17c], we study indefinite nonlinear Neumann problems that hold the characteristic to have the primitive of the nonlinearity with an oscillatory behavior. Nonlinearities of this type have been already introduced in Chapter 6 for Dirichlet BVPs. Here we pursue the investigations with the goal to provide multiplicity results of positive solutions with respect to the case of Neumann boundary conditions. More precisely, we deal with Neumann problems of the form

$$(\mathscr{I}\mathscr{N}) \qquad \begin{cases} u'' + w(t)\psi(u) = 0, \\ u'(0) = u'(T) = 0, \end{cases}$$

where $w\colon [0,T] \to \mathbb{R}$ is a sign-changing function and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function still in the frame of Type 1, namely we suppose it satisfies

(Hψ_1) $\psi(0) = 0$ *and* $\psi(\xi) > 0$ *for all* $\xi > 0$.

In addition to that, we consider also the typifying condition of the problem which is about a *oscillatory behavior for the primitive* of ψ. Denoting by $P(\xi) := \int_0^\xi \psi(s)\,ds$, we assume that

(Hψ_4) $P_\infty := \liminf_{\xi\to+\infty} \dfrac{2P(\xi)}{\xi^2} = 0 < P^\infty := \limsup_{\xi\to+\infty} \dfrac{2P(\xi)}{\xi^2}$.

The analysis carried out on the study of positive solutions for problem $(\mathscr{I}\mathscr{N})$, i.e. a solution $u(t)$ of $(\mathscr{I}\mathscr{N})$ such that $u(t) > 0$ for all $t \in [0,T]$, leads to a result of multiplicity as stated in Theorem 7.1.1. In Section 7.1.2, we will prove this result by means of a shooting-type argument that takes advantage of some technical estimates derived in Section 7.1.1. At last, we will extend in Section 7.2 our approach to the study of positive solutions of problem

$$(\mathscr{I}\mathscr{N}_N) \qquad \begin{cases} \Delta u + w(x)\psi(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a radially symmetric domain (for instance either a ball or an annulus) and w has radial symmetry.

Remark 7.1. Let us first point out a key comment on the assumption made on the weight term. If we look for positive solutions for problem $(\mathscr{I}\mathscr{N}_N)$ under condition (Hψ_1) and

$w \not\equiv 0$, then the assumption $\psi(\xi) > 0$ for every $\xi > 0$ implies a necessary condition: the function w has to change its sign on Ω (see [BPT88]). In fact, by integrating the differential equation in $(\mathscr{I}\mathscr{N}_N)$ over Ω, we obtain

$$0 = \int_\Omega \Delta u + w(x)\psi(u)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\,dx + \int_\Omega w(x)\psi(u)\,dx = \int_\Omega w(x)\psi(u)\,dx.$$

Hence, the involvement of an indefinite weight term comes very natural in this framework when we assume Neumann boundary conditions. ◁

Overview on differential problems with oscillatory potential

Just to motivate the forthcoming results, we stress the fact that in literature there are a great deal of works on BVPs associated to PDEs with ψ satisfying $(\mathrm{H}\psi_4)$ when the boundary conditions are of Dirichlet type. On the contrary, for Neumann BVPs, there are lot of multiplicity results for problems having super-linear or sub-linear nonlinearities, but the case of ψ satisfying $(\mathrm{H}\psi_1)$ and $(\mathrm{H}\psi_4)$ looks still not completely explored, even in the case of one-dimension that is the main topic of this chapter.

Let us justify this preface. Interest, in Dirichlet BVP with an oscillatory potential can be traced back to 1930 with a classical paper of Hammerstein [Ham30]. In that work, the Author proved the existence of solutions to a nonlinear integral equation (nowadays called "Hammerstein equation") of the form

$$\varphi(x) = \int_B K(x,y) f(y,\varphi(y))\,dy,$$

under a linear growth assumption on the function f defined on $B \times \mathbb{R}$ and a non-resonance condition. In our context, such non-resonance condition can be equivalently written as

$$\limsup_{u\to\pm\infty} \frac{2F(x,u)}{u^2} < \lambda_1, \text{ uniformly for } x \in B,$$

where $F(x,u) := \int_0^u f(x,\xi)\,d\xi$, B is a one-dimensional or multi-dimensional bounded domain, $K(x,y)$ is a bounded symmetric and positive definite kernel and λ_1 is the first eigenvalue of the associated linear problem.

The pioneering work of Hammerstein stimulated further researches about the solvability of nonlinear boundary value problems "below the first eigenvalue", by imposing conditions on the primitive of the nonlinearity (see [FG88; GO92; GO95; MWW86]). Applications to the Dirichlet problem, involving these kind of conditions, guarantee the existence of at least one solution for

$$(\mathscr{D}_N) \qquad \begin{cases} \Delta u + \psi(u) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if $h \in L^\infty(\Omega)$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function with a suitable polynomial growth (depending on the Sobolev embeddings) such that

$$\limsup_{\xi\to\pm\infty} \frac{2P(\xi)}{\xi^2} < \lambda_1^{\mathcal{D}}(-\Delta;\Omega),$$

where $\Omega \subseteq \mathbb{R}^N$ is assumed to be a bounded domain with a sufficiently smooth boundary.

In the one-dimensional case $\Omega =]0,T[$, an improvement of this result was obtained in [FOZ89, Theorem 1], by replacing the Hammerstein type condition with

$$\liminf_{s\to\pm\infty} \frac{2P(\xi)}{\xi^2} < \left(\frac{\pi}{T}\right)^2 = \lambda_1^{\mathcal{D}}(-\Delta;\Omega).$$

Moreover, in that paper, the study of the one-dimensional Dirichlet BVP, under the assumptions $\psi(\xi) \to +\infty$ for $\xi \to +\infty$ and

$$P_\infty < \left(\frac{\pi}{T}\right)^2 < P^\infty,$$

leads to the existence of infinitely many solutions $u(t) > 0$ for all $t \in]0,T[$ (see [FOZ89, Theorem 3]). Concerning the multiplicity of positive solutions for Dirichlet problems, further investigations have been performed from different points of view, considering also in [MZ93; NOZ00; OZ96] more general (nonlinear) differential operators.

For an indefinite weight nonlinear problem on a general bounded domain $\Omega \subseteq \mathbb{R}^N$

$$(\mathscr{I}\mathscr{D}_N) \qquad \begin{cases} \Delta u + w(x)\psi(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

an application of [OO06, Theorem 2.2] yields the existence of two sequences of solutions $(u_n)_n$ and $(v_n)_n$ which are strictly positive on Ω and such that $\lim_{n\to+\infty} u_n(x)/\mathrm{dist}(x,\partial\Omega) = \lim_{n\to+\infty} v_n(x)/\mathrm{dist}(x,\partial\Omega) = +\infty$.

On the contrary, dealing with Neumann boundary conditions, the treatment of these kind of problems presents some peculiar features that it is useful to highlight. For the following Neumann problem

$$(\mathscr{N}_N) \qquad \begin{cases} \Delta u + \psi(u) = h(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

the Hammerstein non-resonance condition with respect to the first eigenvalue, namely $\lambda_1^{\mathcal{N}}(-\Delta;\Omega) = 0$, becomes $\limsup_{\xi\to\pm\infty} 2P(\xi)/\xi^2 < 0$. This fact implies the existence of two sequences of real numbers $(w_n)_n$ and $(v_n)_n$ such that $w_n \to -\infty$ and $\psi(w_n) \to +\infty$, as well as, $v_n \to +\infty$ and $\psi(v_n) \to -\infty$. Hence, given any $h \in L^\infty(\Omega)$ we can find a pair (α,β) of constant lower- and upper-solutions with $\alpha < 0 < \beta$. This way, the problem becomes easily affordable via the theory of lower- and upper-solutions [DCH06]. The interesting and more challenging question arises, whether the solvability of the Neumann problem occurs under a Hammerstein type non-resonance condition with respect to the second eigenvalue $\lambda_2^{\mathcal{N}}(-\Delta;\Omega)$ which is the first positive one. Existence results in this direction were carried out in [MWW86, Theorem 2] for a nonlinearity of the form $f(x,u)$ which satisfies a Hammerstein condition without the need of uniformity and in [GO92; GO95] for problem $(\mathscr{N}_N)$ under non-resonance conditions with respect to the eigenvalue $\lambda_2^{\mathcal{N}}(-\Delta;\Omega)$ involving a combination of hypotheses on $\psi(\xi)/\xi$ and $2P(\xi)/\xi^2$.

We report that several results of multiplicity can be found for Neumann problems associated with $\Delta u - k(x)u + w(x)\psi(u) = 0$, where $k(x) > 0$, or even for more general p-Laplacian type equations (see [BD09] and the references therein). In any case, the structure of this latter equation is however completely different to the one treated here. As far as we know, in literature there aren't works about multiple positive solutions for the analogous of problem $(\mathscr{I}\mathscr{D}_N)$ with Neumann boundary conditions. More precisely, the study of an indefinite Neumann problem as in $(\mathscr{I}\mathscr{N}_N)$ with ψ satisfying $(\mathrm{H}\psi_1)$ and $(\mathrm{H}\psi_4)$ and $u(x) > 0$ for every $x \in \overline{\Omega}$, is still an open problem.

7.1 Multiplicity of positive solutions: one-dimension

We are interested in problems $(\mathscr{I}\mathscr{N}_N)$ with a sign-indefinite weight and a positive nonlinearity with an oscillatory potential, that, for $N = 1$ and $\Omega =]0,T[$, lead to problem $(\mathscr{I}\mathscr{N})$. For ease of discussion, we will focus our study to the simplified situation where the weight has a "positive hump" followed by a "negative hump". Actually we can consider more general cases, by allowing the existence of subintervals where the weight function identically vanishes. Namely, to fix our framework, we assume that there exists $\sigma \in]0,T[$ such that

$(\mathrm{H}w_4)$ $\quad w(t) \geq 0,\ w \not\equiv 0$ *for a.e.* $t \in [0,\sigma]$, $\ w(t) \leq 0,\ w \not\equiv 0$ *for a.e.* $t \in [\sigma,T]$.

Generally speaking, not any sign-indefinite weight is suitable to guarantee the existence of solutions to $(\mathscr{I}\mathscr{N})$. For instance, if ψ is continuously differentiable in $\mathbb{R}_0^+$, with $\psi'(\xi) > 0$ for all $\xi > 0$, it is a well-known fact that a positive solution of the Neumann problem on $[0,T]$ may exist only if $\int_0^T w(t)\,dt < 0$. Moreover, other features connected to the graph of ψ, can require further conditions on the positive or negative part of w. Hence, it is convenient to consider a weight of the form

$$w_{\lambda,\mu}(t) := \lambda w^+(t) - \mu w^-(t),$$

for λ and μ given real positive parameters. A weight term of this type are not new in literature and the starting interest can be traced back to the works by López-Gómez [LG97; LG00].

In this manner, problem $(\mathscr{IN})$ reads as

$$(\mathscr{IN}_{\lambda,\mu}) \qquad \begin{cases} u'' + w_{\lambda,\mu}(t)\psi(u) = 0, \\ u'(0) = u'(T) = 0. \end{cases}$$

Through this section, we assume that $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies $(\mathrm{H}\psi_1)$. We also tacitly extend ψ to the whole real line, by setting $\psi(\xi) = 0$ for all $\xi < 0$ (this extension is still denoted by ψ). Furthermore, we suppose that $w \in L^1(0,T)$ is such that conditions in $(\mathrm{H}w_4)$ are satisfied. Therefore, solutions of $(\mathscr{IN}_{\lambda,\mu})$ will be considered in the Carathéodory sense. We are now in position to state our main result of existence and multiplicity of positive solutions for problem $(\mathscr{IN}_{\lambda,\mu})$.

Theorem 7.1.1. *Let $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying $(\mathrm{H}\psi_1)$ and $(\mathrm{H}\psi_4)$ with $\xi \mapsto \psi(\xi)/\xi$ upper bounded in a right neighborhood of 0. Let $w \in L^1(0,T)$ satisfies $(\mathrm{H}w_4)$ with $w^+ \in L^\infty(0,\sigma)$. Suppose also that the interval $[t_1,t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $w^+(t) \geq \delta$ for a.e. $t \in [t_1,t_2]$. Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda,r,k) > 0$ such that for each $\mu > \mu^*$ the problem $(\mathscr{IN}_{\lambda,\mu})$ has at least $2k$ solutions which are nonincreasing on $[0,T]$ and satisfy $0 < u(t) \leq r$ for each $t \in [\sigma,T]$.*

The method of the proof is based on a careful analysis of the trajectories of the associated phase-plane system

$$(\mathrm{S}_{\lambda,\mu}) \qquad \begin{cases} x' = y, \\ y' = -(\lambda w^+(t) - \mu w^-(t))\psi_M(x), \end{cases}$$

where, given a fixed constant $M > 0$, we have denoted by $\psi_M(x)$ the truncated function

$$\psi_M(x) = \begin{cases} 0, & \text{if } x < 0, \\ \psi(x), & \text{if } 0 \leq x \leq M, \\ \psi(M), & \text{if } x > M. \end{cases} \tag{7.1.1}$$

Positive solutions of the Neumann problem will be obtained by means of the shooting-type method applied to system $(\mathrm{S}_{\lambda,\mu})$, starting from the positive half-axis

$$X^+ := \{(x,0) : x > 0\}$$

and hitting again X^+ at the time $t = T$ (cf. Section 3.1 at p. 41). Notice that, by construction, the solutions $(x(t),y(t))$ we find are such that $x'(t) = y(t) \leq 0$ on $[0,T]$. Hence, $u(t) = x(t)$ is nonincreasing on $[0,T]$ and therefore is a solution of $(\mathscr{IN}_{\lambda,\mu})$ provided that $u(0) \leq M$. We will divide the proof of Theorem 7.1.1 into two parts. In Section 7.1.1 we will provide some estimates for the solutions of $(\mathrm{S}_{\lambda,\mu})$, while in Section 7.1.2 we will obtain the desired multiplicity result of positive solutions.

7.1.1 Technical lemmas

The results in this subsection will be done preliminarly for a locally Lipschitz continuous function $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(\mathrm{H}\psi_1)$. To perform our analysis of $(\mathrm{S}_{\lambda,\mu})$, we prove some technical lemmas for understanding the behavior of the solutions of the equations

$u'' + \lambda w^+(t)\psi(u) = 0$ and $u'' - \mu w^-(t)\psi(u) = 0$, separately. This way, we are going to study system $(\mathrm{S}_{\lambda,\mu})$, firstly on the interval $[0,\sigma]$, and then on the interval $[\sigma, T]$.

Let $(x(\cdot\,; t_0, x_0, y_0), y(\cdot\,; t_0, x_0, y_0))$ be the solution of the system

$$(\mathrm{S}^+_\lambda) \qquad \begin{cases} x' = y, \\ y' = -\lambda w^+(t)\psi(x), \end{cases}$$

satisfying the initial condition $x(t_0) = x_0$, and $y(t_0) = y_0$, for $t_0 \in [0,\sigma]$. By the concavity of $x(t)$ and the assumption $\psi(\xi) = 0$ for $\xi < 0$, it is straightforward to check that the solution $(x(t), y(t))$ is globally defined on $[0,\sigma]$.

Lemma 7.1.2. *Let $r > 0$ be fixed. If $(x(t), y(t))$ is any solution of (S^+_λ) with $x(0) > r$ and $y(0) = 0$, then $y(t) \leq 0$ for all $t \in [0,\sigma]$. Furthermore, there exists $\bar{t} \in [0,\sigma[$ such that $y(t) = 0$ for all $0 \leq t \leq \bar{t}$ and $y(t) < 0$ for all $t \in]\bar{t},\sigma]$. If, moreover, $x(0) > (1+\sigma)r$, then*

$$x(t)^2 + y(t)^2 > r^2, \quad \forall\, t \in [0,\sigma].$$

Proof. To prove the first part of the claim, it is sufficient to observe that

$$x'(t) = y(t) = -\lambda \int_0^t w^+(s)\psi(x(s))\, ds \leq 0, \quad \forall\, t \in [0,\sigma].$$

Furthermore, if $x(t^o) = 0$ for some $t^o \in]0,\sigma]$, there exists $\xi^o \in]0, t^o[$ such that $x'(s^o) < 0$ and therefore, $y(t) \leq y(s^o) < 0$ for all $t \in [s^o, \sigma]$. On the other hand, if $x(t) > 0$ for all $t \in [0,\sigma]$, then the same conclusion holds since $\int_0^\sigma w^+(s)\, ds > 0$. Thus, our assertion follows by taking $\bar{t} := \inf\{t \in]0,\sigma] : y(t) < 0\}$.

To prove the last part of the claim, suppose, by contradiction, that there exists $t^\# \in]0,\sigma]$ such that $x(t^\#)^2 + y(t^\#)^2 \leq r^2$. Given $B(0,r) := \{(x,y) : x^2 + y^2 < r^2\}$, since $(x(0), y(0)) \notin \mathrm{cl}B(0,r)$, let $\tilde{t} \in]0,\sigma]$ be the minimum of the t such that $(x(t), y(t)) \in \partial B(0,r)$. This way, $(x(\tilde{t}), y(\tilde{t})) \in \partial B(0,r)$ and $(x(t), y(t)) \notin \mathrm{cl}B(0,r)$ for all $t \in [0,\tilde{t}[$. Recalling that $\psi(\xi) = 0$ for $\xi < 0$, we easily deduce that $x(t) \geq 0$ for all $t \in [0,\tilde{t}]$. The monotonicity of $y(t)$ implies that $|y(t)| \leq |y(\tilde{t})| \leq r$ for all $t \in [0,\tilde{t}]$. From $x' = y$, we have

$$x(t) = x(0) + \int_0^t y(s)\, ds > (1+\sigma)r - \int_0^\sigma |y(s)|\, ds \geq (1+\sigma)r - \sigma r = r, \quad \forall\, t \in [0,\tilde{t}].$$

Hence, for $t = \tilde{t}$, we obtain the contradiction $r \geq x(\tilde{t}) > r$. The result is thus proved. □

Lemma 7.1.2 does not require any special condition on w^+ and ψ. On the contrary, in the next results qualitative information about the solutions will be provided under additional hypotheses on the weight and the nonlinearity.

Lemma 7.1.3. *Suppose that there exist an interval $[t_1, t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $w^+(t) \geq \delta$ for a.e. $t \in [t_1, t_2]$. If*

$$\lambda\delta P^\infty > \left(\frac{\pi}{2(t_2 - t_1)}\right)^2, \tag{7.1.2}$$

then, for any fixed constant ρ with

$$\lambda\delta P^\infty > \lambda\delta\rho > \left(\frac{\pi}{2(t_2 - t_1)}\right)^2, \tag{7.1.3}$$

there exists an increasing sequence of positive real numbers $(d_j)_j$ with $d_j \nearrow +\infty$ for which the following property holds: If $(x(t), y(t))$ is any solution of (S^+_λ) with $x(0) \geq d_j$, $y(0) = 0$ and $x(t_1) = d_j$, then there is $\tilde{t} \in]t_1, t_2[$ such that

- $x(\tilde{t}) = 0$,
- $(y(t)^2/\lambda\delta\rho) + x(t)^2 \geq d_j^2, \;\; \forall\, t \in [t_1, \tilde{t}]$.

Proof. By fixing in (7.1.3) a positive constant ρ with $\rho < P^{\infty}$, from [FOZ89] we know that by $\limsup_{\xi\to+\infty}(2P(\xi)-\rho\xi^2)=+\infty$, there exists an increasing sequence of positive real numbers $(d_j)_j$ with $d_j \nearrow +\infty$ such that the following inequality holds

$$2\left(P(d_j)-P(\xi)\right) > \rho(d_j^2-\xi^2), \quad \forall\, \xi \in [0, d_j[. \tag{7.1.4}$$

Assume that $(x(t), y(t))$ is a solution of (S_λ^+) with $x(0) \geq d_j$, $y(0)=0$ and $x(t_1)=d_j$. Note also that $y(t_1) \leq 0$ (cf. Lemma 7.1.2). Let $[t_1, \tilde{t}] \subseteq [t_1, t_2]$ be the maximal closed subinterval of $[t_1, t_2]$ where $x(t) \geq 0$ (and, necessarily, also $y(t) \leq 0$). From system (S_λ^+), using the fact that $w^+(\xi) \geq \delta$ for a.e. $\xi \in [t_1, t]$, we have

$$yy' + \lambda\delta\psi(x)x' \geq 0, \quad \text{a.e in } [t_1, \tilde{t}],$$

which yields a map $\xi \mapsto \frac{1}{2}y(\xi)^2 + \lambda\delta P(x(\xi))$ nondecreasing in $[t_1, \tilde{t}]$. This in turn implies that, for all $s \in [t_1, \tilde{t}]$,

$$y(s)^2 + 2\lambda\delta P(x(s)) \geq y(t_1)^2 + 2\lambda\delta P(x(t_1)) \geq 2\lambda\delta P(x(t_1)) = 2\lambda\delta P(d_j).$$

Using (7.1.4), in the above inequality, we obtain

$$x'(s)^2 = y(s)^2 \geq \lambda\delta\rho(d_j^2 - x(s)^2), \quad \forall\, s \in [t_1, \tilde{t}] \tag{7.1.5}$$

and, as a further consequence, we also deduce

$$\int_{x(\tilde{t})}^{d_j} \frac{1}{\sqrt{d_j^2 - x^2}}\, dx = \int_{t_1}^{\tilde{t}} \frac{-x'(s)}{\sqrt{d_j^2 - x(s)^2}}\, ds \geq (\tilde{t} - t_1)\sqrt{\lambda\delta\rho}.$$

Notice that $x(s) < d_j$ for all $t_1 < s \leq \tilde{t}$ as $x' = y$ is strictly decreasing on $[t_1, \tilde{t}]$ and hence also $x(t)$ is strictly decreasing as $y(t_1) \leq 0$.

We claim that $\tilde{t} < t_2$. Indeed, otherwise,

$$\int_0^{d_j} \frac{1}{\sqrt{d_j^2 - x^2}}\, dx \geq \int_{x(t_2)}^{d_j} \frac{1}{\sqrt{d_j^2 - x^2}}\, dx \geq (t_2 - t_1)\sqrt{\lambda\delta\rho}.$$

This provides a contradiction because $\int_0^{d_j} 1/\sqrt{d_j^2 - x^2}\, dx = \pi/2$, while, according to the choice of ρ in (7.1.3), we have $(t_2 - t_1)\sqrt{\lambda\delta\rho} > \pi/2$.

We have thus proved that $x(t)$ vanishes at some time $\tilde{t} \in]t_1, t_2[$. The inequality $y(t)^2/\lambda\delta\rho + x(t)^2 \geq d_j^2$, for all $t \in [t_1, \tilde{t}]$, follows from (7.1.5). □

Lemma 7.1.4. *Suppose that $w^+ \in L^\infty([0,\sigma])$ and let $P_\infty = 0$. For any fixed $0 < \theta < 1$ and $0 < \nu < \pi/2$, there exists an increasing sequence of positive numbers $(\beta_j)_j$ with $\lim \beta_j = +\infty$ for which the following property holds: If $(x(t), y(t))$ is any solution of (S_λ^+) with $x(0) = \beta_j$, $y(0) = 0$, then*

- $\theta\beta_j \leq x(t) \leq \beta_j, \;\; \forall\, t \in [0, \sigma]$,
- $\tan(|y(t)|/x(t)) < \tan(\nu), \;\; \forall\, t \in [0, \sigma]$.

Proof. Let $\theta \in]0,1[$ and $\nu \in]0, \pi/2[$ be two fixed constants. The assumption $P_\infty = 0$ implies that $\limsup_{\xi\to+\infty}(\varepsilon\xi^2 - 2P(\xi)) = +\infty$, for every $\varepsilon > 0$. Hence, following [FOZ89], there exists an increasing sequence of positive real numbers $(\beta_j^\varepsilon)_j$ with $\beta_j^\varepsilon \nearrow +\infty$ such that the following inequality holds

$$2\left(P(\beta_j^\varepsilon) - P(\xi)\right) < \varepsilon((\beta_j^\varepsilon)^2 - \xi^2), \quad \forall\, \xi \in [0, \beta_j^\varepsilon[\,. \tag{7.1.6}$$

Assume that $(x(t), y(t))$ is a solution of (S_λ^+) with $x(0) = \beta_j^\varepsilon$ and $y(0) = 0$. Recall from Lemma 7.1.2 also that $y(t) \leq 0$ for all $t \in [0, \sigma]$, so that $x(t) \leq \beta_j^\varepsilon$ for all $t \in [0, \sigma]$.

We claim that $x(t) \geq \theta\beta_j^\varepsilon$ for all $t \in [0,\sigma]$. To prove this claim, suppose, by contradiction that there exists a maximal interval $[0,\hat{t}] \subset [0,\sigma[$ such that

$$\theta\beta_j^\varepsilon \leq x(s) \leq \beta_j^\varepsilon, \quad \forall\, s \in [0,\hat{t}], \quad \text{with } x(\hat{t}) = \theta\beta_j^\varepsilon\,. \tag{7.1.7}$$

From system (S_λ^+), using the fact that $w^+(s) \leq \|w^+\|_\infty$ for a.e. $s \in [0,\sigma]$, we have

$$yy' + \lambda\|w^+\|_\infty\psi(x)x' \leq 0, \quad \text{a.e in } [0,\hat{t}],$$

which yields a map $s \mapsto \frac{1}{2}y(s)^2 + \lambda\|w^+\|_\infty P(x(s))$ nonincreasing in $[0,\hat{t}]$. This in turn implies that, for all $s \in [0,\hat{t}]$,

$$y(s)^2 + 2\lambda\|w^+\|_\infty P(x(s)) \leq y(0)^2 + 2\lambda\|w^+\|_\infty P(x(0)) = 2\lambda\|w^+\|_\infty P(\beta_j^\varepsilon).$$

Using (7.1.6), in the above inequality, we obtain that

$$x'(s)^2 = y(s)^2 \leq \lambda\|w^+\|_\infty\varepsilon((\beta_j^\varepsilon)^2 - x(s)^2) \tag{7.1.8}$$

holds for all $s \in [0,\hat{t}]$. As a further consequence, we have

$$\int_{\theta\beta_j^\varepsilon}^{\beta_j^\varepsilon} \frac{1}{\sqrt{(\beta_j^\varepsilon)^2 - x^2}}\,dx = \int_0^{\hat{t}} \frac{-x'(s)}{\sqrt{(\beta_j^\varepsilon)^2 - x(s)^2}}\,ds \leq \hat{t}\sqrt{\lambda\|w^+\|_\infty\varepsilon} \leq \sigma\sqrt{\lambda\|w^+\|_\infty\varepsilon}.$$

Since the left integral in the above inequality can be explicitly computed, as $(\pi/2) - \arcsin\theta$ (independently on β_j^ε), we obtain

$$\frac{\pi}{2} < \arcsin\theta + \sigma\sqrt{\lambda\|w^+\|_\infty\varepsilon},$$

which is clearly false if ε is chosen sufficiently small, namely

$$0 < \varepsilon < \frac{(\frac{\pi}{2} - \arcsin\theta)^2}{\sigma^2\lambda\|w^+\|_\infty}. \tag{7.1.9}$$

For such a choice of $\varepsilon > 0$ we can find a sequence $(\beta_j^\varepsilon)_j$ such that $\theta\beta_j^\varepsilon \leq x(t) \leq \beta_j^\varepsilon$ for all $t \in [0,\sigma]$. As a consequence, we also know that condition (7.1.8) holds for all $s \in [0,\sigma]$ and therefore, recalling that $y(t) \leq 0$, we deduce

$$|y(t)| \leq \beta_j^\varepsilon\sqrt{\lambda\|w^+\|_\infty\varepsilon}, \quad \forall\, t \in [0,\sigma].$$

This in turn implies that $\tan(|y(t)|/x(t)) < \tan(\nu)$, for all $t \in [0,\sigma]$, provided that

$$0 < \varepsilon < \frac{(\theta\tan(\nu))^2}{\lambda\|w^+\|_\infty}. \tag{7.1.10}$$

This way the theorem is proved by choosing a sequence $(\beta_j^\varepsilon)_j$ for a constant ε satisfying (7.1.9) and (7.1.10). □

Lemma 7.1.5. *Given $w^+ \in L^\infty([0,\sigma])$, suppose that there exist an interval $[t_1,t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $w^+(t) \geq \delta$ for a.e. $t \in [t_1,t_2]$. Assume also $(\mathrm{H}\psi_4)$ and let $\lambda > 0$ be such that (7.1.2) holds. Let also $0 < \theta < 1$, $0 < \nu < \pi/2$ be fixed. Then, there exist two increasing sequences of positive numbers $(\alpha_j)_j$ and $(\beta_j)_j$ with $\lim\alpha_j = \lim\beta_j = +\infty$ and*

$$r < \alpha_1 < \theta\beta_1 < \beta_1 < \alpha_2 < \ldots \alpha_j < \theta\beta_j < \beta_j < \alpha_{j+1} < \ldots \tag{7.1.11}$$

for which the following properties hold:

- $x(t;0,\alpha_j,0)$ *vanishes at some* $t < t_2$,
- $\theta\beta_j \leq x(t;0,\beta_j,0) \leq \beta_j$, $\tan(|y(t;0,\beta_j,0)|/x(t;0,\beta_j,0)) < \tan(\nu)\ \forall t \in [0,\sigma]$

Proof. We choose a constant $\rho > 0$ in accord to (7.1.3) and consider a corresponding sequence $(d_j)_j$ as in Lemma 7.1.3. Next, we apply Lemma 7.1.4 and find a sequence $(\beta_j)_j$. We can also suppose that

$$r < d_1 < \theta\beta_1 < \beta_1 < d_2 < \ldots d_j < \theta\beta_j < \beta_j < d_{j+1} < \ldots$$

up to a subsequence, if necessary. By the intermediate value theorem and the continuous dependence of the solutions on the initial data, for each j, there exists α_j with $d_j \leq \alpha_j < \beta_j$ such that $x(t_1; 0, \alpha_j, 0) = d_j$. At this point, a direct application of Lemma 7.1.3 and Lemma 7.1.4 allows to conclude the proof of the theorem. □

Until now we have analyzed the behavior of the solutions in the interval $[0, \sigma]$ where $w_{\lambda,\mu}(t) \geq 0$ for a.e. t. As a next step, we are going to consider the solutions on the interval $[\sigma, T]$. Due to the sign of $w_{\lambda,\mu}(t)\psi(x(t))$ which implies the convexity of $x(t)$ in the interval $[\sigma, T]$, in general, we cannot guarantee that the solutions are defined on the whole interval. For this reason, we introduce a truncation on the nonlinear term of the form

$$\psi_M(x) = \begin{cases} \psi(x) & \text{if } x \leq M, \\ \psi(M) & \text{if } x > M, \end{cases}$$

where $M > 0$ is a given constant. Accordingly, we study the system

$$(\mathrm{S}_\mu^-) \qquad \begin{cases} x' = y, \\ y' = \mu w^-(t)\psi_M(x), \end{cases}$$

on the interval $[\sigma, T]$. In the foregoing results we shall require a further technical condition on the weight function, namely that $w(t)$ is not identically zero a.e. in each right neighborhood of σ. This can be equivalently expressed by the following condition:

$$W^-(t) > 0, \quad \forall\, t \in]\sigma, T],$$

where we have set

$$W^-(t) := \int_\sigma^t w^-(s)\, ds. \tag{7.1.12}$$

This hypothesis is not restrictive in view of ($\mathrm{H}w_4$) (see [BZ12, Remark 2.2] where an analogous situation is treated). In this framework, we obtain the following result.

Lemma 7.1.6. *For any fixed $r > 0$, $q \in]0, 1[$ and $C > 0$, there is a constant $\hat{\mu} > 0$ such that for each $\mu > \hat{\mu}$ the following holds: If $(x(t), y(t))$ is any solution of (S_μ^-) with $x(\sigma) = r$ and $0 > y(\sigma) \geq -C$, then*

- *$x(t) > qr$ for all $t \in [\sigma, T]$,*
- *$y(t)$ vanishes at some $t \in]\sigma, T[$.*

Proof. First of all, notice that there exists $0 < \varepsilon \leq r(1 - q)/C$ such that $x(t) > qr$ for all $t \in [\sigma, \sigma + \varepsilon[$. Indeed,

$$\begin{aligned} x(t) = x(\sigma) + \int_\sigma^t y(s)\, ds \geq r - \int_\sigma^t C\, ds = r - C(t - \sigma) \\ > r - C\varepsilon \geq qr, \quad \forall t \in [\sigma, \sigma + \varepsilon[. \end{aligned}$$

Therefore, let us fix ε as above and assume by contradiction that there is $\tilde{t} \in [\sigma + \varepsilon, T]$ such that $x(\tilde{t}) = qr$ and $x(t) > qr$ for all $t \in [\sigma, \tilde{t}[$. By denoting with $\kappa_{\psi,r} := \min\{\psi_M(\xi) : qr \leq \xi \leq r\}$, we have

$$x''(t) = y'(t) = \mu w^-(t)\psi(x(t)) \geq \mu w^-(t)\kappa_{\psi,r}, \quad \text{for a.e. } t \in [\sigma, \tilde{t}].$$

After a first integration on $[\sigma, t]$, we get

$$x'(t) = y(t) \geq y(\sigma) + \mu\kappa_{\psi,r}W^-(t) \geq -C + \mu\kappa_{\psi,r}W^-(t), \quad \forall\, t \in [\sigma, \tilde{t}].$$

Integrating again in the same interval we have

$$\begin{aligned} x(t) &\geq x(\sigma) - C(t-\sigma) + \mu\kappa_{\psi,r}\int_\sigma^t W^-(s)\,ds \\ &\geq r - C(T-\sigma) + \mu\kappa_{\psi,r}\int_\sigma^{\sigma+\varepsilon} W^-(s)\,ds. \end{aligned}$$

The evaluation of the above inequality for $t = \tilde{t}$ yields to a contradiction if μ is sufficiently large, namely

$$\mu \geq \mu_1 := \frac{C(T-\sigma)}{\kappa_{\psi,r}\int_\sigma^{\tilde{t}} W^-(s)\,ds}.$$

At this step, we have proved that $x(t) > qr$ for all $t \in [\sigma, T]$.

Suppose now, by contradiction that $y(t)$ never vanishes on $]\sigma, T]$. Then, since $y(\sigma) < 0$, we have $x'(t) = y(t) < 0$ for all $t \in [\sigma, T]$. Hence the function $x(t)$ is decreasing on $[\sigma, T]$ and, therefore, $qr < x(t) < r$ for all $t \in]\sigma, T]$. Accordingly, the inequality $y'(t) \geq \mu w^-(t)\kappa_{\psi,r}$ holds for a.e. $t \in [\sigma, T]$. With an integration on $[\sigma, t]$ we obtain

$$y(t) \geq -C + \mu\kappa_{\psi,r}W^-(t), \quad \forall\, t \in [\sigma, T].$$

So that

$$0 > y(T) \geq -C + \mu\kappa_{\psi,r}W^-(T).$$

A contradiction occurs whenever μ is sufficiently large, namely

$$\mu \geq \mu_2 := \frac{C}{\kappa_{\psi,r}\int_\sigma^{\sigma+T} w^-(s)\,ds}.$$

At this point, the conclusion follows by taking $\hat{\mu} \geq \max\{\mu_1, \mu_2\}$. □

7.1.2 Multiplicity result

In this subsection we prove Theorem 7.1.1. Our method of proof is based on the shooting method and therefore we need to analyze the Poincaré map associated with the planar system

$$(\mathrm{S}_{\lambda,\mu}) \qquad \begin{cases} x' = y, \\ y' = -(\lambda w^+(t) - \mu w^-(t))\psi_M(x), \end{cases}$$

where ψ_M is defined as in (7.1.1) for a suitable constant $M > 0$. In order to have the Poincaré map well defined, we shall implicitly assume the uniqueness of the solutions for the associated initial value problems. Obviously, this is guaranteed if ψ is locally Lipschitz continuous. However, this condition can be removed and this will be discussed at the end of the proof of Theorem 7.1.1. As in Section 2.2 at p. 20, we recall that, given an interval $[\tau_0, \tau_1] \subseteq [0, T]$, the Poincaré map $\Phi_{\tau_0}^{\tau_1}$ for $(\mathrm{S}_{\lambda,\mu})$ on the interval $[\tau_0, \tau_1]$ is the planar map which, to any point $z_0 = (x_0, y_0) \in \mathbb{R}^2$, associates the point $(x(\tau_1), y(\tau_1))$ where $(x(t), y(t))$ is the solution of $(\mathrm{S}_{\lambda,\mu})$ with $(x(\tau_0), y(\tau_0)) = z_0$.

A solution of $(\mathscr{IN}_{\lambda,\mu})$ can be found by looking for a point $(x_0, 0) \in X^+ := \{(x, 0) : x > 0\}$ such that $x_0 \leq M$ and $\Phi_0^T(x_0, 0) \in X^+$. In this case, the first component $u(t)$ of the map $t \mapsto \Phi_0^t(x_0, 0)$ is a solution of $(\mathscr{P}_{\lambda,\mu})$ with $u(0) = x_0$. More formally, we can state the following lemma.

Lemma 7.1.7. *Suppose that there is $(x_0, 0) \in X^+$ with $x_0 \leq M$ such that $\Phi_0^T(x_0, 0) \in X^+$. Let also $(x(t), y(t))$ be the solution of $(\mathrm{S}_{\lambda,\mu})$ with $(x(0), y(0)) = (x_0, 0)$. Then, $u(t) := x(t)$ is a solution of $(\mathscr{P}_{\lambda,\mu})$ with $u(t) \leq M$ and $u'(t) = y(t) \leq 0$ for all $t \in [0, T]$.*

Proof. Consider at first the solution in the interval $[0,\sigma]$. As $x(t)$ is concave in such interval, we have that $x(t) \le x(0) \le M$ and we also claim that $x(t) > 0$ for all $t \in [0,\sigma]$. Indeed, if by contradiction $x(t)$ vanishes somewhere, we take $\hat{t}$, with $0 < \hat{t} \le \sigma$, as its first zero. As a consequence of the concavity, $y(\hat{t}) = x'(\hat{t}) < 0$ and then, $x'(t) = x'(\hat{t}) < 0$ for all $t \in [\hat{t}, T]$, because $\psi_M(\xi) = 0$ for $\xi \le 0$. Thus, we have the contradiction $\Phi_0^T(x_0, 0) \notin X^+$. From $y'(t) = -\lambda w^+(t)\psi(x(t))$, with $\psi(x(t)) > 0$ for all $t \in [0,\sigma]$ and $w^+ \not\equiv 0$, we deduce that $x'(\sigma) = y(\sigma) < 0$. On the other hand, the function $x(t)$ is convex on $[\sigma, T]$ with $x(T) > 0$ and $x'(T) = 0$. Hence, $0 < x(T) \le x(t) < x(\sigma)$ for all $t \in [\sigma, T]$ and this concludes the proof. □

In view of the hypothesis on the weight function, which states that it assumes different sign on the intervals $[0,\sigma]$ and $[\sigma, T]$, it will be convenient to split the Poincaré map as

$$\Phi_0^T := \Phi_\sigma^T \circ \Phi_0^\sigma,$$

where Φ_0^σ and Φ_σ^T are the Poincaré maps associated with systems (S_λ^+) and (S_μ^-), respectively. Consistently with our notation, we observe that for any point $(x_0, 0) \in X^+$ with $x_0 \le M$, we have

$$\Phi_0^t(x_0, 0) = (x(t; 0, x_0, 0), y(t; 0, x_0, 0)), \quad \forall\, t \in [0,\sigma].$$

To formulate the next result, we introduce the following notation. For any real number η, we denote the negative half-line $x = \eta$ by

$$L_\eta := \{(\eta, y) \in \mathbb{R}^2 : y < 0\}.$$

Given two points $(A, 0), (B, 0) \in X^+$, the segment contained in X^+ and joining the two points is denoted by $\overline{AB}$.

Proposition 7.1.8. *Given $w^+ \in L^\infty([0,\sigma])$, suppose that there exist an interval $[t_1, t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $w^+(t) \ge \delta$ for a.e. $t \in [t_1, t_2]$. Assume also* $(H\psi_4)$ *and let $\lambda > 0$ be such that* (7.1.2) *holds. Furthermore, let $r > 0$ be fixed. Then, for any given integer $k \ge 1$ there are constants $M > r$, $C_M > r$ and points*

$$r < A_1' < B_1' < B_1'' < A_1'' < A_2' < \cdots < A_k' < B_k' < B_k'' < A_k'' < M,$$

such that, setting

$$\Gamma_j' := \Phi_0^\sigma\left(\overline{A_j' B_j'}\right), \qquad \Gamma_j'' := \Phi_0^\sigma\left(\overline{B_j'' A_j''}\right),$$

we have

$$\Gamma_j', \Gamma_j'' \subseteq \Big([0, r] \times [-C_M, 0[\Big), \tag{7.1.13}$$

with

$$\Gamma_j' \cap L_0^- \ne \emptyset \ne L_r^- \cap \Gamma_j', \qquad \Gamma_j'' \cap L_0^- \ne \emptyset \ne L_r^- \cap \Gamma_j'', \tag{7.1.14}$$

for all $j = 1, \ldots, k$.

Proof. Given $\lambda > 0$ and $r > 0$, we choose $0 < \theta < 1$ and $0 < \nu < \pi/2$. So, an application of Lemma 7.1.5 provides two sequences $(\alpha_j)_j$ and $(\beta_j)_j$ which satisfy (7.1.11). Moreover, for any integer $k \ge 1$, we take a constant M such that

$$M > \alpha_k\,. \tag{7.1.15}$$

Since M is now fixed, follows that also the vector field in the system $(S_{\lambda,\mu})$ is so. The constant $C_M > 0$ will be chosen so that any possible solution $(x(t), y(t))$ of $(S_{\lambda,\mu})$ with $0 < x(0) \le M$ and $y(0) = 0$, satisfies

$$-C_M \le y(t) \le 0, \quad \forall\, t \in [0,\sigma].$$

Notice that the constant C_M depends on the function w^+ and the constants λ and M, but does not depend on the parameter μ. In fact, we can estimate C_M as follows:

$$C_M := \lambda \|w^+\|_{L^1} \max_{\xi \in [0,M]} \psi(\xi).$$

For the rest of the proof we consider the solutions of the system $(\mathrm{S}_{\lambda,\mu})$ on the interval $[0,\sigma]$, with an initial point $(c,0)$ such that $0 < c \leq M$. These are exactly the solutions $(x(\cdot\,;0,c,0), y(\cdot\,;0,c,0))$ of the system (S^+_λ).

As a first step, for $j = 1, \ldots, k$, we suppose that $\alpha_j \leq c \leq \beta_j$. By Lemma 7.1.5, it follows that

$$x(\sigma; 0, \alpha_j, 0) < 0, \qquad x(\sigma; 0, \beta_j, 0) \geq \theta\beta_j > r.$$

By continuity, we can determine a sub-interval $[A'_j, B'_j] \subseteq \,]\alpha_j, \beta_j[$ such that $x(\sigma; 0, A'_j, 0) = 0$, $x(\sigma; 0, B'_j, 0) = r$ and $0 < x(\cdot\,; 0, c, 0) < r$ for all $c \in \,]A'_j, B'_j[\,$.

As a second step, for $j = 1, \ldots, k$, we suppose that $\beta_j \leq c \leq \alpha_{j+1}$. By Lemma 7.1.5, it follows that

$$x(\sigma; 0, \alpha_{j+1}, 0) < 0, \qquad x(\sigma; 0, \beta_j, 0) \geq \theta\beta_j > r.$$

Again, by continuity, we can determine a sub-interval $[B''_j, A''_j] \subseteq \,]\beta_j, \alpha_{j+1}[$ such that $x(\sigma; 0, B''_j, 0) = r$, $x(\sigma; 0, A''_j, 0) = 0$ and $0 < x(\sigma\,; 0, c, 0) < r$ for all $c \in \,]B''_j, A''_j[\,$. Moreover, $-C_M \leq y(\sigma\,; 0, c, 0) < 0$ (recalling also Lemma 7.1.2).

To conclude, we define

$$\Gamma'_j := \Phi^\sigma_0 \left(\overline{A'_j B'_j}\right), \quad \Gamma''_j := \Phi^\sigma_0 \left(\overline{B''_j A''_j}\right), \quad \forall\, j = 1, \ldots, k.$$

This way each arc, Γ'_j and Γ''_j with $j \in \{1, \ldots, k\}$, satisfies all the desired properties. □

Remark 7.1.9. We observe that the constants β_j are precisely determined in Lemma 7.1.4 by means of (7.1.6), instead of the constants α_j, for which we know only that they belong to $[d_j, \beta_j[\,$. With this respect, it might be more convenient to fix the constant M in terms of the values β_j. For this reason, one could prefer to replace the condition in (7.1.15) with $M > \beta_{k+1}$. Under this latter choice, notice that a further arc, $\Gamma'_{k+1} := \Phi^\sigma_0 \left(\overline{A'_{k+1} B'_{k+1}}\right)$ with $[A'_{k+1} B'_{k+1}] \subseteq \,]\alpha_{k+1}, \beta_{k+1}[$ defined as in the proof, can be determined. Finally, if we assume $M > \beta_{k+1}$, we have $2k+1$ arcs defined as images through the Poincaré map of pairwise disjoint compact sub-intervals of X^+. ◁

The next result deals with the solutions of the system $(\mathrm{S}_{\lambda,\mu})$ in the time interval $[\sigma, T]$, or equivalently, the ones of (S^-_μ). As previously observed, we will suppose that σ is chosen so that $W^-(t) > 0$ for all $\sigma < t \leq T$, where $W^-(t)$ is defined according to (7.1.12).

Proposition 7.1.10. *Given $r > 0$ and $C > r$, there exists a constant $\bar{\mu} > 0$ such that for each $\mu > \bar{\mu}$ the following holds: For any connected set Γ with*

$$\Gamma \subseteq [0, r] \times [-C, 0[, \qquad \Gamma \cap L^-_0 \neq \emptyset \neq L^-_r \cap \Gamma,$$

there exists at least a solution $(x(t), y(t))$ of the system (S^-_μ) with $(x(\sigma), y(\sigma)) \in \Gamma$, $(x(T), y(T)) \in X^+$ such that $r \geq x(t) > 0$ and $y(t) \leq 0$ for all $t \in [\sigma, T]$.

Proof. For r and C given as above, let us fix a parameter q with $0 < q < 1$. From Lemma 7.1.6, we have that for each μ sufficiently large (i.e. $\mu > \hat{\mu}$), any solution $(x(t), y(t))$ of (S^-_μ) with $x(\sigma) = r$ and $-C \leq y(\sigma) < 0$ is such that $x(t) \geq qr$ for all $t \in [\sigma, T]$ and $y(t) = 0$ for some $t \in \,]\sigma, T]$. Let us fix now $\mu > \hat{\mu}$.

We choose a point $Q_1 \in L^-_r \cap \Gamma$ and denote by $(x_{Q_1}(t), y_{Q_1}(t))$ the solution of (S^-_μ) having Q_1 as initial point at the time $t = \sigma$. By Lemma 7.1.6 there exists a first time $t_{Q_1} \in \,]\sigma, T]$ such that $y(t_{Q_1}) = 0$. If $t_{Q_1} = T$, we are done. Otherwise, $y_{Q_1}(t_{Q_1}) = 0$ for $\sigma < t_{Q_1} < T$ and, by the convexity of $x_{Q_1}(t)$ in the interval $[\sigma, T]$, we have $y_{Q_1}(T) \geq y_{Q_1}(t_{Q_1}) = 0$.

Similarly, we select a point $Q_2 \in \Gamma \cap L^-_0$ and denote by $(x_{Q_2}(t), y_{Q_2}(t))$ the solution of (S^-_μ) which has Q_2 as initial point at the time $t = \sigma$. We have $x_{Q_2}(\sigma) = 0$ and

$x'_{Q_2}(\sigma) = y_{Q_2}(\sigma) < 0$. Moreover, $\psi(s) = 0$ for all $s \leq 0$. Hence, $y_{Q_2}(t) = y_{Q_2}(\sigma)$ for all $t \in [\sigma, T]$ and, therefore, $y_{Q_2}(T) < 0$.

The continuous dependence of the solutions on the initial data and the connectedness of Γ imply that there exists a point in $\Gamma \setminus L_0^-$ from which starts (at the time $t = \sigma$) a solution $(x(t), y(t))$ of (S_μ^-) such that $y(T) = 0$. This way, it follows also that $x(t) > 0$ for all $t \in [\sigma, T]$ (in fact, if not, we obtain a contradiction from $\psi(\xi) = 0$ for all $\xi \leq 0$). Finally, we also observe that $y(t) \leq 0$ for all $t \in]\sigma, T]$ (otherwise, if we suppose that $y(t) > 0$ for some $t \in]\sigma, T[$, then a contradiction is reached by a convexity argument). Thus the thesis is achieved by choosing any $\bar{\mu} \geq \hat{\mu}$. □

Proof of Theorem 7.1.1. Our demonstration will be divided into two parts. In the first one we let the shooting method work within its classical framework, by assuming ψ locally Lipschitz continuous. In the second part, we present two possible ways in order to extend the result obtained to the case in which ψ is only continuous.

Under Lipschitz conditions. We suppose that ψ is locally Lipschitz continuous and so it follows immediately that $\xi \mapsto \psi(\xi)/\xi$ upper bounded in a right neighborhood of 0.

First of all, we define a constant $\lambda^* \geq 0$ as $\lambda^* = 0$ if $P^\infty = +\infty$ or $\lambda^* = \pi^2/4(t_2 - t_1)^2\delta P^\infty$ if $P^\infty < +\infty$. In this manner, the inequality in (7.1.2) is satisfied for each $\lambda > \lambda^*$.

We fix now $\lambda > \lambda^*$, $r > 0$ and an integer $k \geq 1$. In accord to Proposition 7.1.8, there are constants $M > r$, $C_M > r$ and points

$$r < A'_1 < B'_1 < B''_1 < A''_1 < A'_2 < \cdots < A'_k < B'_k < B''_k < A''_k < M,$$

such that conditions in (7.1.13) and (7.1.14) are satisfies for the arcs

$$\Gamma'_j := \Phi_0^\sigma\left(\overline{A'_jB'_j}\right), \qquad \Gamma''_j := \Phi_0^\sigma\left(\overline{B''_jA''_j}\right).$$

At this step we apply Proposition 7.1.10 for $C := C_M$ and determine a constant $\bar{\mu}$ such that, for each $\mu > \bar{\mu}$, the following holds: for each Γ'_j, Γ''_j with $j \in \{1, \ldots, k\}$ there exist points $\zeta'_j \in \Gamma'_j$ and $\zeta''_j \in \Gamma''_j$ such that

$$\Phi_\sigma^T(\zeta'_j),\ \Phi_\sigma^T(\zeta''_j) \in X^+.$$

Notice that the constant $\bar{\mu}$ does not depend on the particular choice of the arcs Γ'_j or Γ''_j. It depends only on r and C_M. The last constant, in turn, depends on M and therefore it is derived from λ and k.

On the other hand, ζ'_j and ζ''_j are images through the Poincaré map Φ_0^σ of the initial points $Z'_j \in \overline{A'_jB'_j}$ and $Z''_j \in \overline{B''_jA''_j}$, respectively. Then, we have found $2k$ points $Z'_j, Z''_j \in X^+$ such that $\Phi_0^T(Z'_j), \Phi_0^T(Z''_j) \in X^+$. From Lemma 7.1.7 follows that all the solutions $(x(t), y(t))$ starting from these initial points are such that $0 < x(t) < M$ and $y(t) \leq 0$, for all $t \in [0, T]$. Hence, they are solutions of the system

$$\begin{cases} x' = y, \\ y' = -w_{\lambda,\mu}(t)\psi(x(t)). \end{cases}$$

In particular, they correspond to positive solutions of the problem $(\mathscr{IN}_{\lambda,\mu})$ with initial conditions $(u(0), u'(0)) = Z'_j$ or $(u(0), u'(0)) = Z''_j$, respectively. All these solutions are decreasing in $[0, T]$ by construction and, from Proposition 7.1.10, they satisfy the condition $0 < u(t) \leq r$, for all $t \in [\sigma, T]$. Thus, the result is proved by choosing any $\mu^* \geq \bar{\mu}$ and ψ locally Lipschitz continuous.

Free from Lipschitz conditions. At this point, usually, one can follow two possible ways in order to achieve the result for a nonlinearity ψ which is only continuous. A first approach consists in approximating the nonlinear term ψ with a sequence of locally Lipschitz functions $\psi_n : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(\mathrm{H}\psi_1)$ and such that $\psi_n \to \psi$ uniformly on compact sets, for example, using mollifiers as in [Str80, p. 294]. Then, one can prove that each approximating equation has a solution u_n with $(u_n(t), u_n'(t)) \in \mathcal{K}, \forall\, t \in [0,T]$, where $\mathcal{K}$ is a compact set which can be chosen independently on n. At last, from the Ascoli-Arzelà Theorem we obtain a solution $(u(t), u'(t)) \in \mathcal{K}, \forall\, t \in [0,T]$ of the original equation, passing to the limit along a subsequence. This is a standard procedure well described in the book of Krasnosel'skiĭ [Kra68]. Moreover, this approach is also exploited in [Str80; Zan96] where some specific results of existence and multiplicity of solutions are obtained via the shooting method without uniqueness of the Cauchy problems. In our case, this method can be safely applied by choosing the compact intervals $\overline{A_j'B_j'}$ and $\overline{B_j''A_j''}$ for $j = 1,\dots,k$ pairwise disjoint and observing that the initial points of the solutions of the approximating problems belong to these intervals (at least for n sufficiently large).

A second possible point of view involves a procedure of "shooting without uniqueness", that gives up from the beginning to the hypothesis of uniqueness for the Cauchy problems. In this framework, we can apply a generalized version of the Hukuhara-Kneser result, as presented in [Cop65] or in [DZ07, Section 2]. It is based on the following observation. Let $[\tau_0, \tau_1] \subseteq [0,T]$. Given a set $E_0 \subseteq \mathbb{R}^2$, let us consider the set E_1 made by all the points of $\mathbb{R}^2$ of the form $(x(\tau_1), y(\tau_1))$, where $(x(t), y(t))$ is any solution of the system such that $(x(\tau_0), y(\tau_0)) \in E_0$. Then, E_1 is a compact/connected (or both) provided that E_0 is a compact/connected (or both), respectively (cf. [Cop65, p. 22]). In this context, for all $j = 1,\dots,k$ the sets Γ_j' and Γ_j'' given as in Proposition 7.1.8 are well defined continua (instead of arcs). Moreover, to prove Proposition 7.1.10, instead of using Bolzano Theorem, on the function $y(t)$ we just observe that a connected set Γ at the time $t = \sigma$ is transported into a connected set at the time $t = T$, whose projection on the y-axis contains $y = 0$.

In conclusion, we have found $2k$ non-negative solutions of $(\mathscr{IN}_{\lambda,\mu})$ which are nonincreasing on $[0,T]$ and satisfy $0 \le u(t) \le r$ for each $t \in [\sigma, T]$. Since $\xi \mapsto \psi(\xi)/\xi$ is upper bounded in a right neighborhood of 0, a maximum principle argument applies and the positivity of the solutions on $[0,T]$ is guaranteed. □

Without the condition

$$(\mathrm{H}\psi_5) \qquad \limsup_{\xi \to 0^+} \frac{\psi(\xi)}{\xi} < +\infty$$

we can prove that any solution found satisfies

$$u(t) \ge r,\ \forall\, t \in [0,\sigma] \quad \text{and} \quad 0 \le u(t) \le r\ \forall\, t \in [\sigma, T].$$

Nevertheless, without assuming $(\mathrm{H}\psi_5)$, we cannot guarantee, in general, that $u(t)$ does not vanishes at some point of the interval when the weight is negative. Examples in this direction are given in [BPT88; But78] and they show that $(\mathrm{H}\psi_1)$ along with $(\mathrm{H}\psi_5)$ represent the minimal equipment needed to get the positivity of the solutions. For this reason, the main hypothesis of our result is the "oscillatory condition" $(\mathrm{H}\psi_4)$.

Remark 7.1.11. Let us make some comments on the features assumed for the weight function w. We have considered a weight term that goes from positive to negative values. One could also consider a dual condition instead of $(\mathrm{H}w_4)$, namely

$$(\mathrm{H}w_{4\,\mathrm{bis}}) \quad w(t) \le 0,\ w \not\equiv 0 \textit{ for a.e. } t \in [0,\sigma],\ \ w(t) \ge 0,\ w \not\equiv 0 \textit{ for a.e. } t \in [\sigma, T].$$

In this case, we derive a different version of the Theorem 7.1.1 in which the hypotheses have to be modified by assuming $w^+ \in L^\infty([\sigma,T])$ and $w^+(t) \ge \delta$ for a.e. t in a suitable subinterval of $[\sigma, T]$. As a conclusion, the existence of $2k$ positive solutions to problem $(\mathscr{IN}_{\lambda,\mu})$ is still guaranteed. Such solutions, in this case, are nondecreasing on $[0,T]$ and satisfy $0 < u(t) \le r$ for each $t \in [0,\sigma]$. To prove this assertion, we can either apply Theorem 7.1.1 with the change of variable $t \mapsto T - t$, or apply the shooting method backward in time from $t = T$ to $t = 0$. ◁

We conclude this section with some comments on the choice of linear second order differential operators. The technical tools we have developed for proving Lemma 7.1.3 and

Lemma 7.1.4 rely essentially on time-mapping estimates associated to the autonomous equation

$$u'' + \psi(u) = 0. \tag{7.1.16}$$

This fact suggests different directions along which we could provide extensions of our results. For instance, we can replace the condition (Hψ_4) with an hypothesis of the form

$$0 \le \tau_\infty := \liminf_{c\to+\infty} \tau(c) < \tau^\infty := \limsup_{c\to+\infty} \tau(c) = +\infty, \tag{7.1.17}$$

where, for $c > 0$, $\tau(c)$ is the time-mapping associated to (7.1.16) defined as

$$\tau(c) := 2\int_0^c \frac{d\xi}{\sqrt{2(P(c) - P(\xi))}}.$$

Within (7.1.17), we can deal with more general linear differential operators such as $u'' + m(t)u'$ (see Chapter 6).

7.2 Multiplicity of positive solutions: radial domains

In this section we extend the preceding results to the case of some Neumann problems in $\mathbb{R}^N$, for $N \ge 2$. So, we consider

$$(\mathscr{I}\mathscr{N}_{\lambda,\mu,N}) \qquad \begin{cases} \Delta u + \mathfrak{w}_{\lambda,\mu}(x)\psi(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where the weight function depends on the real positive parameters λ, μ and is defined as

$$\mathfrak{w}_{\lambda,\mu}(x) := \lambda \mathfrak{w}^+(x) - \mu \mathfrak{w}^-(x),$$

for $\mathfrak{w} \in L^1(\Omega)$. We shall focus our study to the case when the domain Ω is an open ball $B(0,R)$ or an open annulus $B(0,R_e) \setminus B[0,R_i]$, where $B[0,r]$ denotes the closed ball of center the origin and radius $r > 0$. As usual, in these situations the problem can be reduced to a Neumann boundary value problem with an ordinary differential equation if $w(x)$ has a radial symmetry. Accordingly, from now on we suppose that

$$\mathfrak{w}(x) = \mathcal{Q}(|x|). \tag{7.2.1}$$

We look for radially symmetric positive solutions of $(\mathscr{I}\mathscr{N}_{\lambda,\mu,N})$, namely solutions of the form

$$u(x) = U(\varrho), \quad \text{with } \varrho := |x|, \tag{7.2.2}$$

and we discuss separately the two cases of our interest.

7.2.1 Neumann problem for an annular domain

Let $R_e > R_i > 0$ be two fixed radii and let us consider the Neumann problem $(\mathscr{I}\mathscr{N}_{\lambda,\mu,N})$ for the domain

$$\Omega := B(0,R_e) \setminus B[0,R_i].$$

We suppose that w is defined as in (7.2.1), with $\mathcal{Q} \in L^1([R_i,R_e])$. By means of (7.2.2) our problem is reduced to the study of

$$\begin{cases} U''(\varrho) + \dfrac{N-1}{\varrho}U'(\varrho) + \mathcal{Q}_{\lambda,\mu}(\varrho)\psi(U(\varrho)) = 0, \\ U'(R_i) = U'(R_e) = 0, \end{cases} \tag{7.2.3}$$

with $U(x) > 0$, for all $\varrho \in [R_i,R_e]$.

By the classical change of variable $t = h(\varrho) := \int_{R_1}^{\varrho} \xi^{1-N}\, d\xi$, $\varrho = \varrho(t) := h^{-1}(t)$, we set

$$v(t) := U(\varrho(t)), \quad w(t) := \varrho(t)^{2(N-1)}\mathcal{Q}(\varrho(t)) \quad \text{and} \quad T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi,$$

this way it follows that problem (7.2.3) is equivalent to

$$\begin{cases} v''(t) + w_{\lambda,\mu}(t)\psi(v(t)) = 0, \\ v'(0) = v'(T) = 0, \end{cases} \tag{7.2.4}$$

with $v(t) > 0$, for all $t \in [0,T]$, see for instance [Bos11; FZ15b]. Hence, we enter in the framework of problem $(\mathscr{IN})$ and we can apply directly Theorem 7.1.1 to the system (7.2.4). Therefore we can state the following result.

Theorem 7.2.1. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$, $(\mathrm{H}\psi_4)$ *and* $(\mathrm{H}\psi_5)$. *Let $\mathcal{Q} \in L^1([R_i, R_e])$ with $\mathcal{Q}^+ \in L^\infty$ and suppose there exists $\sigma \in]R_i, R_e[$ such that*

$$\mathcal{Q}(\varrho) \geq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [R_i, \sigma],\ \mathcal{Q}(\varrho) \leq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [\sigma, R_e].$$

Suppose also that there are an interval $[t_1, t_2] \subseteq [R_i, \sigma]$ and a constant $\delta > 0$ such that $\mathcal{Q}^+(\varrho) \geq \delta$ for a.e. $\varrho \in [t_1, t_2]$. Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda, r, k) > 0$ such that for each $\mu > \mu^*$ the problem $(\mathscr{IN}_{\lambda,\mu,N})$ has at least $2k$ radially symmetric solutions which are nonincreasing in ϱ on $[R_i, R_e]$ and satisfy $0 < u(x) \leq r$ for each x with $|x| \in [\sigma, R_e]$.

7.2.2 Neumann problem for a ball

Let $R > 0$ be a fixed radius and let us consider the Neumann problem $(\mathscr{IN}_{\lambda,\mu,N})$ for the domain

$$\Omega := B(0, R).$$

We suppose that $\mathfrak{w}$ is as in (7.2.1), with $\mathcal{Q} \in L^1([0, R])$. By means of (7.2.2), our problem is reduced to

$$\begin{cases} U''(\varrho) + \dfrac{N-1}{\varrho} U'(\varrho) + \mathcal{Q}_{\lambda,\mu}(\varrho)\psi(U(\varrho)) = 0, \quad 0 < \varrho \leq R, \\ U'(0) = U'(R) = 0, \end{cases} \tag{7.2.5}$$

with $U(x) > 0$, for all $\varrho \in [0, R]$, which has a singularity at $\varrho = 0$. The previous problem is in its turn equivalent to

$$\begin{cases} \left(\varrho^{N-1} U'(\varrho)\right)' + \varrho^{N-1}\mathcal{Q}_{\lambda,\mu}(\varrho)\psi(U(\varrho)) = 0, \quad 0 < \varrho \leq R, \\ U'(0) = U'(R) = 0, \end{cases}$$

with $U(x) > 0$, for all $\varrho \in [0, R]$. In this case, the following result holds.

Theorem 7.2.2. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying* $(\mathrm{H}\psi_1)$, $(\mathrm{H}\psi_4)$ *and* $(\mathrm{H}\psi_5)$. *Let $\mathcal{Q} \in L^1([0, R])$ with $\mathcal{Q}^+ \in L^\infty$ and suppose there exists $\sigma \in]0, R[$ such that*

$$\mathcal{Q}(\varrho) \geq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [0, \sigma],\ \mathcal{Q}(\varrho) \leq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [\sigma, R].$$

Suppose also that there are an interval $[t_1, t_2] \subseteq [0, \sigma]$ and a constant $\delta > 0$ such that $\mathcal{Q}^+(\varrho) \geq \delta$ for a.e. $\varrho \in [t_1, t_2]$. Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda, r, k) > 0$ such that for each $\mu > \mu^*$ the problem $(\mathscr{IN}_{\lambda,\mu,N})$ has at least $2k$ radially symmetric solutions which are nonincreasing in ϱ on $[0, R]$ and satisfy $0 < u(x) \leq r$ for each x with $|x| \in [\sigma, R]$.

Proof. Our proof follows verbatim that of Theorem 7.1.1. For this reason, we focus our attention only to those points which require some technical adjustments due to the presence of the singularity at $\rho = 0$. In particular, we will split our proof in two steps.

Under Lipschitz conditions. Let us truncate ψ as in (7.1.1) at the level $M > 0$, so that the differential equation in (7.2.5) can be read in the phase-plane equivalently as

$$\begin{cases} x' = y, \\ y' = -\frac{N-1}{t} y - \mathcal{Q}_{\lambda,\mu}(t)\psi_M(x), \end{cases} \tag{7.2.6}$$

with $t = \varrho > 0$. Notice that the associated initial value problem has a local solution which is unique and it can globally extended to the all interval $[0, R]$, since ψ is a locally Lipschitz continuous function and ψ_M is bounded. Therefore, the shooting method can be applied also in this context (cf. [CK87]).

With the scheme proposed in the previous section in mind, we discuss now the qualitative behavior of the solutions in both the intervals $[0, \sigma]$ and $[\sigma, R]$.

We start with the analysis of the solutions for $t \in [0, \sigma]$ and, without loss of generality, we suppose that $[t_1, t_2] \subseteq]0, \sigma]$. From

$$x'(t) = y(t) = -\lambda \frac{\int_0^t \xi^{N-1} w^+(\xi)\psi_M(x(\xi))\, d\xi}{t^{N-1}},$$

we obtain $y(t) \le 0$ for all $t \in [0, \sigma]$. Furthermore, analogously as in Lemma 7.1.2, there exists $\bar{t} \in [0, \sigma[$ such that $y(t) = 0$ for all $0 \le t \le \bar{t}$ and $y(t) < 0$ for all $t \in]\bar{t}, \sigma]$. We also find immediately a constant $C_M > 0$ such that any possible solution $(x(t), y(t))$ of (7.2.6) with $0 < x(0) \le M$ and $y(0) = 0$, satisfies

$$-C_M \le y(t) \le 0, \quad \forall\, t \in [0, \sigma].$$

Now we give an analogous result of Lemma 7.1.3. Indeed, within the same framework of that lemma and, in particular for d_j and ρ satisfying (7.1.4), we proceed as follows. Suppose that $(x(t), y(t))$ is a solution of (7.2.6) with $M \ge x(0) \ge d_j$, $y(0) = 0$ and $x(t_1) = d_j$. As in Lemma 7.1.3, we denote by $[t_1, \tilde{t}] \subseteq [t_1, t_2]$ the maximal closed subinterval of $[t_1, t_2]$ where $x(\cdot) \ge 0$ (and, necessarily, also $y(\cdot) \le 0$). From the equation

$$x'' + \frac{N-1}{t} x' + \lambda w^+(t)\psi(x) = 0, \tag{7.2.7}$$

with the position $z(t) := y(t)t^{N-1}$, we have

$$z'z + \lambda w^+(t) t^{2(N-1)} \psi(x) x' = 0.$$

Hence, it follows

$$z'(t)z(t) + \lambda\delta {t_1}^{2(N-1)} \psi(x) x' \ge 0, \quad \text{for a.e. } t \in [t_1, t_2],$$

which implies that the function $\xi \mapsto z(\xi)^2 + 2\lambda\delta {t_1}^{2(N-1)} P(x(\xi))$ is nondecreasing in $[t_1, \tilde{t}]$. From this, we obtain

$$-x'(\xi) = |y(\xi)| \ge \left(\frac{t_1}{t_2}\right)^{N-1} \sqrt{\lambda\delta\rho}\sqrt{d_j^2 - x(\xi)^2}, \quad \forall\, \xi \in [t_1, \tilde{t}].$$

Apart from a multiplicative constant, notice that the above inequality is like the one in (7.1.5), so that the same conclusion is achieved, if λ is taken sufficiently large, namely

$$\lambda\delta P^\infty > \left(\frac{t_2}{t_1}\right)^{2(N-1)} \left(\frac{\pi}{2(t_2 - t_1)}\right)^2.$$

Finally, we give an analogous result of Lemma 7.1.4. Indeed, within the same framework of that lemma and, in particular for a given $\vartheta \in]0, 1[$, and for ε and β_j^ε satisfying (7.1.6), we proceed as follows. Assume that $(x(t), y(t))$ is a solution of (7.2.6) with $0 < x(0) = \beta_j^\varepsilon \le M$ and $y(0) = 0$. As in Lemma 7.1.3 we suppose by contradiction that it is not true that $x(t) \ge \theta\beta_j^\varepsilon$ and then consider a maximal interval $[0, \hat{t}] \subset [0, \sigma]$ such that (7.1.7) holds.

From the equation (7.2.7), we obtain

$$x''x' + \lambda\|w^+\|_\infty \psi(x)x' \le x''x' + \lambda w^+(t)\psi(x)x' = -\frac{N-1}{t}(x')^2 \le 0$$

which implies that the function $\xi \mapsto x'(\xi)^2 + 2\lambda\|w^+\|_\infty P(x(\xi))$ is nonincreasing in $[0, \hat{t}]$. From now on we have only to repeat the same proof of Lemma 7.1.4. With these results at hand and since the shooting method is working, we can recover Lemma 7.1.5 and Proposition 7.1.8.

At this point we consider the analysis of the solutions for $t \in [\sigma, R]$. In this case, we are far from the singularity (which is at $t = 0$) and so we can repeat a similar analysis previously performed in Lemma 7.1.6 and so we recover Proposition 7.1.10.

At last, by Proposition 7.1.8 and Proposition 7.1.10, we get the same conclusion of the proof of the Step I in Theorem 7.1.1.

Free from Lipschitz conditions. Assume now that ψ is only continuous. Then, we can apply the standard techniques recalled in the proof of Theorem 7.1.1. □

Finally, if one is interested in differential equations involving nonlinear differential operators, such as p-Laplacians a condition analogous to $(\mathrm{H}\psi_4)$ is considered in [BD09] for Neumann problems in the p-Laplacian setting. In this respect, we remark that our technique can be also adapted to study the problem

$$\begin{cases} (\phi(u'))' + w(t)\psi(u) = 0, \\ u'(0) = u'(T) = 0, \end{cases}$$

with $u(t) > 0$ for all $t \in [0, T]$, using information about the time-mapping associated with

$$(\phi(u'))' + \psi(u) = 0.$$

In this case, estimates for the time-mappings are already done in [GHMZ11; MZ93; NOZ00; OZ96] and could be fruitfully exploited to extend Theorem 7.1.1 as well as Theorem 7.2.1 and Theorem 7.2.2 to the case of more general differential operators, such as p-Laplacians or ϕ-Laplacians.

8. Nonlinearities arising in population genetics

In this chapter, based on [Sov17; FS18], we present some results of multiplicity of positive solutions for indefinite Neumann problems of the form

$$(\mathscr{I\!N}) \qquad \begin{cases} u'' + w(t)\psi(u) = 0, \\ u'(\omega_1) = u'(\omega_2) = 0, \end{cases}$$

where $\omega_1, \omega_2 \in \mathbb{R}$ with $\omega_1 < \omega_2$, the weight $w\colon [\omega_1, \omega_2] \to \mathbb{R}$ is a sign-changing function and $\psi\colon [0,1] \to \mathbb{R}^+$ is a continuous function satisfying

$$(\mathrm{H}\psi_{1\,\mathrm{bis}}) \qquad \psi(0) = \psi(1) = 0, \quad \psi(\xi) > 0 \textit{ for every } \xi \in]0,1[.$$

This way, we enter in the frame of the study case given by nonlinearities of Type 2 introduced in Chapter 5.

Indefinite Neumann problems with nonlinearities ψ satisfying $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ are a very important issue in the field of population genetics, starting from the pioneering works [BH90; Fle75; Hen81; Sen83]. Still in this topic, in Section 8.1, we solve a conjecture from [LN02] which, adapted to the one-dimensional case, states what follows.

Conjecture. *Given $\Omega =]\omega_1, \omega_2[$, suppose that w changes sign in Ω and satisfies*

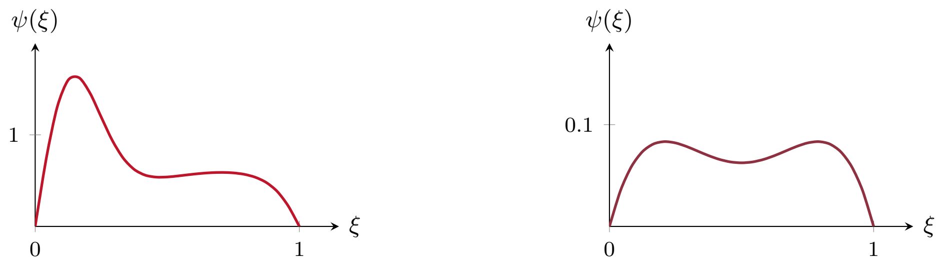
$$(\mathrm{H}w_5) \qquad \bar{w} = \int_\Omega w(t)\,dt < 0.$$

If the nonlinearity ψ satisfies $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and, moreover,

$(\mathrm{H}\psi_{\mathrm{conj}})$ *ψ is not concave and $\xi \mapsto \psi(\xi)/\xi$ is monotone decreasing $]0,1[$,*

then problem $(\mathscr{I\!N})$ has at most one non-trivial positive solution u with $0 < u(t) < 1$ for every $t \in \overline{\Omega}$, which, if it exists, is globally asymptotically stable (cf. [LNN13, p. 4364]).

In more detail, for problem $(\mathscr{I\!N})$, we build up two examples with weights verifying $(\mathrm{H}w_5)$ and nonlinearities ψ satisfying $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and $(\mathrm{H}\psi_{\mathrm{conj}})$, giving this way a negative answer to this conjecture. Indeed, we give evidence of the existence of at least three positive solutions to that problems. Accordingly, also the other question marks in Table 5.1 are solved. This return would not have been possible without previous investigations on Dirichlet problems associated with sublinear elliptic equations (cf. Chapter 6). In



(a) Graph of the nonlinear term $\psi(\xi)$ defined as in Section 8.1.1.

(b) Graph of the nonlinear term $\psi(\xi)$ defined as in Section 8.1.2.

Figure 8.1: Examples of nonlinearities satisfying $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and $(\mathrm{H}\psi_{\mathrm{conj}})$ considered in Section 8.1.

Figure 8.1 we report the graphs of the functions ψ involved, that clearly show the non concave feature of the nonlinearities.

As far as we know, in order to achieve both results of uniqueness and multiplicity, lot of attention has been given to the proprieties of the nonlinearity ψ (see for instance [LN02; LNN13; LNS10]). For example, dealing with a nonlinearity ψ satisfying $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and assuming the further condition

$$(\mathrm{H}\psi_6) \qquad \lim_{\xi \to 0^+} \frac{\psi(\xi)}{\xi} = 0,$$

in [LNS10] a multiplicity result of positive solutions for $(\mathscr{IN})$ is proved. Actually, the result is more general since it involves the case of Neumann BVPs associated with PDEs. Nevertheless, in our simplified case, it states that: if w verifies condition $(\mathrm{H}w_5)$ and ψ satisfies $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and $(\mathrm{H}\psi_6)$ along with $\lim_{\xi\to 0^+}\psi(\xi)/\xi^k > 0$ for some $k > 1$, then the Neumann problem associated with $du'' + w(t)\psi(u) = 0$ has at least two positive solutions for $d > 0$ sufficiently small.

On the other hand, beside ψ, also the shape of the graph of the weight w can give surprises. Indeed, in Section 8.2, we perform our analysis of $(\mathscr{IN})$ paying attention on the weight term. By considering a different dispersal parameter d with respect to that assumed in [LNS10], we study the effects that an indefinite weight has on the dynamics of problem $(\mathscr{IN})$ when the nonlinearity ψ satisfies $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and $(\mathrm{H}\psi_6)$. At last, we prove in Theorem 8.2.1 that the dynamics could be richer than the ones suggested in [LNS10].

Overview on population genetics

Population genetics is a field of the biology concerning the genetic structure inside the populations and studies the changes in the genetic sequence. The genome evolution is influenced by selection, recombination, harmful and beneficial mutations, among others.

Mathematical models of population genetics can be described by relative genotypic frequencies or relative allelic frequencies, that may depend on both space and time. A common assumption is that individuals mate at random in a habitat (which can be bounded or not) with respect to the locus under consideration. Furthermore, the population is usually considered large enough so that frequencies can be treated as deterministic. This way, a probability is associated to the relative frequencies of genotypes/alleles. The dynamics of gene frequencies are the result of some genetic principles along with several environmental influences, such as selection, segregation, migration, mutation, recombination and mating, that lead to different evolutionary processes like adaptation and speciation [Bür14].

Among these influences, by *natural selection* we mean that some genotypes enjoy a survival or reproduction advantage over other ones. This way, the genotypic and allelic frequencies change in accord to the proportion of progeny to the next generation of the various genotypes which is named fitness. Thinking to model real-life populations, we have to take into account which is unusual that the selection factor acts alone. Since every organism lives in environments that are heterogeneous, another considerable factor is the natural subdivision of the population that mate at random only locally. Thus, *migration* is often considered as a factor that affects the amount of genetic change. If the population size is sufficiently large and the selection is restricted to a single locus with two alleles, then deterministic models continuous in time and space lead to mathematical problems which involve a single nonlinear partial differential equation of reaction-diffusion type, as introduced in Chapter 5.

In this direction, a seminal paper was given by [Fis37]. In that work, the Author studied the frequency of an advantageous gene for a uniformly distributed population in a one-dimensional habitat which spreads through an intensity constant selection term. Accordingly, a mathematical model of a *cline* was built up as a *non-constant stationary solution* of the nonlinear diffusion equation in question. The term cline was coined by [Hux38]: "Some special term seems desirable to direct attention to variation within groups, and I propose the term cline, meaning a gradation in measurable characters." One of the major causes of cline's occurrence is the migration or the selection which favors an allele in a region of the habitat and a different one in another region. The *steepness of a cline* is considered as an indicating character of the level of the geographical variation. Another contribution comes from [Hal48], who has studied the cline's stability by considering as a selection term a stepwise function which depends on the space and changes its sign.

Some meaningful generalizations of these models have been performed, for example, in [Fis50] by introducing a linear spatial dependence in the selection term; in [Sla73] by considering a different diffusion term that can model barriers and in [Nag75] by taking into account population not necessarily uniformly distributed and terms of migration-selection that depend on both space and time. During the past decade, these mathematical treatments have opened the door to a great amount of works that investigated the existence, uniqueness and stability of clines (see [Con75; Fle75; Nag76; FP77; Nag78; Pel78] for the earliest contributions).

Understanding the processes that act in order to have non-constant genetic polymorphisms is an important challenge in population genetics. Among several models proposed within this field, let us focus on migration-selection models, continuous in space and in time, so that we recall some basics on population genetics in which the genetic diversity occurs in one locus with two alleles, A_1 and A_2, that leads to reaction-diffusion equations (cf. [Fle75] and [Hen81]).

By considering a population continuously distributed in a bounded habitat, say Ω, we assume that the genetic diversity is the result only of the joint action of dispersal within Ω and selective advantage for some genotype. The genetic structure of the population is measured by the frequencies $u(x,t)$ and $(1-u(x,t))$ at time t and location $x \in \Omega$ of A_1 and A_2, respectively.

Thus the mathematical formulation of this kind of migration-selection model leads to the following semilinear parabolic PDE:

$$\frac{\partial u}{\partial t} = \Delta u + \lambda w(x)\psi(u) \quad \text{in } \Omega\times]0,\infty[, \tag{8.0.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$ is a bounded domain with boundary $\partial\Omega$ of class C^2. The term $\lambda w(x)\psi(u)$ models the effect of the natural selection. More in detail, the real parameter $\lambda > 0$ plays the role of the ratio of the selection intensity and the function $w \in L^\infty(\Omega)$ represents the local selective advantage (if $w(x) > 0$), or disadvantage (if $w(x) < 0$), of the gene at the position $x \in \Omega$. Moreover, following [Fle75] and [Hen81], we consider a function $\psi\colon [0,1] \to \mathbb{R}$ of class C^2 satisfying

($\mathrm{H}\psi_7$) $\psi(0) = \psi(1) = 0, \psi(\xi) > 0$ *for every* $\xi \in]0,1[$ *and* $\psi'(0) > 0 > \psi'(1)$,

which is a particular case of ($\mathrm{H}\psi_{1\,\mathrm{bis}}$). We also impose that there is no-flux of genes into or out of the habitat Ω, namely we assume that

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega\times]0,\infty[. \tag{8.0.2}$$

Since $u(x,t)$ is a frequency, then we are interested only in positive solutions of (8.0.1)-(8.0.2) such that $0 \le u \le 1$.

By the analysis developed in [Hen81], we know that if ψ satisfies (Hψ_7) and $0 \le u(\cdot,0) \le 1$ in Ω, then $0 \le u(x,t) \le 1$ for all $(x,t) \in \Omega \times]0,\infty[$ and equation (8.0.1) defines a dynamical system in

$$X := \{u \in H^1(\Omega) : 0 \le u(x) \le 1, \text{ a.e. in } \Omega\}.$$

Moreover, the stability of the solutions is determined by the equilibrium solutions in the space X. Clearly, a stationary solution of the problem (8.0.1)-(8.0.2) is a solution u of

$$(\mathscr{I}\mathscr{N}_{\lambda,N}) \qquad \begin{cases} \Delta u + \lambda w(x)\psi(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < u(x) < 1$ for all $x \in \overline{\Omega}$. Notice that $u \equiv 0$ and $u \equiv 1$ are constant trivial solutions to problem $(\mathscr{I}\mathscr{N}_{\lambda,N})$, that correspond to monomorphic equilibria, namely when, in the population, the allele A_2 or A_1, respectively, is gone to fixation. So, the maintenance of genetic diversity is examined by seeking for the existence of polymorphic (i.e. non-constant) stationary solutions/clines, that are solutions u to system

By Remark 7.1 (Chapter 7), we stress again the fact that assumption $\psi(\xi) > 0$ for every $\xi \in]0,1[$ lead to a necessary condition for positive solutions of problem $(\mathscr{I}\mathscr{N}_{\lambda,N})$, namely the function w attains both positive and negative values. Furthermore, it is a well-known fact that the existence of positive solutions of $(\mathscr{I}\mathscr{N}_{\lambda,N})$ depends on the sign of

$$\bar{w} := \int_\Omega w(x)\,dx. \tag{8.0.3}$$

Indeed, for the linear eigenvalue problem $-\Delta u(x) = \lambda w(x)u(x)$, under Neumann boundary condition on Ω, the following facts hold: if $\bar{w} < 0$, then there exists a unique positive eigenvalue having an associated eigenfunction which does not change sign; on the contrary, if $\bar{w} \ge 0$ such an eigenvalue does not exist and 0 is the only non-negative eigenvalue for which the corresponding eigenfunction does not vanish [BL80, Theorem 3.13].

Furthermore, under the additional assumption of concavity for the nonlinearity:

$$(\mathrm{H}\psi_8) \qquad \psi''(\xi) \le 0, \quad \forall \xi > 0,$$

it follows that, if $\bar{w} < 0$, then there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$ problem (8.0.1)-(8.0.2) has a unique positive non-constant stationary solution which is asymptotically stable [Hen81, Theorem 10.1.6].

After these works a great deal of contributions appeared in order to complement these results of existence and uniqueness on population genetics, [BPT88; BLT89; BH90]; or to consider also unbounded habitats [FP81]; or even to treat more general uniformly elliptic operators [SH82; Sen83]. Taking into account these works, in [LN02] the migration-selection model with an isotropic dispersion, that is identified with the Laplacian operator, was generalized to an arbitrary migration, which involves a strongly uniformly elliptic differential operator of second order (see also [Nag89; Nag96] for the derivation of this model as a continuous approximation of the discrete one).

By modeling single locus diallelic populations, there is an interesting family of nonlinearities which satisfies (Hψ_7) and allows to consider different phenotypes of alleles, A_1 and A_2. This family can be obtained by considering the map $\psi_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\psi_k(\xi) := \xi(1-\xi)(1+k-2k\xi), \tag{8.0.4}$$

where $-1 \le k \le 1$ represents the degree of dominance of the alleles independently of the space variable [Nag75]. In this special case, if $k = 0$ then the model does not present any kind of dominance, instead, if $k = 1$ or $k = -1$ then the allelic dominance is relative to A_1, in the first case, and to A_2 in the second one (the last is also equivalent to said that A_1 is recessive). In view of this, we can make the following two observations.

Remark 8.1. In the case of no dominance, i.e. $k = 0$, from (8.0.4) we have $\psi_0(\xi) = \xi(1-\xi)$ which is a concave function. Therefore, we can enter in the settings considered by [Hen81]. So if $w(x) > 0$ on a set of positive measure in Ω and $\bar{w} < 0$, then for λ sufficiently large there exists a unique positive non-trivial equilibrium of the equation $\partial u/\partial t = \Delta u + \lambda w(x)u(1-u)$ for every $(x,t) \in \Omega \times]0,\infty[$ under the boundary condition (8.0.2).

Remark 8.2. In the case of completely dominance of allele A_2, i.e. $k = -1$, from (8.0.4) we have $\psi_{-1}(\xi) = 2\xi^2(1-\xi)$ which is not a concave function. Thanks to the results in [LNS10], if $w(x) > 0$ on a set of positive measure in Ω and $\bar{w} < 0$, then for λ sufficiently large there exist at least two positive non-trivial equilibrium of the equation $\partial u/\partial t = \Delta u + \lambda w(x)2u^2(1-u)$ for every $(x,t) \in \Omega \times]0,\infty[$ under the boundary condition (8.0.2).

We observe that the map $\xi \mapsto \psi_0(\xi)/\xi$ is *strictly decreasing* with $\psi_0(\xi)$ *concave*. On the contrary, the map $\xi \mapsto \psi_{-1}(\xi)/\xi$ is *not strictly decreasing* with $\psi_{-1}(\xi)$ *not concave*. Thus, from Remark 8.1 and Remark 8.2, it arises a natural question which involves the possibility to weaken the concavity assumption (Hψ_8) further to the monotonicity of the map $\xi \mapsto \psi(\xi)/\xi$, in order to get uniqueness results of non-trivial equilibria for problem (8.0.1)-(8.0.2). This dichotomy, already set out in Chapter 6 for Dirichlet BVPs $(\mathscr{I}\mathscr{D}_{\lambda,N})$, comes also in this context as an open question, firstly appeared in [LN02, Conjecture 5.1], known as the "conjecture of Lou and Nagylaki".

8.1 Answer to a conjecture of Lou and Nagylaki

In this section we focus on a conjecture stated in [LN02; LNN13] and so, from now on we tacitly consider a function $\psi\colon [0,1] \to \mathbb{R}$ of class C^2 which satisfies (Hψ_7) and (Hψ_{conj}). We concentrate into the one-dimensional case $N = 1$, with the intent to show that there exist more than one non-trivial stationary solution for equation:

$$\frac{\partial u}{\partial t} = u'' + \lambda w(x)\psi(u). \tag{8.1.1}$$

We take as a habitat an open interval $\Omega :=]\omega_1, \omega_2[$ with $\omega_1, \omega_2 \in \mathbb{R}$ and such that $\omega_1 < 0 < \omega_2$. This type of habitats, confined to one-dimensional spaces, have an intrinsic interest in modeling phenomena which occur, for example, in neighborhoods of rivers, sea shores or hills [Nag78]. As in [Nag75; Nag78], we assume that the weight term w is step-wise. At last, as usual, we indicate by $x = t$ the independent variable and we study the indefinite Neumann problem

$$(\mathscr{I}\mathscr{N}_\lambda) \qquad \begin{cases} u'' + \lambda w(t)\psi(u) = 0, \\ u'(\omega_1) = u'(\omega_2) = 0, \end{cases}$$

with $0 < u(t) < 1$ for all $t \in [\omega_1, \omega_2]$, where

$$w(t) := \begin{cases} -\alpha & x \in [\omega_1, 0[, \\ 1 & x \in [0, \omega_2]. \end{cases} \tag{8.1.2}$$

This way, we have

$$\bar{w} = -\omega_1\alpha + \omega_2,$$

with $\bar{w}$ defined as in (8.0.3) and we assume that $\bar{w}$ satisfies (Hw_5).

We will consider two particular functions ψ in order to provide a negative reply to the conjecture under examination. This answer follows a similar topological argument performed in Section 3.1 or in Section 6.1. More precisely, we are going to use the shooting method and, with the aid of some numerical computations, we give evidence of multiplicity of positive solutions for the corresponding problems in $(\mathscr{I}\mathscr{N}_\lambda)$. The shooting method relies on the study of the deformation of planar continua under the action of the vector field associated to the second order scalar differential equation in $(\mathscr{I}\mathscr{N}_\lambda)$, whose formulation, in the phase-plane $(x,y)=(u,u')$, is equivalent to the first order planar system

$$\begin{cases} x' = y, \\ y' = -\lambda w(t)\psi(x). \end{cases} \tag{8.1.3}$$

Solutions u of problem $(\mathscr{I}\mathscr{N}_\lambda)$ we are looking for are also solutions $(x(\cdot),y(\cdot))$ of system (8.1.3), such that $y(\omega_1)=0=y(\omega_2)$.

We set the interval $[0,1]$ contained in the x-axis as follows

$$X_{[0,1]} := \{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ y=0\}.$$

This way, as a real parameter r ranges between 0 and 1, we are interested in the solution, $(x(\cdot\,;\omega_1,r,0), y(\cdot\,;\omega_1,r,0))$, of the Cauchy problem with initial conditions

$$x(\omega_1)=r, \quad y(\omega_1)=0, \tag{8.1.4}$$

such that $(x(\omega_2;\omega_1,r,0), y(\omega_2;\omega_1,r,0))\in X_{[0,1]}$. Hence, let us consider the planar continuum Γ obtained by shooting $X_{[0,1]}$ forward from ω_1 to ω_2, namely

$$\Gamma := \{(x(\omega_2;\omega_1,r,0), y(\omega_2;\omega_1,r,0))\in\mathbb{R}^2 : r\in[0,1]\}.$$

We define the set of the intersection points between this continuum and the segment $[0,1]$ contained in the x-axis, as

$$\mathcal{S} := \Gamma\cap X_{[0,1]}.$$

Then, there exists an injection from the set of the solutions u of $(\mathscr{I}\mathscr{N}_\lambda)$ such that $0<u(t)<1$ for all $t\in[\omega_1,\omega_2]$ and the set $\mathcal{S}\setminus(\{(0,0)\}\cup\{(1,0)\})$. Recall that, for any $\tau_1,\tau_2\in[\omega_1,\omega_2]$, the Poincaré map for system (8.1.3), denoted by $\Phi_{\tau_1}^{\tau_2}$, is the planar map which at any point $z_0=(x_0,y_0)\in\mathbb{R}^2$ associates the point $(x(\tau_2),y(\tau_2))$ where $(x(\cdot),y(\cdot))$ is the solution of (8.1.3) with $(x(\tau_1),y(\tau_1))=z_0$. Notice that $\Phi_{\tau_1}^{\tau_2}$ is a global diffeomorphism of the plane onto itself. This way, the solution u of the Neumann problem with $u(\omega_1)=c$ is found looking at the first component of the map $t\mapsto\Phi_{\omega_1}^{t}(c,0)=(x(t),y(t))$, since, by construction, $u'(\omega_1)=y(\omega_1)=0$ and $u'(\omega_2)=y(\omega_2)=0$. This means that the set $\mathcal{S}$ is made by points such that, each of them determines univocally an initial condition, of the form (8.1.4), for which the solution $(x(\cdot),y(\cdot))$ of the Cauchy problem associated to (8.1.3) verifies $y(\omega_1)=0=y(\omega_2)$.

The study of the uniqueness of the positive solutions is based on the study, in the phase-plane (x,y), of the qualitative properties of the shape of the continuum Γ which is the image of $X_{[0,1]}$ under the action of the Poincaré map $\Phi_{\omega_1}^{\omega_2}$. More in detail, we are interested in find real values $c\in]0,1[$ such that

$$\Phi_{\omega_1}^{\omega_2}(c,0)\in\Phi_{\omega_1}^{\omega_2}\left(X_{[0,1]}\right)\cap X_{[0,1]}=\mathcal{S}.$$

Indeed, if Γ crosses the x-axis more than one time, out of the points $(0,0)$ and $(1,0)$, then $\#(\mathcal{S}\setminus(\{(0,0)\}\cup\{(1,0)\}))>1$ and so, we expect a result of non-uniqueness of positive solutions for equation (8.1.1).

8.1.1 First example

Taking into account the nonlinearities considered in [Fle75; Nag75] (cf. definition of functions in (8.0.4)), given a real parameter $h>0$, let us consider the family of maps $\hat{\psi}_h:[0,1]\to\mathbb{R}$ of class C^2 such that

$$\hat{\psi}_h(\xi) := \xi(1-\xi)(1-h\xi+h\xi^2).$$

By definition $\hat{\psi}(0)=0=\hat{\psi}(1)$. Moreover, to have $\xi\mapsto\hat{\psi}_h(\xi)/\xi$ monotone decreasing in $]0,1[$ it is sufficient to assume $0<h\le 3$. If the parameter h ranges in $]2,3]$, then it is straightforward to check that $\hat{\psi}_h$ is not concave and $\hat{\psi}_h(\xi)>0$ for every $\xi\in]0,1[$.

Let us fix $h=3$. Then, in this case, $\hat{\psi}_3(\xi)=\xi(1-\xi)(1-3\xi+3\xi^2)$ satisfies conditions $(\mathrm{H}\psi_7)$ and $(\mathrm{H}\psi_{\mathrm{conj}})$, see Figure 8.1 (a). As a consequence, we point out the following result of multiplicity.

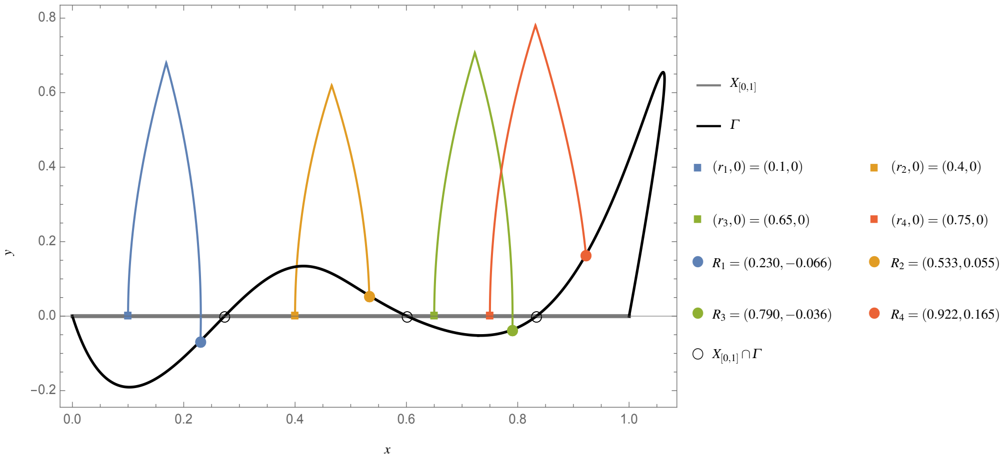


Figure 8.2: In the phase-plane (x, y): intersections between $X_{[0,1]}$ and $\Gamma = \Phi_{\omega_1}^{\omega_2}(X_{[0,1]})$; solutions of the Cauchy problem with initial conditions given by $(x(\omega_1), y(\omega_1)) = (r_i, 0)$ and numerical approximation of the values $R_i = (x(\omega_2\,; \omega_1, (r_i, 0)), y(\omega_2\,; \omega_1, (r_i, 0)))$ with $i = 1, \dots, 4$. The problem's setting $u'' + \lambda w(t)\psi(u) = 0$ is defined as in Proposition 8.1.1.

Proposition 8.1.1. *Let $\psi\colon [0,1] \to \mathbb{R}$ be such that*

$$\psi(\xi) := \xi(1-\xi)(1-3\xi+3\xi^2). \tag{8.1.5}$$

Assume $w : [\omega_1, \omega_2] \to \mathbb{R}$ be defined as in (8.1.2) with $\alpha = 1$, $\omega_1 = -0.21$ and $\omega_2 = 0.2$. Then, for $\lambda = 45$ the problem $(\mathscr{I}\mathscr{N}_\lambda)$ has at least 3 solutions such that $0 < u(t) < 1$ for all $t \in [\omega_1, \omega_2]$.

Notice that $\bar{w} = -0.01 < 0$, so we are in the hypotheses of the conjecture since (Hw_5) holds. Now we follow the scheme of the shooting method, in order to detect three non-trivial stationary solutions for the equation (8.1.1). This approach, with the help of numerical estimates, will enable us to prove Proposition 8.1.1.

In the phase-plane (x, y), Figure 8.2 shows the existence of at least four points $(r_i, 0) \in X_{[0,1]}$ with $i = 1, \dots, 4$ such that, by defining their images through the Poincaré map $\Phi_{\omega_1}^{\omega_2}$ as $R_i := ({R_i}^x, {R_i}^y) = \Phi_{\omega_1}^{\omega_2}(r_i, 0) \in \Gamma$ for every $i \in \{1, \dots, 4\}$, the following conditions

$${R_i}^x < 0 \text{ for } i = 1, 3, \quad {R_i}^y > 0 \text{ for } i = 2, 4,$$

are satisfied. This is done, for example, with the choice of the values $r_1 = 0.1$, $r_2 = 0.4$, $r_3 = 0.65$ and $r_4 = 0.75$. The solutions of the Cauchy problems associated to system (8.1.3), with initial conditions $(r_i, 0)$ for $i = 1, \dots, 4$, take at $t = \omega_2$ the values $R_1 = (0.230, -0.066)$, $R_2 = (0.922, 0.165)$, $R_3 = (0.790, -0.036)$ and $R_4 = (0.533, 0.055)$, truncated at the third significant digit. Therefore, we have that ${R_1}^v < 0 < {R_2}^v$, ${R_2}^v > 0 > {R_3}^v$ and ${R_3}^v < 0 < {R_4}^v$. Then, by a continuity argument (that means an application of the Mean Value Theorem), there exist at least three real values c_1, c_2 and c_3 such that

$$r_j < c_j < r_{j+1} \quad \text{and} \quad C_j := \Phi_{\omega_1}^{\omega_2}(c_j, 0) \in \mathcal{S} \setminus (\{(0,0)\} \cup \{(1,0)\}), \tag{8.1.6}$$

for every $j \in \{1, \dots, i-1\}$. So, let us see how to find such values.

The curve Γ is obtained integrating several systems of differential equations (8.1.3), with initial conditions z_0 taken within a uniform discretization of the interval $[0, 1]$, and then interpolating the approximated values of each solution $(x(t\,; \omega_1, z_0), y(t\,; \omega_1, z_0))$ at $t = \omega_2$. Hence, Γ represents the approximation of the image of $X_{[0,1]}$ under the action of the Poincaré map $\Phi_{\omega_1}^{\omega_2}$.

As Figure 8.2 suggests, the projection of Γ on its first component is not necessarily contained in the interval $[0, 1]$, which includes the only values of biological pertinence. Nonetheless, this does not avoid the existence of solutions u of the problem $(\mathscr{I}\mathscr{N}_\lambda)$ such

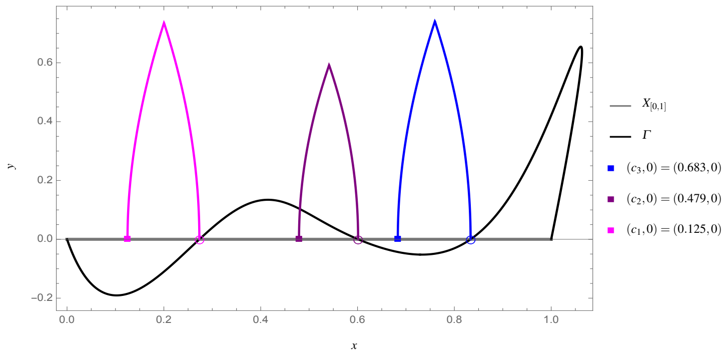


Figure 8.3: In the phase-plane (x, y): solutions of the Cauchy problem associated to system (8.1.3) with initial conditions given by $(x(\omega_1), y(\omega_1)) = (c_j, 0)$ with $j = 1, 2, 3$. The problem's setting $u'' + \lambda w(t)\psi(u) = 0$ is defined as in Proposition 8.1.1.

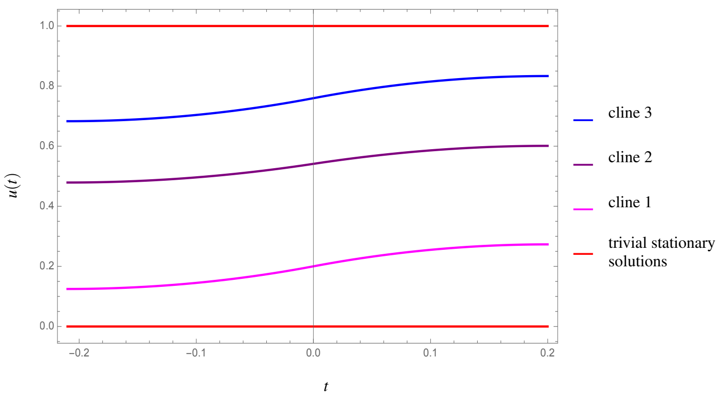


Figure 8.4: Non-constant stationary solutions and trivial stationary solutions ($u \equiv 0$ and $u \equiv 1$) for equation (8.1.1), found as positive solutions of the Neumann problem $u'' + \lambda w(t)\psi(u) = 0$ satisfying the framework of Proposition 8.1.1.

that $0 < u(t) < 1$ for all $t \in [\omega_1, \omega_2]$. This way, by means of a fine discretization of $X_{[0,1]}$, we have found the approximate values of the intersection points $C_j \in \Gamma \cap X_{[0,1]}$, with $j = 1, 2, 3$. In this case they are: $C_1 = (0.273, 0)$, $C_2 = (0.601, 0)$ and $C_3 = (0.833, 0)$, truncated at the third significant digit (see Figure 8.2). The intersection points between $X_{[0,1]}$ and its image Γ through the Poincaré map $\Phi_{\omega_1}^{\omega_2}$, namely C_j with $j = 1, 2, 3$, are in agreement with the previous predictions.

At last, we computed the values $c_1 = 0.125$, $c_2 = 0.479$ and $c_3 = 0.683$, which verify the required conditions (8.1.6). For $j = 1, 2, 3$, in Figure 8.3 are represented the trajectories of the solutions of the initial value problem

$$\begin{cases} u'' + \lambda w(t)\psi(u) = 0, \\ u(\omega_1) = c_j, \\ u'(\omega_1) = 0, \end{cases} \tag{8.1.7}$$

that, by construction, satisfy $u'(\omega_2) = 0$.

We observe also that the values of each solution u of the three different initial value problems range in $]0, 1[$ as desired. Once found the values c_j with $j = 1, 2, 3$, a numerically result of multiplicity of positive solutions is achieved. Indeed, in Figure 8.4, we display the approximation of the three non-trivial stationary solutions u of equation (8.1.1) that are identified by the points $C_j \in (\mathcal{S} \setminus (\{(0, 0)\} \cup \{(1, 0)\}))$, with $j = 1, 2, 3$.

We conclude with some remarks regarding the dependence of the number of positive solutions of $(\mathscr{I}\mathscr{N}_\lambda)$ with respect to the parameter λ. With this aim, we take into account the bifurcation diagram in Figure 8.5, which plots initial data against selection intensity rate. Numerical evidence suggests the existence of a range of λ where one could find results of multiplicity of positive solutions. Accordingly, one could argue that there exist at least two real values λ_*, $\lambda^* > 0$ such that for each $\lambda \in]\lambda_*, \lambda^*[$ there exist at least three non-trivial stationary solutions u of equation (8.1.1). It is interesting to notice that such kinds of bifurcation diagrams, presenting an "isola" coupled with an unbounded branch, are not new in literature and have been observed by [LGMM05; LGT14; LGTZ14] for reaction-diffusion equations with different nonlinearities and boundary conditions than those treated here.

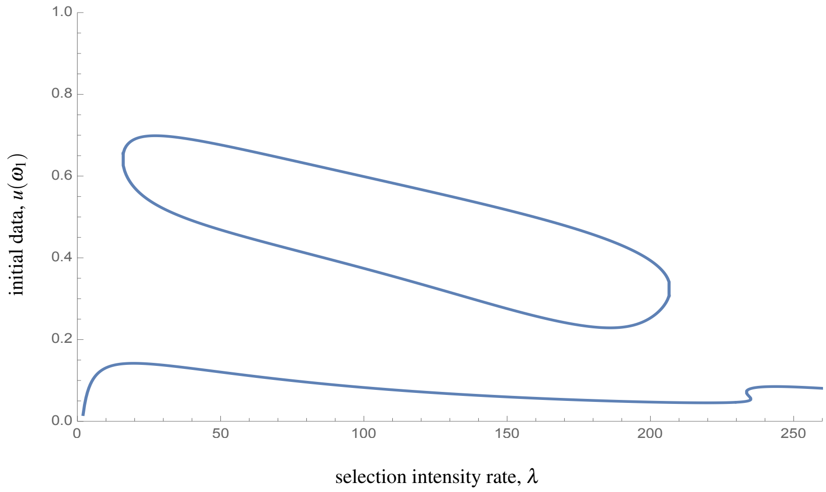


Figure 8.5: In the framework of Proposition 8.1.1, bifurcation diagram for the Neumann problem associated with $u'' + \lambda w(t)\psi(u) = 0$.

8.1.2 Second example

We refer now to the application given in Proposition 6.1.6 at p. 58 and we adapt it to our purposes. So we consider, the nonlinear term $\tilde{\psi} : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\tilde{\psi}(\xi) := \left(10\xi e^{-25\xi^2} + \frac{\xi}{|\xi| + 1}\right)$$

that, as already observed, satisfies $(\mathrm{H}\psi_{\mathrm{conj}})$. Moreover, $\tilde{\psi}(0) = 0$ and $\tilde{\psi}(\xi) > 0$ for every $\xi > 0$, but $\tilde{\psi}$ does not take value zero in $\xi = 1$, since $\tilde{\psi}(1) = 10e^{-25} + 1 \neq 0$. To satisfy all the conditions in $(\mathrm{H}\psi_7)$, it is sufficient to multiply $\tilde{\psi}$ by the term $\arctan(m(1-x))$ with $m > 0$, see Figure 8.1 (b). This way, the following result holds.

Proposition 8.1.2. *Let $\psi \colon [0, 1] \to \mathbb{R}$ be such that*

$$\psi(\xi) := \left(10\xi e^{-25\xi^2} + \frac{\xi}{|\xi| + 1}\right) \arctan(10 - 10\xi). \tag{8.1.8}$$

Assume $w : [\omega_1, \omega_2] \to \mathbb{R}$ be defined as in (8.1.2) with $\alpha = 2.4$, $\omega_1 = -0.255$ and $\omega_2 = 0.6$. Then, for $\lambda = 3$ the problem $(\mathscr{I}\mathscr{N}_\lambda)$ has at least 3 solutions such that $0 < u(t) < 1$ for all $t \in [\omega_1, \omega_2]$.

Notice that, under the assumptions of Proposition 8.1.2, the hypotheses of the conjecture are now all satisfied since $\bar{w} = -0.012 < 0$.

So, our main interest is in finding real values $r_i \in]0, 1[$ with $i \in \mathbb{N}$ such that, given $R_i := ({R_i}^u, {R_i}^v) = \Phi_{\omega_1}^{\omega_2}(r_i, 0)$, it follows

$${R_i}^v < 0 \text{ for } i = 2\ell + 1, \quad {R_i}^v > 0 \text{ for } i = 2\ell, \text{ with } \ell \in \mathbb{N}.$$

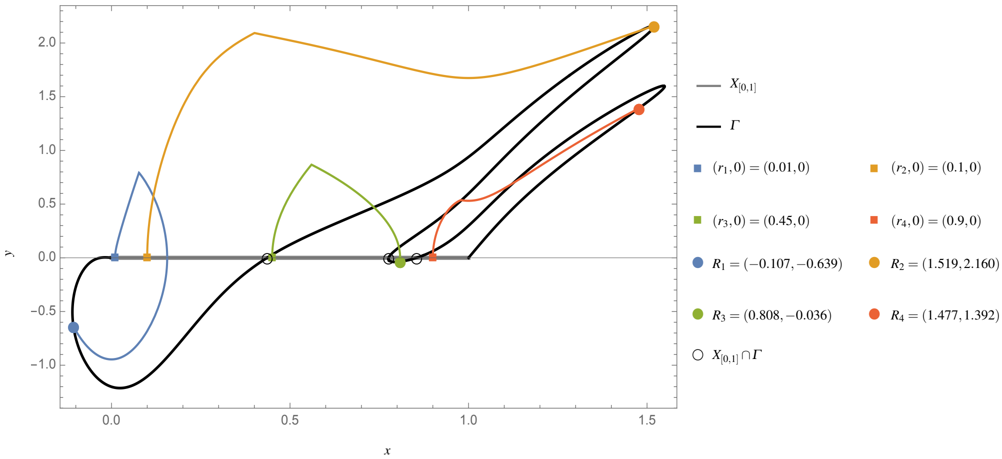


Figure 8.6: In the phase-plane (x,y): intersections between $X_{[0,1]}$ and $\Gamma = \Phi_{\omega_1}^{\omega_2}(X_{[0,1]})$; solutions of the Cauchy problem with initial conditions given by $(x(\omega_1), y(\omega_1)) = (r_i, 0)$ and numerical approximation of the values $R_i = (x(\omega_2\,; \omega_1, (r_i, 0)), y(\omega_2\,; \omega_1, (r_i, 0)))$ with $i = 1, \ldots, 4$. The problem's setting $u'' + \lambda w(t)\psi(u) = 0$ is defined as in Proposition 8.1.2.

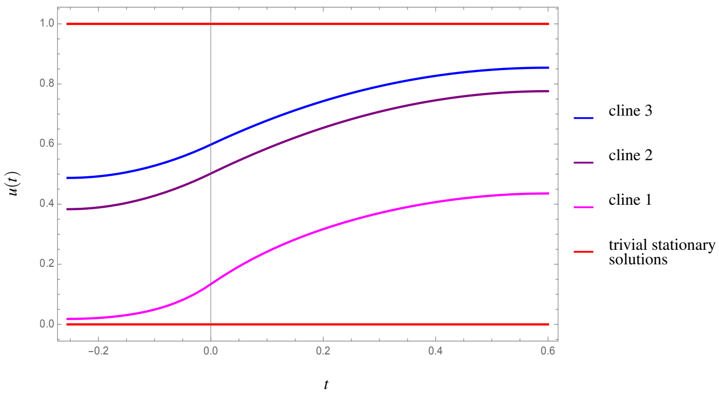


Figure 8.7: Non-constant stationary solutions and trivial stationary solutions ($x \equiv 0$ and $y \equiv 1$) for equation (8.1.1), found as positive solutions of the Neumann problem $u'' + \lambda w(t)\psi(u) = 0$ satisfying the framework of Proposition 8.1.2.

Looking at Figure 8.6, we notice the existence of more than one intersection point between the continuum Γ and the u-axis such that their abscissa is contained in the open interval $]0,1[$.

This way, the previous observation suggests us the following analysis. By choosing the values $r_1 = 0.01$, $r_2 = 0.1$, $r_3 = 0.45$ and $r_4 = 0.9$, we compute the points R_i for $i = 1, \ldots, 4$. All the results achieved are truncated at the third significant digit and so we obtain ${R_1}^v = -0.639 < 0$, ${R_2}^v = 2.160 > 0$, ${R_3}^v < -0.036$ and ${R_4}^v = 1.392 > 0$. The numerical details are thus represented in Figure 8.6.

At this point, an application of the Intermediate Value Theorem guarantees the existence of at least three initial conditions $(c_j, 0)$ with $j = 1, 2, 3$, such that each respective solution of the initial value problem (8.1.7) is also a positive solution of the Neumann problem $(\mathscr{I}\mathscr{N}_\lambda)$ we are looking for. Indeed, the values $c_1 = 0.436$, $c_2 = 0.776$ and $c_3 = 0.854$ satisfy the conditions in (8.1.6). Finally, we display the approximation of the three non-trivial stationary solutions u of equation (8.1.1) in Figure 8.7.

We now direct our attention to the influence of the selection intensity rate on the number of positive solutions of the Neumann problem $(\mathscr{I}\mathscr{N}_\lambda)$. As previously observed, we could find, at least numerically, a range of multiplicity of positive solutions with respect to the parameter λ. This is due to the presence of both an isolated bounded component ("isola") and an unbounded branch, as it is shown in Figure 8.8, for the resulting bifurcation diagram.

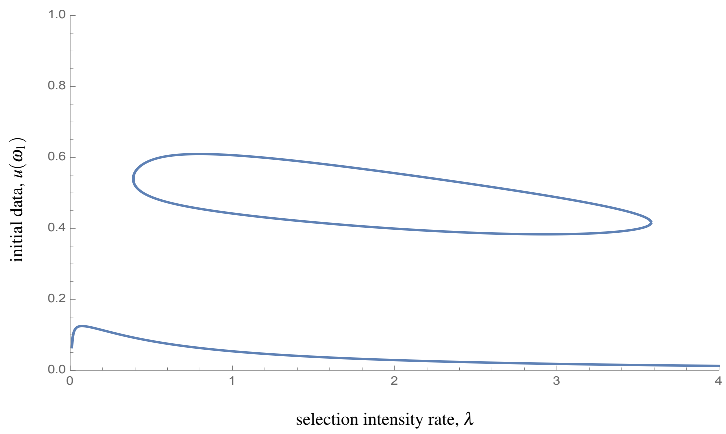


Figure 8.8: In the framework of Proposition 8.1.2, bifurcation diagram for the Neumann problem associated with $u'' + \lambda w(t)\psi(u) = 0$.

Remark 8.1.3. Finally, uniqueness of positive solutions in general is not guaranteed for indefinite Neumann problems $(\mathscr{I\!N}_\lambda)$ whose indefinite weight w is defined on a bounded domain Ω such that verifies condition $(\mathrm{H}w_5)$ and the nonlinear term ψ is a function satisfying $(\mathrm{H}\psi_7)$ and $(\mathrm{H}\psi_{\mathrm{conj}})$. Moreover, since from the concavity of $\psi(\xi)$ follows also the concavity of $\psi(1-\xi)$, one could argue whether the uniqueness of a nontrivial positive solution is guaranteed under the extra condition that the map $\xi \mapsto \psi(1-\xi)/\xi$ is decreasing. Thanks to our first example, which involves (8.1.5), we actually observe that this additional hyphotesis is not sufficient for achieve a result of uniqueness. The approach suggested in the present paper allows to consider also different sign-changing weights satisfying condition $(\mathrm{H}w_5)$ or even $\int_\Omega w(t)\,dt \geq 0$. ◁

8.2 Three positive solutions for a class of Neumann problems

In this section we study the indefinite Neumann problem $(\mathscr{I\!N})$ paying more attention to the dynamical effects produced by the weight term instead of the ones that are produced by the nonlinearity. Our investigations are motivated by the results achieved in [BGH05; Bos11; BFZ18; FZ15a; FZ15b; FZ17; GHZ03; GRLG00; LG00] where the authors established multiplicity results of positive solutions in relation to the nodal behavior of the weight w, dealing with different BVPs compared to the one treated here.

Through this section we tacitly assume that $\psi\colon [0,1] \to \mathbb{R}^+$ is a locally Lipschitz continuous function which satisfies $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and $(\mathrm{H}\psi_6)$. Without loss of generality, in the sequel we suppose $[\omega_1, \omega_2] := [0, T]$.

In our context, a solution $u(t)$ of problem

$$(\mathscr{I\!N}) \qquad \begin{cases} u'' + w(t)\psi(u) = 0, \\ u'(0) = u'(T) = 0, \end{cases}$$

is meant in the Carathéodory's sense and is such that $0 \leq u(t) \leq 1$ for all $t \in [0,T]$.

In analogy with Section 7.1, we assume here that the weight term has a "positive hump" followed by a "negative hump" and another "positive hump". Hence, we suppose that

$$(\mathrm{H}w_6) \qquad \begin{aligned} &\exists\, \sigma, \tau \text{ with } 0 < \sigma < \tau < T \text{ such that} \\ &w^+(t) \succ 0, \quad w^-(t) \equiv 0, \quad \text{on } [0,\sigma], \\ &w^+(t) \equiv 0, \quad w^-(t) \succ 0, \quad \text{on } [\sigma,\tau], \\ &w^+(t) \succ 0, \quad w^-(t) \equiv 0, \quad \text{on } [\tau,T], \end{aligned}$$

where, following a standard notation, $w(t) \succ 0$ means that $w(t) \geq 0$ almost everywhere on a given interval with $w \not\equiv 0$ on that interval. Moreover, given two real positive parameters λ and μ, we will consider the function

$$w(t) = w_{\lambda,\mu}(t) := \lambda w^+(t) - \mu w^-(t), \tag{8.2.1}$$

with $w^+(t)$ and $w^-(t)$ denoting the positive and the negative part of the function $w(t)$, respectively. In our framework, the dispersal parameter is thus modulated by the coefficients λ and μ. A weight term defined as in (8.2.1) is already addressed in different contexts (cf. [LG97; LG00; BFZ18] and Chapter 7).

With the above notation, problem $(\mathscr{IN})$ reads as follows

$$(\mathscr{IN}_{\lambda,\mu}) \qquad \begin{cases} u'' + \big(\lambda w^+(t) - \mu w^-(t)\big)\psi(u) = 0, \\ u'(0) = u'(T) = 0. \end{cases}$$

We are now in position to state our result of multiplicity of positive solutions to problem $(\mathscr{IN}_{\lambda,\mu})$.

Theorem 8.2.1. *Let $\psi\colon [0,1] \to \mathbb{R}^+$ be a locally Lipschitz continuous function satisfying $(\mathrm{H}\psi_{1\,\mathrm{bis}})$ and $(\mathrm{H}\psi_6)$. Let $w\colon [0,T] \to \mathbb{R}$ be an L^1-function satisfying $(\mathrm{H}w_6)$. Then, there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ there exists $\mu^*(\lambda) > 0$ such that for every $\mu > \mu^*(\lambda)$ problem $(\mathscr{IN}_{\lambda,\mu})$ has at least three positive solutions.*

Let us illustrate the dynamics of the parameter-dependent problem $(\mathscr{IN}_{\lambda,\mu})$ by means of the following example.

Example 8.2.2. Consider a nonlinearity similar to that of Remark 8.2, defined as

$$\psi(\xi) := \xi^2(1-\xi), \quad \xi \in [0,1], \tag{8.2.2}$$

and we take a weight term

$$w(t) := w_1 \mathbb{1}_{[0,\sigma]}(t) - w_2 \mathbb{1}_{]\sigma,\tau[}(t) + w_3 \mathbb{1}_{[\tau,T]}(t), \quad t \in [0,T], \tag{8.2.3}$$

where $w_1, w_2, w_3 \in \,]0,+\infty[$ are some fixed values (see Figure 8.9 (a)–(b) for a representation of these functions). The resulting problem $(\mathscr{IN}_{\lambda,\mu})$ is in the setting of Theorem 8.2.1 (see Figure 8.9 (c) for the numerical evidence of the existence of three positive solutions to problem $(\mathscr{IN}_{\lambda,\mu})$ for λ and μ sufficiently large). ◁

The strategy we follow to prove Theorem 8.2.1 is based on the shooting method. With this respect, we will study problem $(\mathscr{IN}_{\lambda,\mu})$ in the phase-plane $(x,y) = (u,u')$. Accordingly, the differential equation in $(\mathscr{IN}_{\lambda,\mu})$ can be equivalently written as a planar system in the following form

$$(\mathrm{S}_{\lambda,\mu}) \qquad \begin{cases} x' = y, \\ y' = -\big(\lambda w^+(t) - \mu w^-(t)\big)\psi(x). \end{cases}$$

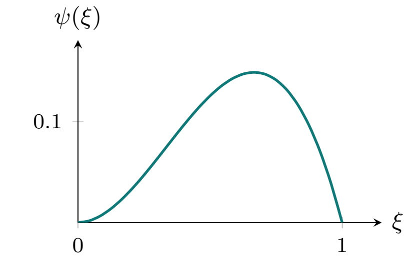
Thus, we consider the vector field associated with $(\mathrm{S}_{\lambda,\mu})$ in order to look at the corresponding deformation of the set $X_{[0,1]} := [0,1] \times \{0\}$. In particular, we look for intersection points between two planar continua: the one obtained from shooting the set $X_{[0,1]}$ forward in time over $[0,\tau]$ with the other one obtained from shooting again the same set $X_{[0,1]}$ backward in time over $[\tau,T]$.

8.2.1 Technical lemmas

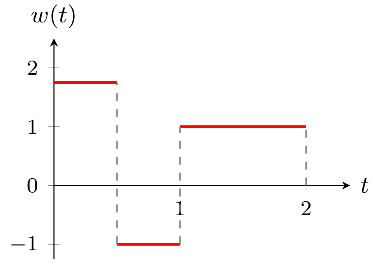
Before passing to the proof of Theorem 8.2.1, we first develop some estimates for the solutions of the Cauchy problems associated with $(\mathrm{S}_{\lambda,\mu})$.

First of all, we extend the function ψ continuously to the whole real line, by setting

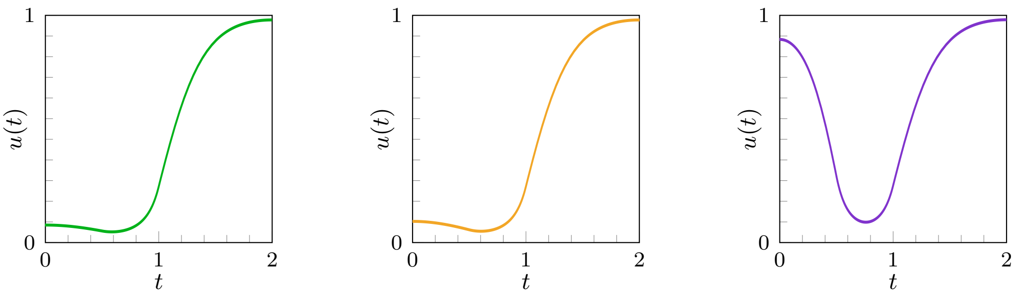
$$\psi(\xi) = 0, \quad \text{for } \xi \in \,]-\infty,0[\,\cup\,]1,+\infty[.$$



(a) Graph of the nonlinear term ψ defined as in (8.2.2).



(b) Graph of the weight term $w(t)$ defined as in (8.2.3) with $\sigma = 0.5$, $\tau = 1$, $T = 2$, $w_1 = 1.75$, $w_2 = 1$, $w_3 = 1$.



(c) Three positive solutions of $(\mathscr{IN}_{\lambda,\mu})$ for $\lambda = 25$ and $\mu = 500$.

Figure 8.9: Multiplicity of positive solutions for the indefinite Neumann problem $(\mathscr{IN}_{\lambda,\mu})$ as in the framework of Example 8.2.2.

The extension is still denoted by ψ. In this manner, any solution of a Cauchy problem associated with $(\mathrm{S}_{\lambda,\mu})$ is globally defined on $[0,T]$.

Secondly, it is convenient to introduce the following notation:

$$W^{\pm}(t',t'') := \int_{t'}^{t''} w^{\pm}(\xi)\,d\xi, \quad t',t'' \in [0,T] \text{ with } t' \leq t''.$$

Moreover, we set

$$\psi_*(\kappa',\kappa'') := \min_{\xi\in[\kappa',\kappa'']} \psi(\xi), \quad \kappa',\kappa'' \in [0,1] \text{ with } \kappa' < \kappa''.$$

In the interval $[0,\sigma]$ system $(\mathrm{S}_{\lambda,\mu})$ reduces to

$$\begin{cases} x' = y, \\ y' = -\lambda w^+(t)\psi(x). \end{cases} \tag{8.2.4}$$

Since $W^+(0,0) = 0$, $W^+(0,\sigma) > 0$ and $t \mapsto W^+(0,t)$ is a continuous non-decreasing map on $[0,\sigma]$, without loss of generality, we can suppose that

$$W^+(0,t) > 0, \quad \forall\, t \in \,]0,\sigma].$$

Otherwise, there exists a maximal interval $[0,t_0]$ where $W(0,t) = 0$ for all $t \in [0,t_0]$ and the study of system (8.2.4) can be performed in the interval $[t_0,\sigma]$.

Through the following lemmas we will show that, for every initial condition $(x_0,0)$ with $x_0 \in \,]0,1[$, the solution $(x(t),y(t))$ of the Cauchy problem associated with (8.2.4) at time $t = \sigma$ belongs to $]-\infty,0] \times \,]-\infty,0[$ for λ sufficiently large.

Lemma 8.2.3. *Let $\lambda > 0$, $\kappa_1 \in \,]0,1[$ and $t_1 \in \,]0,\sigma[$. For every $\gamma_1 \geq \kappa_1/(\sigma - t_1)$, any solution $(x(t),y(t))$ of (8.2.4) with $x(t_1) \leq \kappa_1$ and $y(t_1) \leq -\gamma_1$ satisfies $x(\sigma) \leq 0$ and $y(\sigma) \leq -\gamma_1$.*

Proof. Let λ, κ_1, t_1 and γ_1 be fixed as in the statement. Let $(x(t), y(t))$ be a solution of (8.2.4) with $x(t_1) \leq \kappa_1$ and $y(t_1) \leq -\gamma_1$. Since $y'(t) \leq 0$ on $[0, \sigma]$, we immediately obtain that

$$y(t) \leq y(t_1) \leq -\gamma_1, \quad \text{for all } t \in [t_1, \sigma],$$

and, consequently, we have

$$x(\sigma) = x(t_1) + \int_{t_1}^{\sigma} y(\xi)\, d\xi \leq \kappa_1 - \gamma_1(\sigma - t_1) \leq 0.$$

From the above inequalities the thesis follows. □

Lemma 8.2.4. *Let* κ_0, κ_1 *be such that* $0 < \kappa_1 < \kappa_0 < 1$ *and* $t_1 \in \,]0, \sigma[$. *Given*

$$\lambda^\star(\kappa_0, \kappa_1, t_1) := \frac{\kappa_0 - \kappa_1}{g_*(\kappa_1, \kappa_0) \int_0^{t_1} W^+(0, \xi)\, d\xi} \tag{8.2.5}$$

and $0 < \gamma_1 \leq (\kappa_0 - \kappa_1)/t_1$, *then, for every* $\lambda > \lambda^\star(\kappa_0, \kappa_1, t_1)$, *the solution* $(x(t), y(t))$ *of* (8.2.4) *with initial conditions* $x(0) = \kappa_0$ *and* $y(0) = 0$ *satisfies* $x(t_1) < \kappa_1$ *and* $y(t_1) < -\gamma_1$.

Proof. Let $\kappa_0, \kappa_1, t_1, \gamma_1$ and $\lambda^\star(\kappa_0, \kappa_1, t_1)$ be fixed as in the statement. For $\lambda > \lambda^\star(\kappa_0, \kappa_1, t_1)$ consider the solution $(x(t), y(t))$ of (8.2.4) with $x(0) = \kappa_0$ and $y(0) = 0$.

First, we suppose by contradiction that $x(t_1) \geq \kappa_1$. Consequently, by the monotonicity of $x(t)$ in $[0, \sigma]$, we have

$$0 < \kappa_1 \leq x(t) \leq \kappa_0 < 1, \quad \text{for all } t \in [0, t_1].$$

Since $y'(t) \leq -\lambda w^+(t)\psi_*(\kappa_1, \kappa_0)$ on $[0, t_1]$, we obtain

$$y(t) \leq -\lambda \psi_*(\kappa_1, \kappa_0) W^+(0, t), \quad \text{for all } t \in [0, t_1].$$

Then

$$x(t) \leq x(0) - \lambda \psi_*(\kappa_1, \kappa_0) \int_0^t W^+(0, \xi)\, d\xi, \quad \text{for all } t \in [0, t_1],$$

and, since $\lambda > \lambda^\star(\kappa_0, \kappa_1, t_1)$, in particular we have

$$x(t_1) \leq \kappa_0 - \lambda \psi_*(\kappa_1, \kappa_0) \int_0^{t_1} W^+(0, \xi)\, d\xi < \kappa_1,$$

a contradiction.

Secondly, we suppose by contradiction that

$$y(t) \geq -\gamma_1, \quad \text{for all } t \in [0, t_1].$$

By integrating, we have

$$x(t_1) = \kappa_0 + \int_0^{t_1} y(\xi)\, d\xi \geq \kappa_0 - \gamma_1 t_1 \geq \kappa_1.$$

A contradiction is achieved as above and the lemma is proved. □

Notice that hypothesis $(H\psi_6)$ is not required in the previous lemmas. On the contrary, in the next lemma this condition will be the crucial one. Our goal is now to show that for any fixed $\lambda > 0$, taking an initial condition $(x(0), y(0)) \in \,]0, \delta] \times \{0\}$ with $\delta > 0$ small, then the solution $(x(t), y(t))$ of the Cauchy problem associated with system (8.2.4) at time $t = \sigma$ belongs to $]0, 1[\times]-\infty, 0[$. In more detail, we are going to prove that for such initial condition the corresponding solution does not leave a small angular region contained in $]0, 1[\times]-\infty, 0[$.

Lemma 8.2.5. *Let $\lambda > 0$, $\nu \in]0, \pi/2[$ and $\kappa_1 \in]0,1[$. Then, there exists $\hat{\varepsilon} = \hat{\varepsilon}(\lambda, \nu) > 0$ such that for any $\varepsilon \in]0, \hat{\varepsilon}[$ there exists $\delta_\varepsilon \in]0, \kappa_1[$ such that the following holds: for any fixed $\kappa \in]0, \delta_\varepsilon]$, the solution $(x(t), y(t))$ of (8.2.4) with initial conditions $x(0) = \kappa$ and $y(0) = 0$ satisfies*

$$x(t) > 0 \quad \textit{and} \quad -\nu \leq \arctan\left(\frac{y(t)}{x(t)}\right) \leq 0, \quad \textit{for all } t \in [0, \sigma]. \tag{8.2.6}$$

Proof. Let λ, ν and κ_1 be fixed as in the statement. Let $\hat{\varepsilon} = \hat{\varepsilon}(\lambda, \nu) > 0$ be such that

$$\arctan\Big(\sqrt{\lambda \|w^+\|_\infty \varepsilon}\, \tan\big(\sigma \sqrt{\lambda \|w^+\|_\infty \varepsilon}\big)\Big) < \nu, \quad \text{for all } \varepsilon \in]0, \hat{\varepsilon}[, \tag{8.2.7}$$

where, as usual, we denote the supremum norm by $\|\cdot\|_\infty$. From hypothesis (Hψ_6), for all $\varepsilon > 0$ there exists $\delta_\varepsilon \in]0, \kappa_1[$ such that

$$\psi(\xi) \leq \varepsilon \xi, \quad \text{for all } \xi \in [0, \delta_\varepsilon].$$

For $\kappa \in]0, \delta_\varepsilon]$, we consider the solution $(x(t), y(t))$ of (8.2.4) with $x(0) = \kappa$ and $y(0) = 0$.

First of all, we write the solution in polar coordinates

$$x(t) = \rho(t) \cos(\vartheta(t)), \quad y(t) = \rho(t) \sin(\vartheta(t)).$$

We claim that $x(t) > 0$ for all $t \in [0, \sigma]$. By contradiction, let us suppose that there exists $\sigma_1 \in]0, \sigma]$ such that $x(t) > 0$ for all $t \in [0, \sigma_1[$ and $x(\sigma_1) = 0$. At this point, we observe that

$$\vartheta(t) = \arctan\left(\frac{y(t)}{x(t)}\right)$$

is well defined for all $t \in [0, \sigma_1[$. Thanks to the positivity of $x(t)$ on $[0, \sigma_1[$, since $\vartheta(0) = 0$ and

$$\vartheta'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x^2(t) + y^2(t)} = \frac{-\lambda w^+(t)\psi(x(t))x(t) - y^2(t)}{\rho^2(t)} \leq 0,$$

we have

$$-\frac{\pi}{2} < \vartheta(t) \leq 0, \quad \text{for all } t \in [0, \sigma_1[.$$

Let $\varepsilon \in]0, \hat{\varepsilon}[$, then

$$\begin{aligned} -\vartheta'(t) &= \frac{\lambda w^+(t)\psi(x(t))x(t) + y^2(t)}{\rho^2(t)} \leq \frac{\lambda w^+(t)\varepsilon x^2(t) + y^2(t)}{\rho^2(t)} \\ &\leq \lambda \|w^+\|_\infty \varepsilon \cos^2(\vartheta(t)) + \sin^2(\vartheta(t)), \quad \text{for all } t \in [0, \sigma_1[. \end{aligned}$$

By integrating on $[0, t] \subseteq [0, \sigma_1[$, we obtain

$$-\int_{\vartheta(0)}^{\vartheta(t)} \frac{d\zeta}{\lambda \|w^+\|_\infty \varepsilon \cos^2(\zeta) + \sin^2(\zeta)} \leq \int_0^t d\xi = t \leq \sigma_1 \leq \sigma, \quad \text{for all } t \in [0, \sigma_1[.$$

The first term can be equivalently written as

$$\begin{aligned} &-\int_{\vartheta(0)}^{\vartheta(t)} \frac{d\zeta}{\lambda \|w^+\|_\infty \varepsilon \cos^2(\zeta) + \sin^2(\zeta)} = \\ &= \int_{\vartheta(t)}^{0} \frac{d\zeta}{\cos^2(\zeta)\big(\lambda \|w^+\|_\infty \varepsilon + \tan^2(\zeta)\big)} \\ &= -\int_{\tan(\vartheta(t))}^{0} \frac{dz}{\lambda \|w^+\|_\infty \varepsilon + z^2} \\ &= \frac{1}{\sqrt{\lambda \|w^+\|_\infty \varepsilon}} \arctan\left(\frac{\tan|\vartheta(t)|}{\sqrt{\lambda \|w^+\|_\infty \varepsilon}}\right), \quad \text{for all } t \in [0, \sigma_1[. \end{aligned}$$

Consequently

$$|\vartheta(t)| \leq \arctan\Big(\sqrt{\lambda\|w^+\|_\infty\varepsilon}\,\tan\big(\sigma\sqrt{\lambda\|w^+\|_\infty\varepsilon}\big)\Big), \quad \text{for all } t \in [0,\sigma_1[.$$

By the choice of $\varepsilon \in]0,\hat{\varepsilon}[$, it follows that

$$-\nu < \vartheta(t) \leq 0, \quad \text{for all } t \in [0,\sigma_1[.$$

Therefore, by the continuity of $\vartheta(t)$, we conclude that $\vartheta(\sigma_1) \geq -\nu > -\pi/2$ and so $x(\sigma_1) > 0$, a contradiction. Accordingly, $x(t) > 0$ for all $t \in [0,\sigma]$. The thesis follows from the above computations. □

System $(\mathrm{S}_{\lambda,\mu})$ in the interval $[\tau,T]$ can be equivalently written as (8.2.4). Since $W^+(T,T) = 0$, $W^+(\tau,T) > 0$ and $t \mapsto W^+(t,T)$ is a continuous non-increasing map on $[\tau,T]$, without loss of generality, we can suppose that

$$W^+(t,T) > 0, \quad \forall\, t \in [\tau,T[.$$

Otherwise, there exists a maximal interval $[t_T,T]$ where $W^+(t,T) = 0$ for all $t \in [t_T,T]$ and the study of system (8.2.4) can be performed in the interval $[\tau,t_T]$.

In this context, the situation is exactly symmetric to the one described in Lemma 8.2.3 and Lemma 8.2.4. We collect here the corresponding results, omitting the proofs since they are analogous to the previous ones.

Lemma 8.2.6. *Let $\lambda > 0$, $\kappa_3 \in]0,1[$ and $t_3 \in]\tau,T[$. For every $\gamma_3 \geq \kappa_3/(t_3-\tau)$, any solution $(x(t),y(t))$ of (8.2.4) with $x(t_3) \leq \kappa_3$ and $y(t_3) \geq \gamma_3$ satisfies $x(\tau) \leq 0$ and $y(\tau) \geq \gamma_3$.*

Lemma 8.2.7. *Let κ_3,κ_T be such that $0 < \kappa_3 < \kappa_T < 1$ and $t_3 \in]\tau,T[$. Given*

$$\lambda^{\star\star}(\kappa_3,\kappa_T,t_3) := \frac{\kappa_T - \kappa_3}{\psi_*(\kappa_3,\kappa_T)\int_{t_3}^{T} W^+(\xi,T)\,d\xi} \tag{8.2.8}$$

and $0 < \gamma_3 \leq (\kappa_T-\kappa_3)/(T-t_3)$, then, for every $\lambda > \lambda^{\star\star}(\kappa_3,\kappa_T,t_3)$, the solution $(x(t),y(t))$ of (8.2.4) with initial conditions $x(T) = \kappa_T$ and $y(T) = 0$ satisfies $x(t_3) < \kappa_3$ and $y(t_3) > \gamma_3$.

Consider now the interval $[\sigma,\tau]$, where system $(\mathrm{S}_{\lambda,\mu})$ reduces to

$$\begin{cases} x' = y, \\ y' = \mu w^-(t)\psi(x). \end{cases} \tag{8.2.9}$$

Without loss of generality, we can suppose that $W^-(\sigma,t) > 0$ for all $t \in]\sigma,\tau]$. Indeed, it is always possible to choose a suitable σ as in $(\mathrm{H}w_6)$ that satisfies this additional hypothesis, as pointed out in Chapter 7 at p. 84 (see also [BFZ18; FZ15b; FZ17]).

Our purpose is to determine the initial conditions $(x(\sigma),y(\sigma))$ such that the corresponding solution $(x(t),y(t))$ of the Cauchy problem associated with system (8.2.9) belongs to $[1,+\infty[\times]0,+\infty[$ at time $t=\tau$, for μ sufficiently large.

Lemma 8.2.8. *Let $\mu > 0$, $\kappa_2 \in]0,1[$ and $t_2 \in]\sigma,\tau[$. For every $\omega \geq (1-\kappa_2)/(\tau-t_2)$, any solution $(x(t),y(t))$ of (8.2.9) with $x(t_2) \geq \kappa_2$ and $y(t_2) \geq \omega$ satisfies $x(\tau) \geq 1$ and $y(\tau) \geq \omega$.*

Proof. Let μ,κ_2,t_2 and ω be fixed as in the statement. Let $(x(t),y(t))$ be a solution of (8.2.9) with $x(t_2) \geq \kappa_2$ and $y(t_2) \geq \omega$. Since $y'(t) \geq 0$ on $[\sigma,\tau]$, we immediately obtain that $y(t) \geq y(t_2) \geq \omega$ for every $t \in [t_2,\tau]$. In particular, it follows that $y(\tau) \geq \omega$. Moreover, we have

$$x(\tau) = x(t_2) + \int_{t_2}^{\tau} y(\xi)\,d\xi \geq \kappa_2 + \omega(\tau-t_2) \geq 1.$$

The thesis follows. □

Lemma 8.2.9. *Let κ_σ, κ_2 be such that $0 < \kappa_\sigma < \kappa_2 < 1$ and $\omega_\sigma > 0$. Given*

$$\sigma < t_2 \leq \min\left\{\sigma + \frac{\kappa_\sigma}{2\omega_\sigma}, \tau\right\}, \quad 0 < \omega \leq \frac{\kappa_2 - \kappa_\sigma}{t_2 - \sigma},$$

and

$$\mu^\star(\kappa_2, \kappa_\sigma, t_2, \omega_\sigma) := \frac{\kappa_2 - \kappa_\sigma + (t_2 - \sigma)\omega_\sigma}{\psi_*(\kappa_\sigma/2, \kappa_2)\int_\sigma^{t_2} W^-(\sigma, \xi)\, d\xi}, \tag{8.2.10}$$

then, for every $\mu > \mu^\star(\kappa_2, \kappa_\sigma, t_2, \omega_\sigma)$, any solution $(x(t), y(t))$ of (8.2.9) *with $x(\sigma) = \kappa_\sigma$ and $y(\sigma) \geq -\omega_\sigma$ satisfies $x(t_2) > \kappa_2$ and $y(t_2) > \omega$.*

Proof. Let $\kappa_\sigma, \kappa_2, \omega_\sigma, t_2, \omega$ and $\mu^\star(\kappa_2, \kappa_\sigma, t_2, \omega_\sigma)$ be fixed as in the statement. For $\mu > \mu^\star(\kappa_2, \kappa_\sigma, t_2, \omega_\sigma)$, let $(x(t), y(t))$ be a solution of (8.2.9) with $x(\sigma) = \kappa_\sigma$ and $y(\sigma) \geq -\omega_\sigma$.

First, we suppose by contradiction that $x(t_2) \leq \kappa_2$. This way, by the convexity of the function $x(t)$ in $[\sigma, \tau]$ and the assumption $\kappa_2 > \kappa_\sigma$, we easily deduce that

$$x(t) \leq \kappa_2, \quad \text{for all } t \in [\sigma, t_2].$$

Since $y'(t) \geq 0$ on $[\sigma, \tau]$ and $y(\sigma) \geq -\omega_\sigma$, we derive that

$$x(t) \geq -\omega_\sigma t + \kappa_\sigma + \omega_\sigma \sigma, \quad \text{for all } t \in [\sigma, \tau],$$

and, by the condition on the point t_2, we obtain that

$$x(t) \geq \frac{\kappa_\sigma}{2}, \quad \text{for all } t \in [\sigma, t_2].$$

By an integration of (8.2.9), for every $t \in [\sigma, t_2]$, we have

$$y(t) = y(\sigma) + \int_\sigma^t y'(\xi)\, d\xi = y(\sigma) + \mu \int_\sigma^t w^-(\xi)\psi(x(\xi))\, d\xi$$

and

$$x(t) = x(\sigma) + \int_\sigma^t y(\xi)\, d\xi = \kappa_\sigma + (t - \sigma)y(\sigma) + \mu \int_\sigma^t \int_\sigma^z w^-(\xi)\psi(x(\xi))\, d\xi dz.$$

Then, by the choice of $\mu > \mu^\star(\kappa_2, \kappa_\sigma, t_2, \omega_\sigma)$, it follows that

$$\kappa_2 \geq x(t_2) \geq \kappa_\sigma - (t_2 - \sigma)\omega_\sigma + \mu\psi_*(\kappa_\sigma/2, \kappa_2)\int_\sigma^{t_2} W^-(\sigma, \xi)\, d\xi > \kappa_2,$$

a contradiction.

Secondly, we suppose by contradiction that $y(t_2) \leq \omega$ and thus that $y(t) \leq \omega$ for all $t \in [\sigma, t_2]$. Then

$$x(t_2) \leq \kappa_\sigma + \omega(t_2 - \sigma) \leq \kappa_2$$

and a contradiction is achieved as above. This concludes the proof. □

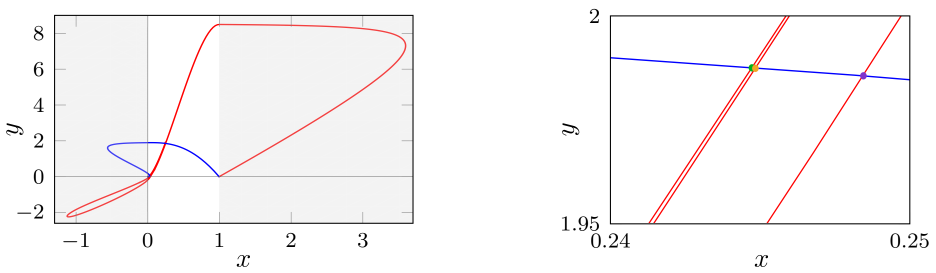
8.2.2 Multiplicity of positive solutions

The working hypotheses assumed through this section guarantee the uniqueness and the global existence of the solution $(x(\cdot; \alpha, x_\alpha, y_\alpha), y(\cdot; \alpha, x_\alpha, y_\alpha))$ to system $(\mathrm{S}_{\lambda,\mu})$ satisfying the initial conditions $x(\alpha) = x_\alpha$, $y(\alpha) = y_\alpha$. Consequently, as in Section 8.1, we introduce (for every fixed couple of parameters λ and μ) the Poincaré map Φ_α^β associated to $(\mathrm{S}_{\lambda,\mu})$ in the interval $[\alpha, \beta] \subseteq [0, T]$ (which is a global diffeomorphism of the plane onto itself) and we will describe the deformation in the phase-plane (x, y) of the interval $X_{[0,1]}$ through the Poincaré map. Since we are interested in positive solutions for problem $(\mathscr{IN}_{\lambda,\mu})$, we look for a point

$$C \in \Phi_0^\tau(X_{[0,1]}) \cap \Phi_T^\tau(X_{[0,1]})$$

which in turns determine univocally a solution $(x(t; \tau, C), y(t; \tau, C))$ of system $(\mathrm{S}_{\lambda,\mu})$ satisfying the Neumann boundary conditions $y(0; \tau, C) = y(T; \tau, C) = 0$. Hence, $u(t) := x(t; \tau, C)$ is a solution of problem $(\mathscr{IN}_{\lambda,\mu})$.

In Figure 8.10 we illustrate this approach by means of numerical simulations in the case of Example 8.2.2.



(a) Shooting of $X_{[0,1]}$ forward over the interval $[0,\tau]$ (red) and shooting of $X_{[0,1]}$ backward over the interval $[\tau,T]$ (blue).

(b) Zooming on three intersection points in $\Phi_0^\tau(X_{[0,1]}) \cap \Phi_T^\tau(X_{[0,1]})$ which identify three solutions of $(\mathscr{IN}_{\lambda,\mu})$.

Figure 8.10: In the phase-plane (x,y): dynamics of the Poincaré maps Φ_0^τ and Φ_T^τ associated to system $(\mathrm{S}_{\lambda,\mu})$ as in the framework of Example 8.2.2 with $\sigma = 0.5$, $\tau = 1$, $T = 2$, $w_1 = 1.75$, $w_2 = 1$, $w_3 = 1$, for $\lambda = 25$ and $\mu = 500$.

Proof of Theorem 8.2.1. Now we are ready to pass to the proof of Theorem 8.2.1 which is divided into four steps. First of all, we will study system $(\mathrm{S}_{\lambda,\mu})$ separately in the three intervals: $[0,\sigma]$, $[\sigma,\tau]$ and $[\tau,T]$ and then we will combine the dynamics of system $(\mathrm{S}_{\lambda,\mu})$ on the whole interval $[0,T]$.

Step I. Dynamics on $[0,\sigma]$**.** Let us fix $0 < \kappa_1 < \kappa_0 < 1$ and $0 < t_1 \leq \sigma(1-\kappa_1/\kappa_0)$. In this manner, we have that $\kappa_1/(\sigma - t_1) \leq (\kappa_0 - \kappa_1)/t_1$ and so we can apply Lemma 8.2.3 together with Lemma 8.2.4. Then, for $\lambda > \lambda^\star(\kappa_0,\kappa_1,t_1)$ (cf. (8.2.5)) and an arbitrary $\mu > 0$, we obtain that

$$x(\sigma;0,\kappa_0,0) \leq 0, \quad y(\sigma;0,\kappa_0,0) < 0.$$

We stress that this conclusion does not depend on μ. Next, we notice that $\Phi_0^\sigma(1,0) = (1,0)$ and, by the concavity of x in $[0,\sigma]$, that $\Phi_0^\sigma([0,1]\times\{0\}) \subseteq\,]-\infty,1]\times\,]-\infty,0]$. Thus, from the continuous dependence of the solutions upon the initial data and the Intermediate Value Theorem, the following fact holds. There exists an interval $[l_1,1] \subseteq [\kappa_0,1]$ such that $\Phi_0^\sigma([l_1,1]\times\{0\}) \subseteq [0,1]\times\,]-\infty,0]$, $\Phi_0^\sigma(l_1,0) \in \{0\}\times\,]-\infty,0[$ and $x(t;0,\xi,0) \in\,]0,1[$ for all $t \in [0,\sigma]$, $\xi \in\,]l_1,1[$.

Furthermore, by Lemma 8.2.5 there exits $\kappa_4 \in\,]0,\kappa_1[$ such that $\Phi_0^\sigma(]0,\kappa_4]\times\{0\}) \subseteq\,]0,1[\,\times\,]-\infty,0]$. Then, recalling that $\Phi_0^\sigma(\kappa_0,0) \in\,]-\infty,0]\times\,]-\infty,0[$, from the same previous arguments of continuity, there exists an interval $[0,r_1] \subseteq [0,\kappa_0]$ (with $r_1 > \kappa_4$) such that $\Phi_0^\sigma([0,r_1]\times\{0\}) \subseteq [0,1[\,\times\,]-\infty,0]$, $\Phi_0^\sigma(r_1,0) \in \{0\}\times\,]-\infty,0[$ and $x(t;0,\xi,0) \in\,]0,1[$ for all $t \in [0,\sigma]$, $\xi \in\,]0,r_1[$.

Step II. Dynamics on $[\tau,T]$**.** Analogously to Step I, let us fix $0 < \kappa_3 < \kappa_T < 1$ and $0 < t_3 \leq \tau + (T-\tau)\kappa_3/\kappa_T$. Given $\lambda > \lambda^{\star\star}(\kappa_3,\kappa_T,t_3)$ (cf. (8.2.8)) and an arbitrary $\mu > 0$, from Lemma 8.2.6 and Lemma 8.2.7 we have that

$$x(\tau;T,\kappa_T,0) \leq 0, \quad y(\tau;T,\kappa_T,0) > 0.$$

Furthermore, we notice that $\Phi_T^\tau(1,0) = (1,0)$ and $\Phi_T^\tau([0,1]\times\{0\}) \subseteq\,]-\infty,1]\times[0,+\infty[$. Consequently, by the continuous dependence of the solutions upon the initial data and the Intermediate Value Theorem, there exists an interval $[l_2,1] \subseteq [\kappa_T,1]$ such that $\Phi_T^\tau([l_2,1]\times\{0\}) \subseteq [0,1]\times[0,+\infty[$, $\Phi_T^\tau(l_2,0) \in \{0\}\times\,]0,+\infty[$ and $x(t;T,\xi,0) \in\,]0,1[$ for all $t \in [\tau,T]$, $\xi \in\,]l_2,1[$.

Step III. Dynamics on $[\sigma,\tau]$**.** Let us define

$$\lambda^* := \max\bigl\{\lambda^\star(\kappa_0,\kappa_1,t_1), \lambda^{\star\star}(\kappa_3,\kappa_T,t_3)\bigr\}$$

and fix $\lambda > \lambda^*$.

First of all, we observe that, for any $x_0 \in \mathbb{R}$, the solution $(x(t), y(t))$ to system $(S_{\lambda,\mu})$ with initial values $x(0) = x_0$ and $y(0) = 0$ satisfies

$$y(\sigma) = y(0) + \lambda \int_0^\sigma w^+(\xi)\psi(x(\xi))\, d\xi \geq -\omega_\sigma,$$

where $\omega_\sigma := \lambda^* W^+(0,\sigma) \max_{s\in[0,1]} g(s)$.

Let us take $p_1 \in]0, r_1[$ and $p_2 \in]l_1, 1[$. We define

$$\kappa_{\sigma,i} := x(\sigma; 0, p_i, 0), \quad \text{for } i = 1, 2.$$

From the properties of the continua $\Phi_0^\sigma([0, r_1] \times \{0\})$ and $\Phi_0^\sigma([l_1, 1] \times \{0\})$ achieved in Step I, it follows that $\kappa_{\sigma,i} \in]0, 1[$ for $i = 1, 2$. Next, for $i = 1, 2$, we fix $\kappa_{2,i} \in]\kappa_{\sigma,i}, 1[$ and choose $t_{2,i}$ such that

$$\sigma < t_{2,i} \leq \min\left\{\sigma + \frac{\kappa_{\sigma,i}}{2\omega_\sigma}, \frac{\sigma(1-\kappa_{2,i}) + \tau(\kappa_{2,i} - \kappa_{\sigma,i})}{1-\kappa_{\sigma,i}}\right\}$$

and $\omega_i \geq (1 - \kappa_{2,i})/(\tau - t_{2,i})$. In this manner, we enter in the setting of Lemma 8.2.8 and Lemma 8.2.9. For $i = 1, 2$, taking $\mu > \mu^\star(\kappa_{2,i}, \kappa_{\sigma,i}, t_{2,i}, \omega_\sigma)$ (cf. (8.2.10)), we obtain that

$$x(\tau; 0, p_i, 0) \geq 1, \quad y(\tau; 0, p_i, 0) > \omega_i > 0, \quad \text{for } i = 1, 2. \tag{8.2.11}$$

We remark now that, for any choice of $t_0 \in [0, T]$ and $y_0 < 0$, if $(x(t), y(t))$ is the solution of the Cauchy problem associated with system $(S_{\lambda,\mu})$ satisfying the initial conditions $x(t_0) = 0$ and $y(t_0) = y_0$, then

$$x(t; t_0, 0, y_0) < 0, \quad y(t; t_0, 0, y_0) < 0, \quad \text{for all } t \in]t_0, T].$$

Indeed, let $]t_0, t^*[\subseteq]t_0, T]$ be the maximal open interval such that $y(t) < 0$ for all $t \in]t_0, t^*[$. By an integration of $x' = y$, we have $x(t) < 0$ for all $t \in]t_0, t^*[$. Assume now, by contradiction, that $t^* < T$. Then, $0 = y(t^*) = y_0 < 0$ and we have a contradiction. The claim follows.

Consequently, we deduce that

$$x(\tau; 0, r_1, 0) < 0, \quad y(\tau; 0, r_1, 0) < 0, \tag{8.2.12}$$

and

$$x(\tau; 0, l_1, 0) < 0, \quad y(\tau; 0, l_1, 0) < 0. \tag{8.2.13}$$

At this point, taking into account (8.2.11), (8.2.12), (8.2.13) and $\Phi_0^\tau(0, 0) = (0, 0)$, thanks to the continuous dependence of the solutions upon the initial data and the Intermediate Value Theorem, we deduce what follows. There exist three intervals

$$[q_{1,1}, q_{2,1}] \subseteq [0, p_1], \quad [q_{1,2}, q_{2,2}] \subseteq [p_1, r_1], \quad [q_{1,3}, q_{2,3}] \subseteq [l_1, p_2],$$

such that, for each $j \in \{1, 2, 3\}$, $\Phi_0^\tau([q_{1,j}, q_{2,j}] \times \{0\}) \subseteq [0, 1] \times \mathbb{R}$ with

$$\Phi_0^\tau(q_{1,j}, 0) \in \{0\} \times]-\infty, 0], \quad \Phi_0^\tau(q_{2,j}, 0) \in \{1\} \times]0, +\infty[,$$

and

$$x(t; 0, \xi, 0) \in]0, 1[, \quad \text{for all } t \in [0, \tau],\ \xi \in]q_{1,j}, q_{2,j}[.$$

We conclude that there exist three sub-continua of $\Phi_0^\tau(X_{[0,1]})$ connecting $\{0\} \times]-\infty, 0]$ with $\{1\} \times]0, +\infty[$. We stress that the three sub-continua do not intersect each other, due to the uniqueness of the solutions to the initial value problems associated with $(S_{\lambda,\mu})$.

Step IV. Conclusion. Let us take

$$\mu > \mu^*(\lambda) := \max_{i\in\{1,2\}} \mu^\star(\kappa_{2,i}, \kappa_{\sigma,i}, t_{2,i}, \omega_\sigma).$$

Then, from Step II, we deduce the existence of a sub-continuum in $\Phi_T^\tau(X_{[0,1]})$ connecting $\{0\} \times]0,+\infty[$ with $(1,0)$. On the other hand, from Step I and Step III, we deduce the existence of three pairwise disjoint sub-continua in $\Phi_0^\tau(X_{[0,1]})$ connecting $\{0\} \times]-\infty, 0]$ with $\{1\} \times]0,+\infty[$. This way, from a standard connectivity argument, it follows the existence of three distinct intersection points:

$$C_j \in \Phi_0^\tau(]q_{1,j}, q_{2,j}[\times \{0\}) \cap \Phi_T^\tau(]l_2, 1[\times \{0\}), \quad j = 1, 2, 3.$$

See Figure 8.10 for a graphical representation. For each $j \in \{1, 2, 3\}$, given the solution $(x(t), y(t))$ of the Cauchy problem associated with system $(S_{\lambda,\mu})$ with initial data at time $t = \tau$ the point C_j, then we have a positive solution to problem $(\mathscr{IN}_{\lambda,\mu})$ defined by $u(t) := x(t; \tau, C_j)$. Moreover, from a straightforward argument by contradiction, it follows that

$$\begin{aligned} &\Phi_0^t(\xi, 0) \in]0, 1[\times \mathbb{R}, \quad \text{for all } t \in]q_{1,j}, q_{2,j}[, \ \xi \in [0, \tau], \\ &\Phi_T^t(\xi, 0) \in]0, 1[\times \mathbb{R}, \quad \text{for all } t \in]l_1, 1[, \ \xi \in [\tau, T], \end{aligned}$$

and so we have that $0 < u(t) < 1$ for all $t \in [0, T]$. Then, Theorem 8.2.1 is proved. □

From the study of problem $(\mathscr{IN}_{\lambda,\mu})$ it could be interesting consider more general domains. A classical application in this direction regards a radially symmetric Neumann BVP defined on an annular domain of $\mathbb{R}^N$ for $N \geq 2$. More precisely, let $R_e > R_i > 0$ be two fixed radii and consider the open annular domain given by $\Omega = B(0, R_e) \setminus B[0, R_i]$. Hence, as in Section 7.2, we deal with

$$(\mathscr{IN}_{\lambda,\mu,N}) \qquad \begin{cases} \Delta u + \mathfrak{w}_{\lambda,\mu}(x)\psi(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where the weight function depends on the real positive parameters λ, μ and is defined by

$$\mathfrak{w}_{\lambda,\mu}(x) := \lambda \mathfrak{w}^+(x) - \mu \mathfrak{w}^-(x).$$

Assuming that the weight term has radial symmetry, namely $\mathfrak{w}(x) = \mathcal{Q}(|x|)$ for all $x \in \Omega$ with $\mathcal{Q}\colon [R_i, R_e] \to \mathbb{R}$, we look for radially symmetric positive solutions to problem $(\mathscr{IN}_{\lambda,\mu,N})$, i.e. solutions of the form $u(x) = U(|x|)$ where $U\colon [R_i, R_e] \to \mathbb{R}$.

Accordingly, our study can be reduced to the search of positive solutions of the Neumann boundary value problem (7.2.3) which, via a standard change of variable, is equivalent to a Neumann problem of the form

$$\begin{cases} v''(t) + w_{\lambda,\mu}(t)\psi(v(t)) = 0, \\ v'(0) = v'(T) = 0, \end{cases}$$

and so, a direct consequence of Theorem 8.2.1 is the following.

Corollary 8.2.10. *Let $\psi\colon [0, 1] \to \mathbb{R}^+$ be a locally Lipschitz continuous function satisfying $(H\psi_{1\,\mathrm{bis}})$ and $(H\psi_6)$. Let $\mathcal{Q} \in L^1([R_i, R_e])$ such that there exist σ, τ with $0 < \sigma < \tau < T$ for which the following holds*

$$\begin{aligned} &\mathcal{Q}^+(t) \succ 0, \quad \mathcal{Q}^-(t) \equiv 0, \quad \text{on } [R_i, \sigma], \\ &\mathcal{Q}^+(t) \equiv 0, \quad \mathcal{Q}^-(t) \succ 0, \quad \text{on } [\sigma, \tau], \\ &\mathcal{Q}^+(t) \succ 0, \quad \mathcal{Q}^-(t) \equiv 0, \quad \text{on } [\tau, R_e], \end{aligned}$$

and let $\mathfrak{w}(x) := \mathcal{Q}(|x|)$, for $x \in \Omega$. Then, there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ there exists $\mu^*(\lambda) > 0$ such that for every $\mu > \mu^*(\lambda)$ problem $(\mathscr{IN}_{\lambda,\mu,N})$ has at least three radially symmetric positive solutions.

9. Further developments from Part II

In Chapter 6 and in Chapter 8, we have considered both indefinite nonlinear Dirichlet and Neumann BVPs

$$\begin{cases} u'' + \lambda w(t)\psi(u) = 0, \\ u(0) = u(T) = 0, \end{cases} \qquad \begin{cases} u'' + \lambda w(t)\psi(u) = 0, \\ u'(0) = u'(T) = 0, \end{cases}$$

where $\lambda > 0$, the weight $w\colon [0,T] \to \mathbb{R}$ is a sign-changing function and the nonlinearity satisfies either

$$\psi : \mathbb{R}^+ \to \mathbb{R}^+ \textit{ such that } \psi(0) = 0,\ \psi(\xi) > 0 \textit{ for all } \xi > 0 \textit{ and sublinear at } \infty \text{ (Type 1)},$$

or

$$\psi : [0,1] \to \mathbb{R}^+ \textit{ such that } \psi(0) = \psi(1) = 0,\ \psi(\xi) > 0 \textit{ for all } \xi \in]0,1[\text{ (Type 2)}.$$

Among other results presented, we have discussed the number of positive solutions in relation with the comparison between the concavity of ψ versus a condition about monotonicity of $\xi \mapsto \psi(\xi)/\xi$. We have found suitable nonlinearities/weights and we gave examples of the multiplicity of positive solution for both problems. As far as we know, the mathematical literature lacks of rigorous multiplicity results involving a minimal equipment of hypotheses to guarantee either the multiplicity or the uniqueness of a non-trivial positive solution for nonlinearities ψ both of Type 1 and Type 2.

Focusing on nonlinearities of Type 2, the main assumption considered in our examples to solve a conjecture of Lou and Nagylaki [LN02] is that ψ has a strict local minimum in $]0,1[$. On the other hand, the presence of a nonlinearity ψ with a unique critical point in $]0,1[$ could produce, for such kinds of indefinite Neumann problems, different behaviors in the number of positive solutions. Indeed, a still open problem is stated in [LNN13] and it asks, considering the case $\int_\Omega w(x)\,dx = 0$, whether in this case the resulting Neumann problem has a unique non-trivial stationary solution for every $\lambda > 0$.

Still in this framework, another natural question that arises is whether, in analogy with the result in [BFZ18] obtained for indefinite Neumann problems with "super-sublinear nonlinearities", there exist more than three non-trivial positive solutions for a Neumann problem with a nonlinearity ψ of Type 2 superlinear at zero (as suggested by numerical simulations and in accord with stability arguments). Actually, one could guess for certain weight terms, that there exist at least eight positive solutions.

In conclusion, looking back on our results in the one-dimensional case and in the radial cases, we hopefully were been the first steps towards the understanding of the structure of the solutions set of challenging problems, even in a higher dimension.

Appendices

A. Mawhin's coincidence degree

In this appendix we recall the basics on *coincidence degree theory* needed to treat some issues in the present thesis. This theory is a powerful tool introduced by J. Mawhin in [Maw79] that it turns out be a very useful technique in the study of nonlinear BVPs. We refer also to [GM77; Maw93] for the proofs of the results collected below as well as a complete discussion of this topic.

Let X and Z be real normed spaces with Ω an open bounded set in X. We consider now the coincidence equation of the form

$$Lu = Nu, \quad u \in \mathrm{dom}L \cap \Omega, \tag{A.0.1}$$

where we assume that

$$L \colon X \supseteq \mathrm{dom}L \to Z$$

is a linear Fredholm mapping of index zero, namely $\mathrm{Im}L$ is a closed subspace of Z with finite $\dim(\ker L) = \mathrm{codim}(\mathrm{Im}L)$, and

$$N \colon X \to Z$$

is a nonlinear operator. In this setting, there exist two linear and continuous projections

$$P \colon X \to \ker L, \quad Q \colon Z \to \mathrm{Im}L,$$

as well as the continuous right inverse of L, denoted by

$$K_P \colon \mathrm{Im}L \to \mathrm{dom}L \cap X_0\,,$$

where $X_0 := \ker P \equiv X/\ker L$ is a complementary subspace of $\ker L$ in X. Notice that equation (A.0.1) is equivalent to the fixed point problem

$$u = \Phi(u) := Pu + JQNu + K_P(Id - Q)Nu, \quad u \in \Omega, \tag{A.0.2}$$

where $J \colon \mathrm{coker}L = \mathrm{Im}Q \equiv Z/\mathrm{Im}L \to \ker L$ is a linear isomorphism.

We further suppose that N is a continuous operator which maps bounded sets to bounded sets and such that, for any bounded set B in X, the set $K_P(Id - Q)N(B)$ is relatively compact, namely N is L-completely continuous [Maw93]. These assumptions imply that the operator Φ, defined in (A.0.2), is completely continuous.

If we suppose that

$$Lu \neq Nu, \quad \forall\, u \in \mathrm{dom}L \cap \partial\Omega,$$

then also $Id - \Phi$ never vanishes on $\partial\Omega$ and, therefore, we can define the *coincidence degree*

$$D_L(L-N,\Omega) := \deg_{LS}(Id - \Phi, \Omega, 0),$$

where "$\deg_{LS}$" denotes the Leray-Schauder degree. To avoid ambiguity of sign, sometimes the convention is to consider only $|D_L(L-N,\Omega)|$. Otherwise, we can fix an orientation on $\ker L$ and $\mathrm{coker} L$. So that, we choose J in the class of orientation preserving isomorphisms (see [Maw93]). In any case, for our application, the choice of P, Q and J is obvious and no ambiguity will arise.

We also point out that the classical properties of the Leray-Schauder degree (such as additivity/excision, homotopic invariance) hold also in this framework. For completeness, we list these basic properties as follows (see [GM77]).

1. Existence theorem: if $D_L(L-N,\Omega) \neq 0$, then $0 \in (L-N)(\mathrm{dom}L \cap \Omega)$.
2. Excision property: if $\Omega_0 \subseteq \Omega$ is an open set such that $(L-N)^{-1}(0) \in \Omega_0$, then $D_L(L-N,\Omega) = D_L(L-N,\Omega_0)$.
3. Additivity property: if $\Omega_1 \cup \Omega_2 = \Omega$ with Ω_1, Ω_2 open and such that $\Omega_1 \cap \Omega_2 = \emptyset$, then $D_L(L-N,\Omega) = D_L(L-N,\Omega_1) + D_L(L-N,\Omega_2)$.
4. Homotopic invariance: if the operator $\mathcal{H}\colon \overline{\Omega}\times[0,1] \to Z$ is L-compact in $\overline{\Omega}\times[0,1]$ and such that for every $\lambda \in [0,1]$, $0 \notin \big(L-\mathcal{H}(\cdot,\lambda)\big)(\mathrm{dom}L \cap \partial\Omega)$, then $D_L(L-\mathcal{H}(\cdot,\lambda),\Omega)$ is independent of λ in $[0,1]$. In particular, $D_L(L-\mathcal{H}(\cdot,0),\Omega) = D_L(L-\mathcal{H}(\cdot,1),\Omega)$.

We now conclude with a key result for the computation of the coincidence degree on an open bounded set in X. Indeed, by denoting with "$\deg_B$" the finite dimensional Brouwer degree, the following result holds in accord to the Mawhin's continuation theorem (see [Maw69; Maw72b]).

Theorem A.1. *Let L and N be as above and let $\Omega \subseteq X$ be an open and bounded set. Suppose that*

$$Lx \neq \lambda N x, \quad \forall\, x \in \mathrm{dom}\, L \cap \partial\Omega, \quad \forall\, \lambda \in \,]0,1],$$

and

$$QN(x) \neq 0, \quad \forall\, x \in \partial\Omega \cap \ker L.$$

Then,

$$D_L(L-N,\Omega) = \deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0).$$

As a consequence, if $\deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, *then* (A.0.1) *has at leat one solution.*

B. Basics on chaotic dynamics

Proving that a dynamical system is "chaotic" is very far from suggesting that it could be so. Indeed, a unique working method to all nonlinear problems does not exist, since there exist in mathematical literature many and different definitions of chaos (see for instance [AK01; KS89]). This appendix is devoted to introduce the reader through the topic of chaotic dynamics with special emphasis of the notions considered in the present thesis.

In order to detect complex behaviors, several approaches are available in accord with the definition taken into account. Nevertheless, there is not a general tool which is applicable without give reference to a particular notion of chaos. Despite this, we notice that there is a common feature in the definitions considered by several authors, which is usually associated with the concept of *deterministic chaos*, namely the possibility to reproduce all the possible outcomes of a coin-tossing experiment, varying the initial conditions within the system.

> "The laws of chance, with good reason, have traditionally been expressed in terms of flipping a coin. Guessing whether heads or tails is the outcome of a coin toss is the paradigm of pure chance." (Stephen Smale, [Sma98]).

This observation leads to the so called chaos in the sense of the coin-tossing. Mathematically, one can express this concept by means of the symbolic dynamics of the *shift map* (also called Bernoulli shift or shift automorphism) on the sets of two-sided sequences of m symbols. In more detail, given a collection of $m \geq 2$ symbols, namely $\{0, \dots, m-1\}$, we denote by $\Sigma_m := \{0, \dots, m-1\}^{\mathbb{Z}}$ the set of all two-sided sequences $S = (s_i)_{i \in \mathbb{Z}}$ with $s_i \in \{0, \dots, m-1\}$ for each $i \in \mathbb{Z}$. The set Σ_m is endowed with a standard metric that makes it a compact space with the product topology. Within this setting, the shift map $\sigma : \Sigma_m \to \Sigma_m$ is defined by $\sigma(S) = S' = (s'_i)_{i \in \mathbb{Z}}$ with $s'_i = s_{i+1}$ for all $i \in \mathbb{Z}$. The shift map on Σ_m is important in this topic since it can be considered as a model for chaotic dynamics. In fact, it contains many of the features which usually characterize the concept of "chaos" as a whole, for example: transitivity, density of periodic points, positive topological entropy (see [AKM65; Dev89; GH83; Wig03]).

At this point, whenever one is interested in showing the presence of chaotic behaviors for a map φ on a metric space, a possible approach is to prove the existence of a compact invariant set $\Lambda \subseteq X$ and a continuous and surjective map $\Pi : \Lambda \to \Sigma_m$ such that $\Pi \circ \varphi(w) = \sigma \circ \Pi(w)$, for all $w \in \Lambda$. If this occurs, we say that φ is *semiconjugate* to the shift map on m symbols. If, moreover, the map Π is one-to-one we say that φ is *conjugate* to the shift map on m symbols. In this manner, when the map φ is semiconjugate to the shift map on m symbols, then φ restricted to Λ inherits all the topological properties of the shift map.

On the other hand, a prototypical example of chaotic dynamics arises by the geometric structure associated with the *Smale horseshoe*. Technically, the Smale's construction deals with a planar diffeomorphism acting on a square, whose image has the shape of a horseshoe that crosses the square in a suitable manner (see [Sma65; Sma67]). The main pattern of the Smale horseshoe is essentially a succession of actions of stretching and folding of the square. This Smale's construction became an important technique in the study of chaotic dynamics since it implies the embedding of the Bernoulli shift map on two symbols into the dynamics of the diffeomorphism. More in detail, the Smale horseshoe map presents a hyperbolic compact invariant set on which it is conjugate to the shift map on two symbols. This is, for instance, the case considered in the frame of Melnikov's theory where a Smale horseshoe occurs for some iterates of the Poincaré map as a consequence of the Smale-Birkhoff theorem. In fact, such theorem considers a diffeomorphism φ possessing a transversal homoclinic point q to a hyperbolic saddle point p. Then, for some N, φ has a hyperbolic invariant set Λ on which the N-th iterate φ^N is conjugate to the shift map on two symbols (see [Hol90]). Accordingly here a natural definition of chaos.

Definition B.1 (Smale's horseshoe occurrence)**.** We say that *a Smale horseshoe occurs* if there is a hyperbolic compact invariant set on which a given map φ is conjugate to (Σ_m, σ) for $m \geq 2$.

In a wide variety of dynamical systems the Smale horseshoe has been detected, however, Smale's conditions are difficult to verify practically and in some cases the location of a horseshoe is a hard task. Then, some weaker notions were derived by keeping more features of Smale's chaotic systems as possible but relaxing some technical conditions (see [BW95; CKM00; MM95; SW97; Srz00; Szy96; Zgl96; ZG04]). An interesting point of view that gets the idea of Smale and moves to a more general topological context is the so-called concept of "topological horseshoe", introduced by J. Kennedy and J. A. Yorke in [KY01]. Given this context, we introduce another definition of chaos as follows.

Definition B.2 (Topological's horseshoe occurrence)**.** We say that *a topological horseshoe occurs* if there is a compact invariant set on which a given map φ is semiconjugate to (Σ_m, σ) for $m \geq 2$ and, moreover, for each periodic sequence $S \in \Sigma_m$, there is at least one periodic point $w \in \Lambda$ with the same period and such that $\Pi(w) = S$.

Notice that also in this case the topological entropy is positive. This way, the topological horseshoes with symbolic dynamics provide a powerful tool to describe time evolution of chaotic dynamics which have inspired some different techniques. One of these is for instance the Stretching Along the Paths (SAP) method, that owes its name to the fact that it treats maps which are expansive only along one direction and compressive in the other ones (we refer to [MPZ09; PZ04] for a presentation of the method and to [Sovbm] for its review). The SAP method is an easy criterion that judges whether a dynamical system is chaotic, for example, in the sense of the coin-tossing. Moreover, its application is made in the practice without involved constructions and it can be straightforwardly sketched via computer simulations. The ease of treatment is due to the fact that no differentiability neither one-to-one conditions are required for the map describing the dynamical system which one would analyze. The requirement that such a map has to satisfy is the continuity on some subsets belonging in its domain. Furthermore, we observe that close to this method there are also other approaches based on the topological horseshoe (see e.g. [Zgl96; Zgl01; ZG04]).

Let us recall the basics on the SAP method, borrowing the notations and definitions from [MPZ09].

Definition B.3. Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a set homeomorphic to $[0,1] \times [0,1]$. The pair $\tilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ is called *oriented topological rectangle* if $\mathcal{R}^- = \mathcal{R}_l^- \cup \mathcal{R}_r^-$, where $\mathcal{R}_l^-$ and $\mathcal{R}_r^-$ are two disjoint compact arcs contained in $\partial\mathcal{R}$.

Definition B.4 (SAP property)**.** Given two topological oriented rectangles $\tilde{\mathcal{M}} := (\mathcal{M}, \mathcal{M}^-)$, $\tilde{\mathcal{N}} := (\mathcal{N}, \mathcal{N}^-)$ and a continuous map $\varphi : \operatorname{dom}\varphi \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, we say that φ *stretches* $\tilde{\mathcal{M}}$ *to* $\tilde{\mathcal{N}}$ *along the paths* and we write

$$(\mathcal{K}, \varphi) \colon \tilde{\mathcal{M}} \Longleftrightarrow\!\!\!\!\rightarrow \tilde{\mathcal{N}}$$

if $\mathcal{K}$ is a compact subset of $\mathcal{M} \cap \operatorname{dom}\varphi$ and for each path $\gamma \colon [0,1] \to \mathcal{M}$ such that $\gamma(0) \in \mathcal{M}_l^-$ and $\gamma(1) \in \mathcal{M}_r^-$ (or vice-versa), there exists $[t', t''] \subseteq [0,1]$ such that

- $\gamma(t) \in \mathcal{K}$ for all $t \in [t', t'']$,
- $\varphi(\gamma(t)) \in \mathcal{N}$ for all $t \in [t', t'']$,
- $\varphi(\gamma(t'))$ and $\varphi(\gamma(t''))$ belong to different components of $\mathcal{N}^-$.

Given a positive integer m, we say that φ *stretches* $\tilde{\mathcal{M}}$ *to* $\tilde{\mathcal{N}}$ *along the paths with crossing number* m and we write

$$(\mathcal{K}, \varphi) \colon \tilde{\mathcal{M}} \Longleftrightarrow\!\!\!\!\rightarrow^m \tilde{\mathcal{N}}$$

if there exist m pairwise disjoint compact sets $\mathcal{K}_0, \dots, \mathcal{K}_{m-1} \subseteq \mathcal{M} \cap \operatorname{dom}\varphi$ such that $(\mathcal{K}_i, \varphi) \colon \tilde{\mathcal{M}} \Longleftrightarrow\!\!\!\!\rightarrow \tilde{\mathcal{N}}$ for each $i \in \{0, \dots, m-1\}$.

Definition B.5. Let $\varphi : \operatorname{dom}\varphi \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a map and let $\mathcal{D} \subseteq \operatorname{dom}\varphi$ be a nonempty set. We say that φ *induces chaotic dynamics on* $m \geq 2$ *symbols on a set* $\mathcal{D}$ if there exist m nonempty pairwise disjoint compact sets $\mathcal{K}_0, \dots, \mathcal{K}_{m-1} \subseteq \mathcal{D}$ such that for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0, \dots, m-1\}^{\mathbb{Z}}$ there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ such that

$$w_i \in \mathcal{K}_{s_i} \text{ and } w_{i+1} = \varphi(w_i) \text{ for all } i \in \mathbb{Z}, \tag{B.1}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence for some $k \geq 1$ there exists a k-periodic sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (B.1).

For applications point of view, the case $m \geq 2$ is more interesting because, the bigger m is, the richer symbolic dynamic structure becomes. Moreover, Definition B.5 is inspired by the definition of chaos in the sense of coin-tossing or in the sense of Block-Coppel [BC92] and it allows us to assert that a topological horseshoe occurs (see for instance [MPZ09; MRZ10; PPZ08]). Indeed, for a one-to-one map φ, it ensures the existence of a nonempty compact invariant set $\Lambda \subseteq \cup_{i=0}^{m-1}\mathcal{K}_i \subseteq \mathcal{D}$ such that $\varphi_{|\Lambda}$ is semiconjugate to the Bernoulli shift map on $m \geq 2$ symbols by a continuous surjection Π. Moreover, it guarantees that the set of the periodic points of φ is dense in Λ and, for all two-sided periodic sequence $S \in \Sigma_m$, the preimage $\Pi^{-1}(S)$ contains a periodic point of φ with the same period.

Finally, in order to detect chaos, an useful topological tool in the framework of switched systems is the following (see [MRZ10, Th. 2.1]).

Theorem B.6 (SAP method)**.** *Let $\nu : \operatorname{dom}\nu \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \operatorname{dom}\eta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be continuous maps. Let $\tilde{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$ and $\tilde{\mathcal{N}} = (\mathcal{N}, \mathcal{N}^-)$ be oriented rectangles in $\mathbb{R}^2$. Suppose that*

- *there exist $n \geq 1$ pairwise disjoint compact subsets of $\mathcal{M} \cap \operatorname{dom}\nu$, $\mathcal{Q}_0, \dots, \mathcal{Q}_{n-1}$, such that $(\mathcal{Q}_i, \nu) \colon \tilde{\mathcal{M}} \Longleftrightarrow\!\!\!\!\rightarrow \tilde{\mathcal{N}}$ for $i = 0, \dots, n-1$,*
- *there exist $m \geq 1$ pairwise disjoint compact subsets of $\mathcal{N} \cap \operatorname{dom}\eta$, $\mathcal{K}_0, \dots, \mathcal{K}_{m-1}$, such that $(\mathcal{K}_i, \eta) \colon \tilde{\mathcal{N}} \Longleftrightarrow\!\!\!\!\rightarrow \tilde{\mathcal{M}}$ for $i = 0, \dots, m-1$.*

If at least one between n and m is greater or equal than 2, then the map $\varphi = \eta \circ \nu$ induces chaotic dynamics on $n \times m$ symbols on

$$\mathcal{Q}^* = \bigcup_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \mathcal{Q}_i \cap \nu^{-1}(\mathcal{K}_j).$$

According to Theorem B.6, the trick to ensure that a topological horseshoe occurs is only the verification of some stretching properties for the continuous maps ν and η.

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