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Qualitative resonance of feedback-controlled chaotic oscillators

Fabio Della Rossa [*,**] **Fabio Dercole** [**]

[*] *Department of Mechanical Engineering, University of New Mexico,*
Albuquerque, New Mexico 87131, USA (e-mail: fdellarossa@unm.edu)
[**] *Department of Electronics, Information, and Bioengineering, Politecnico di Milano,*
Milan, 20133, Italy (e-mail: fabio.dercole@polimi.it)

Abstract: The qualitative resonance of feedback-controlled chaotic oscillators is the ability of the control system to qualitatively synchronize with a reference signal similar to one of the unstable periodic orbits embedded in the open-loop attractor. This property, discovered by O. De Feo (2004a; 2004b) while studying Shilnikov-type attractors, was explained in terms of the random-like rephasing mechanism characterizing the oscillator's dynamics, so to guarantee the eventual in-phase looking with the reference forcing. We experimentally show that the phenomenon works more in general, even in the absence of a rephasing mechanism. Intuitively, the forcing by the target cycle, or by a qualitative approximation of it, is sufficient to bring in the in-phase condition. Our results can make chaos control more practicable than so far imagined, as a qualitative control can be achieved with no a-priori knowledge about the target solution.

Keywords: Chaos control, Nonlinear systems, Qualitative resonance, Stability, Synchronization.

1. INTRODUCTION

The phenomenon of *qualitative resonance* of feedback-controlled chaotic oscillators is a property introduced by O. De Feo (2004a; 2004b) for nonlinear oscillators characterized by an open-loop Shilnikov-type attractor—the chaotic attractor organized around a homoclinic bifurcation to a saddle-focus equilibrium (Shil'nikov et al., 1998, Sect. 13.5). When forced by the discrepancy between a signal similar to an observable of the open-loop attractor and the same observable of the system's state, the oscillator reduces its complexity by qualitatively synchronizing with the reference forcing. In contrast, when the forcing is uncorrelated with the endogenous open-loop dynamics, the system's complexity is amplified—anti-resonance. Note that this phenomenon is conceptually different from the open-loop entrainment of the oscillator by an external forcing (Pikovsky et al., 2001).

When targeting one of the unstable periodic orbits—the *target* cycle—embedded in the open-loop attractor, the resonance becomes "perfect" and can be locally explained by classical linear-periodic control theory (Callier and Desoer, 1991; Brogan, 1991). The control scheme is reported in Fig. 1. An observable of the target cycle is injected as reference (w) for the control and the error (e) w.r.t. the same observable (y) of the system's state is fed back into the system through a constant gain (α). If the state observable and the gain are suitably chosen—under some controllability assumption—the target cycle in phase with the control reference is a stable orbit of the closed-loop system.

The global convergence to the synchronous solution was explained by De Feo (2004a,b) exploiting a rephasing mechanism typical of Shilnikov attractors. These type of chaotic attractors are indeed characterized by a sensitive phase-in-phase-out mapping close to the saddle equilibrium organizing the homoclinic chaos. By means of this random-like rephasing, the system's state eventually gets in close phase with the reference forcing, so to converge to the locally stabilized target cycle. De Feo (2004a,b) further noted—and explained in terms of the dynamical features of Shilnikov-type attractors—the qualitative nature of the imperfect resonance when a noisy or approximated observable of the target cycle is used as control reference. The control system thus behaves as a resonator, qualitatively synchronizing with signals similar to those endogenously generated by the open-loop oscillator, whereas amplifying the complexity when different signals are used.

We experimentally show that the random-like rephasing mechanism of Shilnikov attractors is not necessary for the resonant behavior of the control system of Fig. 1. That is, the oscillator qualitatively synchronizes with the control reference whenever the latter resembles a typical pattern of the open-loop attractor. We test this claim on several well-known non-Shilnikov chaotic oscillators and even on a coherent oscillator built by periodically forcing a Rossler oscillator. Coherent chaotic attractors—attractors producing oscillations with chaotic amplitude and very regular frequency (Pikovsky et al., 2001)—lack any significant rephasing mechanism and are therefore benchmark tests for our claim. Intuitively, in the case of perfect resonance, the unstable modes of the open-loop attractor are sufficient to eventually bring in the in-phase condition and trigger the convergence to the stabilized target solution.
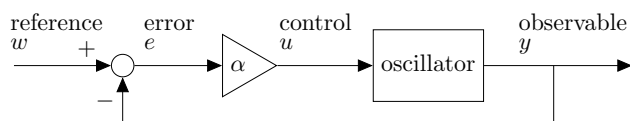

Fig. 1. The continuous output-feedback control scheme.

Our results broaden the range of applications in which the phenomenon of qualitative resonance can be usefully exploited. The feedback-controlled oscillator could be used as a test for the recognition of the temporal patterns represented by the open-loop attractor, with particular interest in neuroscience (Hopfield et al., 1995; Fang et al., 2010). More in general, synchronization phenomena are responsible of information processing and propagation in biological networks (Getting, 1989; Boccaletti et al., 2010), and phenomena of network qualitative resonance might emerge because of the interconnections acting as feedback. E.g., the complexity reduction and qualitative synchronization in excited networks has been used to model epileptic seizures (Barbieri et al., 2012b,a).

With respect to control problems, our results make the applicability of chaos control (Fradkov and Pogromsky, 1998; Boccaletti et al., 2000; González-Miranda, 2004; Schöll and Schuster, 2008) wider than so far imagined. When the aim of control is the complexity reduction of an otherwise chaotic motion, rather than the targeting of a specific solution, a qualitative approximation of the typical orbit—sometimes even a simple harmonic forcing at the average frequency of the open-loop attractor—can be used as reference. No a-priori knowledge of the target solution—the major limit of the two most common control methods (Ott et al., 1990; Pyragas, 1992)—is required.

2. METHODS

2.1 The control scheme

The continuous-time scheme for feedback chaos control is reported in Fig. 1, with reference to the stabilization of a specific target cycle $\gamma = \{\mathbf{x} = \mathbf{x}_T(t) \in \mathbf{R}^n, t \in [0,T], \mathbf{x}_T(0) = \mathbf{x}_T(T)\}$ of period T embedded in the open-loop attractor. The open-loop system is described by a set of n autonomous nonlinear ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), \quad \mathbf{x}(t) \in \mathbf{R}^n \qquad (1)$$

and is assumed to be chaotic, i.e., a chaotic attractor exists for system (1) and the initial state $\mathbf{x}(0)$ is always assumed within its basin of attraction. When the control u is applied, the system is described by a non-autonomous set of ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), u(t)), \quad u(t) \in \mathbf{R}, \quad \mathbf{F}(\mathbf{x}, 0) = \mathbf{f}(\mathbf{x}), \quad (2)$$

e.g., the affine system $\mathbf{F}(\mathbf{x}, u) = \mathbf{f}(\mathbf{x}) + \mathbf{b}u$ where $\mathbf{b} \in \mathbf{R}^n$ is a constant input vector. The control is proportional to the mismatch between the system's output

$$y(t) = g(\mathbf{x}(t)), \quad y(t) \in \mathbf{R}, \qquad (3)$$

and the same observable

$$w(t) = g(\mathbf{x}_T(t)) \qquad (4)$$

of the target cycle, i.e.,

$$u(t) = \alpha(w(t) - y(t)), \qquad (5)$$

where α is the control gain. The choice of $\mathbf{x}_T(0)$ on γ sets the initial phase of the target cycle.

The control system (2–5) is non-autonomous, due to the reference forcing. To discuss orbital stability, it is convenient to imagine the reference w as generated by a set of ODEs, with state $\mathbf{z}$, for which the cycle γ is a stable orbit. We therefore add the virtual equations

$$\dot{\mathbf{z}}(t) = \mathbf{h}(\mathbf{z}(t)), \quad \mathbf{z}(t) \in \mathbf{R}^n, \quad \mathbf{z}(0) = \mathbf{x}_T(0), \qquad (6)$$

$$w(t) = g(\mathbf{z}(t)), \qquad (7)$$

where $\mathbf{h}(x) = \mathbf{f}(x)$ for any $\mathbf{x}$ on γ. The aim of control is then to make the cycle $\Gamma = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} = \mathbf{z} \in \gamma\}$ a stable orbit of the extended autonomous system (2–7) (where Eq. (7) equivalently overwrites Eq. (4)).

2.2 Local stability

Linearization of the control system (2–7) around the cycle Γ gives the periodic linear system

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}(t)\delta\mathbf{x}(t) - \alpha\mathbf{b}(t)\delta y(t), \qquad (8)$$

$$\delta y(t) = \mathbf{c}(t)^\top \delta\mathbf{x}(t), \qquad (9)$$

with $\delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{z}(t)$, $\delta y(t) = y(t) - w(t)$, and

$$\mathbf{A}(t) = \tfrac{\partial}{\partial\mathbf{x}}\mathbf{f}(x)|_{\mathbf{x}=\mathbf{x}_T(t)}, \qquad (10a)$$

$$\mathbf{b}(t) = \tfrac{\partial}{\partial u}\mathbf{F}(\mathbf{x}_T(t), u)|_{u=0}, \qquad (10b)$$

$$\mathbf{c}(t)^\top = \tfrac{\partial}{\partial\mathbf{x}}g(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_T(t)}. \qquad (10c)$$

Classical results of periodic control theory (Callier and Desoer, 1991; Brogan, 1991) ensure that if the pair $(\mathbf{A}(t), \mathbf{b}(t))$ is controllable and the pair $(\mathbf{A}(t), \mathbf{c}(t)^\top)$ observable, the Floquet multipliers of system (8–10) can be arbitrarily assigned by means of suitable T-periodic input vector $\mathbf{b}(t)$ and observation vector $\mathbf{c}(t)^\top$, with $\alpha \neq 0$ closing the loop. Moreover, if $\mathbf{b}(t) = \mathbf{b}$ and $\mathbf{c}(t) = \mathbf{c}$ are constant vectors, as in all examples that we consider, there exists at least one stabilizing choice, i.e., $(\mathbf{b}, \mathbf{c}, \alpha)$ such that the solution $\delta\mathbf{x}(t)$ of system (8–10) goes to zero from any initial condition (Brunovskỳ, 1970; Aeyels and Willems, 1995; Colaneri et al., 1998).

In all our examples, the state observable is, for simplicity, one of the state variables (i.e., $\mathbf{c} = \mathbf{e}^{(i)}$ with $\mathbf{e}_j^{(i)} = 1$ if $i = j$, 0 otherwise). In all cases, a stabilizing control has been found by directly acting on the same variable's rate of change (i.e., $\mathbf{b} = \mathbf{c}$). Fig. 2 shows two examples on the Colpitts oscillator (see Tab. 1 for the model's equations and parameters). The Floquet multipliers of the linearized system (8–10) are plotted versus the control gain α (the target cycle is unstable for $\alpha = 0$). Note that these are only $n = 3$ of the $2n - 1 = 5$ multipliers of the extended cycle Γ, the other two (stable by assumption, though irrelevant for the purpose of the analysis) are virtually obtained by linearizing Eq. (6) around the target cycle γ (and discarding the trivial unit-multiplier).

2.3 Perfect resonance

We first consider the case of perfect resonance, in which the reference w is exactly observed along a target cycle γ. As noted in Sects. 2.1 and 2.2, the stabilizing control law ensures the convergence to γ only if control is switched on when the oscillator's state $\mathbf{x}$ is close to γ and in close phase with the reference w (i.e., if the state $(\mathbf{x}, \mathbf{z})$ of the extended system (2–7) is close to the cycle Γ).

The need to start from (or eventually reach) an in-phase condition between the reference w and the oscillator's state $\mathbf{x}$ led De Feo (2004a,b) to consider chaotic oscillators with Shilnikov-type open-loop attractors, i.e., attractors
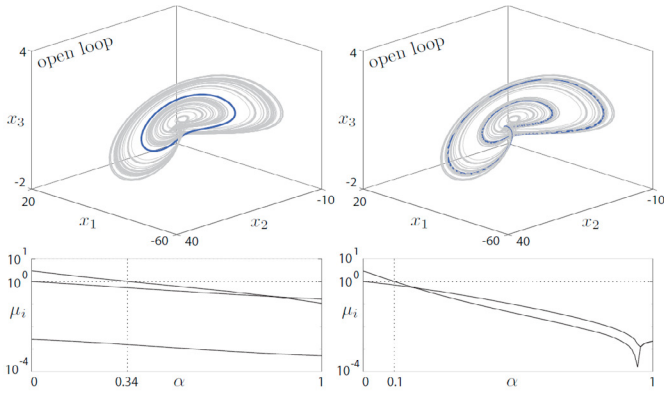
Fig. 2. Closed-loop stability of two cycles embedded in the Colpitts attractor. The cycle (blue) and the open-loop attractor ($\alpha = 0$, gray) are shown in the top panel. The Floquet multipliers $\mu_{1,2,3}$ of system (8–10) are shown in the bottom panel ($\mu_3 \approx 10^{-6}$ is not visible in the right panel). The corresponding reference signals w are shown in Fig. 3 (bottom panels, left and right). In both cases the cycle turns stable for sufficiently large control gain α.

Fig. 3. Perfect resonance in the Colpitts oscillator. The target cycles (green, left and right; corresponding reference signals w in the bottom panels) are the same of Fig. 2, stabilized for $\alpha = 0.5$. The initial condition of the closed-loop dynamics is taken at random in the shown box of the state space.

obtained with the system's parameters in the vicinity of a saddle-focus homoclinic bifurcation (Shil'nikov et al., 1998; Kuznetsov, 2004). These attractors are characterized by a "random-like" rephasing of the oscillations that works as follows. The ergodicity of the attractor ensures repeated passing close to the saddle-focus equilibrium, close to which the time spent by the orbit in-between two local, inward and outward, Poincaré sections—the first intersected by the flow when approaching the equilibrium, the second when leaving—is large and highly sensitive to the first intersection point. As a result, the phase of the oscillator at the outward section is weakly dependent from the phase at the inward section, as if a random re-phasing acts when passing close to the saddle-focus.

Then, the convergence of the oscillator's state $\mathbf{x}$ to the target cycle γ is granted by the following arguments. First, the rephasing mechanism is assumed to work also in the closed-loop attractor. This should be guaranteed, by continuity, for sufficiently small α (a small norm of the matrix $\alpha\,\mathbf{b}\,\mathbf{c}^\top$ is actually required, see Sect. 2.2). Second, continuity also guarantees that the state $\mathbf{x}$ repeatedly passes close to γ, being however most of the time out-of-phase w.r.t. the reference w (i.e., $\mathbf{x}$ close to γ at some time t but at the same time far from the state $\mathbf{z}$ of the virtual oscillator (6)). Third, by the rephasing mechanism, the state $\mathbf{x}$ will eventually get close to γ in close phase with w (i.e., $\mathbf{x}(t)$ close to the point $\mathbf{z}(t)$ of γ). The closed-loop local stability will then ensure convergence.

Thus, independently of the initial phase set for the reference w (i.e., independently of the initial condition $\mathbf{z}(0)$ of the virtual oscillator (6)), the state $\mathbf{x}$ of the control system in Fig. 1 should synchronize with the reference forcing. This has been numerically confirmed by De Feo (2004a) for several chaotic oscillators of Shilnikov-type and the experiment is repeated in Fig. 3 for the Colpitts oscillator (targeting the two cycles of Fig. 2).
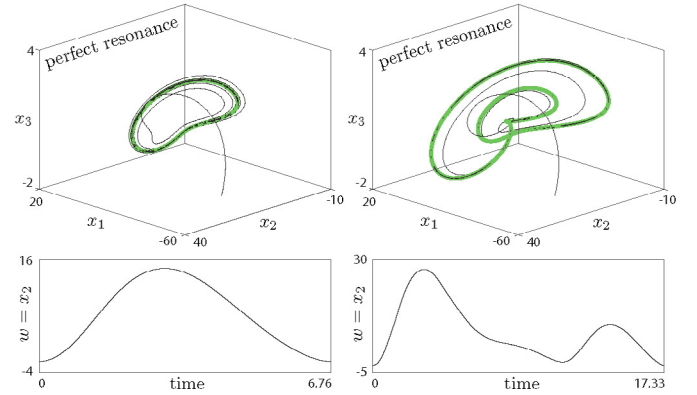
Phenomenologically, the synchronization with the reference forcing can be seen as a resonant behavior of the control system. When the observable w of the target cycle is injected as reference, the power spectrum of the system's output peaks at the frequency of the target cycle, whereas it is distributed over a full range of frequencies in the open-loop chaotic behavior.

2.4 Qualitative resonance

To test the qualitative resonance, De Feo (2004a) considered approximated observables w of the target cycle γ. Piecewise-linear and noisy approximations have been tested, and even a simple sinusoidal forcing at the average frequency of the open-loop attractor. Sufficiently good piecewise-linear approximations are able to synchronize the control system to a periodic behavior. Otherwise, piecewise-linear and noisy approximations give a qualitative resonance in the form of a thin, coherent, and weakly-chaotic closed-loop attractor, an "almost-periodic" motion on a sort of "thick" limit cycle (not to be confused with a quasi-periodic attractor). In some cases, the sinusoidal forcing is able to synchronize the control system to a periodic behavior. Otherwise, a quasi-periodic behavior or a wide chaotic attractor (anti-resonance) is observed. Some tests on the Colpitts oscillator are reported in Fig. 4. The theoretical foundations of the phenomenon for Shilnikov-type attractors are detailed in De Feo (2004b).

3. RESULTS

We claim that the phenomenon of qualitative resonance is a generic property of the feedback-controlled chaotic oscillator of Fig. 1. In particular, it does not require the dynamical features of Shilnikov-type attractors to which the phenomenon was originally attributed (De Feo, 2004a,b).

We provide in this paper the numerical support of the above claim. We do this by analyzing four well-known examples of non-Shilnikov chaotic attractors, see Tab. 1 for the oscillators' equations, parameter settings, and control.

Oscillator	Equations	Parameters		Control
Colpitts	$\dot{x}_1 = g(-e^{-x_2} + 1 + x_3)/(Q(1-k))$ $\dot{x}_2 = gx_3/(Qk)$ $\dot{x}_3 = -Qk(1-k)(x_1+x_2)/g - x_3/Q$	$g = 3$ $Q = 1.4$ $k = 0.5$		$\mathbf{b} = \mathbf{c} = \mathbf{e}^{(2)}$ $\alpha = 0.5$
Lorenz	$\dot{x}_1 = \sigma(x_2 - x_1)$ $\dot{x}_2 = x_1(\rho - x_3) - x_2$ $\dot{x}_3 = x_1 x_2 - \beta x_3$	$\sigma = 28$ $\rho = 10$ $\beta = 8/3$		$\mathbf{b} = \mathbf{c} = e^{(2)}$ $\alpha = 4$
Rossler	$\dot{x}_1 = -(x_2 + x_3)$ $\dot{x}_2 = x_1 + ax_2$ $\dot{x}_3 = b + x_3(x_1 - c)$	$a = 0.1$ $b = 0.1$ $c = 18$		$\mathbf{b} = \mathbf{c} = e^{(1)}$ $\alpha = 0.3$
Jansen & Rit Neural Mass	$\ddot{x}_1 = O(x_2 + x_3 + 3.36) - 2\dot{x}_1 - x_1 \quad O(x) = (1 + \gamma e^{-x})^{-1}$ $\ddot{x}_2 = \frac{4}{5}aO(ax_1) - 2\dot{x}_2 - x_2$ $\ddot{x}_3 = -\frac{11}{13}aO(\frac{1}{4}ax_1 + \zeta e^{-2\delta \cos^2 \omega t}) - 2\beta\dot{x}_3 - \beta^2 x_3$	$a = 12.285$ $\beta = 0.5$ $\gamma = 28.7892$	$\delta = 110$ $\zeta = 1.5$ $\omega = \pi/25$	$\mathbf{b} = \mathbf{c} = e^{(1)}$ $\alpha = 2$
Coherent Rossler	$\dot{x}_1 = -(x_2 + x_3 + \varepsilon \cos \omega t)$ $\dot{x}_2 = x_1 + ax_2$ $\dot{x}_3 = b + x_3(x_1 - c)$	$a = 0.1$ $b = 0.1$ $c = 18$	$\varepsilon = 0.5$ $\omega = 1.0345$	$\mathbf{b} = \mathbf{c} = e^{(1)}$ $\alpha = 0.3$

Table 1. Oscillators' equations, parameters, and control.

The results are graphically summarized in Fig. 5. Panel a shows a target cycle embedded in the open-loop attractor, panel b the perfect resonance, obtained using the exact observable of the target cycle as control reference, panels c–f four examples of qualitative resonance.

Specifically, the first is the Lorenz system, with the paradigmatic butterfly-shaped chaotic attractor, that is
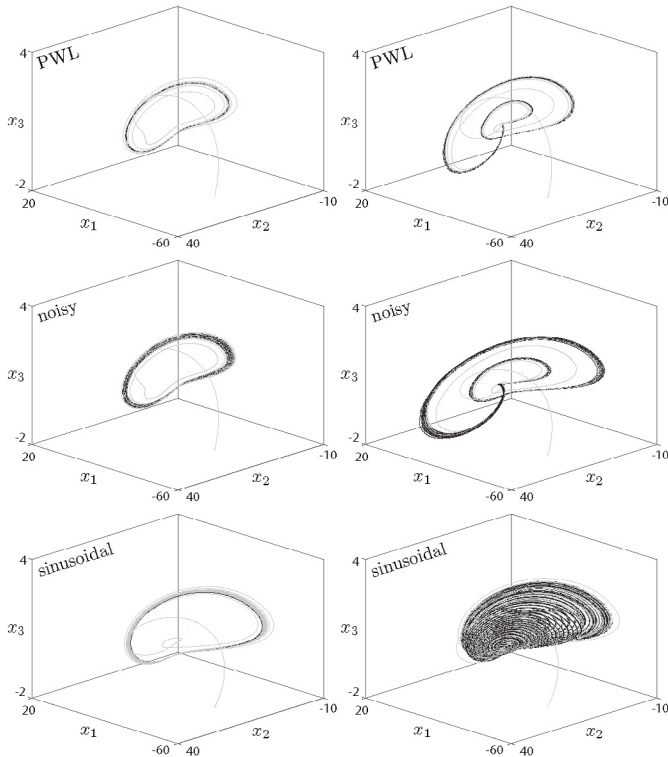

Fig. 4. Qualitative resonance in the Colpitts oscillator when targeting the two cycles (left and right) of Fig. 2. PWL: 10 intervals; noisy: 10% random amplitude modulation $w(1+\text{WGN}(0,0.01))$; sinusoidal: average amplitude and frequency of $w = x_2$ in the open-loop attractor. The closed-loop attractor is shown black, together with the transient (gray) from a random initial condition.

generated through a cascade of symmetric homoclinic bifurcations to a real saddle (hence, of non-Shilnikov type (Shil'nikov et al., 1998, Chap. 13)). The second is the Rossler oscillator. While the Rossler attractor can be of Shilnikov-type (Shil'nikov et al., 1998, Appx. C.7), we analyze it for a parameter setting for which the chaotic behavior is organized by a Feigenbaum cascade of period doubling bifurcations (Rössler, 1976). The third system is a neural-mass model describing the mean electrical activity of a macro-area of the brain cortex (Jansen and Rit, 1995; Spiegler et al., 2010). It is known to have a chaotic attractor organized by a homoclinic bifurcation to a saddle node, that shows a weak rephasing mechanism because of the "gost" of the disappeared saddle-node equilibrium. Finally, the last system is an artificially designed example of coherent chaotic oscillator—a periodically forced Rossler system.

Note that in all cases, the sinusoidal forcing is able to synchronize the control system to a periodic behavior, even though the obtained closed-loop cycle looks rather different from those embedded in the open-loop attractor. This is especially true in the Lorentz and Jansen & Rit systems, since they produce open-loop oscillations that are far from harmonic. Remarkably, qualitative resonance works fine even in the periodically forced Rossler system.

In all cases, we have tested a new form of qualitative resonance. We identify the most typical reference signal produced by the open-loop oscillator with the following heuristic. Among many finite signals $y_k(t)$, $t \in [0, T_k]$, obtained with respect to a Poincaré section at constant $y = x_i$, we select the one $y^*(t)$, $t \in [0, T^*]$, that includes the largest number of the others within a prescribed neighborhood (a $\pm\varepsilon$-band in the phase domain, where $\varepsilon = 0.3\|y^*\|_\infty = 0.3\max_{t \in [0,T^*]}|y^*(t)|$ and the phase φ in $[0,1]$ is defined by normalizing the time-domain of the signals, $y_k(t) = y_k(T_k\varphi)$. We then use as reference w the periodic signal obtained by repeating the period $y^*(t)$. Panels f in Fig. 5 show the qualitative resonance obtained with this heuristic.

1. Lorenz
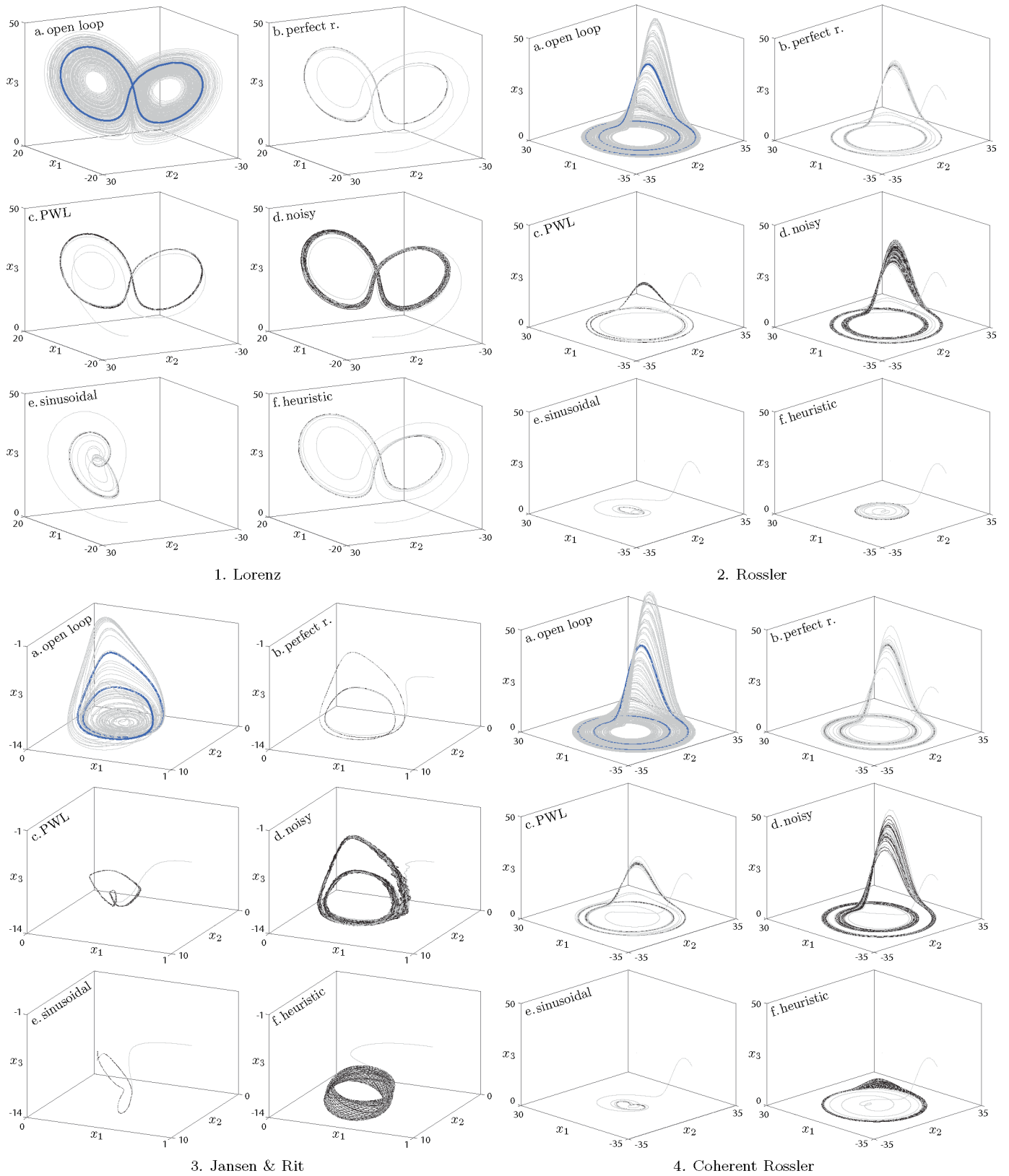
2. Rossler

3. Jansen & Rit

4. Coherent Rossler

Fig. 5. Perfect and qualitative resonance in four non-Shilnikov chaotic attractors. a, the target cycle (blue) embedded in the open-loop attractor (gray); b, perfect resonance from a random initial condition; c–e, qualitative resonance as in Fig. 4; f, qualitative resonance to the typical open-loop signal (heuristic parameter $\varepsilon = XXX$). b–f, the closed-loop attractor is shown black, together with the transient (gray) from a random initial condition.

4. DISCUSSION AND CONCLUSIONS

We have numerically shown that the qualitative resonance of feedback-controlled chaotic oscillators—the ability of the control system of Fig. 1 to qualitatively synchronize with a reference similar to the signals endogenously generated by the open-loop oscillator—is a general property of chaotic oscillators, not limited to those of Shilnikov type (Shil'nikov et al., 1998, Sect. 13.5) for which it was originally introduced and justified (De Feo, 2004a,b).

We have tested our claim on four well-known non-Shilnikov chaotic oscillators (see Tab. 1): the classical Lorenz and Rossler systems (the latter for a parameter setting showing a non-Shilnikov attractor), the Jansen & Rit neural-mass model used to study brain function and pathology, and a coherent chaotic oscillator obtained by periodically forcing the Rossler system. The last example particularly supports our claim, because free of the random-like rephasing mechanism of Shilnikov attractors, to which the property of qualitative resonance was originally attributed.

Besides the general value of our results for the understanding of resonance and synchronization phenomena, here we focus on their implications to the control of chaos (Fradkov and Pogromsky, 1998; Boccaletti et al., 2000; González-Miranda, 2004; Schöll and Schuster, 2008)—the set of control techniques aimed at rendering an otherwise chaotic motion more stable and predictable. Feedback controllers (Ott et al., 1990; Pyragas, 1992) are typically designed to stabilize one of the saddle periodic orbits—the target cycle—embedded in the open-loop attractor, as too strong control actions would be required to stabilize an equilibrium. The determination of the saddle solution to be stabilized—a difficult step, both numerically and experimentally—is therefore required and this has limited the applicability of chaos control. We have essentially shown that this undesirable step can be avoided. When the stabilization of a particular orbit is not the aim of control, the complexity reduction can be achieved with the control scheme of Fig. 1, using as reference a typical signal recorded during the free motion of the oscillator and tuning the control gain.

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