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Original Article

Multiset proximity spaces

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Abstract A multiset is a collection of objects in which repetition of elements is essential. This paper is an attempt to explore the theoretical aspects of multiset by extending the notions of compact, proximity relation and proximal neighborhood to the multiset context. Examples of new multiset topologies, open multiset cover, compact multiset and many identities involving the concept of multiset have been introduced. Further, an integral examples of multiset proximity relations are obtained. A multiset topology induced by a multiset proximity relation on a multiset M has been presented. Also the concept of multiset δ- neighborhood in the multiset proximity space which furnishes an alternative approach to the study of multiset proximity spaces has been mentioned. Finally, some results on this new approach have been obtained and one of the most important results is: every T_4-multiset space is semi-compatible with multiset proximity relation δ on M (Theorem 5.10).

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1. Introduction

Classical set theory is a basic concept to represent various situations in mathematical notation where repeated occurrences of elements are not allowed. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset [1] or bag [2], for short). Actually, various circumstances repetition of elements become mandatory to the system. For example, a graph with loops, there are many hydrogen atoms, many water molecules, many strands of identical DNA etc. This leads to effectively three possible relations between any two physical objects; they are different, they are the same but separate, or they are coinciding and identical. For example, ammonia NH_3, with three hydrogen atoms, say H, H and H, and one nitrogen atom, say N. Clearly H and N are different. However H, H and H are the same but separate, while H and H are coinciding and identical. There are many other examples, for instance, carbon dioxide CO_2, sulfuric acid H_2SO_4, and water H_2O etc.

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This paper is an attempt to generalize the notions of compact, proximity relation and proximal neighborhood in the multiset context. Section 2 has a collection of all basic definitions and notions for further study. In Section 3, new multiset topologies, compact multiset topology and some identities involving the concept of multiset are mentioned. A multiset proximity relations and an integral examples of multiset proximity are obtained in Section 4. In Section 5, a multiset topology induced by a multiset proximity relation on a multiset M is presented. In addition to this point, the essential properties of this new multiset topology via multiset proximity are mentioned.

2. Preliminaries and basic definitions

In this section, a brief survey of the notion of msets as introduced by Yager [2], Blizard [1,3] and Jena et al. [4] have been collected. Furthermore, the different types of collections in mset context introduced by Girish and John [5] are presented.

Definition 2.1. A collection of elements containing duplicates is called an mset. Formally, if X is a set of elements, an mset M drawn from the set X is represented by a function count M or C_M defined as $C_M : X \to \mathbb{N}$, where $\mathbb{N}$ represents the set of nonnegative integers.

Let M be an mset from the set $X = \{x_1, x_2, \dots, x_n\}$ with x appearing n times in M. It is denoted by $x \in {}^n M$. The mset M drawn from the set X is denoted by $M = \{k_1/x_1, k_2/x_2, \dots, k_n/x_n\}$ where M is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. In Definition 2.1, $C_M(x)$ is the number of occurrences of the element x in the mset M. However those elements which are not included in the mset M have zero count. An mset M is a set if $C_M(x) = 0$ or $1 \forall x \in X$.

Definition 2.2. A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times. The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset.

Let $M, N \in [X]^m$. Then, the following are defined:

(1) M is a submset of N denoted by $(M \subseteq N)$ if $C_M(x) \leq C_N(x) \forall x \in X$.
(2) $M = N$ if $M \subseteq N$ and $N \subseteq M$.
(3) M is a proper submset of N denoted by $(M \subset N)$ if $C_M(x) \leq C_N(x) \forall x \in X$ and there exists at least one element $x \in X$ such that $C_M(x) < C_N(x)$.
(4) $P = M \cup N$ if $C_P(x) = \max\{C_M(x), C_N(x)\}$ for all $x \in X$.
(5) $P = M \cap N$ if $C_P(x) = \min\{C_M(x), C_N(x)\}$ for all $x \in X$.
(6) Subtraction of M and N results in a new mset $P = M \ominus N$ such that $C_P(x) = \max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$, where $\oplus$ and $\ominus$ represent mset addition and mset subtraction, respectively.
(7) An mset M is empty if $C_M(x) = 0 \forall x \in X$.
(8) The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$; that is, M^* is an ordinary set and it is also called root set.
(9) The cardinality of an mset M drawn from a set X is Card(M)= $\sum_{x \in X} C_M(x)$.
(10) M and N are said to be equivalent if and only if Card(M)= Card(N).

Axioms of multiset theory 2.3. [1]

(1) If $x \in^{k_1} M$ and $x \in^{k_2} M$ then $k_1 = k_2$. In other words, the multiplicity with which an element belongs to an mset is unique.
(2) If $x \in {}^k M$ iff $x \in {}^k N$ for all x, then $M = N$. If two msets have exactly the same elements occurring with exactly the same multiplicities, then they are equal.
(3) There exists mset M such that $x \notin^k M$ for all $x \in X, k \in \mathbb{N}$. It asserts the existence of at least one mset that does not contain any element. Indeed, the mset M is unique by Axiom −2. This unique mset M is denoted by the symbol ϕ.
(4) For any mset M and any number n, there is a unique (by Axiom 2) mset N containing exactly n copies of M and nothing else.
(5) For any two distinct msets M and N and any numbers n and m, there exists a unique (by Axiom 2) mset P containing exactly n copies of M, m copies of N, and nothing else.

Definition 2.4. Let $N \in [X]^m$. Then the complement N^c of N in $[X]^m$ is an element of $[X]^m$ such that

$$C_{N^c}(x) = m - C_N(x) \forall x \in X.$$

Definition 2.5. A submset N of M is a whole submset of M with each element in N having full multiplicity as in M; that is, $C_N(x) = C_M(x)$ for every $x \in N^*$.

Definition 2.6. Let $M \in [X]^m$. The power whole mset of M denoted by $PW(M)$ is defined as the set of all whole submsets of M.

Definition 2.7. Let $M \in [X]^m$. The power mset $P(M)$ of M is the set of all submsets of M. We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \phi$, then $N \in {}^1P(M)$; and if $N \neq \phi$, then $N \in {}^kP(M)$ such that $k = \prod_z \binom{|[M]_z|}{|[N]_z|}$, the product $\prod_z$ is taken over distinct elements of the mset N and $|[M]_z| = m$ *iff* $z \in {}^m M$, $|[N]_z| = n\, iff\, z \in^n N$, then

$$\binom{|[M]_z|}{|[N]_z|} = \binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$. The following theorem shows the cardinality of the power set of an mset.

Definition 2.8. Let $M \in [X]^m$ and $\tau \subseteq P^*(M)$. Then τ is called an mset topology if τ satisfies the following properties.

(1) ϕ and M are in τ,
(2) The union of the elements of any sub-collection of τ is in τ,
(3) The intersection of the elements of any finite sub-collection of τ is in τ.

An mset topological space is a pair (M, τ) consisting of an mset M and an mset topology τ on M. Note that τ is an ordinary set whose elements are msets and the mset topology is abbreviated as an M-topology. Also, a submset U of M is an open mset of M if U belongs to the collection τ. Moreover, a submset N of M is closed mset if $M \ominus N$ is open mset.

Definition 2.9. Let (M, τ) be an M-topological space and N be a submset of M. Then the interior of N is defined as the mset union of all open msets contained in N and is denoted by N^o; that is,

$$N^o = \cup\{V \subseteq M : V \text{ is an open mset and } V \subseteq N\} \quad \text{and}$$
$$C_{N^o}(x) = \max\{C_V(x) : V \subseteq N\}.$$

Definition 2.10. Let (M, τ) be an M-topological space and N be a submset of M. Then the closure of N is defined as the mset intersection of all closed msets containing N and is denoted by $\overline{N}$; that is,

$$\overline{N} = \cap\{K \subseteq M : K \text{ is a closed mset and } N \subseteq K\} \quad \text{and}$$
$$C_{\overline{N}}(x) = \min\{C_K(x) : N \subseteq K\}.$$

Remark 2.11. The symbol $\in_+$ was first introduced by Singh et al. [6]. For $x \in M^*$, $x \in_+ N$ means that x belongs to N at least one time. Thus, $C_N(x) = 0$ implies $x \notin N$, and $x \in_+^k N$ implies x belongs to N at least k times, however $x \in {}^k N$ means x belongs k times to N.

Definition 2.12. An mset M is called simple if all its elements are the same. For example, $\{3/x\}$ is simple mset.

Definition 2.13. Let M be a nonempty mset. Two submsets N_1 and N_2 are called similar, denoted by $N_1 \cong N_2$, if $N_1^* = N_2^*$.

3. On multiset topologies

Theorem 3.1. *Let N_1 and N_2 be two submsets of an mset M. Then*

(1) *If $C_{(N_1 \cap N_2)}(x) = 0$ for all $x \in M^*$, then $C_{N_1}(x) \leq C_{(M \ominus N_2)}(x)$ for all $x \in M^*$,*
(2) *$C_{N_1}(x) \leq C_{N_2}(x) \Leftrightarrow C_{(M \ominus N_2)}(x) \leq C_{(M \ominus N_1)}(x)$ for all $x \in M^*$.*

Proof.

(1) Let $C_{(N_1 \cap N_2)}(x) = 0$ for all $x \in M^*$. Since $C_{(N_1 \cap N_2)}(x) = \min\{C_{N_1}(x), C_{N_2}(x)\}$, then $C_{N_1}(x) = 0$ or $C_{N_2}(x) = 0$ for all $x \in M^*$. It follows that $C_{N_1}(x) + C_{N_2}(x) \leq C_M(x)$ for all $x \in M^*$, and hence $C_{N_1}(x) \leq C_M(x) - C_{N_2}(x) = C_{(M \ominus N_2)}(x)$ for all $x \in M^*$. Hence the result.
(2) $C_{N_1}(x) \leq C_{N_2}(x) \Leftrightarrow -C_{N_2}(x) \leq -C_{N_1}(x) \Leftrightarrow C_M(x) - C_{N_2}(x) \leq C_M(x) - C_{N_1}(x) \Leftrightarrow C_{(M \ominus N_2)}(x) \leq C_{(M \ominus N_1)}(x)$ for all $x \in M^*$. □

The following example shows that the converse of Theorem 3.1 part (1) is not true in general.

Example 3.2. Let $M = \{2/a, 4/b, 5/c\}$, $N_1 = \{1/a, 1/b, 2/c\}$ and $N_2 = \{1/a, 1/b\}$. Hence $M \ominus N_2 = \{1/a, 3/b, 5/c\}$. It's clear that $C_{N_1}(x) \leq C_{M \ominus N_2}(x)$ for all $x \in M^*$, but $C_{(N_1 \cap N_2)}(x) > 0$ for some $x \in M^*$.

The following example shows that $N_1 \ominus N_2 \neq N_1 \cap (M \ominus N_2)$ in general.

Example 3.3. Let $M = \{3/x, 4/y\}$, $N_1 = \{2/x, 3/y\}$ and $N_2 = \{1/x, 2/y\}$. Hence $M \ominus N_2 = \{2/x, 2/y\}$, $N_1 \ominus N_2 = \{1/x, 1/y\}$, and $N_1 \cap (M \ominus N_2) = \{2/x, 2/y\}$.

Definition 3.4. Let X be an infinite set. Then $M = \{k_\alpha/x_\alpha : \alpha \in \Lambda\}$ be an infinite mset drawn from X. That is, the infinite mset M drawn from X is denoted by $M = \{k_1/x_1, k_2/x_2, k_3/x_3, \ldots\ldots\}$.

Notation 3.5. The mset space $[X]^m_\infty$ is the set of all infinite msets whose elements are in X such that no element in the mset occurs more than m times.

It may be noted that the following examples of mset topologies are not tackled before.

Example 3.6. Let $M \in [X]^m_\infty$ and $\{k_0/x_0\}$ be a simple submset of M. Then the collection

$$\tau_{(k_0/x_0)} = \{V \subseteq M : C_V(x_0) \geq k_0\} \cup \{\emptyset\}$$

is an M-topology on M called the particular point M-topology.

Example 3.7. Let $M \in [X]^m_\infty$ and $\{k_0/x_0\}$ be a simple submset of M. Then the collection

$$\tau_{k_0/x_0} = \{V \subseteq M : C_V(x_0) < k_0\} \cup \{M\}$$

is an M-topology on M called the excluded point M-topology.

Example 3.8. Let $M \in [X]^m_\infty$. Then the collection

$$\tau = \{V \subseteq M : M \ominus V \text{ is finite}\} \cup \{\emptyset\}$$

is an M-topology on M called the cofinite M-topology.

Example 3.9. Let $M \in [X]^m_\infty$ and N be a submset of M. Then the collection

$$\tau_{(N)} = \{V \subseteq M : C_N(x) \leq C_V(x) \text{ for all } x \in M^*\} \cup \{\emptyset\}$$

is M-topology on M.

Example 3.10. Let $M \in [X]^m_\infty$ and N be a submset of M. Then the collection

$$\tau_N = \{V \subseteq M : C_N(x) \geq C_V(x) \text{ for all } x \in M^*\} \cup \{M\}$$

is M-topology on M.

Definition 3.11. Let (M, τ) be an M-topological space and N be a nonempty submset of M. Then the collection $\{V_i : i \in I\}$ of submsets of M is called cover of N if

$$C_{[N \cap (\cap_{i \in I}(M \ominus V_i))]}(x) = 0 \text{ for all } x \in M^*.$$

It should be noted that the cover is called open mset cover if its elements are open msets.

Definition 3.12. Let (M, τ) be an M-topological space. A nonempty submset N of M is called compact mset if every open mset cover of N reduced to finite subcover of N; that is, if $\{V_i: i \in I\}$ is a collection of open msets cover for N, then there exist $V_1, V_2, \ldots, V_n \in \tau$ such that

$$C_{[N \cap (\cap_{i=1}^n (M \ominus V_i))]}(x) = 0 \text{ for all } x \in M^*.$$

It may be noted that an M-topological space (M, τ) is called compact M-topological space if M is compact mset.

Example 3.13. Every finite M-topological space (M, τ) is compact M-topological space. Since the number of possible submsets of M are also finite. Consequently, (M, τ) is compact M-topological space.

Example 3.14. The indiscrete M-topological space (M, τ) is compact M-topological space. Since the only open mset cover of M is M, therefore (M, τ) is compact M-topological space.

Example 3.15. Let $M \in [X]^m_\infty$ and $\tau = P^*(M)$. Then (M, τ) is not compact M-topological space. Since the collection $\mathcal{A} = \{\{k/x\} : x \in^k M\}$ is an open mset cover of M, but can't select a finite subcover from $\mathcal{A}$ to cover M.

Theorem 3.16. *Let (M, τ) be a compact M-topological space and N be a closed submset of M. Then N is compact mset.*

Proof. Let $\{V_i: i \in I\}$ be a collection of open mset cover of N. It follows that $C_{[N\cap(\cap_{i\in I}(M\ominus V_i))]}(x) = 0$ for all $x \in M^*$. Also, $C_{[M\cap(N\cap(\cap_{i\in I}(M\ominus V_i)))]}(x) = 0$ for all $x \in M^*$. Therefore, $M\ominus N$ and $\{V_i: i \in I\}$ are open mset cover of M, and hence there exists finite subcover of open msets to cover M, say $M\ominus N, V_1, V_2, \ldots, V_n$ such that $C_{[M\cap(N\cap(\cap_{i=1}^n(M\ominus V_i)))]}(x) = 0$ for all $x \in M^*$. Since $C_M(x) \geq C_N(x)$ for all $x \in M^*$. Consequently, $C_{[N\cap(\cap_{i=1}^n(M\ominus V_i))]}(x) = 0$ for all $x \in M^*$. Hence N is compact mset. □

4. Multiset proximity relations

Definition 4.1. Let $M \in [X]^m_\infty$. A binary relation δ on $P^*(M)$ is called an mset proximity if it satisfies the following conditions:-

(MP_1) $N_1\ \delta\ N_2 \Rightarrow N_2\ \delta\ N_1$,
(MP_2) $N_1 \delta (N_2 \cup N_3) \Leftrightarrow N_1\ \delta\ N_2$ or $N_1\ \delta\ N_3$,
(MP_3) $N_1 \delta N_2 \Rightarrow C_{N_1}(x) > 0$ and $C_{N_2}(x) > 0$ for some $x \in M^*$,
(MP_4) $N_1 \not\delta N_2 \Rightarrow$ there exists $N_3 \subseteq M$ such that $N_1 \not\delta N_3$ and $(M \ominus N_3) \not\delta N_2$,
(MP_5) $C_{(N_1\cap N_2)}(x) > 0$ for some $x \in M^* \Rightarrow N_1\ \delta\ N_2$.

An mset proximity space (M, δ) consisting of an mset M and an mset proximity relation on M. We shall write $N_1\ \delta\ N_2$ if the submsets N_1 and N_2 of M are δ-related, otherwise we shall write $N_1 \not\delta N_2$. An mset proximity is abbreviated as an M-proximity.

An M-proximity relation δ on an mset M is said to be separated if it satisfies:
(MP_6) $(k/x)\delta(n/y) \Rightarrow k/x = n/y$.

Remark 4.2. An M-proximity relation δ on an mset M is a proximity relation if $C_M(x) = 0$ or $1\ \forall\ x \in M^*$.

Example 4.3. Let $M \in [X]^m_\infty$ and δ be a binary relation on $P^*(M)$ defined as

$N_1\ \delta\ N_2$ iff $C_{N_1}(x) > 0$ and $C_{N_2}(x) > 0$ for some $x \in M^*$.

Then δ is an M-proximity relation. It's clear that δ satisfies (MP_1). To prove that δ satisfies (MP_2) $\forall N_1, N_2, N_3 \in P^*(M)$, $N_1\delta(N_2 \cup N_3) \Leftrightarrow C_{N_1}(x) > 0$ and $C_{(N_2\cup N_3)}(x) > 0$ for some $x \in M^* \Leftrightarrow (C_{N_1}(x) > 0$ and $C_{N_2}(x) > 0)$ for some $x \in M^*$ or $(C_{N_1}(x) > 0$ and $C_{N_3}(x) > 0)$ for some $x \in M^* \Leftrightarrow N_1\ \delta\ N_2$ or $N_1\ \delta\ N_3$. (MP_3) is a direct consequence from the definition of δ. For (MP_4), let $N_1 \not\delta N_2$. Hence $C_{N_1}(x) = 0$ or $C_{N_2}(x) = 0$ for all $x \in M^*$. If $C_{N_1}(x) = 0$ for all $x \in M^*$, then $C_{(M\ominus N_1)}(x) > 0$ for all $x \in M^*$. By taking $N_3 = M \ominus N_1$. It follows that $N_1 \not\delta N_3$ and $(M \ominus N_3) \not\delta N_2$. Similarly, if $C_{N_2}(x) = 0$ for all $x \in M^*$. For (MP_5), Let $N_1, N_2 \in P^*(M)$ such that $C_{(N_1\cap N_2)}(x) > 0$ for some $x \in M^*$. If $C_{(N_1\cap N_2)}(x) = C_{N_1}(x) > 0$ for some $x \in M^*$, then $C_{N_2}(x) > 0$ for some $x \in M^*$. It follows that $N_1\ \delta\ N_2$. Similarly if $C_{(N_1\cap N_2)}(x) = C_{N_2}(x) > 0$ for some $x \in M^*$, and hence $C_{N_1}(x) > 0$ for some $x \in M^*$. Then the result.

Lemma 4.4. *If $N_1\ \delta\ N_2$ and $C_{N_2}(x) \leq C_{N_3}(x)$ for all $x \in M^*$, then $N_1\ \delta\ N_3$.*

Proof. Since $C_{N_2}(x) \leq C_{N_3}(x)$ for all $x \in M^*$, then $C_{(N_2\cup N_3)}(x) = C_{N_3}(x)$ for all $x \in M^*$. That is, $N_2 \cup N_3 = N_3$. This result, combined with (MP_2), implies $N_1\ \delta\ N_3$. □

Definition 4.5. An M-topological space (M, τ) is a T_1-mset space if every simple submset of M is closed mset.

Example 4.6. Let $M \in [X]^m_\infty$. Then $P^*(M)$ is T_1-mset space.

Example 4.7. Let $M \in [X]^m_\infty$. Then the cofinite M-topological space is T_1-mset space.

Theorem 4.8. *If (M, τ) is a T_1-mset space, then for any two simple submsets $\{k/x\}$, $\{n/y\}$ of M with $x \neq y$ there exist two open msets V and W with $C_V(x) \geq k$, $C_W(y) \geq n$, $C_V(y) = 0$, and $C_W(x) = 0$.*

Proof. Straightforward. □

The following example shows that the converse of Theorem 4.8 is not true in general.

Example 4.9. Let M be a nonempty mset. Then $PW(M) \cup \{\phi\}$ satisfies that for any two simple submsets $\{k/x\}$, $\{n/y\}$ of M with $x \neq y$ there exist two open msets V and W with $C_V(x) \geq k$, $C_W(y) \geq n$, $C_V(y) = 0$, and $C_W(x) = 0$, but it is not T_1-mset space.

Definition 4.10. An M-topological space (M, τ) is a normal mset space if whenever any two closed msets N_1 and N_2 such that $C_{(N_1\cap N_2)}(x) = 0$ for all $x \in M^*$, then there are two open msets V and W such that $C_{(V\cap W)}(x) = 0$, $C_{[N_1\cap(M\ominus V)]}(x) = 0$ and $C_{[N_2\cap(M\ominus W)]}(x) = 0$ for all $x \in M^*$.

Definition 4.11. An M-topological space (M, τ) is a T_4-mset space if it is normal and T_1-mset space.

Example 4.12. Let (M, τ) be a normal mset space and δ be a binary relation on $P^*(M)$ defined as:

$$\textit{for all } N_1, N_2 \subseteq M, N_1\delta N_2 \Longleftrightarrow C_{(\overline{N_1}\cap\overline{N_2})}(x) > 0 \text{ for some } x \in M^*. \tag{1}$$

Then δ is an M-proximity relation on M. It's clear that δ satisfies (MP_1). To prove that δ satisfies (MP_2) $\forall N_1, N_2, N_3 \in P^*(M)$ $N_1\delta(N_2 \cup N_3) \Leftrightarrow C_{[\overline{N_1}\cap(\overline{N_2\cup N_3})]}(x) > 0$ for some $x \in M^* \Leftrightarrow C_{[(\overline{N_1}\cap\overline{N_2})\cup(\overline{N_1}\cap\overline{N_3})]}(x) > 0$ for some $x \in M^* \Leftrightarrow C_{(\overline{N_1}\cap\overline{N_2})}(x) > 0$ for some $x \in M^*$ or $C_{(\overline{N_1}\cap\overline{N_3})}(x) > 0$ for some $x \in M^* \Leftrightarrow N_1\ \delta\ N_2$ or $N_1\ \delta\ N_3$. For (MP_3), let $N_1\ \delta\ N_2$. Then $C_{(\overline{N_1}\cap\overline{N_2})}(x) > 0$ for some $x \in M^*$. It follows that $C_{\overline{N_1}}(x) > 0$ for some $x \in M^*$ and $C_{\overline{N_2}}(x) > 0$ for some $x \in M^*$, which implies that $C_{N_1}(x) > 0$ and $C_{N_2}(x) > 0$ for some $x \in M^*$. For (MP_4), let $N_1 \not\delta N_2$. Hence $C_{(\overline{N_1}\cap\overline{N_2})}(x) = 0$ for all $x \in M^*$. It follows that there are two open msets V and W such that $C_{(V\cap W)}(x) = 0$, $C_{[\overline{N_1}\cap(M\ominus V)]}(x) = 0$ and $C_{[\overline{N_2}\cap(M\ominus W)]} = 0$ for all $x \in M^*$. Since $M\ominus V$ and $M\ominus W$ are closed msets, then $N_1 \not\delta (M \ominus V)$ and $(M \ominus W) \not\delta N_2$. This result, combined with $C_{(V\cap W)}(x) = 0$, Theorem 3.1 part (1) and Lemma 4.4, implies $N_1 \not\delta (M \ominus V)$ and $V \not\delta N_2$. For (MP_5), Let $N_1, N_2 \in P^*(M)$ such that $C_{(N_1\cap N_2)}(x) > 0$ for some $x \in M^*$. Since $C_{(\overline{N_1}\cap\overline{N_2})}(x) \geq C_{(N_1\cap N_2)}(x)$ for all $x \in M^*$. It follows that $C_{(\overline{N_1}\cap\overline{N_2})}(x) > 0$ for some $x \in M^*$. Hence $N_1\ \delta\ N_2$.

5. Multiset topology induced by a multiset proximity

Lemma 5.1. *Let N be a submset of an M-proximity space (M, δ). Then the δ-operator*

$$\delta : P^*(M) \longrightarrow P^*(M)$$

defined by

$$N^\delta = \{x \in^k M : (k/x)\delta N\} \tag{2}$$

satisfies the following property

$N_1 \not\delta N_2$ *implies* $C_{(N_2^\delta \cap N_1)}(x) = 0$ *for all* $x \in M^*$.

Proof. Straightforward. □

Theorem 5.2. *Let* (M, δ) *be an* M*-proximity space. Then the* δ*-operator, defined in (2), satisfies Kuratwski's axioms and induces an* M*-topology on* M *denoted by* τ_δ *and given by:*

$$\tau_\delta = \{N \subseteq M : (M \ominus N)^\delta = M \ominus N\}.$$

Proof.

(1) (MP_3) implies that $\phi^\delta = \phi$.
(2) Let $x \in {}^kN$. Hence (MP_5) implies that $(k/x)\ \delta\ N$. So $x \in {}^kN^\delta$, and hence $N \subseteq N^\delta$.
(3) By (MP_2), $x \in {}^k(N_1 \cup N_2)^\delta \iff (k/x)\delta(N_1 \cup N_2) \iff (k/x)\ \delta\ N_1$ or $(k/x)\delta N_2 \iff x \in^k N_1^\delta$ or $x \in^k N_2^\delta \iff x \in^k (N_1^\delta \cup N_2^\delta)$. Consequently, $(N_1 \cup N_2)^\delta = N_1^\delta \cup N_2^\delta$.
(4) To prove that $(N^\delta)^\delta \subseteq N^\delta$, let $(k/x) \notin N^\delta$. Then $(k/x) \not\delta N$, and hence (MP_4) implies that there exists $N_3 \in P^*(M)$ such that $(k/x) \not\delta N_3$ and $(M \ominus N_3) \not\delta N$. Lemma 5.1 and Theorem 3.1 part (1) imply that $N^\delta \subseteq N_3$. This result, combined with $(k/x) \not\delta N_3$ and Lemma 4.4, implies $(k/x) \not\delta N^\delta$. Then $(k/x) \notin (N^\delta)^\delta$. It follows that $(N^\delta)^\delta \subseteq N^\delta$. □

Lemma 5.3. *For any two submsets* N_1 *and* N_2 *of an* M*-proximity space* (M, δ),

$$N_1 \not\delta N_2 \text{ iff } Cl(N_1) \not\delta Cl(N_2),$$

where the closure is taken with respect to τ_δ.

Proof. Necessity is a trivial consequence of Lemma 4.4. To prove sufficiency, suppose $N_1 \not\delta N_2$. Then (MP_4) implies there exists $N_3 \in P^*(M)$ such that $N_1 \not\delta N_3$ and $(M \ominus N_3) \not\delta N_2$. This result, combined with Lemma 5.1 and Theorem 3.1 part (1), implies $N_2^\delta \subseteq N_3$; that is $Cl(N_2) \subseteq N_3$. It then follows from $N_1 \not\delta N_3$ and Lemma 4.4 that $N_1 \not\delta Cl(N_2)$. This result, combined with (MP_1), completes the proof of the theorem. □

Theorem 5.4. *Let* (M, τ) *be a normal mset space and* δ *is the formula (1). Then* $\tau_\delta \subseteq \tau$.

Proof. To prove the theorem, it suffices to show that $\overline{N} \subseteq Cl(N)$ *for all* $N \subseteq M$. Let $k/x \notin Cl(N)$. It follows that $(k/x) \not\delta N$. Hence formula (1) implies $\overline{\{k/x\}} \cap \overline{N} = \phi$. Hence $k/x \notin \overline{N}$. Then the result □

Definition 5.5. An M-topological space (M, τ) is compatible with an M-proximity relation δ on M, denoted by $\tau \sim \delta$, if $\tau = \tau_\delta$.

The following example shows that a normal M-topological space (M, τ) is not necessary to be compatible with τ_δ.

Example 5.6. Let $M = \{2/x, 3/y\}$, $\tau = \{M, \phi, \{1/x\}, \{2/x\}, \{1/y\}, \{3/y\}, \{1/x, 1/y\}, \{1/x, 3/y\}, \{2/x, 1/y\}\}$ is a normal M-topology, δ is the formula (1), and $N = \{1/y\}$. Then $\overline{N} = \{2/y\}$ and $Cl(N) = \{3/y\}$. So $\tau \neq \tau_\delta$.

The following example shows that a T_4-mset space (M, τ) is not necessary to be compatible with τ_δ.

Example 5.7. Let $M = \{2/x, 3/y\}$, $\tau = P^*(M)$ is a T_4-mset space, δ is the formula (1), and $N = \{1/x\}$. Then $\overline{N} = \{1/x\}$ and $Cl(N) = \{2/x\}$. So $\tau \neq \tau_\delta$.

Definition 5.8. The two M-topologies τ_1, τ_2 on M are called similar, denoted by $\tau_1 \cong \tau_2$, if for all $N \subseteq M$, $\overline{N}^1 \cong \overline{N}^2$.

Definition 5.9. An M-topological space (M, τ) is semi-compatible with an M-proximity relation δ on M, denoted by $\tau \simeq \delta$, if $\tau \cong \tau_\delta$.

Theorem 5.10. *Let* (M, τ) *be a* T_4*-mset space and* δ *is the formula (1). Then* $\tau \simeq \delta$.

Proof. To prove the theorem, it suffices to show that for all $N \subseteq M$, $\overline{N} \cong Cl(N)$; that is, $(\overline{N})^* = (Cl(N))^*$. Let $x \in (Cl(N))^*$. Hence $C_{Cl(N)}(x) > 0$. Suppose that $C_{Cl(N)}(x) = k$ such that $k > 0$, which implies that $(k/x)\ \delta\ N$. Hence formula (1) implies $C_{\overline{[\{k/x\}} \cap \overline{N}]}(x) > 0$ for some $x \in M^*$. Since (M, τ) be a T_1-mset space, then $C_{[\{k/x\} \cap \overline{N}]}(x) > 0$ for some $x \in M^*$. This result implies that $x \in_+ \overline{N}$. Hence $x \in (\overline{N})^*$. It follows that $(Cl(N))^* \subseteq (\overline{N})^*$. The other inclusion is a direct consequence of Theorem 5.4. □

Definition 5.11. A submset N_1 of an M-proximity space (M, δ) is called an mset δ-neighborhood of N_2, denoted by $N_2 \ll N_1$, if $N_2 \not\delta (M \ominus N_1)$.

Lemma 5.12. *Let* (M, δ) *be an* M*-proximity space and let* $Cl(N)$ *and* $Int(N)$ *denote, respectively, the closure and interior of* N *in* τ_δ. *Then*

(1) $N_1 \ll N_2$ *implies* $Cl(N_1) \ll N_2$,
(2) $N_1 \ll N_2$ *implies* $N_1 \ll Int(N_2)$.
Consequently $N_1 \subseteq Int(N_2)$, *showing that an mset* δ*-neighborhood is an* M*-topological neighborhood.*

Proof.

(1) Let $N_1 \ll N_2$, then $N_1 \not\delta (M \ominus N_2)$. Hence Lemma 5.3 implies $Cl(N_1) \not\delta (M \ominus N_2)$; that is, $Cl(N_1) \ll N_2$.
(2) $N_1 \not\delta (M \ominus N_2)$ implies $N_1 \not\delta Cl(M \ominus N_2)$. Equivalently, $N_1 \not\delta (M \ominus Int(N_2))$, i.e. $N_1 \ll Int(N_2)$. □

Theorem 5.13. *Let* N_1 *and* N_2 *be two submsets of an* M*-proximity space* (M, δ). *Then* (MP_4) *is equivalent to*

$$N_1 \not\delta N_2 \Rightarrow \exists N_3,\quad N_4 \subseteq M \text{ such that } N_1 \not\delta (M \ominus N_3), \quad (M \ominus N_4) \not\delta N_2 \text{ and } N_3 \not\delta N_4. \tag{3}$$

Proof. Let (MP_4) holds. Then $N_1 \not\delta N_2 \Rightarrow$ there exists an mset N_3 such that $N_1 \not\delta N_3$ and $(M \ominus N_3) \not\delta N_2$. Furthermore, there exists a submset N_4 such that $N_1 \not\delta (M \ominus N_4)$ and $N_4 \not\delta N_3$. On the other hand, let (3) holds. Then $N_3 \not\delta N_4$ and (MP_5) imply $C_{N_3 \cap N_4}(x) = 0$ for all $x \in M^*$. Hence Theorem 3.1 part (1) implies $N_3 \subseteq (M \ominus N_4)$. Setting $N = M \ominus N_3$, we have $N_1 \not\delta N$ and $(M \ominus N) \not\delta N_2$. □

Theorem 5.14. *Let* δ *be a semi-compatible* M*-proximity relation and* (M, τ) *be a* T_4*-mset space. If* N *compact mset,* V *is closed mset, and* $C_{(N \cap V)}(x) = 0$ *for all* $x \in M^*$, *then* $N \not\delta V$.

Proof. For all $x \in {}^kN$, $C_V(x) = 0$. Then $x \notin V^*$. Since V is closed mset and δ be a semi-compatible M-proximity relation on M. It follows that $x \notin V^* = (Cl(V))^*$; that is, $C_{Cl(V)}(x) = 0$, so $(k/x) \not\delta V$. This result implies there exists $H \subseteq M$ such that $(k/x) \not\delta (M \ominus H)$ and $H \not\delta V$; that is, $k/x \ll H$ and $H \ll (M \ominus V)$. This result, combined with (MP_5) and Theorem 3.1 part (1), implies $x \in {}^kM \ominus V$ which is open submset of M.

Lemma 5.12 (2) implies $k/x \ll Int(H) \subseteq H \ll (M \ominus V)$. Let $O_{k/x} = Int(H)$. Hence $O_{k/x} \not\delta V$. Now $\{O_{k/x}: x \in {}^k N\}$ is an open msets in τ_δ cover of N. Theorem 5.4 implies that $\{O_{k/x}: x \in {}^k N\}$ is an open msets in τ cover of N. Since N is compact mset. It follows that there is finite subcover $\{O_{k_1/x_1}, O_{k_2/x_2}, \ldots, O_{k_n/x_n}\}$ of N. Thus $C_{[N \cap (\cap_{i=1}^{n} (M \ominus O_{k_i/x_i}))]}(x) = 0$ for all $x \in M^*$. It follows that $C_N(x) \leq C_O(x)$ for all $x \in M^*$ such that $O = \cup_{i=1}^{n} O_{k_i/x_i}$. Moreover, (MP_2) implies $O \not\delta V$. Hence Lemma 4.4 implies $N \not\delta V$. Hence the result. □

Lemma 5.15. *Let (M, τ) be a normal mset space and δ is the formula (1). Then for any two submsets N_1 and N_2 of an M-proximity space (M, δ),*

$$N_1 \not\delta N_2 \; iff \; \overline{N_1} \not\delta \overline{N_2},$$

where the closure is taken with respect to τ.

Proof. Let $\overline{N_1} \not\delta \overline{N_2}$. Then Lemma 4.4 implies $N_1 \not\delta N_2$. On the other hand, let $N_1 \not\delta N_2$. It follows that there exists $N \subseteq M$ such that $N_1 \not\delta N$ and $(M \ominus N) \not\delta N_2$. Hence Lemma 5.1 implies $C_{[(M \ominus N) \cap Cl(N_2)]}(x) = 0$ for all $x \in M^*$. Then Theorem 3.1 part (1) implies $C_{Cl(N_2)}(x) \leq C_N(x)$ for all $x \in M^*$ and by Theorem 5.4, $C_{\overline{N_2}}(x) \leq C_{Cl(N_2)}(x) \leq C_N(x)$ for all $x \in M^*$. This result, combined with $N_1 \not\delta N$ and Lemma 4.4, implies $N_1 \not\delta \overline{N_2}$. Then it follows from (MP_1) that $\overline{N_1} \not\delta \overline{N_2}$. □

Theorem 5.16. *Every compact M-topological space which is T_4 has a unique M-proximity relation, defined in formula (1), satisfies $\tau_\delta \cong \tau$.*

Proof. Let γ be any semi-compatible M-proximity relation on M and $N_1 \, \delta \, N_2$; that is, $C_{(\overline{N_1} \cap \overline{N_2})}(x) > 0$ for some $x \in M^*$. Hence (MP_5) implies $\overline{N_1} \gamma \overline{N_2}$. Then Lemma 5.15 implies $N_1 \gamma N_2$. Hence $\delta \leq \gamma$. On the other hand, let $N_1 \not\delta N_2$. Hence $C_{(\overline{N_1} \cap \overline{N_2})}(x) = 0$ for all $x \in M^*$. Since $\overline{N_2}$ is closed submset of a compact mset M. Then Theorem 5.14 implies $\overline{N_1} \not\gamma \overline{N_2}$, and hence Lemma 5.15 implies $N_1 \not\gamma N_2$. So $\gamma \leq \delta$. Then δ is a unique M-proximity relation, formula (1), satisfies $\tau_\delta \cong \tau$. □

6. Conclusion

Proximity relations are helpful in solving problems based on human perception [7] that arise in areas such as image analysis [8] and face recognition [9]. In addition, best proximity point [10] is among the popular topic in the fixed point theory. Kandil et al. [11–13] introduced new approaches of proximity relations based on ideals and soft set notions. For further results and applications of proximity relations (See [14–16]).

A multiset is a collection of objects in which repetition of elements is essential. The main goal of this paper is to introduce new approach of proximity relations in the multiset context. Many properties of this new approach have been mentioned. In addition, a multiset topology induced by a multiset proximity relation on a multiset M is presented. Finally, many properties of this new multiset topology are studies.

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