

# Solving the Noether procedure for cubic interactions of higher spins in (A)dS

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ABSTRACT: The Noether procedure represents a perturbative scheme to construct all possible consistent interactions starting from a given free theory. In this note we describe how cubic interactions involving higher spins in any constant-curvature background can be systematically derived within this framework.

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## 1 Introduction

String theory (ST) and Vasiliev's equations (VE) [1, 2] are the only known examples of consistent theories of interacting higher-spin (HS) particles.<sup>1</sup> Although their current formulations provide mathematically elegant descriptions involving infinitely many auxiliary fields, some important aspects, as the number of derivatives involved in the cubic vertices or the possible (non-)local nature of higher-order interactions, are hidden. Indeed, the cubic vertices involving fields in the first Regge trajectory of the open bosonic string have been obtained only recently in [5, 6]. Leaving aside Chan-Paton factors, their illuminating form is given by

$$|V_3\rangle = \exp \left\{ \frac{1}{2} \sum_{i \neq j} \sqrt{2\alpha'} a_i^\dagger \cdot p_j + a_i^\dagger \cdot a_j^\dagger \right\} |0\rangle_{123} \quad \left[ (a^\mu)^\dagger \equiv \alpha_{-1}^\mu \right], \quad (1.1)$$

where  $i, j = 1, 2, 3$  label the Fock spaces associated to the interacting particles. From the latter expression, the  $s_1 - s_2 - s_3$  interactions, as well as the corresponding coupling constants, can be easily extracted via a Taylor expansion of the exponential function. Besides reflecting the world-sheet (Gaussian integral) origin of ST, the latter fulfills all requirements dictated by the compatibility with both the string spectrum and the corresponding global symmetries. Hence, many key properties can be deduced by investigating the consistent cubic vertices. Concerning VE, one might ask what is the form of the cubic vertices and what can one learn from them. Given the analogy to ST, we expect to understand how the global HS symmetry constrains the massless  $s_1 - s_2 - s_3$  interactions making the entire spectrum of VE as a single immense multiplet. This question has been partly addressed in [7–9], where it has been shown that, starting from VE, the extraction of the cubic vertices requires infinitely many field redefinitions making the analysis very involved.

On the other hand, moving from a top-down to a bottom-up viewpoint, one may ask oneself which HS cubic interactions lead to fully nonlinear HS theories and whether ST and VE are the only solutions or not. This question (called Gupta or Fronsdal program) can be tackled solving the Noether procedure for HS fields. The latter is a perturbative scheme (order by order in the number of fields) whose aim is to classify all consistent interactions starting from a given free theory. The first step of such procedure is to find out the most general couplings of three massive or massless HS particles in an arbitrary constant-curvature background. For the case of symmetric HS fields, this problem has been addressed in [10, 11], where, making use of the ambient-space formalism, all possible  $s_1 - s_2 - s_3$  interactions were provided.<sup>2</sup>

Notice that the aforementioned program is closely related to the classification of all consistent CFTs in arbitrary dimensions. More precisely, in the context of AdS/CFT, the Noether procedure seems to share many analogies with the problem of classifying all possible consistent OPEs of HS operators  $\mathcal{O}_i(x)$ . In turn, this is tantamount to enumerating

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<sup>1</sup> See [3, 4] for some recent reviews.

<sup>2</sup> See [12, 13] for a frame-like approach to the problem of massless interactions, and [9, 14–19] for other works on HS cubic interactions in (A)dS.

all possible tensor structures of three-point functions  $\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \rangle$ , from which all conformal blocks can be computed.<sup>3</sup> In this respect, it would be interesting to understand the dictionary between the Noether procedure requirements on the bulk side and the conformal symmetry on the boundary side. This perspective can possibly clarify the role of Lagrangian locality, usually assumed in the bulk, or of possible alternatives, and may provide a new look into the AdS/CFT correspondence itself. Moreover, it is worth mentioning that the CFT results (3-pt functions) can be obtained *a priori* from the AdS results (cubic vertices) attaching the Boundary-to-Bulk propagators to the vertices.

In the present paper we present the construction of HS cubic interactions in (A)dS along the lines of [10, 11]. We shall show that this problem is equivalent to finding polynomial solutions of a rather simple set of linear PDEs. Each solution is in one-to-one correspondence to a consistent cubic interaction. Let us stress that, since the solution space is linear, an arbitrary linear combination of these cubic vertices is also consistent, leaving their relative coupling constants unfixed (at this order). The latter are constrained within the Noether procedure either by compatibility with the global symmetries of the free theory and/or by consistency of quartic interactions. Let us stress once again the connection to the conformal bootstrap program which may entail key (still unclear) requirements dictated by the Noether procedure.

The organization of the paper is the following: in Section 2 we introduce the Noether procedure which represents the main tool of our construction. In Section 3 we briefly review how to apply such a scheme to derive cubic interactions in flat space. Section 4 is devoted to the formulation of the free theories in the ambient space. The ambient-space action at the cubic level is discussed in Section 5. Finally, in Sections 6 and 7 we present the solution to the cubic order of the Noether procedure in (A)dS.

## 2 Noether procedure

The aim of the Noether procedure is to find all consistent (at least classically) interacting structures associated to a given set of particles, order by order in the number of fields.<sup>4</sup> In the case of massless spin 1 or spin 2 particles ( $A_\mu$  or  $h_{\mu\nu}$ ), this corresponds to identifying the consistent interactions starting from Maxwell or Fierz-Pauli Lagrangians. Arbitrary vertices involving  $A_\mu$  or  $h_{\mu\nu}$  would mostly cause a propagation of *unphysical* DOFs, which, at the free level, are removed by the linear gauge symmetries:  $\delta^{(0)} A_\mu = \partial_\mu \alpha$  and  $\delta^{(0)} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ . Hence, a key condition for the consistency of the interacting theories is the existence of gauge symmetries which are nonlinear deformations of the linear ones.

Let us consider an arbitrary set of gauge fields  $\varphi^a$  (where  $a$  labels different fields) with free action  $S^{(2)}$  and linear gauge symmetries  $\delta^{(0)} \varphi^a$ . The problem is to find the corresponding nonlinear action  $S$  together with the non-linear gauge symmetries  $\delta \varphi^a$ . For

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<sup>3</sup> See [20–25] for the correlation functions of three conserved currents.

<sup>4</sup> See [26] for a detailed discussion.

this purpose, one can consider the following perturbative expansions:

$$S = S^{(2)} + S^{(3)} + S^{(4)} + \dots, \quad (2.1)$$

$$\delta \varphi^a = \delta^{(0)} \varphi^a + \delta^{(1)} \varphi^a + \delta^{(2)} \varphi^a + \dots, \quad (2.2)$$

where the superscript ( $n$ ) denotes the number of fields involved. Taking the variation of the action (2.1) under the gauge transformation (2.2), one ends up with a system of gauge invariance conditions:

$$\delta^{(0)} S^{(2)} = 0, \quad (2.3)$$

$$\delta^{(0)} S^{(3)} + \delta^{(1)} S^{(2)} = 0, \quad (2.4)$$

$$\delta^{(0)} S^{(4)} + \delta^{(1)} S^{(3)} + \delta^{(2)} S^{(2)} = 0, \quad (2.5)$$

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The first equation implies the linear gauge invariance of the free theory. The second equation is a condition for both the cubic interactions  $S^{(3)}$  and the first-order gauge deformations  $\delta^{(1)} \varphi^a$ , and so on. Since the second term in (2.4) is proportional to the free EOM, condition (2.4) implies

$$\delta^{(0)} S^{(3)} \approx 0, \quad (2.6)$$

where  $\approx$  means equivalence up to the free EOM. Solving this equation one can identify all cubic interactions consistent with the linear gauge symmetries. In the case of massless spin 1, one finds two independent interaction terms which schematically read

$$S^{(3)} = \lambda_1^{1-1-1} \int A A F + \lambda_0^{1-1-1} \int F F F. \quad (2.7)$$

The first is the one-derivative YM vertex while the second is the three-derivative Born-Infeld one. In the massless spin 2 case, there are three independent interactions:

$$S^{(3)} = \lambda_2^{2-2-2} \int h \partial h \partial h + \lambda_1^{2-2-2} \int R R + \lambda_0^{2-2-2} \int R R R, \quad (2.8)$$

where the first is the two-derivative gravitational minimal coupling while the other two come from the expansions of  $(\text{Riemann})^2$  and  $(\text{Riemann})^3$  and involve four and six derivatives respectively. As one can see from these lower-spin examples, the general solutions to eq. (2.6) are  $s_1 - s_2 - s_3$  vertices with different number of derivatives associated with the coupling constants  $\lambda_n^{s_1 - s_2 - s_3}$ . It is worth noticing that these coupling constants are independent at this level.

The next step consists in solving (2.4) for the first-order gauge transformations  $\delta^{(1)} \varphi^a$  associated to each solution  $S^{(3)}$  found at the previous step. In the lower-spin cases, only the first vertices in (2.7) and (2.8) lead to nontrivial deformations:

$$\delta^{(1)} A = \lambda_1^{1-1-1} A \alpha, \quad \delta^{(1)} h = \lambda_2^{2-2-2} (h \partial \xi - \partial h \xi), \quad (2.9)$$

of the linear gauge transformations. The latter correspond to the standard non-Abelian YM gauge transformations and to diffeomorphisms respectively. Although not deforming the gauge symmetries, the remaining vertices can be completed to the full non-linear order

keeping consistency with the gauge transformations (2.9). They form the first elements of a class of higher-derivative gauge or diffeomorphism invariants, where the remaining elements appear at higher orders  $S^{(n)}$ . General (HS) gauge theories present as well two types of cubic vertices: the ones deforming the linear gauge symmetries, and the ones giving rise to possible higher-derivative gauge invariants. Although the former define the deformed non-Abelian gauge algebra, the second are also relevant since they provide possible (quantum) counter-terms. Hence, if no independent non-deforming vertices survive at higher-orders, then no counter-terms would be available and the theory would be UV finite. This issue can be initially addressed solving eq. (2.5) which involves quartic interactions as well as the spectrum of the theory [27, 28].

So far we have only considered the case of gauge theories. Constructing interactions of massive HS fields also raises similar problems. Arbitrary interaction vertices would mostly violate the Fierz conditions resulting in the propagation of unphysical DOFs. One way to proceed is to introduce Stueckelberg symmetries into the massive theory and perform the Noether procedure similarly to the massless case.<sup>5</sup>

### 3 Flat-space case

In this section we consider flat-space cubic vertices since their construction reveals some of the key ideas also used in the (A)dS case. See the review [3] for an exhaustive list of references on the cubic interactions, and [29–32] for more recent developments.

In order to deal with arbitrary HS fields, it is useful to introduce auxiliary variables  $u^\mu$ , which are the analogue of the string oscillators ( $\alpha_{-1}^\mu \leftrightarrow u^\mu$ ), and define the generating function:

$$\varphi^A(x, u) = \frac{1}{s!} \varphi_{\mu_1 \dots \mu_s}^A(x) u^{\mu_1} \dots u^{\mu_s}. \quad (3.1)$$

Here, the superscript  $A$  labels different HS fields, and in the following we use  $A = a$  for massless fields and  $A = \alpha$  for massive ones. In this notation, the most general form of the cubic vertices is

$$S^{(3)} = \int d^d x C_{A_1 A_2 A_3}(\partial_{x_i}, \partial_{u_i}) \varphi^{A_1}(x_1, u_1) \varphi^{A_2}(x_2, u_2) \varphi^{A_3}(x_3, u_3) \Big|_{\substack{x_i=x \\ u_i=0}}, \quad (3.2)$$

where  $i = 1, 2, 3$ . Different functions  $C_{A_1 A_2 A_3}$  describe different vertices, embodying the coupling constants of the theory. Notice that  $C_{A_1 A_2 A_3}$  plays the same role as the state  $|V_3\rangle$  in the BRST (String field theory) approach. Restricting the attention to the parity invariant interactions, the dependence of  $C_{A_1 A_2 A_3}$  on the six vectors  $\partial_{x_i}$  and  $\partial_{u_i}$ , is through the 21 (6+9+6) Lorentz scalars  $\partial_{x_i} \cdot \partial_{x_j}$ ,  $\partial_{u_i} \cdot \partial_{x_j}$  and  $\partial_{u_i} \cdot \partial_{u_j}$ . For instance, a vertex of the form:

$$\varphi_{\mu\nu\rho\lambda} \partial^\mu \partial^\nu \varphi^{\rho\sigma} \partial_\sigma \partial_\tau \varphi^{\tau\lambda}, \quad (3.3)$$

is encoded by

$$C = (\partial_{u_1} \cdot \partial_{x_2})^2 (\partial_{u_1} \cdot \partial_{u_2}) (\partial_{u_1} \cdot \partial_{u_3}) (\partial_{u_2} \cdot \partial_{u_2}) (\partial_{u_3} \cdot \partial_{x_3}). \quad (3.4)$$

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<sup>5</sup> See e.g. [15] for some explicit constructions.

Notice that not all  $C_{A_1 A_2 A_3}$ 's are physically distinguishable but there exist two kinds of ambiguities. The first is due to the triviality of total derivative terms (or integrations by parts). This ambiguity can be fixed by removing  $\partial_{u_i} \cdot \partial_{x_{i-1}}$  in terms of the other Lorentz scalar operators as

$$\partial_{u_i} \cdot \partial_{x_{i-1}} = -\partial_{u_i} \cdot \partial_{x_{i+1}} - \partial_{u_i} \cdot \partial_{x_i} + \partial_{u_i} \cdot \partial_x \quad [i \simeq i+3]. \quad (3.5)$$

The second ambiguity is related to the possibility of performing non-linear field redefinitions which can create *fictive* interaction terms. However, these vertices are all proportional to the linear EOM so that the corresponding ambiguity can be fixed by disregarding the on-shell vanishing vertices. This amounts to neglect the dependence of the function  $C_{A_1 A_2 A_3}$  on the  $\partial_{x_i} \cdot \partial_{x_j}$ 's, since the latter can be expressed as

$$\partial_{x_i} \cdot \partial_{x_{i-1}} = \frac{1}{2} \left( \partial_{x_{i+1}}^2 - \partial_{x_i}^2 - \partial_{x_{i-1}}^2 \right) + \frac{1}{2} \partial_x \cdot (\partial_{x_i} + \partial_{x_{i-1}} - \partial_{x_{i+1}}), \quad (3.6)$$

and, up to EOM, the  $\partial_{x_i}^2$ 's can be replaced by the  $\partial_{u_i} \cdot \partial_{x_i}$ 's and the  $\partial_{u_i}^2$ 's. For instance, the (Fronsdal's) massless HS EOM reads

$$\left[ \partial_x^2 - u \cdot \partial_x \partial_u \cdot \partial_x + \frac{1}{2} (u \cdot \partial_x)^2 \partial_u^2 \right] \varphi^a(x, u) \approx 0. \quad (3.7)$$

Taking into account the latter ambiguities, the vertex function  $C_{A_1 A_2 A_3}$  can only depend on 12 (3+3+6) Lorentz scalars:  $\partial_{u_i} \cdot \partial_{x_{i+1}}$ ,  $\partial_{u_i} \cdot \partial_{x_i}$  and  $\partial_{u_i} \cdot \partial_{u_j}$ . It is worth noticing that at the cubic order, there are no non-local vertices since the  $\partial_{x_i} \cdot \partial_{x_j}$ 's have been removed while the other scalar operators can only enter with positive powers (otherwise the tensor contractions do not make sense).

When a gauge field, say  $\varphi^{a_1}$ , enters the interaction, the function  $C_{a_1 A_2 A_3}$  is further constrained by the condition (2.6). In this notation, the linear HS gauge transformations,  $\delta^{(0)} \varphi_{\mu_1 \dots \mu_s}^a = \partial_{(\mu_1} \varepsilon_{\mu_2 \dots \mu_s)}^a$ , read

$$\delta^{(0)} \varphi^a(x, u) = u \cdot \partial_x \varepsilon^a(x, u). \quad (3.8)$$

Hence the cubic-order gauge consistency condition (2.6) gives

$$\left[ C_{a_1 A_2 A_3}(\partial_{u_i} \cdot \partial_{x_{i+1}}, \partial_{u_i} \cdot \partial_{x_i}, \partial_{u_i} \cdot \partial_{u_j}), u_1 \cdot \partial_{x_1} \right] \approx 0, \quad (3.9)$$

where  $\approx$  means equivalence modulo the Fronsdal equation (3.7). In order to tackle the above equation, it is convenient to split the function  $C_{a_1 A_2 A_3}$  into two parts: the one which does not involve any divergence,  $\partial_{u_i} \cdot \partial_{x_i}$ , trace,  $\partial_{u_i}^2$  or auxiliary fields (we call it the *transverse and traceless* (TT) part), and the one which does (DTA part). Let us notice that the TT part is precisely what survives after eliminating unphysical DOF. Indeed, besides the mass-shell condition, the Fierz system:

$$\text{Fierz system : } (\partial_x^2 - m^2) \varphi^A = 0, \quad \partial_u \cdot \partial_x \varphi^A = 0, \quad \partial_u^2 \varphi^A = 0, \quad (3.10)$$

involves also the transverse and traceless conditions.<sup>6</sup> Therefore, the TT part of the action plays a key role encoding the on-shell content of the theory. On the other hand, the part

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<sup>6</sup> When  $m = 0$ , one has to quotient the system by the gauge symmetries (3.8) with parameters  $\varepsilon^a$  satisfying the same conditions (3.10).

containing divergences, traces or auxiliary fields vanishes after gauge fixing.<sup>7</sup> The next question is whether it is possible to determine the TT part of the vertex without using any information about the other part. From a physical point of view, this ought to be possible since the physical (on-shell) interactions cannot depend on the unphysical ones.<sup>8</sup> Concerning massive fields, the TT conditions already assure the propagation of the correct DOF and no further constraints has to be imposed on the TT parts of the cubic interactions.

Let us also comment on the remaining parts of the cubic interactions involving divergences and traces. For massless fields, the latter turns out to be completely determined by their TT part enforcing gauge invariance [6, 34]. Similarly, when massive fields are involved, after introducing Stueckelberg fields into the TT part (see [11] for details), one may in principle determine the remaining parts of the action requiring the consistency with Stueckelberg gauge symmetries.

In the following, we show how to determine the TT parts of  $C_{A_1 A_2 A_3}$  from eq. (3.9). First, after removing all the ambiguities through eqs. (3.5, 3.6), any functional  $F$  can be *univocally* written in terms of its TT part and the remaining part as  $F = [F]_{\text{TT}} + [F]_{\text{DTA}}$ . Hence, eq. (2.6) can be split into two equations:

$$[\delta^{(0)} S^{(3)}]_{\text{TT}} \approx 0, \quad [\delta^{(0)} S^{(3)}]_{\text{DTA}} \approx 0, \quad (3.11)$$

where, henceforth,  $\approx$  means equivalence modulo the Fierz system (3.10). Second, as the gauge variations of divergences, traces or auxiliary fields are proportional to themselves up to  $\partial_{x_i}^2$ -terms:  $[\delta^{(0)} [S^{(3)}]_{\text{DTA}}]_{\text{TT}} \approx 0$ , the first equation in (3.11) gives an independent condition for the TT parts,  $[S^{(3)}]_{\text{TT}}$ , of the interactions:

$$[\delta^{(0)} S^{(3)}]_{\text{TT}} = [\delta^{(0)} \{ [S^{(3)}]_{\text{TT}} + [S^{(3)}]_{\text{DTA}} \}]_{\text{TT}} \approx [\delta^{(0)} [S^{(3)}]_{\text{TT}}]_{\text{TT}} \approx 0. \quad (3.12)$$

At this point,  $[S^{(3)}]_{\text{TT}}$  can be expressed as in eq. (3.2) through a function  $C_{A_1 A_2 A_3}^{\text{TT}}(y_i, z_i)$  of 6 variables:

$$y_i = \partial_{u_i} \cdot \partial_{x_{i+1}}, \quad z_i = \partial_{u_{i+1}} \cdot \partial_{u_{i-1}}. \quad (3.13)$$

Then, assuming the first field to be massless,  $A_1 = a_1$ , eq. (3.12) gives a condition for  $C_{a_1 A_2 A_3}^{\text{TT}}$  analogous to (3.9). Using the Leibniz rule, we obtain a rather simple differential equation:

$$[y_2 \partial_{z_3} - y_3 \partial_{z_2} + \frac{1}{2} (m_2^2 - m_3^2) \partial_{y_1}] C_{a_1 A_2 A_3}^{\text{TT}} = 0, \quad (3.14)$$

where  $m_i$  is the mass of the  $i$ -th field. When two or three massless fields are involved in the interactions, one has respectively two or three differential equations given by the cyclic permutations of eq. (3.14).

Depending on the cases, the corresponding solutions  $C_{A_1 A_2 A_3}^{\text{TT}}$  are constrained to depend on some of the  $y_i$ 's and the  $z_i$ 's only through the *building blocks*:

$$g = y_1 z_1 + y_2 z_2 + y_3 z_3, \quad (3.15)$$

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<sup>7</sup> Indeed the light-cone interaction vertices [33] can be obtained solely from the TT part by going to the light-cone gauge.

<sup>8</sup> The light-cone gauge approach is consistent in its own without calling for some additional conditions on its corresponding covariant off-shell description.

$$h_i = y_{i+1} y_{i-1} + \frac{1}{2} [m_i^2 - (m_{i+1} + m_{i-1})^2] z_i. \quad (3.16)$$

As an example, we consider the interactions of three massless HS fields where the consistent cubic interactions are encoded in an arbitrary function:

$$C_{a_1 a_2 a_3}^{\text{TT}} = \mathcal{K}_{a_1 a_2 a_3}(y_1, y_2, y_3, g). \quad (3.17)$$

Leaving aside Chan-Paton factors, the latter can be expanded as

$$\mathcal{K} = \sum_{n=0}^{\min\{s_1, s_2, s_3\}} \lambda_n^{s_1 - s_2 - s_3} g^n y_1^{s_1 - n} y_2^{s_2 - n} y_3^{s_3 - n}, \quad (3.18)$$

where the  $\lambda_n^{s_1 - s_2 - s_3}$ 's are independent coupling functions that ought to be fixed by the quest for consistency of higher order interactions. From the latter expression it is straightforward to see that the number of consistent couplings is  $\min\{s_1, s_2, s_3\} + 1$ , while the number of derivatives contained in each vertex is  $s_1 + s_2 + s_3 - 2n$ . In particular, focussing on the 2-2-2 case, eq. (3.18) gives

$$\mathcal{K} = \lambda_2^{2-2-2} g^2 + \lambda_1^{2-2-2} g y_1 y_2 y_3 + \lambda_0^{2-2-2} y_1^2 y_2^2 y_3^2, \quad (3.19)$$

that exactly reproduces eq. (2.8).

#### 4 Ambient-space formalism for HS

In order to address the HS interaction problem around an arbitrary constant-curvature background (*i.e.* (A)dS space), one can still rely on the Noether procedure introduced in Section 2. However, in this case the starting point are the HS free theories in (A)dS, where besides massive and massless particles, new types of particles (called partially-massless) appear [35].<sup>9</sup> Moreover, the cubic interactions built on top of the free theories would involve (A)dS covariant derivatives whose non-commuting nature makes the construction cumbersome. Their commutators give rise to lower-derivative pieces proportional to the cosmological constant, making the vertices inhomogeneous in the number of derivatives. The ambient-space formalism proves to be a convenient tool in dealing with free (A)dS HS, and hence, it represents a natural framework in order to construct (A)dS cubic interactions. Furthermore, recently it has been intensively used in the context of *Mellin amplitude* in the computations of Witten diagrams [37, 38].

The ambient-space formalism [39, 40] consists in regarding the  $d$ -dimensional (A)dS space as the hyper-surface  $X^2 = \epsilon L^2$  embedded into a  $(d+1)$ -dimensional flat-space. In our convention the ambient metric is  $\eta = (-, +, \dots, +)$ , so that AdS ( $\epsilon = -1$ ) is Euclidean while dS ( $\epsilon = 1$ ) is Lorentzian. Focussing on the region  $\epsilon X^2 > 0$ , there exists an isomorphism between symmetric tensor fields in (A)dS,  $\varphi_{\mu_1 \dots \mu_s}$ , and those in ambient space,  $\Phi_{M_1 \dots M_s}$ , satisfying the *homogeneity* and *tangentiality* (HT) conditions:

$$\begin{aligned} \text{Homogeneity :} & \quad (X \cdot \partial_X - U \cdot \partial_U + 2 + \mu) \Phi(X, U) = 0, \\ \text{Tangentiality :} & \quad X \cdot \partial_U \Phi(X, U) = 0. \end{aligned} \quad (4.1)$$

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<sup>9</sup> In the case of mixed-symmetry HS fields in AdS, even the notion of massless-ness changes with respect to the flat-space case [36].



Here we have used the auxiliary-variable notation for the ambient-space fields:

$$\Phi(X, U) = \frac{1}{s!} \Phi_{M_1 \dots M_s}(X) U^{M_1} \dots U^{M_s}. \quad (4.2)$$

The degree of homogeneity  $\mu$  parametrizes the (A)dS mass-squared term appearing in the (A)dS Lagrangian:

$$m^2 = \frac{(-\epsilon)}{L^2} [(\mu - s + 2)(\mu - s - d + 3) - s], \quad (4.3)$$

so that  $\mu = 0$  corresponds to the massless case. In the ambient-space formalism, the EOM of both massless and massive HS fields are given by the Fronsdal ones (3.7) after replacing  $(x, u)$  by  $(X, U)$ . Let us remind the reader that the concept of *massless-ness* in (A)dS is not related to the vanishing of the mass term but rather to the appearance of gauge symmetries. In fact, if one postulates the latter to be of the form:

$$\delta^{(0)} \Phi(X, U) = U \cdot \partial_X E(X, U), \quad (4.4)$$

then the compatibility with the HT conditions (4.1) alone restricts both the possible values of  $\mu$  and the normal(radial) components of  $E$ . In particular, when  $\mu = 0, 1, \dots, s - 1$ , then there exist compatible higher-derivative gauge symmetries:

$$\delta^{(0)} \Phi(X, U) = (U \cdot \partial_X)^{\mu+1} \Omega(X, U) \quad [E = (U \cdot \partial_X)^\mu \Omega], \quad (4.5)$$

with the gauge parameters  $\Omega$  satisfying

$$(X \cdot \partial_X - U \cdot \partial_U - \mu) \Omega(X, U) = 0, \quad X \cdot \partial_U \Omega(X, U) = 0. \quad (4.6)$$

On the other hand, for other values of  $\mu$ , no gauge symmetries (in absence of auxiliary fields) are allowed, implying that the corresponding fields are massive. Notice that the massless field,  $\mu = 0$ , is the first member of a class of representations where the other members, with  $\mu = 1, 2, \dots, s - 1$ , are called partially-massless. However, partially-massless fields describe unitary representations only in dS.

Before closing this section, let us discuss the flat limit from the ambient-space viewpoint. The latter consists first in translating the coordinate system as  $X^M \rightarrow X^M + L N^M$ , where  $N$  is a constant vector satisfying  $N^2 = \epsilon$ , and second, in taking the  $L \rightarrow \infty$  limit. As a result, the HT conditions (4.1) reduces to

$$(N \cdot \partial_X - \sqrt{-\epsilon} M) \Phi(X, U) = 0, \quad N \cdot \partial_U \Phi(X, U) = 0, \quad (4.7)$$

where the flat mass  $M$  is related to the (A)dS *mass*  $\mu$  as

$$\sqrt{-\epsilon} M = - \lim_{L \rightarrow \infty} \frac{\mu}{L}. \quad (4.8)$$

Notice that in this limit, all (A)dS representations become massless, while, in order to recover massive representations in flat space one should consider the  $\mu \rightarrow \infty$  limit.

## 5 Ambient-space action

In the previous section we have shown how to describe HS fields in (A)dS making use of the ambient-space language. In Section 6 we shall use the latter framework in order to solve the Noether procedure. For this purpose, one needs to know first of all how to express the (A)dS action in terms of ambient-space quantities. As far as the Lagrangian is concerned, no subtleties arise since, together with the isomorphism between (A)dS and ambient-space fields, there is an analogous one between (A)dS-covariant derivatives  $\nabla_\mu$  and ambient-space ones  $\partial_{X^M} - (X_M/X^2) X \cdot \partial_X$ . Hence, any Lagrangian  $\mathcal{L}_{(A)dS}$  written in terms of (A)dS intrinsic fields is in one-to-one correspondence with the ambient-space one  $\mathcal{L}_{\text{Amb}}$ . More precisely, considering a single term in the Lagrangian, the two descriptions are related by

$$\mathcal{L}_{\text{Amb}}(\Phi, \partial\Phi, \partial\partial\Phi, \dots) = \left(\frac{R}{L}\right)^\Delta \mathcal{L}_{(A)dS}(\varphi, \nabla\varphi, \nabla\nabla\varphi, \dots), \quad (5.1)$$

where  $\Delta$  is a constant depending on the spins and the  $\mu$ -values of the fields as well as on the number of derivatives entering  $\mathcal{L}_{\text{Amb}}$ . Regarding the action, the first attempt would be to consider

$$\int d^{d+1}X \mathcal{L}_{\text{Amb}} = \left(\int_0^\infty dR \left(\frac{R}{L}\right)^{d+\Delta}\right) \times \left(\int_{(A)dS} d^d x \sqrt{-\epsilon g} \mathcal{L}_{(A)dS}\right). \quad (5.2)$$

However, the latter contains a diverging radial integral so that controlling its gauge invariance becomes ambiguous. A way of solving this problem would be to introduce a cut-off in order to regulate the radial integral, or similarly, a boundary for the ambient space. Then, the presence of the boundary breaks gauge invariance which can be restored only by adding boundary (total-derivative) terms in the action. The latter are the analogue of the Gibbons-Hawking-York boundary term needed in order to amend the Einstein-Hilbert action in manifolds with boundary.

Another equivalent way is suggested by the fact that the ambient space can be considered as a tool to rewrite intrinsic  $d$ -dimensional (A)dS quantities in a manifestly  $SO(1, d)$ -covariant form. In this respect, with the aid of a delta function, one can simply rewrite the (A)dS action in the ambient-space language as

$$S = \int d^d x \sqrt{-\epsilon g} \mathcal{L}_{(A)dS} = \int d^{d+1}X \delta(\sqrt{\epsilon X^2} - L) \mathcal{L}_{\text{Amb}}. \quad (5.3)$$

As a candidate for the Lagrangian  $\mathcal{L}_{\text{Amb}}$ , one may think to use the flat  $d$ -dimensional one where all the  $d$ -dimensional quantities are replaced by  $(d+1)$ -dimensional ones. However, in general this way does not lead to a consistent (A)dS action. The reason is that, because of the delta function, total-derivative terms in  $\mathcal{L}_{\text{Amb}}$  no longer vanish but contribute as

$$\delta(\sqrt{\epsilon X^2} - L) \partial_{X^M}(\dots) = -\delta'(\sqrt{\epsilon X^2} - L) \frac{\epsilon X_M}{\sqrt{\epsilon X^2}}(\dots) \neq 0. \quad (5.4)$$

Thereby, in order to compensate these terms, the Lagrangian  $\mathcal{L}_{\text{Amb}}$  has to be amended by additional total-derivative contributions. It is worth noticing that the latter vertices contain a lower number of derivatives compared to the initial vertices in  $\mathcal{L}_{\text{Amb}}$ . Actually,

this is the ambient-space analogue of what happens in the intrinsic formulation: the replacement of ordinary derivatives by covariant ones requires the inclusion of additional lower-(covariant) derivative vertices in the Lagrangian.

As previously discussed, a consistent (A)dS action consists of vertices containing terms with different number of derivatives sized by proper powers of  $L^{-2}$ :

$$\mathcal{L}_{\text{Amb}} = \mathcal{L}_{\text{Amb}}(L^{-2}). \quad (5.5)$$

In the ambient-space formalism, it is convenient to rather size such contributions by different derivatives of the delta function:

$$\delta^{[n]}(\sqrt{\epsilon X^2} - L) \quad \left[ \delta^{[n]}(R - L) = \left(\frac{1}{R} \frac{d}{dR}\right)^n \delta(R - L) \right], \quad (5.6)$$

since the latter naturally appear in the terms (5.4) that need to be compensated. Indeed, thanks to the following identity:

$$\delta^{[n]}(R - L) R^\lambda = \frac{(-2)^n [(\lambda - 1)/2]_n}{(L^2)^n} \delta(R - L) R^\lambda, \quad (5.7)$$

arbitrary powers of  $L^{-2}$  can be always absorbed into derivatives of the delta function. Therefore, the ambient-space Lagrangian can be expanded as

$$\delta(\sqrt{\epsilon X^2} - L) \mathcal{L}_{\text{Amb}}(L^{-2}) = \sum_{n \geq 0} \delta^{[n]}(\sqrt{\epsilon X^2} - L) \mathcal{L}_{\text{Amb}}^{[n]}, \quad (5.8)$$

where the  $\mathcal{L}_{\text{Amb}}^{[n]}$ 's do not involve any power of  $L^{-2}$ . In order to conveniently handle the above series, it is useful to express  $\delta^{[n]}$  by means of an auxiliary variable  $\hat{\delta}$  as

$$\delta^{[n]}(R - L) = \exp\left(\frac{\epsilon L}{R} \frac{d}{dR} \frac{d}{d\hat{\delta}}\right) \delta(R - L) \left(\epsilon \frac{\hat{\delta}}{L}\right)^n \Big|_{\hat{\delta}=0}. \quad (5.9)$$

For simplicity, in the following we work with the rule  $\delta^{[n]}(R - L) = \delta(R - L) (\epsilon \hat{\delta}/L)^n$ .<sup>10</sup> The advantage of introducing the auxiliary variable  $\hat{\delta}$  lies on the simple rule in dealing with total derivatives:

$$\delta(\sqrt{\epsilon X^2} - L) \partial_{X^M}(\dots) = -\delta(\sqrt{\epsilon X^2} - L) \frac{\hat{\delta}}{L} X_M(\dots). \quad (5.10)$$

Moreover, it also allows one to factorize the delta function in the series (5.8) and rewrite the Lagrangian as a polynomial function in  $\hat{\delta}/L$ :

$$\mathcal{L}_{\text{Amb}} = \mathcal{L}_{\text{Amb}}\left(\frac{\hat{\delta}}{L}\right). \quad (5.11)$$

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<sup>10</sup> The  $1/L$  in the definition of  $\hat{\delta}$  has been introduced to provide a well-defined flat limit: the corresponding rule in flat space becomes  $\delta(N \cdot X) \partial_{X^M}(\dots) = -\delta(N \cdot X) \hat{\delta} N_M(\dots)$ .

## 6 Construction of HS cubic interactions in (A)dS

In this section we present the solution to the Noether procedure at the cubic level of for arbitrary symmetric HS fields in (A)dS. The logic is the same as in the flat-space case discussed in Section 3, thus in the following we mainly focus on those points wherein the peculiarities of (A)dS arise.

Apart from the presence of the delta function, the discussions which led to the most general form of the TT parts of the cubic vertices still hold. The only subtleties are related to the total-derivative terms in (3.5) and (3.6) which no longer vanish. However, as we shall explain below, their contributions can be reabsorbed into redefinitions of the cubic vertices. Hence, the most general expression for the TT parts of the cubic vertices reads

$$[S^{(3)}]_{\text{TT}} = \int d^{d+1}X \delta(\sqrt{\epsilon X^2} - L) C_{A_1 A_2 A_3}^{\text{TT}}\left(\frac{\hat{\delta}}{L}; Y_i, Z_i\right) \times \\ \times \Phi^{A_1}(X_1, U_1) \Phi^{A_2}(X_2, U_2) \Phi^{A_3}(X_3, U_3) \Big|_{\substack{X_i=X \\ U_i=0}}, \quad (6.1)$$

where the  $Y_i$ 's and the  $Z_i$ 's are defined analogously to (3.13). Let us remind the reader that, as we have discussed in the previous section, the inhomogeneity of the vertices in the number of derivatives is encoded in the  $\hat{\delta}/L$ -dependence of the function  $C_{A_1 A_2 A_3}^{\text{TT}}$ .

Whenever a gauge field joins the interactions, the cubic vertices are constrained to satisfy the gauge compatibility condition (3.12) associated to that field. Assuming the first field to be (partially-)massless (*i.e.*  $\mu_1 \in \{0, 1, \dots, s_1 - 1\}$ ), one gets

$$\left[ C_{a_1 A_2 A_3}^{\text{TT}}\left(\frac{\hat{\delta}}{L}; Y, Z\right), (U_1 \cdot \partial_{X_1})^{\mu_1+1} \right] \Big|_{U_1=0} \approx 0, \quad (6.2)$$

where  $\approx$  means again equivalence modulo the  $\partial_{X_i}^2$ 's,  $\partial_{U_i} \cdot \partial_{X_i}$ 's and  $\partial_{U_i}^2$ 's, *i.e.* modulo the ambient-space Fierz system. Aside from the higher-derivative nature of the gauge transformations, the key difference with respect to the flat case is the non-triviality of the total-derivative terms arising from the commutations of  $U_1 \cdot \partial_{X_1}$  with the  $Y_i$ 's and the  $Z_i$ 's. Let us sketch how these total-derivative terms can be dealt with:

- Because of the identity (5.10), the total-derivatives terms give rise to contributions of order  $\hat{\delta}/L$  and proportional to the operators  $X \cdot \partial_{X_i}$  or  $X \cdot \partial_{U_i}$ .
- Appearing right after the delta function, the latter can be replaced by  $X_i \cdot \partial_{X_i}$  and  $X_i \cdot \partial_{U_i}$  respectively.
- Pushing these operators to the right and making them act on the fields, one can use the HT conditions (4.1) to replace  $X_i \cdot \partial_{X_i}$  by the corresponding homogeneity degrees and  $X_i \cdot \partial_{U_i}$  by zero.

All in all, one can recast the condition (6.2) into a higher-order partial differential equation of the form:

$$\prod_{n=0}^{\mu_1} \left[ Y_2 \partial_{Z_3} - Y_3 \partial_{Z_2} + \frac{\hat{\delta}}{L} \left( Y_2 \partial_{Y_2} - Y_3 \partial_{Y_3} - \frac{\mu_1 + \mu_2 - \mu_3 - 2n}{2} \right) \partial_{Y_1} \right] C_{a_1 A_2 A_3}^{\text{TT}}\left(\frac{\hat{\delta}}{L}; Y, Z\right) = 0. \quad (6.3)$$

It is worth noticing that the *masses* of the other two fields,  $\mu_2$  and  $\mu_3$ , enter the equation as *effective masses*,  $\mu_2 - 2Y_2 \partial_{Y_2}$  and  $\mu_3 - 2Y_3 \partial_{Y_3}$ , dressed by number operators. Therefore, even in the massless case ( $\mu_2 = \mu_3 = 0$ ) a mass-like term survives. Again, depending on the number of (partially-)massless fields involved in the interactions, one can have a system of (up to three) differential equations given by the cyclic permutations of eq. (6.3).

## 7 Solutions of HS cubic interactions in (A)dS

In this section we discuss the polynomial solutions of the system of PDEs given by eq. (6.3) and possible cyclic permutations thereof. Indeed, since the generating function  $\Phi(X, U)$  is a formal series in  $U^M$ , the latter are the only relevant ones. Our discussion mainly focuses on the interactions involving three massless fields which are of capital importance due to their connections to VE.

### 7.1 Three massless case

In the three massless case ( $\mu_i = 0$ ,  $i = 1, 2, 3$ ), one has a system of three second order PDEs of the form:

$$\left[ Y_{i+1} \partial_{Z_{i-1}} - Y_{i-1} \partial_{Z_{i+1}} + \frac{\hat{\delta}}{L} (Y_{i+1} \partial_{Y_{i+1}} - Y_{i-1} \partial_{Y_{i-1}}) \partial_{Y_i} \right] C_{a_1 a_2 a_3}^{\text{TT}} \left( \frac{\hat{\delta}}{L}; Y, Z \right) = 0, \quad (7.1)$$

where  $[i \simeq i + 3]$ . The latter can be solved via standard techniques (the Laplace transform and the method of characteristics), and its solutions are given by

$$C_{a_1 a_2 a_3}^{\text{TT}} = \exp \left\{ -\frac{\hat{\delta}}{L} [Z_1 \partial_{Y_2} \partial_{Y_3} + Z_1 Z_2 \partial_{Y_3} \partial_G + \text{cyclic} + Z_1 Z_2 Z_3 \partial_G^2] \right\} \times \mathcal{K}_{a_1 a_2 a_3}(Y_1, Y_2, Y_3, G), \quad (7.2)$$

where  $\mathcal{K}_{a_1 a_2 a_3}$  is an arbitrary polynomial function of the  $Y_i$ 's and  $G = Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3$ . Notice that in the flat limit, one recovers the coupling (3.17). The exponential function provides the correct lower-derivative tails needed for the consistency of the corresponding (A)dS interactions. For instance, considering the lowest-derivative 4-4-4 interaction,  $\mathcal{K} = \lambda_4^{4-4-4} G^4$ , one gets

$$\begin{aligned} C^{\text{TT}} &= \lambda_4^{4-4-4} \left[ G^4 - 12 \frac{\hat{\delta}}{L} Z_1 Z_2 Z_3 G^2 + 12 \left( \frac{\hat{\delta}}{L} \right)^2 Z_1^2 Z_2^2 Z_3^2 \right] \\ &= \lambda_4^{4-4-4} \left[ G^4 + \frac{12\epsilon(d+3)}{L^2} Z_1 Z_2 Z_3 G^2 + \frac{12(d+3)(d+5)}{L^4} Z_1^2 Z_2^2 Z_3^2 \right], \end{aligned} \quad (7.3)$$

where, in the second line we have used the identity (5.7) in order to replace the powers of  $\hat{\delta}/L$  by those of  $L^{-2}$ .

It is worth mentioning another way of presenting the solution (7.2). It consists in encoding all the  $\hat{\delta}$  contributions into total derivatives as

$$C_{a_1 a_2 a_3}^{\text{TT}} = \mathcal{K}_{a_1 a_2 a_3}(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_2, \tilde{G}), \quad (7.4)$$

where the  $\tilde{Y}_i$ 's and  $\tilde{G}$  are the (A)dS deformations:

$$\begin{aligned}\tilde{Y}_i &= Y_i + \alpha_i \partial_{U_i} \cdot \partial_X, & \tilde{G} &= \sum_{i=1}^3 (Y_i + \beta_i \partial_{U_i} \cdot \partial_X) Z_i, \\ (\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3) &= \left( \alpha, -\frac{1}{\alpha+1}, -\frac{\alpha+1}{\alpha}; \beta, -\frac{\beta+1}{\alpha+1}, -\frac{\alpha-\beta}{\alpha} \right),\end{aligned}\quad (7.5)$$

of the flat-space building blocks  $Y_i$ 's and  $G$ . The equivalence between the two representations (7.2) and (7.4) of the cubic interactions can be shown carrying out the integration by parts of all the total-derivative terms present in (7.4). Notice that the freedom of  $\alpha$  and  $\beta$  reflects a redundancy in expressing the building blocks in terms of total derivatives. Finally, let us conclude the discussion on the interactions of massless HS fields providing the example (7.3) in terms of ambient-space tensor contractions:

$$\begin{aligned}[S^{(3)}]_{\text{TT}} &= \lambda_4^{4-4} \int d^{d+1} X \delta(\sqrt{\epsilon X^2} - L) \\ &\times \left[ \Phi_{MNPQ} \partial^M \partial^N \partial^P \partial^Q \Phi_{RSTV} \Phi^{RSTV} + 8 \Phi_{MNPQ} \partial^M \partial^N \partial^P \Phi_{RSTV} \partial^V \Phi^{RSTQ} \right. \\ &\quad + 6 \Phi_{MNTV} \partial^M \partial^N \Phi_{PQRS} \partial^R \partial^S \Phi^{PQTV} + 12 \partial_S \Phi_{MNR}{}^T \partial^M \partial^N \Phi_{PQRT} \partial^R \Phi^{PQRS} \\ &\quad + \frac{12\epsilon(d+3)}{L^2} \left( \Phi_{MNS}{}^T \partial^M \partial^N \Phi_{PQRT} \Phi^{PQRS} + 2 \Phi_{MRS}{}^T \partial^M \Phi_{NPQT} \partial^N \Phi^{PQRS} \right) \\ &\quad \left. + \frac{4(d+3)(d+5)}{L^4} \Phi_{MNPQ} \Phi^{MNR S} \Phi^{PQ}{}_{RS} \right].\end{aligned}\quad (7.6)$$

## 7.2 General cases

The interactions of three massless fields represent a subclass of the interactions one can envisage depending on the values of the  $\mu_i$ 's. Let us notice that in the general cases the solutions are given by intersections of the solution spaces of the PDE (6.3) and its cyclic permutations. Therefore, we start our discussion from the solutions of one PDE, for which it is instructive to first analyze the corresponding equation in flat space (3.14). The latter exhibits a singular point in correspondence of the value  $m_2 = m_3$ . Indeed, aside from this value, a rescaling of  $m_2^2 - m_3^2$  is tantamount to a rescaling of  $y_1$ . Therefore, any polynomial solution with  $m_2 \neq m_3$  can be smoothly deformed to a solution with  $m_2 = m_3$ , while the opposite is not true. Consequently, the solution space with  $m_2 = m_3$  is always bigger than (or equal to) the one with  $m_2 \neq m_3$ . Indeed, an explicit analysis shows that the  $m_2 \neq m_3$  solutions  $\mathcal{K}(y_2, y_3, h_2, h_3, z_1)$  can be always expressed in terms of the  $m_2 = m_3$  solutions  $\mathcal{K}(y_1, y_2, y_3, g, z_1)$ , while the opposite is not true.

The latter phenomenon has a richer counterpart in (A)dS, where the role of  $m_2^2 - m_3^2$  in eq. (3.14) is played by the combinations:

$$\mu_1 + \mu_2 - \mu_3 - 2(Y_2 \partial_{Y_2} - Y_3 \partial_{Y_3} + n) \quad [n = 0, \dots, \mu_1], \quad (7.7)$$

in eq. (6.3). Indeed, because of the number operator  $Y_2 \partial_{Y_2} - Y_3 \partial_{Y_3}$ , eq. (7.7) may have several vanishing points for  $\mu_1 + \mu_2 - \mu_3 \in 2\mathbb{Z}$ . More precisely, in correspondence of the latter values, one can consider an ansatz of the form:

$$C_{a_1 a_2 a_3}^{\text{TT}} \left( \frac{\hat{\delta}}{L}; Y, Z \right) = Y_{2,3}^{|M|} \bar{C}_{a_1 a_2 a_3}^{\text{TT}} \left( \frac{\hat{\delta}}{L}; Y, Z \right) \quad \left[ M = \frac{\mu_1 + \mu_2 - \mu_3 - 2n}{2} \right], \quad (7.8)$$

where we use  $Y_2$  for  $M > 0$  and  $Y_3$  for  $M < 0$ . Plugging this ansatz into the original equation (6.3), one ends up with an analogous equation for  $\bar{C}_{a_1 A_2 A_3}^{\text{TT}}$ , whose  $n$ -th factor coincides with the operator appearing in the massless case (7.1). Therefore, the solutions of the massless equation provide solutions of the original equation through the ansatz (7.8). Notice that, when  $\mu_1 + \mu_2 - \mu_3 \notin 2\mathbb{Z}$ , the aforementioned solutions are no longer available since they become non-polynomial. In all the cases which can not be covered by the ansatz (7.8), the solutions can be expressed as arbitrary functions of the building blocks:

$$\tilde{H}_i = \partial_{U_{i-1}} \cdot \partial_{X_{i+1}} \partial_{U_{i+1}} \cdot \partial_{X_{i-1}} - \partial_{X_{i+1}} \cdot \partial_{X_{i-1}} Z_i, \quad (7.9)$$

which are the (A)dS deformations of the flat-space building blocks  $h_i$  (3.16). It is worth noticing that this pattern is similar to what happens in flat space where the  $h_i$ -type solutions exist independently on the mass values, while the massless-type ones (involving  $g$ ) only appear for particular values of the  $m_i$ 's.

Moving to the cases in which more than one equation is involved, one has to consider intersections of the corresponding solution spaces. Since in flat space the only *enhancement point* arises for  $m_i = m_{i+1}$ , one is led to five different cases: (1) three massless ( $m_1 = m_2 = m_3 = 0$ ), (2) two massless and one massive ( $m_1 = m_2 = 0, m_3 \neq 0$ ), (3) one massless and two massive with different masses ( $m_1 = 0, m_2 \neq m_3$ ), (4) one massless and two massive with equal masses ( $m_1 = 0, m_2 = m_3$ ), (5) three massive. On the other hand, due to the presence of a richer pattern of enhancement points ( $\mu_i + \mu_{i+1} - \mu_{i-1} \in 2\mathbb{Z}$ ), more combinations appear in (A)dS. The analysis of the above cases goes beyond the scope of the present letter, and we refer to the forthcoming paper [41] for the detailed discussion.

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