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Research Article

Conditions on the Energy Market Diversification from Adaptive Dynamics

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We study a mathematical model based on ordinary differential equations to describe the dynamic interaction in the market of two types of energy called standard and innovative. The model consists of an adaptation of the generalized Lotka-Volterra system in which the parameters are assumed to depend on a quantitative and continuous attribute characteristic of energy generation. Using the analysis of the model the fitness function for the innovative energy is determined, from which conditions of invasion can be established in a market dominated by the conventional power. The canonical equation of the adaptive dynamics is studied to know the long-term behavior of the characteristic attribute and its impact on the market. Then we establish conditions under which evolutionary ramifications occur, that is to say, the requirements of coexistence and divergence of the characteristic attributes, whose occurrence leads to the origin of diversity in the energy market.

1. Introduction

The energy market is a complex system in a rapidly varying context in which decision-making is difficult. Its complexity is due to a large number of physical and economic factors involved. In particular, physical factors may be related to climatic conditions and have an unpredictable medium- and long-term behavior, as well as an unknown effect on aspects of the market such as supply, demand, and price. Market regulation and public policies generate causal relationships between all these elements producing highly complex interactions. Other factors associated with technological and social changes, such as innovations in energy generation or changes in consumption patterns, which are not predictable in the medium or long term, are also determinants [1].

Technological innovation is one of the most important components to drive development. Technological change and technological diversity are two intimately linked concepts that represent both the means and the results of economic development [2–5]. The processes of technological change are dynamic and three important stages can be characterized: the emergence, the substitution, and the possible coexistence of technologies, constituting what has been called the technological cycle, which finally aims to understand the emergence of technological innovations and its subsequent evolution. When a technological innovation is successful, the new technology can invade the market, and it remains in it for some time until a new competitive technology emerges and challenges its domain [3, 6]. Different phenomena can originate in this point, on the one hand, when the adoption of the new technology implies that the previous technologies became obsolete, which configures a substitution scenario in the market, or that the competing technologies do not become obsolete but share the market without replacing each other; this scenario leads to market diversification [3, 7, 8].

In recent years there has been a significant development of alternative energy generation technologies, as reported in [9], who find that the EU ETS has increased low-carbon innovation among regulated firms by as much as 10%, while not crowding out patenting for other technologies. They also find evidence that the EU ETS has not affected licensing beyond the set of regulated companies. These results imply that the ETS accounts for nearly a 1% increase in European low-carbon patenting compared to a counterfactual scenario. In this context, it is necessary to study the energy market and, in particular, the dynamics that arise after the introduction of innovative technologies, using mathematical tools that help to describe the inherent complexity of the system. In the study of energy markets, it is necessary to take into account some intrinsic characteristics or attributes, such as generation source, emission reduction, final consumer price, generation technologies, generation capacity, and level of investment. Also, it is essential to describe how its dynamics in the long-term influences the conditions of interaction between agents established in the market and those who consider themselves innovative. In [10] they define environmental innovations as a product, process, marketing, and organizational changes leading to a noticeable reduction of environmental burdens. Positive ecological effects can be explicit goals or side-effects of innovations.

The adaptive dynamics (AD) constitute a theoretical background originating in evolutionary biology that link demographic dynamics to evolutionary changes and allow describing evolutionary dynamics in the long-term when considering mutations as small and rare events in the demographic time scale [11–15]. AD describes evolution through an ordinary differential equation known as the canonical equation of the adaptive dynamics. This approach focuses on the long-term evolutionary dynamics of continuous (quantitative) adaptive traits and overlooks genetic detail through the use of asexual demographic models, which is justified under different demographic and evolutionary timescales. This approach considers interactions to be the evolutionary driving force and takes into account the feedback between evolutionary change and the selection forces that agents undergo [11, 16, 17].

Analogies between the ecological processes of competition and collaboration with the dynamics of markets are powerful conceptual tools when used in the appropriate contexts. Nair et al. [18], based on real cases in the industry, argue that the complexity of technological change and the ecological and institutional dynamics can allow regimes of coexistence of competing technologies [3]. Cooperative interactions are studied in [19] through adaptive dynamics framework, where the authors show that asymmetrical competition within species for the commodities offered by mutualistic partners provides a simple and testable ecological mechanism that can account for the long-term persistence of mutualism. In the competition context, in [20] a model devoted to the study of an evolutionary system where similar individuals are competing for the same resources is presented. Examples can be found in predator-prey dynamics, evolution of dispersal, dynamics in allele space, and cannibalistic interactions. In addition, detailed mathematical developments of the theory can be found; particularly in [11] a thorough review of the theoretical aspects and applications is made.

Using the theoretical framework of adaptive dynamics, the canonical equation, corresponding to an ordinary differential equation, is presented to describe the behavior over time of the characteristic attribute as a result of innovation processes. Besides biology, the theoretical framework of adaptive dynamics has been recently used to model a varied spectrum of nongenetic innovations, in particular social [21] or technological innovations [7, 8]. In the technological context, the authors explore the emergence of technological diversity arising from market interaction and technological innovation. Particularly, existing products compete with the innovative ones resulting in a slow and continuous evolution of the underlying technological characteristics of successful products.

In the present work a mathematical model based on ordinary differential equations is studied, to describe the dynamics of a market dominated by a standard energy (SE) generation technology in interaction with an innovative energy IE generation technology. Initially, the model consists of an adaptation of the Lotka-Volterra equations under the consideration that interaction between both types of energy can occur in a market based on competition or cooperation as interaction strategies, as described in [22] in a cross-country study on the relationship between diffusion of wind and photovoltaic solar technology. In both cases, SE and IE are measured with the cumulative generation capacity (CGC) as a nonnegative real number defining its level of penetration into the market. Under those scenarios, we determine conditions for IE to invade and establish the market, giving rise to diversification. The model parameters are defined as functional coefficients depending on the values of a characteristic quantitative and continuous trait to determine some relevant aspects of energy generation. In general, adaptive dynamics theory allows us to study the long-term evolutionary dynamics of the quantitative attributes that characterize both energy CGSs and to describe how they affect the interaction dynamics in the short-term market timescale. On the other hand, it also allows us to establish how the conditions of interaction in the market influence the evolutionary dynamics of the attributes and, ultimately, to determine which innovative characteristics can invade or which attributes disappear definitively.

In the second section of this paper, the reader will find a description of the adaptation made to the Lotka-Volterra model to describe the interaction between two similar types of energy. Local stability is described and invasion conditions are determined. In the third section, an explicit definition of the coefficients of the model according to the standard and innovative attributes is stated, to consider some particular aspects of the market and later to determine how they influence the conditions of invasion of the innovative energy. The canonical equation is described and, from it, the long-term evolutionary dynamics of the characteristic attributes follows. In particular, there are conditions under which there is evolutionary branching that allow market diversification. We illustrate the situation with numerical simulations. Finally the conclusions and the references are shown.

2. Model Description

2.1. Innovative-Standard Model. Some technical assumptions on the model are the following: (a) we consider two types of energy generation, which are differentiated by the technology used (we refer to them in this paper as *energy generation technology*). (b) Each energy generation technology is characterized by the value of a given characteristic attribute quantifiable by means of a real number, i.e., a measure of the technology. It can be assumed that a higher attribute value is related to more advanced technologies, although innovations are not necessarily preferred by consumers. (c) In the absence of innovations, the generation of established energy reaches a specific equilibrium value on a time scale that we call the "market time scale". (d) Innovations are rare events on the market time scale; i.e., they occur on a much longer time scale that we call "evolutionary time scale." This separation of the scales allows us to assume that the market is in equilibrium when an innovation occurs and that the market is affected by a single innovation at the same time [11]. (e) Finally, it is assumed that the innovations are small; that is, the innovative attribute only differs somewhat from the established quality; this corresponds to considering marginal innovations that give origin to energies similar to those established.

Consider an energy market dominated by a *standard energy* generation technology (SE), with *cumulative generation capacity* (CGC) $n_1 = n_1(t)$ at any time t, and assume there is some standard characteristic trait x_1 to determine a suitable feature of SE generation. It can be, for example, the final price of energy to the consumer or other characteristics such as energy saving, emission reduction, or generation capacity or level of investment. Suppose a marginal innovation occurs in this characteristic trait, slightly changing the value x_1 to x_2 and leading to the appearance of an *Innovative Energy* generation technology (IE), with CGC $n_2 = n_2(t)$, different from n_1, and characterized by the trait x_2, called innovative characteristic trait from now on.

Generation Growth Rate. Consider the CGC n of a given generation technology to grow at rate $r(x)$, as a function of the characteristic trait x. This function describes how fast n increases depending on the value of x. Growing rate r should be considered as a positive function $r(x) > 0$ for all $x \in \mathbb{R}$.

Maximum Capacity. Let the function $K(x)$ describe the maximum cumulative generation capacity that some generation technology can reach and allocate into the market, as a function of its characteristic trait x. As generation and demand grow, it is realistic to consider K as a nonnegative increasing function of x, bounded above by some maximum value corresponding to technical limitations or imposed normative obeying public police.

Interaction Coefficient. Define the function $c(x_i, x_j)$ to determine the interaction into the energy market between the i generation technology with CGC n_i and the j generation technology with CGC n_j. It corresponds to the rate of increase/decrease of CGC suffered by n_i by the presence of n_j; we assume $c(x_i, x_i) = 1$ to indicate internal competition; i.e., $c(x_1, x_1) = 1$ corresponds to internal competition between SE generation technologies, and, similarly, $c(x_2, x_2) = 1$ corresponds to internal competition between IE generation technologies.

Additional general situations can occur depending on the region of the (x_1, x_2)–plane where the point (x_1, x_2) is located:

(i) If $c(x_i, x_j) > 1$, for $x_i \neq x_j$, external competition predominates internal competition; that is, $c(x_i, x_j) > 1$ implies that the competition between generation technology i and generation technology j is stronger than competition between systems generating the same type of energy.

(ii) If $0 \leq c(x_i, x_j) \leq 1$, for $x_i \neq x_j$, then internal competition predominates external competition. It has to be stronger competition between different SE generation technologies among each other than the competition between SE and IE generation technologies. In particular, if $c(x_i, x_j) = 0$, there is no interaction at all and if $c(x_i, x_j) = 1$, both competitions are equally strong.

(iii) If $c(x_i, x_j) < 0$, for $x_i \neq x_j$, the interaction between generation technologies i and j does not correspond to competition but to cooperation, a situation that can describe the integration of systems. In this case, each one is rewarded by the presence of the other.

In general, it is assumed that x_2 is close to x_1; i.e., the innovation is small and it has a small effect. So doing, such an innovation always compete with the established one and, only after diversification, the market could turn cooperative. Additionally, there might be mixed cases. For instance, when $c(x_1, x_2) > 1$ and $c(x_2, x_1) < 1$ for $x_2 > x_1$, the low-tech energy generation suffers the high-tech more than itself, and, conversely, when $c(x_1, x_2) < 1$ and $c(x_2, x_1) > 1$ for $x_2 > x_1$ the high-tech energy generation suffers the low-tech more than itself.

Under the assumptions described, we propose an interaction Lotka-Volterra model:

$$\begin{aligned} \dot{n}_1 &= n_1 r(x_1)\left(1 - \frac{n_1 + c(x_1, x_2) n_2}{K(x_1)}\right) \\ &= n_1 g(n_1, n_2, x_1, x_2, x_1) \\ \dot{n}_2 &= n_2 r(x_2)\left(1 - \frac{n_2 + c(x_2, x_1) n_1}{K(x_2)}\right) \\ &= n_2 g(n_1, n_2, x_1, x_2, x_2) \end{aligned} \tag{1}$$

defined on the set $\Omega = \{(n_1, n_2) : n_1 \geq 0, n_2 \geq 0\}$. Note that both *relative* growth rates $\dot{n}_1/n_1$ and $\dot{n}_2/n_2$ can be expressed by means of a single function g that in the AD framework is called *fitness generating function*

$$\begin{aligned} &g(n_1, n_2, x_1, x_2, z) \\ &\quad = r(z)\left(1 - \frac{c(x_1, z) n_1 + c(z, x_2) n_2}{K(z)}\right) \end{aligned} \tag{2}$$

Taking into account the fact that we have assumed the condition $c(z, z) = 1$, for all $z \in \mathcal{X}$, in system (1),

$$g(n_1, n_2, x_1, x_2, x_1) = r(x_1)\left(1 - \frac{n_1 + c(x_1, x_2) n_2}{K(x_1)}\right) \tag{3}$$

TABLE 1: Description of state variables and coefficients with their corresponding ranges.

	State variables description	*Units*
$n_1(t)$	CGC* for Standard Energy, characterized by x_1	MW
$n_2(t)$	CGC for Innovative Energy, characterized by x_2	MW
	Parameter description	*Ranges*
x_1	Quantitative continuous characteristic trait defining SE	$x_1 \in \mathbb{R}$
x_2	Quantitative continuous characteristic trait defining IE	$x_2 \in \mathbb{R}$
$r(x_i)$	CGC growing rate as a function of x_i, for $i = 1, 2$	$r > 0$
$K(x_i)$	Maximum CGC as function of x_i, for $i = 1, 2$	$K > 0$ MW
$c(x_1, x_2)$	Interaction coefficient between both CGC as a function of x_1 and x_2	$c \in \mathbb{R}$

*CGC: cumulative generation capacity. **GTs: generation technologies.

represents the *relative* growth rate $\dot{n}_1/n_1$ of SE generation technology. Along the same lines, the relative growth rate of IE, $\dot{n}_2/n_2$, is given by $g(n_1, n_2, x_1, x_2, x_2)$. A more general description of state variables, functional coefficients, and parameter description can be found in Table 1.

2.2. Innovative-Standard Model Local Stability. The importance of the local stability analysis of the model is that it will provide us with relevant information regarding the dynamics of the market of the types of energy that interact and will allow establishing conditions, under which the coexistence is possible, or the definitive disappearance of any of them. In particular, it is essential to know what circumstances EI can invade and remain in the market.

By solving the system $\dot{n}_1 = 0, \dot{n}_2 = 0$ is possible to find four steady states of system (1) given by

(i) $P_0(\overline{n}_1^0, \overline{n}_2^0) = (0, 0)$, corresponding to the absence of SE and IE in the market

(ii) $P_1(0, \overline{n}_2^1(x_2)) = (0, K(x_2))$, corresponding to the exclusion of SE from the market and the IE is dominant

(iii) $P_2(\overline{n}_1^2(x_1), 0) = (K(x_1), 0)$, corresponding to the exclusion of IE from the market and the SE is dominant

(iv) $P_3(\overline{n}_1^3(x_1, x_2), \overline{n}_2^3(x_1, x_2)) = ((c(x_1, x_2)K(x_2) - K(x_1))/(c(x_2, x_1)c(x_1, x_2) - 1), (c(x_2, x_1)K(x_1) - K(x_2))/(c(x_2, x_1)c(x_1, x_2) - 1))$, corresponding to the case when SE and IE are both present and share the market

Notice P_3 can be written as

$$P_3\left(\overline{n}_1^3, \overline{n}_2^3\right) = \left(\frac{K(x_1)(H(x_1, x_2) - 1)}{c(x_2, x_1)c(x_1, x_2) - 1}, \frac{K(x_2)(H(x_2, x_1) - 1)}{c(x_2, x_1)c(x_1, x_2) - 1}\right) \tag{4}$$

where

$$H(x_1, x_2) = \frac{c(x_1, x_2)K(x_2)}{K(x_1)}, \quad \text{and } H(x_2, x_1) = \frac{c(x_2, x_1)K(x_1)}{K(x_2)} \tag{5}$$

Therefore, $P_3 \in \Omega$ if and only if both of its coordinates are nonnegative; this implies two different situations.

Case I. $c(x_2, x_1)c(x_1, x_2) < 1$; then $P_3 \in \Omega$ if and only if

$$H(x_1, x_2) < 1 \quad \text{and } H(x_2, x_1) < 1 \tag{6}$$

In particular, when $H(x_1, x_2) = 1$ and $H(x_2, x_1) < 1$, P_3 coalesce with P_1, and when $H(x_1, x_2) < 1$ and $H(x_2, x_1) = 1$, P_3 coalesce with P_2.

Case II. $c(x_2, x_1)c(x_1, x_2) > 1$; then $P_3 \in \Omega$ if and only if

$$H(x_1, x_2) > 1 \quad \text{and } H(x_2, x_1) > 1 \tag{7}$$

Analogously to the previous case, when $H(x_1, x_2) = 1$ and $H(x_2, x_1) > 1$, P_3 coalesce with P_1, and when $H(x_1, x_2) > 1$ and $H(x_2, x_1) = 1$, P_3 coalesce with P_2.

At this point, local stability analysis will bring some insights into the market dynamics and will help to answer a further question of under what conditions can IE spread into the market and interact or even substitute SE. Indeed, If we consider a market dominated exclusively by SE, the instability of P_2 is related to the possibility for an IE to invade the market, while the existence and stability of P_3 are related to the coexistence of both kinds of energy sharing the market, leading to diversification.

Proposition 1. *The steady state P_0 of system (1) is always unstable.*

Proof. The Jacobian matrix of system (1) at P_0 is given by

$$A(P_0) = \begin{bmatrix} r(x_1) & 0 \\ 0 & r(x_2) \end{bmatrix} \tag{8}$$

Then, the corresponding eigenvalues are $r(x_1)$ and $r(x_2)$, both positive by definition; therefore, steady state P_0 is unstable. □

Proposition 2. *The steady states P_1 and P_2 of system (1) are locally asymptotically stable if and only if $H(x_1, x_2) > 1$ and $H(x_2, x_1) > 1$, respectively.*

Proof. The Jacobian matrix of system (1) at P_1 can be written as

$$A(P_1)=\begin{bmatrix} r(x_1)(1-H(x_1,x_2)) & 0 \\ -r(x_2)c(x_2,x_1) & -r(x_2) \end{bmatrix} \quad (9)$$

Then, the corresponding eigenvalues are $r(x_1)(1-H(x_1,x_2))<0$ if and only if $H(x_1,x_2)>1$, as stated in the proposition, and $-r(x_2)<0$ by definition. On the other hand, the Jacobian matrix of system (1) at P_2 is given by

$$A(P_2)=\begin{bmatrix} -r(x_1) & -r(x_1)c(x_1,x_2) \\ 0 & r(x_2)(1-H(x_2,x_1)) \end{bmatrix} \quad (10)$$

Then, the corresponding eigenvalues are $-r(x_1)<0$ and $r(x_2)(1-H(x_2,x_1))<0$ if and only if $H(x_2,x_1)>1$. This proves the proposition. □

Proposition 3. *Given the steady state P_3, when existing in Ω, its local stability is described in the following way:*

(I) *If $c(x_2,x_1)c(x_1,x_2)<1$, $H(x_1,x_2)<1$ and $H(x_2,x_1)<1$, then P_3 is locally asymptotically stable*

(II) *If $c(x_2,x_1)c(x_1,x_2)>1$, $H(x_1,x_2)>1$ and $H(x_2,x_1)>1$, then P_3 is unstable*

Proof. The Jacobian matrix of system (1) at P_3 can be written as

$$A(P_3)=\begin{bmatrix} -\dfrac{r(x_1)[H(x_1,x_2)-1]}{c(x_2,x_1)c(x_1,x_2)-1} & -\dfrac{[H(x_1,x_2)-1]r(x_1)c(x_1,x_2)}{c(x_2,x_1)c(x_1,x_2)-1} \\ -\dfrac{[H(x_2,x_1)-1]r(x_2)c(x_2,x_1)}{c(x_2,x_1)c(x_1,x_2)-1} & -\dfrac{r(x_2)[H(x_2,x_1)-1]}{c(x_2,x_1)c(x_1,x_2)-1} \end{bmatrix} \quad (11)$$

Let Δ denote the determinant of $A(P_3)$; then, it can be written as

$$\Delta=-\frac{r(x_1)r(x_2)[H(x_1,x_2)-1][H(x_2,x_1)-1]}{c(x_2,x_1)c(x_1,x_2)-1} \quad (12)$$

Similarly, let T be the trace of $A(P_3)$; then

$$T=-\frac{r(x_1)[H(x_1,x_2)-1]+r(x_2)[H(x_2,x_1)-1]}{c(x_2,x_1)c(x_1,x_2)-1} \quad (13)$$

To be consistent, consider the cases when $P_3\in\Omega$. This implies two different situations.

Case I. $c(x_2,x_1)c(x_1,x_2)<1$; then $P_3\in\Omega$ if and only if

$$H(x_1,x_2)<1 \quad \text{and } H(x_2,x_1)<1 \quad (14)$$

In this scenario, $\Delta>0$ and $T<0$. Then P_3 is locally asymptotically stable [23]. As stated above, when $H(x_1,x_2)=1$ and $H(x_2,x_1)<1$, P_3 coalesce with P_1 and transfers its stability to P_1 when $H(x_1,x_2)>1$, case when $P_3\notin\Omega$ although it exists and it is unstable (indeed, $H(x_1,x_2)>1$ and $H(x_2,x_1)<1$ implies $\Delta<0$). Similarly, if $H(x_1,x_2)<1$ and $H(x_2,x_1)=1$, P_3 coalesce with P_2 and transfers its stability. In fact, $H(x_1,x_2)<1$ and $H(x_2,x_1)>1$ imply $\Delta<0$ and $P_3\notin\Omega$ and it is unstable. Both situations correspond to *transcritical bifurcations* [23, 24]. In Table 2 these results are summarized.

Case II. $c(x_2,x_1)c(x_1,x_2)>1$; then $P_3\in\Omega$ if and only if

$$H(x_1,x_2)>1 \quad \text{and } H(x_2,x_1)>1 \quad (15)$$

Notice that, in this case, $\Delta<0$; then $P_3\in\Omega$ but it is unstable (a saddle) and this rules out the possibility of cycles. Analogously to the previous case, when $H(x_1,x_2)=1$ and $H(x_2,x_1)>1$, P_3 collides with P_1, and when $H(x_1,x_2)>1$ and $H(x_2,x_1)=1$, P_3 meets with P_2. Both situations correspond to transcritical bifurcations also (see Table 2). □

It is important to clarify that the last scenario in Table 2 is not possible. If $H(x_1,x_2)<1$ and $H(x_2,x_1)<1$, then $H(x_1,x_2)H(x_2,x_1)<1$ implies $c(x_2,x_1)c(x_1,x_2)<1$, contradicting the case hypothesis of being $c(x_2,x_1)c(x_1,x_2)>1$. A similar situation occurs in the third scenario in Table 2; if $H(x_1,x_2)>1$ and $H(x_2,x_1)>1$, then $H(x_1,x_2)H(x_2,x_1)>1$ which implies $c(x_2,x_1)c(x_1,x_2)>1$, contradicting the case of being $c(x_2,x_1)c(x_1,x_2)<1$.

Another interesting situation that may be considered is to have pure imaginary values for P_3. In this case we should require that $T=0$ and $-4\Delta<0$. From the second condition you get

$$-4\Delta = 4\frac{r(x_1)r(x_2)[H(x_1,x_2)-1][H(x_2,x_1)-1]}{c(x_2,x_1)c(x_1,x_2)-1} < 0 \quad (16)$$

which is only possible, as stated in Table 2, when $c(x_2,x_1)c(x_1,x_2)<1$, $H(x_1,x_2)<1$, and $H(x_2,x_1)<1$. Special cases in which $H(x_1,x_2)=1$ or $H(x_2,x_1)=1$ are not considered, since, in those cases, P_3 collides with some of the equilibria in the axes of the phase plane and transcritical bifurcations occur. Additionally, to have a zero trace it is required that

$$r(x_1)[H(x_1,x_2)-1]=-r(x_2)[H(x_2,x_1)-1] \quad (17)$$

Table 2: Classification of local stability. Scenarios marked with an ∗ correspond to impossible scenarios (see the text for further details). LAS: locally asymptotically stable. U: unstable.

Case	*Condition*	P_0	P_1	P_2	P_3
$c(x_2,x_1)c(x_1,x_2)<1$	$H(x_1,x_2)>1; H(x_2,x_1)<1$	U	LAS	U	$\notin\Omega$
	$H(x_1,x_2)<1; H(x_2,x_1)>1$	U	U	LAS	$\notin\Omega$
	$H(x_1,x_2)>1; H(x_2,x_1)>1*$	U	LAS	LAS	$\notin\Omega$
	$H(x_1,x_2)<1; H(x_2,x_1)<1$	U	U	U	LAS
$c(x_2,x_1)c(x_1,x_2)>1$	$H(x_1,x_2)>1; H(x_2,x_1)<1$	U	LAS	U	$\notin\Omega$
	$H(x_1,x_2)<1; H(x_2,x_1)>1$	U	U	LAS	$\notin\Omega$
	$H(x_1,x_2)>1; H(x_2,x_1)>1$	U	LAS	LAS	U
	$H(x_1,x_2)<1; H(x_2,x_1)<1*$	U	U	U	$\notin\Omega$

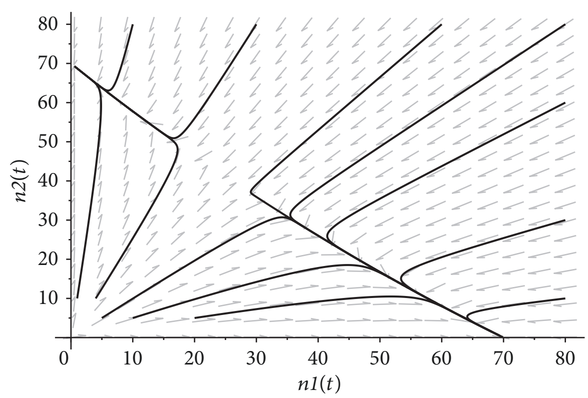

Figure 1: Phase portrait corresponding to the 7^{th} scenario in Table 2, where $c(x_2,x_1)c(x_1,x_2)>1$, $H(x_1,x_2)>1$, and $H(x_2,x_1)>1$. As it can be deduced from the stability propositions and, as it is shown in the table, P_0 and P_3 are unstable and P_1 and P_2 are both locally asymptotically stable. In this case, initial conditions determine which equilibria is going to attract a particular trajectory. For the simulations, we consider x_1 and x_2 in order to have $c(x_1,x_2)=1.1$, $c(x_2,x_1)=1.15$, $K(x_1)=K(x_2)=70$, and $r(x_1)=r(x_2)=0.3$. This is a scenario corresponding to competition favoring the SE; i.e., $c(x_1,x_2)>c(x_2,x_1)$, which could mean, for instance, a bigger taxes imposition on IE.

Let us assume $H(x_1,x_2)<1$ and $H(x_2,x_1)<1$. Then a trivial case for the occurrence of the previous situation is that $x_1=x_2\neq 0$, and then $H(x_1,x_2)=H(x_2,x_1)$. Therefore, $T=0$ if and only if, $r(x_1)=-r(x_2)$, which is not possible since we defined $r(x)>0$ for all x. On the other hand, the previous equality is also admissible when $r(x_1)=r(x_2)=0$, a scenario that lacks practical interest for this model. Taking into account this analysis we can conclude that, under the assumptions considered here, it is not possible for the Jacobian matrix to have pure imaginary eigenvalues in $P_3\in\Omega$, which rules out the occurrence of a Hopf bifurcation in P_3.

In Figure 1, the phase portrait of system (1) is shown, which corresponds to the case $c(x_2,x_1)c(x_1,x_2)>1$, $H(x_1,x_2)>1$, and $H(x_2,x_1)>1$. As stated in Table 2, P_0 and P_3 are unstable and P_1 and P_2 are both locally asymptotically stable. In this case, initial conditions determine which equilibria are going to attract a particular trajectory. Note that this is the unique scenario guaranteeing two simultaneous locally asymptotically stable equilibria. Thus the market final state will depend only on the initial conditions.

Note that condition $H(x_1,x_2)>1$ implies $c(x_1,x_2)K(x_2)/K(x_1)>1$ and then $c(x_1,x_2)>0$. Similarly, condition $H(x_2,x_1)>1$ implies $c(x_2,x_1)K(x_1)/K(x_2)>1$ and thus $c(x_2,x_1)>0$. Such situations can only occur in the case when we have competitive interactions (see the c function description in Section 2.1). On the other hand, $H(x_1,x_2)<1$ and $H(x_2,x_1)<1$ imply $c(x_1,x_2)K(x_2)/K(x_1)<1$ and $c(x_2,x_1)K(x_1)/K(x_2)<1$, respectively. These conditions can be satisfied when c is positive or negative. Therefore, the corresponding interaction scenario can be competition or cooperation.

The local stability analysis implies that the energy market will not crash under any circumstances, guaranteeing a permanent energy supply from any (or both) generation technologies; i.e., there is at least one stable equilibria corresponding to dominance of SE or IE or their coexistence to supply energy demand.

2.3. Standard Energy Model and Invasion Conditions. From the AD theory, invasion is ruled by the sign of the fitness function of the IE, as given by $\dot{n}_2/n_2$, from the g function at $P_2(\overline{n}_1^2(x_1),0)=(K(x_1),0)$. To describe this situation with more detail, we take into account the fact that, just before an innovation occurs, it is assumed that only SE is available to supply energy demand; that is, $n_2=0$. For simplicity, we denote $x_1=x$, $n_1=n$. Therefore, the energy market is modeled by only one differential equation

$$\dot{n}=nr(x)\left(1-\frac{n}{K(x)}\right) \tag{18}$$

corresponding to the classical logistic equation. It is known that (18) has two equilibria given by $\overline{n}^0=0$ which are always unstable, and $\overline{n}^1=K(x)$ is always asymptotically stable under the definitions given to r and K. SE being the only generation technology available in the market, it is assumed that n reaches its maximum capacity $K(x)$ to satisfy the market demand.

Once an innovation occurs, it is interesting to determine whether or not the IE can invade and share the market (coexist) with the SE. Just after the innovation, it is assumed that system (1) is at equilibrium $P_2(\overline{n}_1^2(x_1),0)=(K(x_1),0)$. As discussed in the proof of Proposition 2, the Jacobian matrix at P_2 is given by

$$A(P_2) = \begin{bmatrix} -r(x_1) & -r(x_1)c(x_1,x_2) \\ 0 & r(x_2)(1-H(x_2,x_1)) \end{bmatrix} \quad (19)$$

Define the *fitness function* of the IE as the innovative eigenvalue, also known as the *invasion eigenvalue* in the adaptive dynamics language.

$$\lambda(x_1,x_2) = r(x_2)(1-H(x_2,x_1)) \quad (20)$$

Clearly P_2 stability is determined by the sign of $\lambda(x_1,x_2)$; i.e., if $\lambda(x_1,x_2) > 0$, then P_2 is unstable and therefore IE can invade the market. On the other hand, if $\lambda(x_1,x_2) < 0$, then P_2 is locally asymptotically stable and the IE is going to be excluded indefinitely from the market (see Proposition 2). For a further study of this situation, assume the nondegenerate situation $(\partial\lambda/\partial x_2)(x_1,x_2) \neq 0$. Then the first-order Taylor expansion of $\lambda(x_1,x_2)$ around $x_2 = x_1$ is

$$\lambda(x_1,x_2) = \lambda(x_1,x_1) + (x_2-x_1)\frac{\partial\lambda}{\partial x_2}(x_1,x_1) + O\left(\left|x_2-x_1\right|^2\right) \quad (21)$$

Note that the term $\lambda(x_1,x_1)$ in the previous expansion,

$$\lambda(x_1,x_1) = r(x_1)(1-c(x_1,x_1)) = 0 \quad (22)$$

Therefore, $\lambda(x_1,x_2)$, described as in (21), has opposite sign for $x_2 > x_1$ or $x_2 < x_1$, with x_2 close to x_1. Thus if $(x_2-x_1)(\partial\lambda/\partial x_2)(x_1,x_1)$ is positive, i.e., if

$$\frac{\partial\lambda}{\partial x_2}(x_1,x_1) > 0, \quad x_2 > x_1, \text{ or,} \qquad \frac{\partial\lambda}{\partial x_2}(x_1,x_1) < 0, \quad x_2 < x_1, \quad (23)$$

the invasion eigenvalue $\lambda(x_1,x_2)$ is positive and equilibria P_2 are unstable. In such case, IE invades the market, and vice versa; if $(x_2-x_1)(\partial\lambda/\partial x_2)(x_1,x_1)$ is negative, P_2 is locally asymptotically stable and IE goes extinct. From now on the quantity

$$\frac{\partial\lambda}{\partial x_2}(x_1,x_1) \quad (24)$$

is going to be called *selection gradient* for the innovative energy. In the next section, specific coefficients are established according to the characteristic traits and the meaning and scope of the described invasion condition will be analyzed in more depth.

The question of whether invasion implies substitution of the standard energy requires the study of the global behavior of the standard-innovative model. In Appendix B of [11], the following theorem is proved.

Theorem 4 (invasion implies substitution). *Given x_1 in the evolution set $\mathcal{X}$, if $(x_2-x_1)(\partial/\partial x')\lambda(x_1,x_2) > 0$ and $|x_2-x_1|$ and $|(n_1(0)+n_2(0)-\overline{n}(x_1))|$ are sufficiently small, then the trajectory $(n_1(t),n_2(t))$ of the standard-innovative model (1) tends toward equilibrium P_1 for $t \longrightarrow \infty$.*

3. Evolutionary Dynamics under Cooperation and Competition

3.1. Functional Coefficients. Consider a market where the CGC growing rate r does not depend on the characteristic traits and therefore it is constant.

To define the maximum capacity function K, we consider it as an increasing function of x, for $x \geq 0$, decreasing to zero if $x < 0$, and bounded above by some maximum value k_1 corresponding to technical limitations, imposed normative obeying public policies, or technical or financial restrictions. As an example, if we consider the amount of money invested in new technology as a measure of the technology of energy generation then, very large positive values of x (own resources) or negative values (resources coming from the indebtedness) would allow increasing the maximum generation capacity K. We consider the expression

$$K(x) = \frac{k_1 x^2}{k_2^2 + x^2} \quad (25)$$

such that $K(x) \longrightarrow k_1$ as $x \longrightarrow \pm\infty$ as in Figure 2 (left). Note that $K(x)$ increases [decreases] rapidly when x is small and positive [negative], but at large positive [negative] values of x (larger inversion from own resources [indebtedness], for instance), the maximum capacity grows up [decreases down] slowly to [from] its maximum k_1.

A large value of k_2 implies that it is necessary to invest more resources (large x) to reach the maximum value k_1, while a small value of k_2 implies that the maximum level k_1 is reached with smaller investments (smaller x). Geometrically, $K(x)$ increases rapidly for all $0 < x < \sqrt{3}k_2/3$, (rapidly decreases if $-\sqrt{3}k_2/3 < x < 0$) and increases slowly to k_1, for all $x > \sqrt{3}k_2/3$ (decreases slowly from k_1 if $x < -\sqrt{3}k_2/3$). This corresponds to the fact that the graph of K has two inflection points at $x = \pm\sqrt{3}k_2/3$. Figure 2 (left), shows the plot of K with the parameter values described in the caption.

On the other hand, k_2 is related to the inflection point in the graph of K, changing its increasing speed. Indeed, $K(x)$ increases rapidly for all $0 < x < \sqrt{3}k_2/3$, (rapidly decreases if $-\sqrt{3}k_2/3 < x < 0$) and increases slowly to k_1, for all $x > \sqrt{3}k_2/3$ (decreases slowly from k_1 if $x < -\sqrt{3}k_2/3$). Then a large k_2 implies a less sensitive K to large values of x. For instance, if $x > 0$ denotes monetary investment, then a large k_2 implies the necessity of larger investments to reach the maximum k_1. In Figure 2 (Left), the plot of K with the parameter values described in Table 3 and its caption can be observed.

In the formulation of the interaction function c, we want to consider the symmetry regarding line $x_2 = x_1$ as an important issue. In fact, by definition, $c(x_1,x_2)$ corresponds to the increasing/decreasing rate of CGC suffered by n_1 by the presence of n_2 and, conversely, $c(x_2,x_1)$ corresponds to the increasing/decreasing rate of CGC suffered by n_2 by the presence of n_1 (see Section 2.1). If $c(x_i,x_j) < 0$, for $i,j = 1$ or 2, the interaction described by c corresponds to cooperation, and it corresponds to competition if $c(x_i,x_j) > 0$. If $c(x_1,x_2) = c(x_2,x_1)$ the interaction is called *fair*, and it is called *unfair* in any of the cases $c(x_1,x_2) > c(x_2,x_1)$

Table 3: Parameter description and the corresponding baseline values used at simulations.

	Parameter description	*Value*
k_1	Upper bound for the maximum capacity K, due to technical limitations or imposed public policies	100 MW
k_2	Measure of the speed at which maximum capacity can grow	10
c_1	Subsidies if positive/Taxes if negative or any other similar policy on SE	Varies
c_2	Subsidies if positive/Taxes if negative or any other similar policy on IE	Varies

or $c(x_1, x_2) < c(x_2, x_1)$. Consider the interaction function between both kinds of energies is given by the function

$$c(x_1, x_2) = \frac{\left(c_1^2 + c_2^2\right) x_1 x_2}{c_1^2 x_1^2 + c_2^2 x_2^2} \tag{26}$$

depicted in Figure 2 (right). Note that $c \in \mathbb{R}$, for all x_1 and x_2. Function c corresponds to competition if x_1 and x_2 have the same sign (first and third quadrants of the (x_1, x_2)-plane) and to cooperation if x_1 and x_2 have opposite signs (second and fourth quadrants of the (x_1, x_2)-plane). The coefficient c has a set of maximums on the line $x_2 = (c_1/c_2)x_1$, where the maximum competition takes place and its value is $(c_1^2 + c_2^2)/2c_1c_2$. It has a set of minimums at the line $x_2 = -(c_1/c_2)x_1$, where the cooperation is maxima and its value is $-(c_1^2 + c_2^2)/2c_1c_2$. Symmetric competition occurs when $c_1 = c_2$. In this case the lines of maxima and minima coincide with $x_2 = x_1$ and $x_2 = -x_1$, respectively. On the other hand, $c_2 > c_1$ [conversely $c_2 < c_1$] implies asymmetric interaction in favor of n_2 [conversely n_1].

Symmetric interaction is not likely to occur in almost any market. Therefore, we will consider the asymmetric case by stating $c_1 \neq c_2$. Both parameters can be considered as the effect of market policies in the competition, such as subsidies awarded or any other similar policy when $c_1, c_2 > 0$ or some privative policy as taxes imposition when $c_1, c_2 < 0$. In general, whether an innovation is stimulated or unstimulated depends on if $x_1 > x_2$ or $x_2 > x_1$ and also on whether they are positive or negative. If $c_2 > c_1$ and $x_1 > 0$, a small innovation x_2 is stimulated by interaction if $x_2 < x_1$. Geometrically, the point (x_1, x_2) is below the diagonal (closer to the line of maxima) and $c(x_1, x_2) > 1$, while the point (x_2, x_1) is above the diagonal and $c(x_2, x_1) < 1$. It is unstimulated if $x_2 > x_1$.

3.2. Selection Gradient and Invasion Conditions. A more detailed study of the invasion conditions will be discussed in this subsection. Under the definitions of r, K, and c described above, the IE growing rate, also known as *fitness* function (20), takes the form

$$\lambda(x_1, x_2) = r\left(1 - \frac{\left(c_1^2 + c_2^2\right)\left(k_2^2 + x_2^2\right) x_1^3}{\left(k_2^2 + x_1^2\right)\left(c_1^2 x_2^2 + c_2^2 x_1^2\right) x_2}\right) \tag{27}$$

and the selection gradient is explicitly given by

$$\frac{\partial \lambda}{\partial x_2}(x_1, x_1) = \frac{\left[k_2^2\left(3c_1^2 + c_2^2\right) - \left(c_2^2 - c_1^2\right) x_1^2\right] r}{x_1\left(k_2^2 + x_1^2\right)\left(c_1^2 + c_2^2\right)} \tag{28}$$

The invasion conditions were discussed in the previous section and established in (23). Now, with the explicit expressions for r, K, and c, we will study invasion in the energy market in a more detailed way.

Since $r > 0$ and $(k_2^2 + x_1^2)(c_1^2 + c_2^2) > 0$ in every case, then sign of $(\partial\lambda/\partial x_2)(x_1, x_1)$ is given by

$$\frac{k_2^2\left(3c_1^2 + c_2^2\right) - \left(c_2^2 - c_1^2\right) x_1^2}{x_1} \tag{29}$$

(i) If $c_2^2 - c_1^2 > 0$, then $(\partial\lambda/\partial x_2)(x_1, x_1) < 0$ implies two cases:

$$\begin{gathered} x_1 > 0 \Longleftrightarrow \\ k_2^2\left(3c_1^2 + c_2^2\right) - \left(c_2^2 - c_1^2\right) x_1^2 < 0 \Longleftrightarrow \\ \frac{k_2^2\left(3c_1^2 + c_2^2\right)}{c_2^2 - c_1^2} - x_1^2 < 0 \Longleftrightarrow \\ x_1 \in (x_I, \infty) \end{gathered} \tag{30}$$

Similarly,

$$\begin{gathered} x_1 < 0 \Longleftrightarrow \\ k_2^2\left(3c_1^2 + c_2^2\right) - \left(c_2^2 - c_1^2\right) x_1^2 > 0 \Longleftrightarrow \\ x_1 \in (-x_I, 0) \end{gathered} \tag{31}$$

where $x_I = k_2\sqrt{(3c_1^2 + c_2^2)/(c_2^2 - c_1^2)}$. In any of these cases, innovations with $x_2 > x_1$ invade.

On the other hand, $(\partial\lambda/\partial x_2)(x_1, x_1) > 0$ for $x_1 > 0$ implies $x_1 \in (0, x_I)$and for $x_1 < 0$ implies $x_1 \in (-\infty, -x_I)$; in both cases, innovations with $x_1 > x_2$ invade.

In Figure 3 (left), the schematic structure of the invasion region in the (x_1, x_2)–plane when $c_2^2 - c_1^2 = 0.1025 > 0$ ($c_1 = 1$ and $c_2 = 1.05$ were used) is shown; blue regions above the line $x_2 = x_1$ correspond to negative selection gradients, and gray regions below that line correspond to positive selection gradients.

(ii) If $c_2^2 - c_1^2 < 0$ (equivalently $c_1^2 - c_2^2 > 0$), we can rewrite (29) as

$$\frac{k_2^2\left(3c_1^2 + c_2^2\right) + \left(c_1^2 - c_2^2\right) x_1^2}{x_1} \tag{32}$$

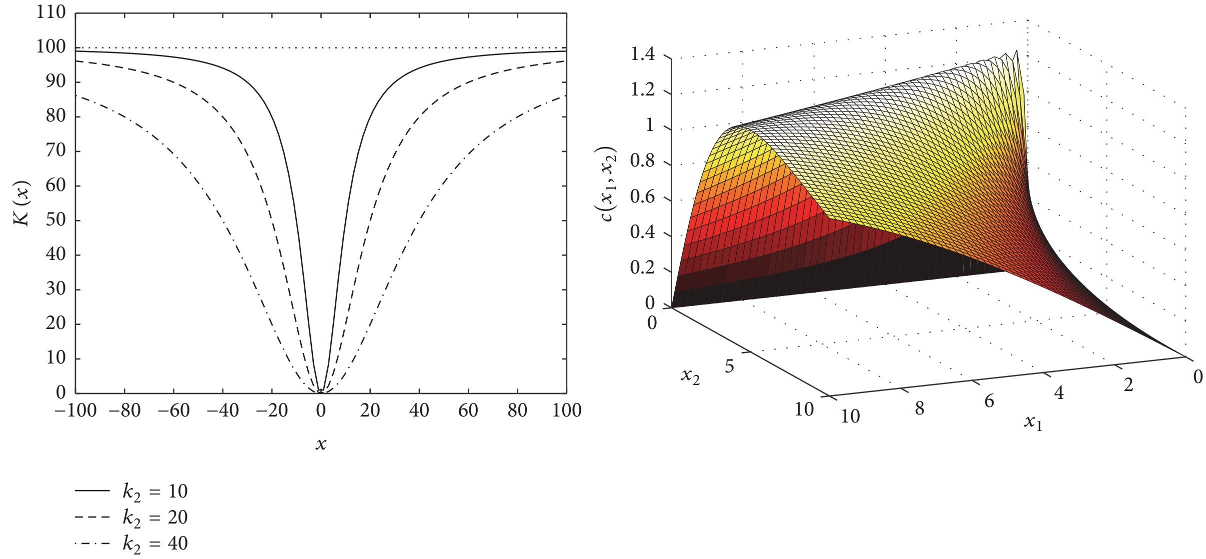

FIGURE 2: Maximum capacity function $K(x) = k_1x^2/(k_2^2 + x^2)$, plotted only for three different values of k_2 (left) and interaction function $c(x_1, x_2) = (c_1^2 + c_2^2)x_1x_2/(c_1^2x_1^2 + c_2^2x_2^2)$, conveniently plotted only for positive values of x_1 and x_2 (right). Parameter values used are $r = 0.3$, $c_1 = 1$, $c_2 = 2$, $k_1 = 100$, $k_2 = 10$ (solid), $k_2 = 20$ (dashed), and $k_2 = 40$ (dash-dot).

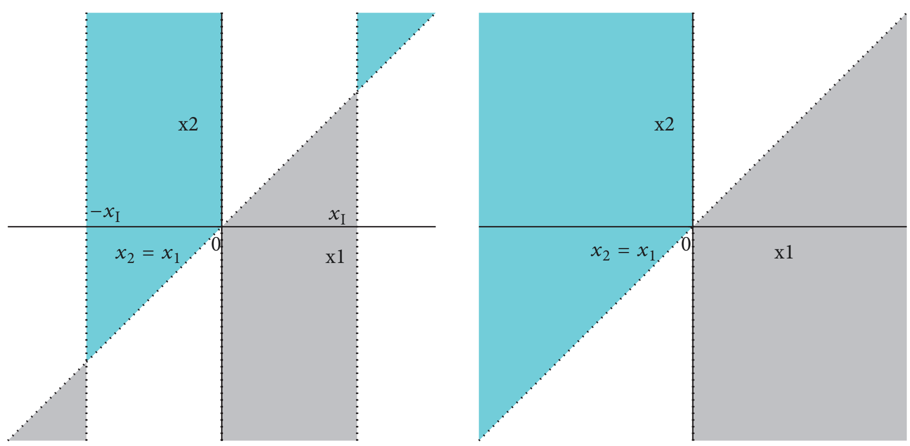

FIGURE 3: Different regions in the (x_1, x_2)−plane where invasion conditions given in (23) are satisfied. Blue regions above the line $x_2 = x_1$ correspond to negative selection gradients, and gray regions below that line correspond to positive selection gradients. **Left:** $c_2^2 - c_1^2 = 0.1025 > 0$ ($c_1 = 1$ and $c_2 = 1.05$ were used). **Right:** $c_2^2 - c_1^2 = -0.1025 < 0$ (with $c_1 = 1.05$ and $c_2 = 1$). Note that a big innovation is required to have a cooperative market just after an innovation in a market dominated by SE generation technology.

Therefore, $x_1 > 0$ implies $(\partial\lambda/\partial x_2)(x_1, x_1) > 0$ and the innovations with $x_1 > x_2$ invade; similarly, $x_1 < 0$ implies $(\partial\lambda/\partial x_2)(x_1, x_1) < 0$ and the innovations with $x_2 > x_1$ invade.

In Figure 3 (right), the schematic structure of the invasion region in the (x_1, x_2)−plane when $c_2^2 - c_1^2 = -0.1025 < 0$ ($c_1 = 1.05$ and $c_2 = 1$ were used) is shown; as in the previous case, blue regions above the line $x_2 = x_1$ correspond to negative selection gradients, and gray regions below that line correspond to positive selection gradients.

Note that although functional parameters r, K, and c are defined for all x_1 and x_2 in $\mathbb{R}$, and also the interaction dynamics from system (1) are well defined for both strategies (cooperation and competition), the invasion conditions determine configurations (specific regions of the (x_1, x_2)−plane) under which the invasion of the innovative attribute is possible and configurations that lead to its disappearance. Additionally, note that a big innovation is required to have a cooperative market just after an innovation in a market dominated by SE generation technology.

3.3. Adaptive Dynamics Canonical Equation. The behavior and long-term evolution of the attribute x_1 that characterizes energy market is now described as a result of advantageous innovations on this attribute that allow the survival of the respective CGC in the market. The goal in this section is to describe the *Canonical Equation of Adaptive Dynamics* briefly. The reader is invited to review [11, 15, 25, 26], to expand the information shown, in particular, regarding the deduction of the equations.

The dynamics of x_1 are given by the ordinary differential equation:

$$\dot{x}_1 = \frac{1}{2}\mu(x_1)\sigma^2(x_1)\overline{n}(x_1)\frac{\partial\lambda}{\partial x_2}(x_1,x_1) \quad (33)$$

In [11], there is a full deduction of this equation. A parameter $\epsilon \longrightarrow 0$ is considered as a scaling factor separating the market timescale (the time considered above in all the derivatives of n_1 and n_2), from the evolutionary timescale for x_1. In fact, a small amount dt of time on the evolutionary timescale corresponds to a large amount of time dt/ϵ on the market timescale. This fact allows affirming that, between one innovation and the next, the market has time enough to find an equilibrium configuration. It is worth clarifying that while n_1 and n_2 are on the market timescale, x is on the evolutionary timescale. All the derivatives concerning the time are represented with *dot* notation.

Equation (33) is known as the *Canonical Equation of Adaptive Dynamics*. In the context of this work, $\mu(x_1)$ is proportional to the probability that an IE entering the market corresponds to an innovation. $\sigma(x_1)$ is proportional to the standard deviation of the measure of the change in the attribute in which innovation occurs. $\overline{n}(x_1)$ represents the market equilibrium before innovation (i.e., $\overline{n}(x_1) = K(x_1)$), and $(\partial\lambda/\partial x_2)(x_1,x_1)$ is the selection gradient of the x_1 attribute on which the innovation is performed.

Denoting $x = x_1$ for simplicity and considering $\mu(x) = \mu$ and $\sigma^2(x) = \sigma^2$ (i.e., they do not depend upon the characteristic trait), the Adaptive Dynamics Canonical Equation is given by

$$\dot{x} = \frac{\mu\sigma^2 k_1 r}{2(c_1^2+c_2^2)} \cdot \frac{\left[(c_1-c_2)(c_1+c_2)x^2 + k_2^2(3c_1^2+c_2^2)\right]x}{(k_2^2+x^2)^2} \quad (34)$$

To study this nonlinear differential equation, it is necessary to find the equilibrium points (*evolutionary equilibria* from now on) by solving $\dot{x} = 0$, to find

$$\begin{aligned} \overline{x}_0 &= 0, \\ \overline{x}_1 &= k_2\sqrt{\frac{3c_1^2+c_2^2}{c_2^2-c_1^2}} = x_I \\ \text{and } \overline{x}_2 &= -k_2\sqrt{\frac{3c_1^2+c_2^2}{c_2^2-c_1^2}} = -x_I \end{aligned} \quad (35)$$

which are real values when $c_2^2 - c_1^2 > 0$. We obtain the region R

$$R = \left\{(c_1,c_2)\in\mathbb{R}^2 : c_2^2 - c_1^2 > 0\right\} \quad (36)$$

Now, to study the stability of equilibria $\overline{x}_i$, for $i = 0,1,2$, define

$$f(x) = \frac{\mu\sigma^2 k_1 r}{2(c_1^2+c_2^2)} \cdot \frac{\left[(c_1-c_2)(c_1+c_2)x^2 + k_2^2(3c_1^2+c_2^2)\right]x}{(k_2^2+x^2)^2} \quad (37)$$

as the right hand of (34); then, linearizing,

$$\frac{df}{dx}(x) = -\frac{\mu\sigma^2 k_1 r}{2(c_1^2+c_2^2)} \cdot \frac{x^4(c_1^2-c_2^2) + 6x^2(c_1^2+c_2^2)k_2^2 - (3c_1^2+c_2^2)k_2^4}{(k_2^2+x^2)^3} \quad (38)$$

and we get

$$\frac{df}{dx}(\overline{x}_0) = \frac{\sigma^2\mu k_1 r(3c_1^2+c_2^2)}{2k_2^2(c_1^2+c_2^2)} > 0 \quad \text{for all } c_1, c_2 \quad (39)$$

and

$$\frac{df}{dx}(\overline{x}_1) = \frac{df}{dx}(\overline{x}_2) = -\frac{\mu\sigma^2 k_1 r(c_1^2-c_2^2)^2(3c_1^2+c_2^2)}{4k_2^2(c_1^2+c_2^2)} < 0 \quad \text{for all } c_1, c_2 \quad (40)$$

Note that $\overline{x}_0$ is always unstable and $\overline{x}_i$, for $i = 1,2$, is always locally asymptotically stable. Thus, in the market, repeated innovations and replacements of generation technologies with new ones drive the attribute x toward any of the equilibrium values $\overline{x}_1$ or $\overline{x}_2$. In Figure 4 some numeric solutions of the canonical equation (34) are shown, with the parameters described in the corresponding caption.

Note that condition $c_2^2 - c_1^2 > 0$ determines scenarios for evolutionary equilibria not only to exist but also to be locally asymptotically stable.

At this point, it is necessary to study evolutionary dynamics in a neighborhood of the evolutive equilibria $\overline{x}_i$, for $i = 1,2$. Since, in the vicinity of the singular strategy, $(\partial\lambda/\partial x_2)(\overline{x}_1,\overline{x}_1) = 0$, then the market and evolutionary dynamics are dominated by the second derivatives of the fitness function.

3.4. Coexistence and Divergence. Geritz et al. [14, 15] showed that if the coexistence condition holds, innovative and standard energies mutually invade each other. This situation implies the instability of both "single trait" equilibria P_1 and P_2 (i.e., coexistence describes the situation when the values of the characteristic traits of the IE and the SE are in the vicinity of the equilibrium $\overline{x}$, defining energies that are similar to each other, and sharing the market, that is, "coexist"). On the other hand, if the coexistence condition does not hold, both energies are mutually excluded and any of the "single trait" equilibria gains stability. In particular ([11], pages 99-104), it is proved that IE-SE coexistence is possible if

$$\frac{\partial^2\lambda}{\partial x_1 \partial x_2}(\overline{x}_i,\overline{x}_i) < 0, \quad i = 1,2 \quad (41)$$

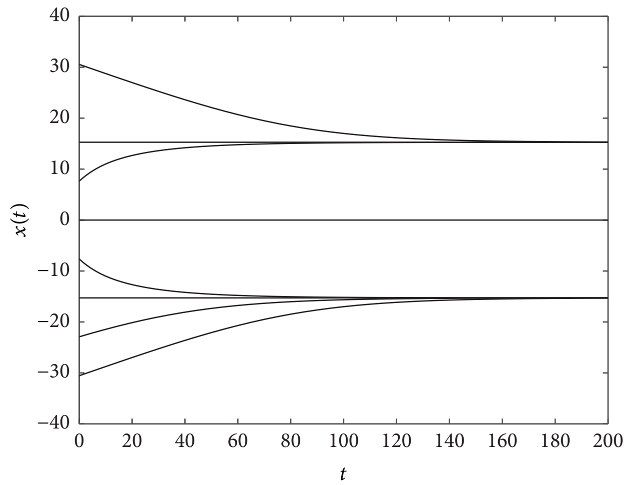

Figure 4: Numeric simulation of evolutionary dynamics of the characteristic trait x described by the ADCE (34), considering $r = 0.3$, $c_1 = 1$, $c_2 = 2$, $k_1 = 100$, $k_2 = 10$, $\mu = 1$, and $\sigma = 1$.

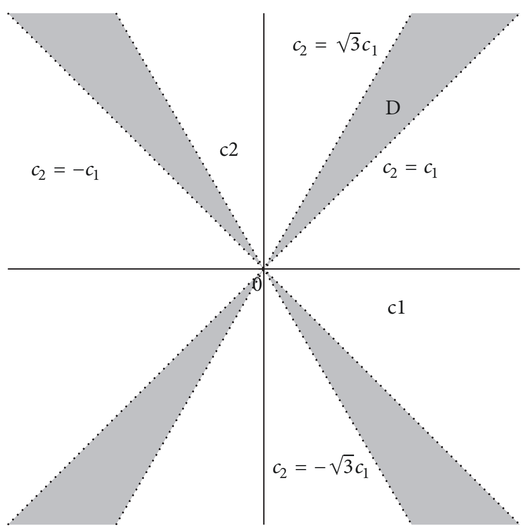

Figure 5: Region D of attribute divergence.

Explicitly we have

$$\frac{\partial^2 \lambda}{\partial x_1 \partial x_2}(\overline{x}_i, \overline{x}_i) = -\frac{4r(c_2^2 - c_1^2) c_2^2 c_1^2}{k_2^2 (c_1^2 + c_2^2)^2 (3c_1^2 + c_2^2)} < 0, \quad i = 1, 2 \tag{42}$$

Note that coexistence condition holds when $c_2^2 - c_1^2 > 0$. This situation corresponds to $(c_1, c_2) \in R$, which was defined above for the existence of $\overline{x}_i$, for $i = 1, 2$ in $\mathbb{R}$. This result can be stated as follows: *evolutionary stability implies coexistence of IE and SE characteristic traits*.

An equally important question as coexistence is whether it can be guaranteed that the two attributes that coexist after the invasion of IE are indeed similar and not identical. That is, if it is not possible to differentiate x_1 from x_2, then the condition of coexistence would only mean that in practice

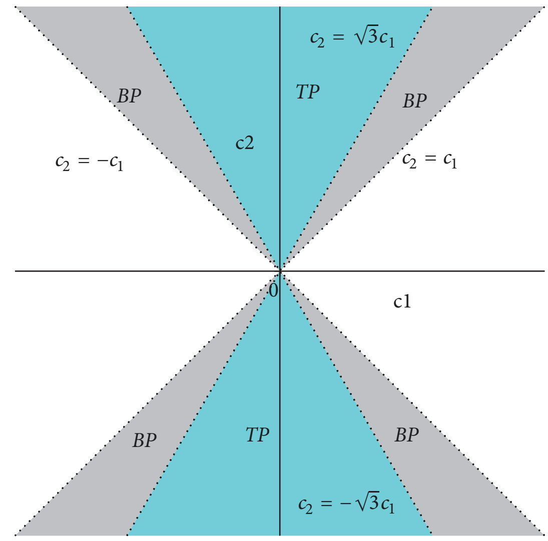

Figure 6: Classification of stable evolutionary equilibria as BP, TP, or BBP.

there is only one type of energy that has been "virtually" separated into two classes. In this way, the divergence is understood as the dissimilarity between the values of the characteristic attribute of SE and IE. This situation allows differentiating one from the other, implying the "origin of diversity" in the market. It is shown in [11] that x_1 and x_2 attributes diverge from each other, when

$$\frac{\partial^2 \lambda}{\partial x_2^2}(\overline{x}_i, \overline{x}_i) > 0, \quad i = 1, 2 \tag{43}$$

Explicitly,

$$\frac{\partial^2 \lambda}{\partial x_2^2}(\overline{x}_i, \overline{x}_i) = \frac{(c_2^2 - c_1^2)(3c_1^2 - c_2^2) r}{k_2^2 (3c_1^2 + c_2^2)(c_1^2 + c_2^2)} > 0, \quad i = 1, 2 \tag{44}$$

A detailed analysis of the inequality proposed by the previous condition leads to confirming that divergence is possible when $(c_1, c_2) \in D$, where D is the portion of the (c_1, c_2)–plane, described by

$$D = \left\{(c_1, c_2) \in \mathbb{R}^2 : (c_2^2 - c_1^2)(3c_1^2 - c_2^2) > 0\right\} \tag{45}$$

as illustrated in Figure 5. Then we can classify evolutionary equilibria in three categories:

(i) **Branching points (BP):** they are locally asymptotically stable evolutionary equilibria in which the attribute can branch, which occurs when both conditions (41) and (43) are satisfied. This implies that the BP occur when $(c_1, c_2) \in D$, as illustrated in Figure 5 and in the gray area in Figure 6 (labelled BP), where $\overline{x}_i \in \mathbb{R}$ is locally asymptotically stable and, in addition, conditions (41) and (43) are satisfied

(ii) **Terminal points (TP):** they are locally asymptotically stable evolutionary equilibria, but they are not branching points. At these points the evolution and

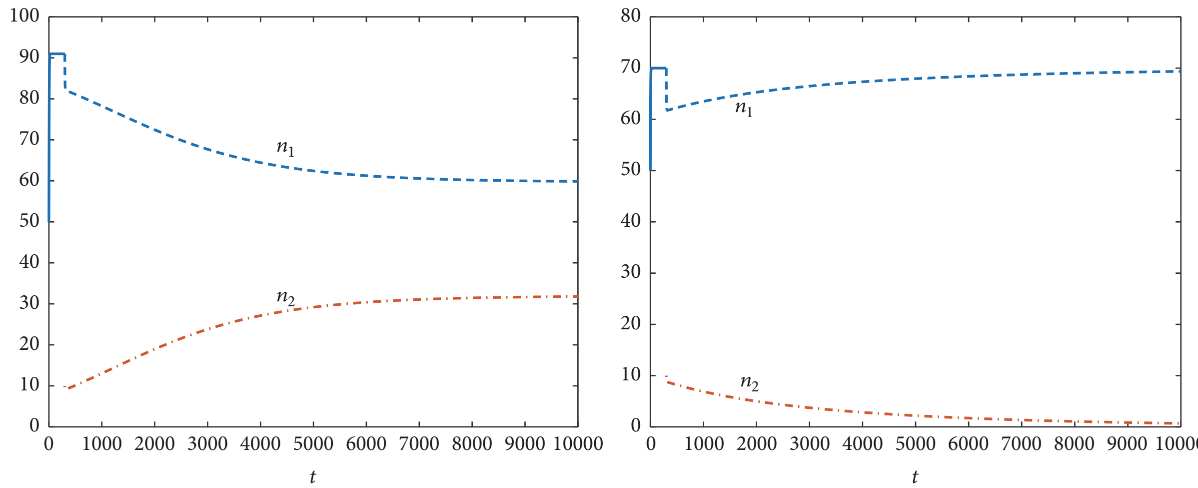

FIGURE 7: Numeric simulation of market dynamics under the influence of trait dependent maximum capacity K and interaction function c. **Left**: shows the market dynamics considering $r = 0.3$, $k_1 = 100$, $k_2 = 10$, $c_1 = 1$, $c_2 = 1.2$, $x = x_1 = \overline{x}_1 = 31.7662$, and $x_2 = 1.1 * x_1$. Since $x_1 > 0$ and $x_2 > 0$, it corresponds to competition in the market. Before the innovation occurs (solid line), the simulation corresponds to the resident model (18) with the initial condition $n(0) = 50$. Under the absence of competition, the equilibrium $\overline{n} = K(x) = 90.9836$ is reached. After the innovation, the simulation corresponds to system (1) with initial conditions $n_1(0) = K(\overline{x}_1) = 90.9836$ (dashed line) and $n_2(0) = 10$ (dash-dot line). Note that $(c_1, c_2) \in R$. Thus the evolutionary equilibrium is a branching point (BP) and market diversification arises. These market dynamics describe a case when IE invades the market but do not substitute SE. Then they share the market. **Right**: corresponds to the same parameter configuration, but with $c_2 = 2$. In this case $x_1 = \overline{x}_1 = 15.2753$ and $x_2 = 1.1x_1$. Then the initial conditions are $n_1(0) = K(\overline{x}_1) = 70$ and $n_2(0) = 10$. Note that $(c_1, c_2) \in T$. Thus the evolutionary equilibrium is a terminal point (TP), and therefore diversification is not possible.

diversification are not possible. We have this situation when any of the conditions (41) or (43) fails. In this case it corresponds to $(c_1, c_2) \in T$, where

$$T = \left\{(c_1, c_2) \in \mathbb{R}^2 : c_2^2 - c_1^2 > 0,\ 3c_1^2 - c_2^2 < 0\right\} \tag{46}$$

This region is shown in Figure 6 (blue color and labelled with TP)

(iii) **Bifurcation branching points (BBP):** this situation corresponds to the border points between branch and end points. In this case we obtain the set of straight lines:

$$BBP = \left\{(c_1, c_2) \in \mathbb{R}^2 : c_2^2 - c_1^2 = 0, \vee, 3c_1^2 - c_2^2 = 0\right\} \tag{47}$$

This bifurcation is unfolded in detail in [27, 28]

An example of competitive market dynamics under asymmetric interaction $c_2 > c_1$ is shown in Figure 7(left). It illustrates the market dynamics under the influence of trait dependent maximum capacity K and interaction function c. The left panel shows the market dynamics considering $r = 0.3$, $k_1 = 100$, $k_2 = 10$, $c_1 = 1$, $c_2 = 1.2$, $x_1 = \overline{x}_1 = 31.7662$, and $x_2 = 1.1x_1$. Since $x_1 > 0$ and $x_2 > 0$, then the interaction corresponds to competition in the market, according to model (1). The initial conditions $n_1(0) = K(\overline{x}_1) = 90.9836$ and $n_2(0) = 10$ were used. This scenario considers $x_2 > x_1$ and gives a coexistence condition $(\partial^2\lambda/\partial x_1\partial x_2)(\overline{x}_1, \overline{x}_1) = -2.8763 \times 10^{-4} < 0$ and a divergence condition $(\partial^2\lambda/\partial x_2^2)(\overline{x}_2, \overline{x}_2) = 1.9008 \times 10^{-4} > 0$. Hence, it corresponds to the case when both conditions (41) and (43) hold and then $\overline{x}_1$ is a branching point (BP). This market dynamics describe a case when IE invades the market but does not substitute SE. Thus they share the market.

By the other hand, Figure 7 (right) illustrates a scenario when the evolutionary equilibrium $\overline{x}_1$ is a terminal point. The same parameters are considered but $c_2 = 2$. Therefore, $x_1 = \overline{x}_1 = 15.2753$ and $x_2 = 1.1x_1$ and the initial conditions $n_1(0) = K(\overline{x}_1) = 70$ (recall $\overline{x}_1$ depends on c_2) and $n_2(0) = 10$ were used. Since $x_1 > 0$ and $x_2 > 0$, then the interaction corresponds also to competition. The coexistence condition is $(\partial^2\lambda/\partial x_1\partial x_2)(\overline{x}_2, \overline{x}_2) = -8.2286 \times 10^{-4} < 0$ and the divergence condition $(\partial^2\lambda/\partial x_2^2)(\overline{x}_2, \overline{x}_2) = -2.5714 \times 10^{-4} < 0$. The last one does not hold as stated by (43). In fact, as $(c_1, c_2) \in T$, the evolutionary equilibrium x_1 corresponds to a terminal point (TP) and evolution has a halt. Thus no market diversification is possible.

After the branching has occurred (i.e., both, coexistence and divergence conditions hold), the IE and SE share the market at the strictly positive equilibrium on the market space $P_3 = (\overline{n}_1(x_1, x_2), \overline{n}_2(x_1, x_2))$. Thus the IE becomes standard (i.e., there are two similar SE generation technologies with CGC n_1 and n_2 and characterized by the trait values x_1 and x_2 respectively). Now, it is possible to consider a new innovation to occur in any of the traits x_1 or x_2 leading to the appearance of a new (similar but slightly different) trait x_1' or x_2'. This situation will be shown in the next two 3×3 systems:

$$\dot{n}_1 = n_1 r(x_1)\left(1 - \frac{n_1 + c(x_1, x_2)\,n_2 + c(x_1, x_1')\,n_1'}{K(x_1)}\right)$$

$$
\begin{aligned}
&= n_1 g\left(n_1, n_2, n_1', x_1, x_2, x_1\right) \\
&\dot{n}_2 \\
&= n_2 r\left(x_2\right)\left(1 - \frac{c\left(x_2, x_1\right) n_1 + n_2 + c\left(x_2, x_1'\right) n_1'}{K\left(x_2\right)}\right) \\
&= n_2 g\left(n_1, n_2, n_1', x_1, x_2, x_2\right) \\
&\dot{n}_1' \\
&= n_1' r\left(x_1'\right)\left(1 - \frac{c\left(x_1', x_1\right) n_1 + c\left(x_1', x_2\right) n_2 + n_1'}{K\left(x_1'\right)}\right) \\
&= n_1' g\left(n_1, n_2, n_1', x_1, x_2, x_1'\right)
\end{aligned} \tag{48}
$$

and

$$
\begin{aligned}
&\dot{n}_1 \\
&= n_1 r\left(x_1\right)\left(1 - \frac{n_1 + c\left(x_1, x_2\right) n_2 + c\left(x_1, x_2'\right) n_2'}{K\left(x_1\right)}\right) \\
&= n_1 g\left(n_1, n_2, n_2', x_1, x_2, x_1\right) \\
&\dot{n}_2 \\
&= n_2 r\left(x_2\right)\left(1 - \frac{c\left(x_2, x_1\right) n_1 + n_2 + c\left(x_2, x_2'\right) n_2'}{K\left(x_2\right)}\right) \\
&= n_2 g\left(n_1, n_2, n_2', x_1, x_2, x_2\right) \\
&\dot{n}_2' \\
&= n_2' r\left(x_2'\right)\left(1 - \frac{c\left(x_2', x_1\right) n_1 + c\left(x_2', x_2\right) n_2 + n_2'}{K\left(x_2'\right)}\right) \\
&= n_2' g\left(n_1, n_2, n_2', x_1, x_2, x_2'\right)
\end{aligned} \tag{49}
$$

After branching, it is irrelevant which one of the SE is called x_1 or x_2, and systems above are equivalent. The AD canonical equation governing the interaction of both SE's characteristic traits can be derived by repeating the analysis shown above. The invasion fitness of the IE's n_1' and n_2' are given by

$$
\begin{aligned}
&\lambda\left(x_1, x_2, x_1'\right) \\
&\quad = g\left(0, \overline{n}_1\left(x_1, x_2\right), \overline{n}_2\left(x_1, x_2\right), x_1', x_1, x_2\right)
\end{aligned} \tag{50}
$$

and

$$
\begin{aligned}
&\lambda\left(x_1, x_2, x_2'\right) \\
&\quad = g\left(0, \overline{n}_1\left(x_1, x_2\right), \overline{n}_2\left(x_1, x_2\right), x_2', x_1, x_2\right)
\end{aligned} \tag{51}
$$

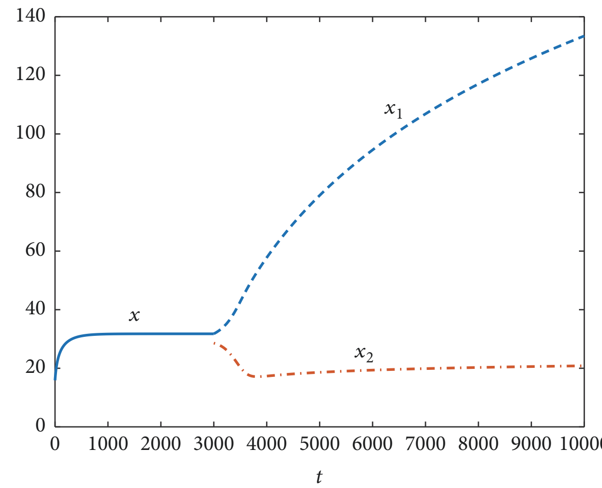

Figure 8: Characteristic traits considering $x_2 < x_1$. It shows the trait dynamics with $r = 0.3$, $k_1 = 100$, $k_2 = 10$, $c_1 = 1$, and $c_2 = 1.2$, corresponding to the branching point $\overline{x}_1 = 31.7662$ shown in Figure 7(Left). The first part of the curve (before the innovation occurs) corresponds to the simulation of (34) with initial condition $x(0) = 15.8831$. After the innovation, the curves correspond to the simulation of (52) and (53) with initial conditions $x_1(0) = \overline{x}_1$ and $x_2(0) = 0.9\overline{x}_1$.

The canonical equation reads, respectively,

$$
\dot{x}_1 = \frac{1}{2}\mu_1 \sigma_1^2 \overline{n}_1\left(x_1, x_2\right) \frac{\partial \lambda_1}{\partial x_1'}\left(x_1, x_2, x_1'\right)\Big|_{x_1' = x_1} \tag{52}
$$

and

$$
\dot{x}_2 = \frac{1}{2}\mu_2 \sigma_2^2 \overline{n}_2\left(x_1, x_2\right) \frac{\partial \lambda_2}{\partial x_2'}\left(x_1, x_2, x_2'\right)\Big|_{x_2' = x_2} \tag{53}
$$

where $\overline{n}_1(x_1, x_2)$ and $\overline{n}_2(x_1, x_2)$ are the coordinates corresponding to the coexistence equilibria P_3; i.e.,

$$
\begin{aligned}
\overline{n}_1\left(x_1, x_2\right) &= \frac{c\left(x_1, x_2\right) K\left(x_2\right) - K\left(x_1\right)}{c\left(x_2, x_1\right) c\left(x_1, x_2\right) - 1}, \text{ and,} \\
\overline{n}_2\left(x_1, x_2\right) &= \frac{c\left(x_2, x_1\right) K\left(x_1\right) - K\left(x_2\right)}{c\left(x_2, x_1\right) c\left(x_1, x_2\right) - 1}
\end{aligned} \tag{54}
$$

The explicit expressions of (52) and (53) are omitted since they are very long. Nevertheless, they can be generated and handled by means of symbolic computation as in Figure 8.

Figure 8 shows the evolutionary dynamics of the characteristic traits under asymmetric interaction ($c_2 > c_1$) and considering $x_2 < x_1$. The first part of the curve (before the innovation occurs) corresponds to the simulation of (34) with initial condition $x(0) = 15.8831$. After the innovation, the curves correspond to the simulation of (52) and (53) considering the parameter configuration described in the corresponding caption. Initially, x grows toward $\overline{x}_1$ until the branching occurs. Then, the dynamics are the result of the interaction between the innovative energy IE and the standard energy SE. The attribute x_1 permanently grows, while

x_2 initially decreases. This makes perfect sense since each of them is being governed by a different canonical equation and therefore coexist under different selection pressures.

Finally, the same scenario is shown in Figure 7, (right), but considering the case when $x_2 > x_1$. It can be observed that now, it is the attribute x_2 of the IE which always grows while the attribute of the SE initially decreases and then stabilizes in values significantly below those of the other attributes.

4. Results and Conclusions

Local stability analysis of model (1) brings information on the market dynamics and helps to answer a further question of under what conditions can IE spread into the market and interact or even substitute SE. Indeed, in a market dominated exclusively by SE, the instability of P_2 is related to the possibility for an IE to invade the market, while the existence and stability of P_3 are related to the coexistence of both kinds of energy sharing the market, leading to diversification. Additionally, stability analysis implies that energy market will not crash under any circumstances, guaranteeing a permanent energy supply from any (or both) of the generation technologies. This situation means that there is at least one stable equilibrium corresponding to the dominance of SE or IE or their coexistence, to supply energy demand. Even more, it shows that both cooperative and competitive strategies are adequate to guarantee market stability.

The interaction function c describes both, competition and cooperation, depending on the values of the characteristic traits: competition on the first and third quadrants of the (x_1, x_2)-plane and cooperation at the second and fourth quadrants of the (x_1, x_2)-plane. However, although functional parameters r, K, and c are defined for all x_1 and x_2 in $\mathbb{R}$ and also the interaction dynamics defined by (1) are well defined for both market configurations, the invasion conditions determine specific regions of the (x_1, x_2)−plane under which the invasion of the innovative attribute is possible and configurations that lead to its disappearance. Furthermore, it was proven that, under convenient configurations of subsidies awarded (or taxes imposed) to both energy generation technologies, it is possible to determine scenarios for evolutionary equilibria to exist, to be locally asymptotically stable, and, also, it was shown that evolutionary stability implies coexistence.

Under the assumptions of our analysis, evolutionary equilibria can be terminal points, where no marginal innovation can invade the market. However, evolutionary equilibria can also be branching points, where innovative energy can penetrate, coexist, and diversify the market, concerning the previously established. In this context, although both parameters c_1 and c_2 describe the dynamics in the market time scale, they finally make a difference in the evolutionary time scale. In fact, the expressions $c_2^2 - c_1^2$ and $3c_1^2 - c_2^2$ are a measure of the strength of diversification through innovation. Taking into account the geometric characteristics of the interaction function c, we can say that diversification occurs in markets that are at least slightly asymmetric and in which IE is stimulated over SE, either by the allocation of subsidies or by the imposition of lower taxes.

In order to understand the evolution of the system after the second branch, it is necessary to repeat the analysis obtaining a three-dimensional canonical equation and then try to verify if the attributes converge to evolutionary solutions where the conditions of coexistence and divergence are satisfied. Due to the complexity of the expressions, it is necessary to perform the verification by computational methods. Finally note that repeated process of innovation can give origin to a rich variety of different and complex kinds of technological evolution [29]. However, it is important to note that these processes of emergence and disappearance of energy generation technologies is influenced by a wide range of external and internal factors, which may exert additional selection processes on innovations. Specific situations should be studied in greater depth and detail in order to achieve an informed decision-making.

Data Availability

There is no significant data in the manuscript since most of the paper is theoretical, even though the authors declare that all the data contained in the paper is available.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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