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Decomposition spaces and restriction species

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Abstract. We show that Schmitt's restriction species (such as graphs, matroids, posets, etc.) naturally induce decomposition spaces (a.k.a. unital 2-Segal spaces), and that their associated coalgebras are an instance of the general construction of incidence coalgebras of decomposition spaces. We introduce *directed restriction species* that subsume Schmitt's restriction species and also induce decomposition spaces. Whereas ordinary restriction species are presheaves on the category of finite sets and injections, directed restriction species are presheaves on the category of finite posets and convex maps. We also introduce the notion of *monoidal (directed) restriction species*, which induce monoidal decomposition spaces and hence bialgebras, most often Hopf algebras. Examples of this notion include rooted forests, directed graphs, posets, double posets, and many related structures. A prominent instance of a resulting incidence bialgebra is the Butcher–Connes–Kreimer Hopf algebra of rooted trees. Both ordinary and directed restriction species are shown to be examples of a construction of decomposition spaces from certain cocartesian fibrations over the category of finite ordinals that are also cartesian over convex maps. The proofs rely on some beautiful simplicial combinatorics, where the notion of convexity plays a key role. The methods developed are of independent interest as techniques for constructing decomposition spaces.

Contents

0. Introduction 2
1. Decomposition spaces 4
2. Two motivating examples and two basic examples 8
3. Simplicial preliminaries 9
4. The decomposition space $\mathbf{I}$ of layered finite sets 12
5. Restriction species 14
6. The decomposition space $\mathbf{C}$ of layered finite posets 21
7. Directed restriction species 23
8. Convex correspondences and 'nabla spaces' 28
9. Sesquicartesian fibrations 30
10. From restriction species to iesq-sesqui 33
11. Decalage and fully faithfulness 38
12. Remarks on strictness 41
References 42

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0. Introduction

The notion of decomposition space was introduced in [**19**] as a very general framework for incidence (co)algebras and Möbius inversion. Let us briefly recount the abstraction steps that led to this notion, taking as starting point the classical theory of incidence algebras of locally finite posets. More extensive introductions can be found in [**19**] and in [**22**]. A very different motivation and formulation of the notion is due to Dyckerhoff and Kapranov [**11**].

The first step is the observation, due to Leroux [**39**], that the Möbius inversion principle for incidence algebras of both locally finite posets (Rota et. al [**28, 48**]) and monoids with the finite decomposition property (Cartier–Foata [**6**]) has a common generalisation to the notion of Möbius category, and that this setting allows for good functorial properties.

The next step is to observe that in many examples where symmetries play a role, a more elegant treatment can be achieved by considering groupoid-enriched categories instead of plain (set-enriched) categories, as illustrated in [**16**]. This involves a homotopical viewpoint, in which the algebraic identities arise as homotopy cardinality of equivalences of groupoids, rather than just ordinary cardinality of bijections of sets. At the same time it becomes clear that the algebraic structures can actually be defined and manipulated at the objective level, postponing the act of taking cardinality, and that structural phenomena can be seen at this level which are not visible at the usual 'numerical' level. For example, at this level of abstraction one can view the algebra of species under the Cauchy tensor product as the incidence algebra of the symmetric monoidal category of finite sets and bijections [**22**]. (The homotopy viewpoint induces one to consider even ∞-groupoids [**18, 19**], but this is not important in the present contribution.)

Finally, considering groupoid-enriched categories as simplicial groupoids via the nerve construction led to the discovery [**19**] that the Segal condition, which essentially characterises category objects among simplicial groupoids, is not actually needed, and that a weaker notion suffices for the theory of incidence (co)algebras and Möbius inversion: this is the notion of *decomposition space*, which can be seen as the systematic theory of decompositions, where categories are the systematic theory of compositions.

While many coalgebras and bialgebras in combinatorics do arise from (groupoid-enriched) categories, there are also many examples that can easily be seen *not* to arise from such categories. Two prominent examples are the Schmitt Hopf algebra of graphs [**50**] (also called the chromatic Hopf algebra [**1**]), and the Butcher–Connes–Kreimer Hopf algebra of rooted trees (see [**9**] and [**7**]). These two examples are reviewed below, where we shall see that they cannot possibly arise directly from categories, but that they do naturally come from decomposition spaces, cf. [**19, 20, 22**]. (They can be obtained indirectly from certain auxiliary categories, by means of a reduction step, cf. Dür [**9**], whose construction we subsume as a decalage, in Example 7.16.)

The aim of the present paper is to fit these two examples into a large class of decomposition spaces. One may say there are two large classes of decomposition spaces, but the first can be regarded as a special case of the second. The first is the class of decomposition spaces coming from Schmitt's restriction species [**49**]—Schmitt already showed that the Hopf algebra of graphs comes from a restriction species. While restriction species are presheaves on the category of finite sets and injections, expressing the ability to decompose combinatorial structures, the new notion of directed restriction species expresses decompositions compatible with an underlying partial order:

Definition. A *directed restriction species* is a presheaf on the category of finite posets and convex maps.

Ordinary restriction species can be regarded as directed restriction species supported on discrete posets.

We show that every directed restriction species defines a decomposition space, and hence a coalgebra. Instead of constructing these simplicial objects by hand, we found it worth taking a slight detour through some more abstract constructions. On one hand, this serves to exhibit the general principles behind the results, and on the other to develop machinery of independent interest for the sake of constructing decomposition spaces. We route the construction through certain sesquicartesian fibrations over $\underline{\mathbb{A}}$ (the category of finite ordinals, including the empty ordinal): they are cocartesian fibrations which are furthermore cartesian over convex maps, satisfying Beck–Chevalley, and subject to one further condition which we refer to as the *iesq* (for '*identity-extension-square*') condition.

The main results can now be organised as follows:

Theorem. (Proposition 10.6 and Corollary 10.8.) *Restriction species and directed restriction species naturally induce iesq sesquicartesian fibrations.*

Theorem 9.7. *Iesq sesquicartesian fibrations naturally induce decomposition spaces.*

Together, and more precisely:

Theorem. (Theorems 11.4 and 11.5.) *There is a functor from restriction species to decomposition spaces CULF over* $\mathbf{I}$*, and this functor is fully faithful. Similarly there is a functor from directed restriction species to decomposition spaces CULF over* $\mathbf{C}$*, also fully faithful.*

Here $\mathbf{I}$ is a certain decomposition space of layered finite sets (§4), and $\mathbf{C}$ is a certain decomposition space of layered finite posets (§6). For CULF functors, see 1.9 below.

Many combinatorial structures which form (directed) restriction species are closed under taking disjoint union in a way compatible with restrictions. We capture this through the notion of monoidal directed restriction species (7.8), and show:

Proposition 7.9. *Monoidal directed restriction species naturally induce monoidal decomposition spaces and hence bialgebras.*

Examples of this notion include rooted forests, directed graphs, posets, double posets, and many related structures. A prominent instance of a resulting incidence bialgebra is the Butcher–Connes–Kreimer Hopf algebra of rooted trees.

Note. This paper was originally posted as Section 6 of the long manuscript *Decomposition spaces, incidence algebras and Möbius inversion* [**17**], which has now been split into six papers, the first five being [**18, 19, 20, 21, 22**]. The relevant definitions and results from these papers (mostly [**19**]) are reviewed below as needed, to render the paper reasonably self-contained.

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1. Decomposition spaces

In this section we briefly recall and motivate the notion of decomposition space.

1.1. Incidence coalgebras of locally finite categories. Recall from Leroux [**39**] that the incidence coalgebra of a locally finite category $\mathscr{C}$ has underlying vector space spanned by the arrows of $\mathscr{C}$ and comultiplication given by

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b. \tag{1}$$

Local finiteness ensures the sum is finite, and coassociativity follows from associativity of composition of arrows. If the category is just a poset or a monoid, this is the classical notion of incidence coalgebra of Rota et al. [**28, 48**] or of Cartier and Foata [**6**] respectively.

1.2. Groupoids, fat nerves, and homotopy viewpoints. In practice one is often interested in combinatorial objects up to isomorphism, while keeping track of automorphisms. This can be accomplished elegantly by replacing $\mathscr{C}$ with its fat nerve, the simplicial groupoid $\mathsf{X} = \mathbf{N}\mathscr{C} : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$ defined formally by

$$\mathsf{X}_n = \mathrm{Map}([n], \mathscr{C}),$$

the groupoid whose objects are functors $[n] \to \mathscr{C}$ (i.e. sequences of n arrows) and whose morphisms are invertible natural transformations between them. Hence X_0 is the groupoid of all objects, X_1 is the groupoid of all arrows, and X_2 is the groupoid of composable pairs of arrows. An up-to-isomorphism-but-keeping-track-of-automorphisms version of (1) is encoded by the canonical span of groupoids

$$\mathsf{X}_1 \xleftarrow{d_1} \mathsf{X}_2 \xrightarrow{(d_2,d_0)} \mathsf{X}_1 \times \mathsf{X}_1.$$

The fibre of d_1 over a given arrow $f \in \mathsf{X}_1$ is the groupoid of composable pairs with composite f, and (d_2, d_0) then returns the two component arrows (a, b). For the best interpretation of this, one works homotopically as long as possible, in the slice category $\boldsymbol{Grpd}_{/\mathsf{X}_1}$, before taking homotopy cardinality to arrive at the vector space $\mathbb{Q}_{\pi_0 \mathsf{X}_1}$. Thus all notions must be homotopy notions, invariant under equivalences of groupoids, and hence well-behaved when taking homotopy cardinality (cf. 1.8 below).

> *Throughout, when we say pullback (resp. fibre), we refer to the homotopy pullback (resp. homotopy fibre).*

Strict pullbacks are *not* in general homotopy invariant, except if one of the maps pulled back along is an iso-fibration; this will be exploited occasionally. Similarly when we talk about simplicial groupoids we may allow pseudo-functors $\mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$, not just strict functors, since this is the homotopy invariant notion. Most of the simplicial groupoids of the present paper can be arranged to be strict, though (cf. §12).

The comultiplication formula resulting from the span construction now concerns isoclasses of arrows, and the sum is over isoclasses of factorisations. In practice this is precisely what one wants. For example, if $\mathscr{C}$ is the category of finite sets and surjections, the incidence coalgebra resulting from the fat nerve is the Faà di Bruno coalgebra [**29**].

1.3. From Segal spaces to decomposition spaces. Fat nerves of categories can be characterised (in part) by the Segal condition, which is the pullback condition $\mathsf{X}_{p+q} \simeq$

$\mathsf{X}_p \times_{\mathsf{X}_0} \mathsf{X}_q$. The first instance is the pullback square

$$\begin{array}{ccc} \mathsf{X}_2 & \xrightarrow{d_0} & \mathsf{X}_1 \\ {\scriptstyle d_2}\downarrow & \lrcorner & \downarrow{\scriptstyle d_1} \\ \mathsf{X}_1 & \xrightarrow[d_0]{} & \mathsf{X}_0 \end{array} \tag{2}$$

which says that X_2 can be identified with the groupoid $\mathsf{X}_1 \times_{\mathsf{X}_0} \mathsf{X}_1$ of composable pairs of arrows. The Segal condition thus expresses the ability to compose.

The decomposition-space axiom, which is weaker, stipulates that certain other squares are (homotopy) pullbacks, the most important cases being

$$\begin{array}{ccc} \mathsf{X}_3 & \xrightarrow{d_2} & \mathsf{X}_2 \\ {\scriptstyle d_0}\downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ \mathsf{X}_2 & \xrightarrow[d_1]{} & \mathsf{X}_1 \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{X}_3 & \xrightarrow{d_1} & \mathsf{X}_2 \\ {\scriptstyle d_3}\downarrow & \lrcorner & \downarrow{\scriptstyle d_2} \\ \mathsf{X}_2 & \xrightarrow[d_1]{} & \mathsf{X}_1. \end{array}$$

This axiom can be interpreted as the expression of the ability to *decompose* (cf. [**19**]).

To define more formally what a decomposition space is—and to construct them—we need the notions of active and inert maps:

1.4. Active and inert maps (generic and free maps). The category $\mathbb{\Delta}$ of nonempty finite ordinals $[n] = \{0, 1, \ldots, n\}$ and monotone maps has a so-called active-inert factorisation system. An arrow $a : [m] \to [n]$ in $\mathbb{\Delta}$ is *active* (also called *generic*) when it preserves end-points, $a(0) = 0$ and $a(m) = n$; we use the special arrow symbol $\twoheadrightarrow\!\!|$ to denote active maps. An arrow $a : [m] \to [n]$ in $\mathbb{\Delta}$ is *inert* (also called *free*) if it is distance preserving, $a(i+1) = a(i) + 1$ for $0 \leq i \leq m-1$; we use the special arrow symbol $\rightarrowtail$. The active maps are generated by the codegeneracy maps $s^i : [n{+}1] \to [n]$ and by the *inner* coface maps $d^i : [n{-}1] \to [n]$, $0 < i < n$, while the inert maps are generated by the *outer* coface maps $d^\perp := d^0$ and $d^\top := d^n$. Every morphism in $\mathbb{\Delta}$ factors uniquely as an active map followed by an inert map. Furthermore, it is a basic fact [**19**] that active and inert maps in $\mathbb{\Delta}$ admit pushouts along each other, and the resulting maps are again active and inert. (The notions of generic and free maps are general categorical notions, important in monad theory [**54, 55**]. We have adopted the more recent 'active/inert' terminology, due to Lurie [**40**], which seems more suggestive of the role the two classes of maps play.)

1.5. Decomposition spaces [19]. A simplicial groupoid $\mathsf{X} : \mathbb{\Delta}^{\text{op}} \to \boldsymbol{Grpd}$ is called a *decomposition space* when it takes active-inert pushouts in $\mathbb{\Delta}$ to pullbacks. The notion of decomposition space is equivalent to the unital 2-Segal spaces of Dyckerhoff and Kapranov [**11**], formulated in terms of triangulations of polygons. Their work shows that the notion is of interest well beyond combinatorics.

The fat nerve of a category is always a decomposition space [**19**, Proposition 3.7].

The following result is the main motivation for the notion of decomposition space.

Theorem 1.6. [**19**] *If* $\mathsf{X} : \mathbb{\Delta}^{\text{op}} \to \boldsymbol{Grpd}$ *is a decomposition space, the slice category* $\boldsymbol{Grpd}_{/\mathsf{X}_1}$ *acquires the structure of a coassociative and counital coalgebra (up to coherent equivalence), with comultiplication and counit*

$$\Delta : \boldsymbol{Grpd}_{/\mathsf{X}_1} \to \boldsymbol{Grpd}_{/\mathsf{X}_1} \otimes \boldsymbol{Grpd}_{/\mathsf{X}_1}, \qquad\qquad \varepsilon : \boldsymbol{Grpd}_{/\mathsf{X}_1} \to \boldsymbol{Grpd}$$

defined by the spans

$$\mathsf{X}_1 \xleftarrow{d_1} \mathsf{X}_2 \xrightarrow{(d_2,d_0)} \mathsf{X}_1 \times \mathsf{X}_1, \qquad\qquad \mathsf{X}_1 \xleftarrow{s_0} \mathsf{X}_0 \longrightarrow 1.$$

Upon taking homotopy cardinality (in suitably finite situations, cf. 1.7 below), this yields a coalgebra in the classical sense.

1.7. Finiteness conditions (cf. [20]). Various finiteness conditions are important for various reasons. They tend to be satisfied in examples coming from combinatorics, and we shall establish them for all restriction species and directed restriction species.

In order to be able to take homotopy cardinality to get a coalgebra in vector spaces, it is necessary to assume that X is *locally finite* (cf. [**20**, §7]). This means first of all that X_1 is a locally finite groupoid (i.e. has finite automorphism groups), and second that each active map is a finite map (i.e. has finite fibres). For a decomposition space X, this can be measured on the two maps

$$\mathsf{X}_0 \xrightarrow{s_0} \mathsf{X}_1 \xleftarrow{d_1} \mathsf{X}_2.$$

For the comultiplication formula to be free of denominators, another condition is required, namely that X must be *locally discrete* (cf. [**22**, §1]), which for a decomposition space amounts to the two displayed maps having discrete fibres.

In order to have a Möbius inversion formula, yet another finiteness condition is needed, which refers to a notion of non-degeneracy which is meaningful for *complete* decomposition spaces (cf. [**20**, §2]), i.e. those for which s_0 is mono. The condition is to have *locally finite length*, and it means (cf. [**20**, §6]) that for each $a \in \mathsf{X}_1$ there is an upper bound on the n for which the map $\mathsf{X}_n \to \mathsf{X}_1$ has non-degenerate elements in the fibre. See op cit. for precision—the upshot is that there are only finitely many ways of splitting an object into non-degenerate pieces.

1.8. Homotopy cardinality. Assuming local finiteness, the groupoid-level incidence coalgebra yields a vector-space level coalgebra by taking homotopy cardinality. We refer to [**18**] for the full story (in the setting of ∞-groupoids) and to [**22**] for some introduction geared towards combinatorics. Very briefly, the homotopy cardinality of a groupoid X is defined to be $|X| := \sum_{x\in\pi_0 X} \frac{1}{|\mathrm{Aut}(x)|}$. The groupoid slice $\boldsymbol{Grpd}_{/B}$ is the objective counterpart of the vector space $\mathbb{Q}_{\pi_0 B}$ spanned by the symbols δ_b denoting isoclasses of objects in B. The homotopy cardinality of an object $X \to B$ is then the formal linear combination $\sum_{b\in\pi_0 B} \frac{|X_b|}{|\mathrm{Aut}(b)|}\,\delta_b$, where $|X_b|$ is the homotopy cardinality of the (homotopy) fibre X_b.

If the groupoids involved are just sets, the automorphism groups are trivial, and the notion reduces to ordinary cardinality. Building the automorphism groups into the definition ensures it behaves well with respect to all the important operations, such as products and sums, (homotopy) pullbacks and (homotopy) fibres, etc.

1.9. CULF functors. For the present purposes, the relevant notion of morphism between decomposition spaces is that of CULF functor [**19**]: CULF functors between decomposition spaces induce coalgebra homomorphisms. A simplicial map is called *ULF* (unique lifting of factorisations) if it is cartesian on active coface maps, and it is called *conservative* if cartesian on codegeneracy maps. We say *CULF* for conservative and ULF, that is, cartesian on all active maps.

Since CULFness is defined in terms of pullbacks, the following useful lemma is immediate from the analogous property of pullbacks.

Lemma 1.10. *Given simplicial maps* $\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y} \xrightarrow{\mathsf{g}} \mathsf{Z}$*, if* g *and* $\mathsf{g} \circ \mathsf{f}$ *are CULF then so is* f.

Since CULFness refers to active maps, just as the finiteness conditions in 1.7, we have the following useful result.

Lemma 1.11. *Let P denote a property of decomposition spaces which is measured on active maps (such as being complete, locally discrete, or of locally finite length). Then if* $\mathsf{f} : \mathsf{Y} \to \mathsf{X}$ *is CULF and* X *has property P, then also* Y *has property P. This is also the case for the property of being locally finite, except we must check additionally that* Y_1 *is locally finite.*

In fact, also:

Lemma 1.12. *A simplicial groupoid CULF over a decomposition space is itself a decomposition space.*

1.13. Monoidal decomposition spaces and bialgebras. There is a natural notion of *monoidal decomposition space* [**19**], leading to bialgebras. Briefly, it is a decomposition space X equipped with a functor $\otimes : \mathsf{X} \times \mathsf{X} \to \mathsf{X}$ required to be a monoidal structure and to be CULF. The homotopy cardinality of this monoidal structure is an algebra structure, and the CULF condition ensures the compatibility with the coalgebra structure to result altogether in a bialgebra. This is important in most applications to combinatorics, where almost always this monoidal structure, and hence the algebra structure, is given by disjoint union. In the present contribution we focus mostly on the comultiplication, but comment on monoidal structure in 5.14–5.15 and 7.8–7.9.

1.14. Remark. There is another strategy for getting bialgebras from decomposition spaces, namely simply to define the multiplication by reading the comultiplication span backwards. It is rarely the case that this gives a bialgebra directly (in general it gives only a *lax* bialgebra [**47**]), but sometimes the discrepancy amounts to a symmetry factor, which can be absorbed into the multiplication (Green's theorem) to get an honest bialgebra, as explained in detail by Dyckerhoff [**10**]. Such 'twisted' bialgebra structures are typical for linear contexts (representation theory), where there is no notion of disjoint union (as in combinatorics).

1.15. Decalage. (See [**27**].) Given a simplicial groupoid X the *lower Dec*, $\mathrm{Dec}_\perp\mathsf{X}$, is a new simplicial groupoid obtained by deleting X_0 and shifting everything one place down, deleting also all d_0 face maps and all s_0 degeneracy maps. It comes equipped with a simplicial map, called the *dec map*, $d_\perp : \mathrm{Dec}_\perp\mathsf{X} \to \mathsf{X}$ given by the original d_0.

In the present contribution, we shall exploit decalage to relate the fat nerve of the Grothendieck construction of a restriction species with its associated decomposition space (Proposition 11.1 and Corollary 11.3).

1.16. Right fibrations and left fibrations. (See [**19**].) A simplicial map $\mathsf{f} : \mathsf{Y} \to \mathsf{X}$ is called a *right fibration* if it is cartesian on all bottom face maps $d_\perp$. This implies that it is also cartesian on all active maps (i.e. is CULF). Similarly, f is called a *left fibration* if it is cartesian on $d_\top$ (and consequently on all active maps also).

Lemma 1.17. *If* $\mathsf{f} : \mathsf{Y} \to \mathsf{X}$ *is a CULF functor between decomposition spaces, then* $\mathrm{Dec}_\perp(\mathsf{f}) : \mathrm{Dec}_\perp\mathsf{Y} \to \mathrm{Dec}_\perp\mathsf{X}$ *is a right fibration of Segal spaces. Similarly,* $\mathrm{Dec}_\top(\mathsf{f}) : \mathrm{Dec}_\top\mathsf{Y} \to \mathrm{Dec}_\top\mathsf{X}$ *is a left fibration.*

2. Two motivating examples and two basic examples

While many important examples of coalgebras in combinatorics come from decomposition spaces which are just (fat nerves of) categories, there are also many examples which do not (directly) come from a category. (Sometimes, a construction can be made, involving a reduction procedure [**9**].)

In this section we first explain the two examples that triggered the present investigations, and then explain the most basic example from the two families they belong to. The first example, Schmitt's Hopf algebra of graphs, is an example of a restriction species. The terminal restriction species is that of finite sets. The second example, the Butcher–Connes–Kreimer Hopf algebra, is an example of a new notion we introduce, directed restriction species, and the terminal such is the example of finite posets.

2.1. The chromatic Hopf algebra (of graphs). The following Hopf algebra of graphs was first studied by Schmitt [**50**]; see also [**1**] and [**26**]. For a graph G with vertex set V, and a subset $U \subset V$, define $G|U$ to be the graph whose vertex set is U, and whose graph structure is induced by restriction (that is, the edges of $G|U$ are those edges of G both of whose incident vertices belong to U). On the vector space spanned by isoclasses of graphs, define a comultiplication by the rule

$$\Delta(G) = \sum_{A+B=V} G|A \otimes G|B.$$

This coalgebra is the cardinality of the coalgebra of a *decomposition space* but not directly of a *category*. Indeed, define a simplicial groupoid $\mathsf{G} : \mathbb{\Delta}^{\text{op}} \to \boldsymbol{Grpd}$ with G_1 the groupoid of graphs, and more generally G_k the groupoid of graphs with an ordered partition of the vertex set V into k parts (possibly empty), i.e. a function $V \to \underline{k}$—this is what we shall call a *layering* of the graph (4.1). In particular, G_0 is the contractible groupoid consisting only of the empty graph. The outer face maps delete the first or last layer, and the inner face maps join adjacent layers. The degeneracy maps insert an empty layer. It is clear that this is not a Segal space (as Square (2) is not a pullback): a graph structure on a given 2-layered set cannot be reconstructed from knowledge of the graph structure of the two layers individually, since this gives no information about the edges connecting the layers. One can easily check that it *is* a decomposition space, hence induces a coalgebra. The operation of taking disjoint union makes this a bialgebra, in fact a Hopf algebra, which is precisely Schmitt's chromatic Hopf algebra. (A picture illustrating the decomposition-space axiom in this case can be found in [**20**].)

2.2. Butcher–Connes–Kreimer Hopf algebra. The Butcher–Connes–Kreimer Hopf algebra of rooted trees [**7**] is the free algebra on the set of isoclasses of rooted trees, with comultiplication defined by summing over certain admissible cuts c:

$$\Delta(T) = \sum_{c \in \text{adm.cuts}(T)} P_c \otimes R_c.$$

An admissible cut c is a splitting of the set of nodes into two subsets, such that the second forms a subtree R_c containing the root node (or is the empty forest); the first subset, the complement 'crown', then forms a subforest P_c, regarded as a monomial of trees. Note that compared to the arbitrary splitting allowed in Schmitt's Hopf algebra of graphs, the admissible cuts are thus required to be compatible with the partial order underlying trees and forests.

We can obtain the Butcher–Connes–Kreimer coalgebra from a decomposition space as follows (cf. also [**19**]): let H_1 denote the groupoid of forests, and let H_2 denote the groupoid of forests with an admissible cut. More generally, H_0 is defined to be a point, and H_k is the groupoid of forests with $k-1$ compatible admissible cuts. These form a simplicial groupoid H in which the inner face maps forget a cut, and the outer face maps project away either the crown or the bottom layer (the part of the forest below the bottom cut). It is clear that H is not a Segal space: a tree with a cut cannot be reconstructed from its crown and its bottom tree, which is to say that H_2 is not equivalent to $\mathsf{H}_1 \times_{\mathsf{H}_0} \mathsf{H}_1$. It is straightforward to check that it *is* a decomposition space, and that its incidence coalgebra is precisely the Butcher–Connes–Kreimer coalgebra. As in the graph example, disjoint union makes this coalgebra into a bialgebra, in fact a Hopf algebra. For comparison with the construction of Dür [**9**], see 7.16 below.

2.3. Getting decomposition spaces from restriction species and directed restriction species. The graph example is just one in a large family of coalgebras (and bialgebras) constructed by Schmitt [**49**], namely coalgebras induced by restriction species (see also [**2**]). We shall show, first of all, that restriction species in the sense of Schmitt [**49**] are examples of decomposition spaces, and that they and their associated coalgebras exemplify the general construction (Proposition 5.13). The example with trees does not come from a restriction species, but we introduce the notion of *directed restriction species* (§7), which covers this examples and many others, and which also define decomposition spaces.

The next two examples are the basic ones.

2.4. The binomial Hopf algebra. Define a comultiplication on the vector space spanned by isoclasses of finite sets by

$$\Delta(A) = \sum_{A_1+A_2=A} A_1 \otimes A_2.$$

Here the sum is over all pairs of subsets of A whose union is A and whose intersection is empty. We shall realise this from a decomposition space I of layered finite sets (§4).

2.5. The Hopf algebra of finite posets. Define a comultiplication on the vector space spanned by isoclasses of finite posets by

$$\Delta(P) = \sum_{c \in \mathrm{cuts}(P)} D_c \otimes U_c.$$

Here the sum is over all admissible cuts of P; an *admissible cut* $c = (D_c, U_c)$ is by definition a way of writing P as the set-theoretic disjoint union of a lower-set D_c and an upper-set U_c. This coalgebra was studied by Aguiar–Bergeron–Sottile [**1**], who trace its origins back to Gessel [**24**]. See also Figueroa–Gracia-Bondía [**12**]. We shall realise this from a decomposition space $\mathbf{C}$ of layered finite posets (§6).

3. Simplicial preliminaries

A key ingredient in our constructions is the beautiful interplay between the topologist's Delta and the algebraist's Delta. After setting up the notation, we establish a certain correspondence between squares in the two categories.

3.1. 'Topologist's Delta'. The category $\mathbb{\Delta}$ is the skeleton of the category of non-empty finite ordered sets and monotone maps.
Notation: its objects are

$$[n] := \{0, 1, \dots, n\}, \qquad n \geq 0.$$

The monotone maps are generated by

- $s^k : [n{+}1] \to [n]$ that repeats the element $k \in [n]$,
- $d^k : [n] \to [n{+}1]$ that skips the element $k \in [n{+}1]$.

Note that $[0]$ is terminal. We denote by $\mathbb{\Delta}_{\text{act}}$ and $\mathbb{\Delta}_{\text{inert}}$ the subcategories with all the objects and the active or inert maps respectively.

3.2. 'Algebraist's Delta'. The category $\underline{\mathbb{\Delta}}$ is the skeleton of the category of finite ordered sets (including the empty set) and monotone maps.
Notation: its objects are

$$\underline{n} := \{1, \dots, n\}, \qquad n \geq 0.$$

The monotone maps are generated by

- $\underline{s}^k : \underline{n{+}1} \to \underline{n}$ that repeats the element $k+1 \in \underline{n}$, $(0 \leq k \leq n-1)$,
- $\underline{d}^k : \underline{n} \to \underline{n{+}1}$ that skips the element $k+1 \in \underline{n{+}1}$, $(0 \leq k \leq n)$.

Note that $\underline{1}$ is terminal, $\underline{0}$ is initial, and the only map with target $\underline{0}$ is the identity.

There is a full inclusion $\mathbb{\Delta} \to \underline{\mathbb{\Delta}}$ which on objects sends $[n] = \{0, \dots, n\}$ to $\underline{n{+}1} = \{1, \dots, n+1\}$. On maps it just does nothing, up to the canonical relabelling of the elements, $[n] \cong \underline{n{+}1}$. Thus it sends d^k to $\underline{d}^k$ and s^k to $\underline{s}^k$.

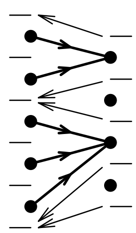
More important is the following duality, which is standard [**30**].

Lemma 3.3. *There is a canonical isomorphism of categories*

$$\mathbb{\Delta}_{\text{act}}^{\text{op}} \cong \underline{\mathbb{\Delta}},$$

- $\underline{n}$ corresponds to $[n]$,
- $\underline{d}^k : \underline{n} \to \underline{n{+}1}$ corresponds to $s^k : [n{+}1] \to [n]$,
- $\underline{s}^k : \underline{n{+}1} \to \underline{n}$ corresponds to the inner coface map $d^{k+1} : [n] \to [n{+}1]$.

The following graphical representation may be helpful. In $\underline{\mathbb{\Delta}}$, draw the elements in $\underline{n}$ as n dots, and in $\mathbb{\Delta}_{\text{act}}$ draw the elements in $[n]$ as $n+1$ walls. A map operates as a function on the set of dots when considered a map in $\underline{\mathbb{\Delta}}$ while it operates as a function on the walls when considered a map in $\mathbb{\Delta}_{\text{act}}$. Here is a picture of a certain map $\underline{5} \to \underline{4}$ in $\underline{\mathbb{\Delta}}$ and of the corresponding map $[5] \leftarrow [4]$ in $\mathbb{\Delta}_{\text{act}}$.



3.4. Ordinal sum. The ordinal sum monoidal structure $(\underline{\mathbb{\Delta}}, +, \underline{0})$ gives a monoidal structure $(\mathbb{\Delta}_{\text{act}}, \vee, [0])$, via Lemma 3.3. The inert maps $[n] \rightarrowtail [n']$ in $\mathbb{\Delta}$ may be expressed uniquely as $[n] \rightarrowtail [a] \vee [n] \vee [b]$.

3.5. Pullbacks in $\underline{\Delta}$. We shall need the following lemmas, whose proofs are straightforward.

Lemma 3.6. *For each $0 \leq k \leq n$, the following square is a pullback in $\underline{\Delta}$:*

$$\begin{array}{ccccc} \underline{n} & \xrightarrow{\underline{d}^k} & \underline{n+1} & \xrightarrow{\underline{d}^k} & \underline{n+2} \\ {\scriptstyle =}\big\downarrow & \lrcorner & & & \big\downarrow{\scriptstyle \underline{s}^k} \\ \underline{n} & & \xrightarrow[\underline{d}^k]{\qquad\qquad} & & \underline{n+1}. \end{array}$$

Lemma 3.7. *For each $0 \leq k \leq n$, the following square is a pullback in $\underline{\Delta}$:*

$$\begin{array}{ccc} \underline{n} & \xrightarrow{=} & \underline{n} \\ {\scriptstyle =}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \underline{d}^k} \\ \underline{n} & \xrightarrow[\underline{d}^k]{} & \underline{n+1}. \end{array}$$

Lemma 3.8. *For $0 < k < n$ and all j the following squares are pullbacks*

$$\begin{array}{ccc} \underline{n} & \xrightarrow{\underline{d}^\top} & \underline{n+1} \\ {\scriptstyle \underline{d}^k}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \underline{d}^k} \\ \underline{n+1} & \xrightarrow[\underline{d}^\top]{} & \underline{n+2} \end{array} \qquad \begin{array}{ccc} \underline{n} & \xrightarrow{\underline{d}^\top} & \underline{n+1} \\ {\scriptstyle \underline{s}^j}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \underline{s}^j} \\ \underline{n-1} & \xrightarrow[\underline{d}^\top]{} & \underline{n} \end{array} \qquad \begin{array}{ccc} \underline{n} & \xrightarrow{\underline{d}^\bot} & \underline{n+1} \\ {\scriptstyle \underline{d}^k}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \underline{d}^{k+1}} \\ \underline{n+1} & \xrightarrow[\underline{d}^\bot]{} & \underline{n+2} \end{array} \qquad \begin{array}{ccc} \underline{n} & \xrightarrow{\underline{d}^\bot} & \underline{n+1} \\ {\scriptstyle \underline{s}^j}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \underline{s}^{j+1}} \\ \underline{n-1} & \xrightarrow[\underline{d}^\bot]{} & \underline{n}. \end{array}$$

3.9. Convex maps. A map j in $\underline{\Delta}$ is called *convex* and written $j : \underline{n} \rightarrowtail \underline{n}'$ if it is distance-preserving: $j(x+1) = j(x) + 1$, for all $x \in \underline{n}$. (In the subcategory $\Delta \subset \underline{\Delta}$ we called these 'inert maps'. We prefer to use different names since they play a different role in the two categories.) Observe that the convex maps are just the canonical inclusions

$$j : \underline{n} \rightarrowtail \underline{a} + \underline{n} + \underline{b},$$

and that, for $k > 0$, there is a canonical bijection

$$\underline{\Delta}_{\text{convex}}(\underline{k}, \underline{n}) \;\; \cong \;\; \underline{\Delta}_{\text{convex}}(\underline{k+1}, \underline{n+1}).$$

Here $\underline{\Delta}_{\text{convex}}$ denotes the subcategory of $\underline{\Delta}$ with all the objects and the convex maps. In combination with the full inclusion $\Delta \subset \underline{\Delta}$, we get

Lemma 3.10. *For $k > 0$, there is a canonical isomorphism*

$$\Delta^{\geq 1}_{\text{inert}} \;\; \cong \;\; \underline{\Delta}^{\geq 1}_{\text{convex}}, \qquad [k] \mapsto \underline{k} \quad (k \geq 1).$$

Note that this does *not* extend to $k \geq 0$ (since $\underline{0}$ is initial but $[0]$ is not).

Lemma 3.11. *Convex maps in $\underline{\Delta}$ admit pullback along any map: given the solid cospan consisting of g and i, with i convex,*

$$\begin{array}{ccc} \underline{n}' & \overset{j}{\leftarrowtail} & \underline{n} \\ {\scriptstyle g}\big\downarrow & \llcorner & \big\downarrow{\scriptstyle f} \\ \underline{k}' & \underset{i}{\leftarrowtail} & \underline{k}\,, \end{array}$$

the pullback exists and j is again convex.

Proposition 3.12. *For $k > 0$, there is a bijection between the set of pullback squares along convex maps in $\underline{\mathbb{\Delta}}$ and the set of commutative squares of active against inert maps in $\mathbb{\Delta}$*

$$\left\{ \begin{array}{ccc} \underline{n}' & \leftarrowtail & \underline{n} \\ \downarrow & \lrcorner & \downarrow \\ \underline{k}' & \leftarrowtail & \underline{k} \end{array} \quad \text{in } \underline{\mathbb{\Delta}} \right\} \quad = \quad \left\{ \begin{array}{ccc} [n'] & \leftarrowtail & [n] \\ \uparrow\!\!\!\!\top & & \uparrow\!\!\!\!\top \\ [k'] & \leftarrowtail & [k] \end{array} \quad \text{in } \mathbb{\Delta} \right\}.$$

The bijection is given by Lemma 3.3 on the vertical maps, and by Lemma 3.10 on the bottom horizontal map.

In the case $k = 0$, we necessarily have $n = 0$ and $n' = k'$, but there is not even a bijection on the bottom arrows in this case.

Proof. The bijection is the composite of the three bijections

$$\left\{ \begin{array}{ccc} \underline{n}' & \leftarrowtail & \underline{n} \\ \downarrow & \lrcorner & \downarrow \\ \underline{k}' & \leftarrowtail & \underline{k} \end{array} \right\} = \left\{ \begin{array}{ccc} \underline{n}' & & \\ \downarrow & & \\ \underline{k}' & \leftarrowtail & \underline{k} \end{array} \right\} = \left\{ \begin{array}{ccc} [n'] & & \\ \uparrow\!\!\!\!\top & & \\ [k'] & \leftarrowtail & [k] \end{array} \right\} = \left\{ \begin{array}{ccc} [n'] & \leftarrowtail & [n] \\ \uparrow\!\!\!\!\top & & \uparrow\!\!\!\!\top \\ [k'] & \leftarrowtail & [k] \end{array} \right\}$$

where the first bijection is by existence of pullbacks along convex maps (Lemma 3.11), the second is by Lemmas 3.3 and 3.10 (here we use that $k > 0$), and the third is by unique active-inert factorisation of the composite $[k] \rightarrowtail [k'] \twoheadrightarrow\!\!\!| \; [n']$. It can be checked that the bijection between the right-hand arrows is again that of Lemma 3.3. In fact, the bijection is

$$\left\{ \begin{array}{ccc} \underline{a}_1 + \underline{n} + \underline{a}_2 & \leftarrowtail & \underline{n} \\ {\scriptstyle \underline{g}_1 + \underline{f} + \underline{g}_2} \downarrow & \lrcorner & \downarrow {\scriptstyle \underline{f}} \\ \underline{b}_1 + \underline{k} + \underline{b}_2 & \leftarrowtail & \underline{k} \end{array} \right\} = \left\{ \begin{array}{ccc} [a_1] \vee [n] \vee [a_2] & \leftarrowtail & [n] \\ {\scriptstyle g_1 \vee f \vee g_2} \uparrow\!\!\!\!\top & & \uparrow\!\!\!\!\top {\scriptstyle f} \\ [b_1] \vee [k] \vee [b_2] & \leftarrowtail & [k] \end{array} \right\}.$$

□

3.13. Identity-extension squares. A square in $\underline{\mathbb{\Delta}}$ is called an *identity-extension square (iesq)* if is it of the form

$$\begin{array}{ccc} \underline{a} + \underline{n} + \underline{b} & \overset{j}{\leftarrowtail} & \underline{n} \\ {\scriptstyle \mathrm{id}_a + f + \mathrm{id}_b} \downarrow & & \downarrow {\scriptstyle f} \\ \underline{a} + \underline{k} + \underline{b} & \underset{i}{\leftarrowtail} & \underline{k}, \end{array} \tag{3}$$

where i and j are convex. Note that an iesq is both a pullback and a pushout.

Lemma 3.14. *Under the correspondence of Proposition 3.12, identity-extension squares in $\underline{\mathbb{\Delta}}$ correspond to active-inert pushouts in $\mathbb{\Delta}$.*

4. The decomposition space **I** of layered finite sets

Let $\mathbb{I}$ be the category of finite sets and injections. We define and study the monoidal decomposition space **I** of *layered finite sets*: finite sets with an ordered partition into any number of possibly empty layers. It is equivalent to the monoidal nerve of the monoidal groupoid of finite sets and bijections, but the layering viewpoint will generalise nicely to the directed case (§6).

4.1. The groupoid of n-layered finite sets. An *n-layering*, or just a layering, of a finite set A is a function $p : A \to \underline{n}$. We refer to the fibres $A_i = p^{-1}(i)$, $i \in \underline{n}$, as *layers*. Layers may be empty. We consider the groupoid $\mathbf{I}_n := \mathbb{I}^{\text{iso}}_{/\underline{n}}$ of all n-layerings of finite sets, whose arrows are commutative triangles,

$$\begin{array}{ccc} A & \xrightarrow{\;\simeq\;} & A' \\ & \searrow \quad \swarrow & \\ & \underline{n}. & \end{array}$$

4.2. The simplicial groupoid of layered finite sets. We now assemble the groupoids of layered finite sets into a simplicial groupoid, exploiting the active-inert factorisation system on $\mathbb{\Delta}$. For an active map $g : [n] \twoheadrightarrow [m]$ of $\mathbb{\Delta}$, consider the map $g^* : \mathbb{I}^{\text{iso}}_{/m} \to \mathbb{I}^{\text{iso}}_{/n}$ given by postcomposition with the corresponding map $\underline{g} : \underline{m} \to \underline{n}$ of $\underline{\mathbb{\Delta}}$ under the correspondence of Lemma 3.3,

$$g^* := \underline{g}_! : \mathbb{I}^{\text{iso}}_{/\underline{m}} \to \mathbb{I}^{\text{iso}}_{/\underline{n}}, \quad (A{\to}\underline{m}) \mapsto (A{\to}\underline{m}\overset{g}{\to}\underline{n}).$$

To define the outer face maps $d_\perp, d_\top : \mathbb{I}^{\text{iso}}_{/\underline{k}} \to \mathbb{I}^{\text{iso}}_{/\underline{k-1}}$, we take $A{\to}\underline{k}$ to the pullbacks

$$\begin{array}{ccc} A' & \hookrightarrow & A \\ {\scriptstyle d_\perp(a):=\underline{d}^{\perp *}(a)}\downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ \underline{k-1} & \xrightarrow[\underline{d}^\perp]{} & \underline{k}, \end{array} \qquad \begin{array}{ccc} A' & \hookrightarrow & A \\ {\scriptstyle d_\top(a):=\underline{d}^{\top *}(a)}\downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ \underline{k-1} & \xrightarrow[\underline{d}^\top]{} & \underline{k}, \end{array}$$

projecting away the first or the last layer. We make the specific choice that the pullbacks are given by subsets; this will ensure that the simplicial object we are defining is strict. More abstractly, for an inert map $f : [n] \to [m]$ of $\mathbb{\Delta}$, the map $f^* : \mathbb{I}^{\text{iso}}_{/\underline{m}} \to \mathbb{I}^{\text{iso}}_{/\underline{n}}$ is defined by pullback along the corresponding convex map $\underline{f} : \underline{n} \to \underline{m}$ in $\underline{\mathbb{\Delta}}$, given for $n \geq 1$ by the correspondence of Lemma 3.10 between inert maps in $\mathbb{\Delta}$ and convex maps in $\underline{\mathbb{\Delta}}$. Note that all maps $[0] \to [n]$ correspond to the unique map $\underline{0} \to \underline{n}$.

Proposition 4.3. *The groupoids $\mathbf{I}_n$ and the maps g^*, f^* above form a simplicial groupoid $\mathbf{I} : \mathbb{\Delta}^{\text{op}} \to$ **Grpd**, which is a Segal space, and hence a decomposition space.*

Proof. The active-active simplicial identities are already known to hold by construction, because they correspond under $\mathbb{\Delta}^{\text{op}}_{\text{act}} \simeq \underline{\mathbb{\Delta}}$ to identities in $\underline{\mathbb{\Delta}}$.

We need to check the following nine simplicial identities involving outer face maps:

$$d_\top \circ d_\perp = d_\perp \circ d_\top$$

$$\begin{array}{ll} d_\perp \circ d_\perp = d_\perp \circ d_1 & \qquad d_\top \circ d_\top = d_\top \circ d_{\top-1} \\ d_\perp \circ s_\perp = \text{id} & \qquad d_\top \circ s_\top = \text{id} \\ s_k \circ d_\perp = d_\perp \circ s_{k+1} & \qquad s_k \circ d_\top = d_\top \circ s_k \\ d_k \circ d_\perp = d_\perp \circ d_{k+1} & \qquad d_k \circ d_\top = d_\top \circ d_k. \end{array}$$

These relations, according to the definitions we have given of outer face maps in $\mathbf{I}$, translate into the following relations between pullback (upperstar) and postcomposition (lowershriek) operations, using the dictionary compiled in Lemma 3.3.

$$\underline{d}^{\top *} \circ \underline{d}^{\perp *} = \underline{d}^{\perp *} \circ \underline{d}^{\top *}$$

$$\mathrm{id}_! \circ \underline{d}^{\perp *} \circ \underline{d}^{\perp *} = \underline{d}^{\perp *} \circ \underline{s}^{\perp}{}_! \qquad \mathrm{id}_! \circ \underline{d}^{\top *} \circ \underline{d}^{\top *} = \underline{d}^{\top *} \circ \underline{s}^{\top}{}_!$$

$$\underline{d}^{\perp *} \circ \underline{d}^{\perp}{}_! = \mathrm{id}_! \circ \mathrm{id}^* \qquad \underline{d}^{\top *} \circ \underline{d}^{\top}{}_! = \mathrm{id}_! \circ \mathrm{id}^*$$

$$\underline{d}^{k}{}_! \circ \underline{d}^{\perp *} = \underline{d}^{\perp *} \circ \underline{d}^{k+1}{}_! \qquad \underline{d}^{k}{}_! \circ \underline{d}^{\top *} = \underline{d}^{\top *} \circ \underline{d}^{k}{}_!$$

$$\underline{s}^{k-1}{}_! \circ \underline{d}^{\perp *} = \underline{d}^{\perp *} \circ \underline{s}^{k}{}_! \qquad \underline{s}^{k-1}{}_! \circ \underline{d}^{\top *} = \underline{d}^{\top *} \circ \underline{s}^{k-1}{}_!.$$

The first of these is induced from a commutative square in $\underline{\mathbb{\Delta}}$. The other eight hold by Beck–Chevalley, since the squares in $\underline{\mathbb{\Delta}}$ are pullbacks by Lemmas 3.6–3.8.

The simplicial identities can be arranged to hold on the nose: the only subtlety is the pullback construction involved in defining the outer face maps, but these pullbacks can all be chosen to be always actual subset inclusions.

Finally, since $\mathbb{I}^{\mathrm{iso}}_{/\underline{0}} \simeq 1$, the Segal condition says (for each m, n) the projection map $\mathbb{I}^{\mathrm{iso}}_{/\underline{m+n}} \to \mathbb{I}^{\mathrm{iso}}_{/\underline{m}} \times \mathbb{I}^{\mathrm{iso}}_{/\underline{n}}$ must be an equivalence (Cf. 1.3). But this is clear, since an inverse is given by sending $(A{\to}\underline{m}, B{\to}\underline{n})$ to $A{+}B \to \underline{m{+}n}$. □

Lemma 4.4. *The decomposition space* $\mathbf{I}$ *is complete, locally finite, locally discrete, and of locally finite length.*

Proof. The checks are straightforward verifications. (Some indications can be found in the similar Lemma 6.13.) □

Proposition 4.5. *There are natural (levelwise) equivalences*

$$\mathrm{Dec}_\perp \mathbf{I} \simeq \mathbf{N}\mathbb{I} \qquad \mathrm{Dec}_\top \mathbf{I} \simeq \mathbf{N}\mathbb{I}^{\mathrm{op}}.$$

The first equivalence identifies a map $A \to \underline{k}$ *with the string of* $k-1$ *injections*

$$A_1 \hookrightarrow A_1 + A_2 \hookrightarrow \ldots \hookrightarrow A_1 + \cdots + A_{k-1} \hookrightarrow A_1 + \cdots + A_k.$$

Proof. This is a straightforward verification, easier than the analogous arguments in the proof of Proposition 6.11, where the case of posets is treated. In fact the current proposition can be deduced from the poset case by Proposition 11.1, using the fact 7.3 that $\mathbb{I}$ is a directed restriction species (see Remark 7.15). □

Lemma 4.6. $\mathbf{I}$ *is a monoidal decomposition space under disjoint union.*

Proof. As the proof of Lemma 6.14, but changing $\mathbb{C}$ to $\mathbb{I}$ and $\mathbf{C}$ to $\mathbf{I}$ everywhere. □

5. Restriction species

5.1. Schmitt's restriction species. Recall that $\mathbb{I}$ denotes the category of finite sets and injections. Schmitt [**49**] defines restriction species to be presheaves on $\mathbb{I}$,

$$\begin{aligned} R : \mathbb{I}^{\mathrm{op}} &\longrightarrow \textbf{\textit{Set}} \\ A &\longmapsto R[A]. \end{aligned}$$

An element X of $R[A]$ is called an *R-structure* on the set A. Compared to a classical species [**29**], a restriction species R is thus functorial not only in bijections but also in injections, meaning that an R-structure on a set A induces also such a structure on every subset $B \subset A$ (denoted with a restriction bar):

$$\begin{array}{rcl} R[A] & \longrightarrow & R[B] \\ X & \longmapsto & X|B. \end{array}$$

A morphism of restriction species is just a natural transformation $R \Rightarrow R'$ of functors $\mathbb{I}^{\text{op}} \to \boldsymbol{Set}$, i.e. for each finite set A a map $R[A] \to R'[A]$, natural in A.

5.2. Schmitt construction. The Schmitt construction [**49**] associates to a restriction species $R : \mathbb{I}^{\text{op}} \to \boldsymbol{Set}$ a (cocommutative) coalgebra structure on the vector space spanned by the isoclasses of R-structures: the comultiplication is

$$\Delta(X) = \sum_{A_1+A_2=A} X|A_1 \otimes X|A_2, \qquad X \in R[A],$$

and the counit sends $X \in R[\varnothing]$ to 1 and other structures to 0.

If $u : R' \Rightarrow R$ is a morphism of restriction species, the assignment $(A, X') \mapsto (A, u_A(X'))$ defines a linear map between the vector spaces underlying the incidence coalgebras, and since the summation in the comultiplication formula only involves the underlying sets, it is clear that this linear map is a coalgebra homomorphism.

A great many (cocommutative) combinatorial coalgebras can be realised by the Schmitt construction (see [**49**] and also [**2**]). For example, graphs (2.1), matroids, simplicial complexes, posets, categories, etc., form restriction species and hence coalgebras. In many cases, disjoint union furthermore defines an algebra structure, and altogether a bialgebra. Finally, in most cases, $R[\varnothing]$ is singleton. This implies that the bialgebra is connected and hence a Hopf algebra. Schmitt actually includes this condition in his definition of restriction species. In the present work, we shall *not* assume $R[\varnothing]$ singleton.

5.3. Groupoid-valued species. In line with our general philosophy, we shall work with groupoids rather than sets, aspiring to a native treatment of symmetries. Groupoid-valued species were first advocated by Baez and Dolan [**3**] (who called them *stuff types*, as opposed to *structure types*, their translation of Joyal's *espèces de structures* [**29**]), for the sake of dealing with symmetries of Feynman diagrams. They showed also that over groupoids (but not over sets), the generating function of a species is the homotopy cardinality of its associated analytic functor. Furthermore, over groupoids, analytic functors are polynomial [**32**], [**23**], meaning that they are given by pullback functors and their adjoints. Since the decomposition-space machinery is based on homotopy pullbacks and homotopy cardinality, we may as well consider groupoid-valued species, which we do from now on.

For the sake of taking homotopy cardinality, it is furthermore natural to require the groupoid values to be locally finite. This means that every object has finite automorphism group. This is usually the case of combinatorial objects. In particular, every set (finite or not) is locally finite. So a classical species is always locally finite.

5.4. Restriction species. A *restriction species* is a groupoid-valued presheaf on $\mathbb{I}$,

$$\begin{array}{rcl} R : \mathbb{I}^{\text{op}} & \longrightarrow & \boldsymbol{Grpd} \\ A & \longmapsto & R[A]. \end{array}$$

A morphism of restriction species is a natural transformation. We actually allow pseudo-functors and pseudo-natural transformations, but make some remarks on the strict case in §12. This defines the category $\boldsymbol{RSp}$ of restriction species.

A restriction species corresponds, by the Grothendieck construction, to a right fibration (i.e. a cartesian fibration with groupoid fibres)

$$\mathbb{R} \to \mathbb{I}.$$

Here $\mathbb{R}$ is the category of elements of R, whose objects are R-structures and whose arrows are structure-preserving injections. More precisely, an object is a pair (A, X) where A is a finite set and $X \in R[A]$, and a morphism $(A', X') \to (A, X)$ is an injection $A' \to A$ in $\mathbb{I}$ and an arrow $X' \xrightarrow{\sim} X|A'$ in the groupoid $R[A']$. The category of restriction species is canonically equivalent to the categories of groupoid-valued presheaves on $\mathbb{I}$, and of right fibrations over $\mathbb{I}$:

$$\boldsymbol{RSp} \simeq \boldsymbol{Grpd}^{\mathbb{I}^{\mathrm{op}}} \simeq \boldsymbol{RFib}_{/\mathbb{I}}.$$

It is sometimes more informative to describe a restriction species by describing the right fibration $\mathbb{R} \to \mathbb{I}$ rather than describing the functor $R : \mathbb{I}^{\mathrm{op}} \to \boldsymbol{Grpd}$, because the description of the category $\mathbb{R}$ already has the specifics about the restrictions, encoded in the arrows of the category. We shall see this in the examples.

We make the standing assumption that restriction species take values in locally finite groupoids.

5.5. Examples of restriction species. (See [**49**] for these and more examples.)

(1) *Graphs.* The species of finite graphs is a restriction species, cf. Example 2.1. It is fruitful to look at it also as a right fibration $\mathbb{G} \to \mathbb{I}$: the category $\mathbb{G}$ is then the category whose objects are finite graphs, and whose morphisms are full graph inclusions. Full means that if two vertices x and y are in the subgraph then all edges between x and y must also be included. (Allowing non-full inclusions, such as $\bullet\ \bullet \hookrightarrow \bullet\!\!-\!\!\bullet$, would prevent $\mathbb{G} \to \mathbb{I}$ from being a right fibration.)

(2) *Matroids.* (See Oxley [**46**] for definitions.) The species of matroids is a restriction species [**49**]. Many important classes of matroids are stable under restriction and are therefore also restriction species. For example, transversal matroids, representable matroids, regular matroids, graphic matroids, bond matroids, planar matroids, and so on.

(3) *Posets.* The species of posets is a restriction species. The corresponding right fibration is $\mathbb{P} \to \mathbb{I}$, where $\mathbb{P}$ is the category of finite posets and full poset inclusions $F \hookrightarrow P$. 'Full' means that for two elements x, y in F we have $x \leq_F y$ if and only if $x \leq_P y$.

In §7 we shall introduce directed restriction species, based on a different category of posets, namely the category $\mathbb{C}$ of finite posets and convex maps. The forgetful functor $\mathbb{C} \to \mathbb{I}$ is *not* a right fibration: there is no convex lift of the set inclusion $\{0, 2\} \hookrightarrow \{0, 1, 2\}$ to the linear order $\{0 \leq 1 \leq 2\}$.

(4) *Categories.* The species of finite categories assigns to a finite set the groupoid of all finite-category structures on that set of objects. In this case the right fibration is $\mathbb{F} \to \mathbb{I}$, where $\mathbb{F}$ is the category of finite categories and full subcategory inclusions (or more precisely, injective-on-objects fully faithful functors). The underlying-set functor $\mathbb{F} \to \mathbb{I}$ is a right fibration because clearly any subset of the object set of a category determines uniquely a full subcategory.

Note: in the examples of graphs and categories we stress the word 'finite': if we allowed an infinite number of edges/arrows between two elements, an infinite automorphism group would result, violating the standing local finiteness assumption (made in 5.4).

5.6. Slices of examples. Recall that for any object x in a category $\mathscr{C}$, the domain projection $\mathscr{C}_{/x} \to \mathscr{C}$ is a right fibration. In particular, if $\mathbb{R} \to \mathbb{I}$ is a restriction species, for any R-structure X, the slice category $\mathbb{R}_{/X}$ is again a restriction species. It is the restriction species of R-substructures of X. See Bergner et al. [**5**] for examples of slices of the decomposition space of graphs. The fact that slicing a restriction species produces again restriction species reflects the local nature of coalgebras: every element in a coalgebra generates a subcoalgebra.

5.7. Restriction species as decomposition spaces over I. From a restriction species R, or a right fibration $\mathbb{R} \to \mathbb{I}$, we shall construct a simplicial groupoid R of layered R-structures, together with a CULF functor $\mathsf{R} \to \mathsf{I}$.

As in §4, the subtlety is that the obvious functoriality is in $\underline{\mathbb{\Delta}} \simeq \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}}$, not in all of $\mathbb{\Delta}^{\mathrm{op}}$. Consider first the functor $\mathbb{I}^{\mathrm{iso}}_{/\_} : \underline{\mathbb{\Delta}} \to \boldsymbol{Grpd}$ and form the pullbacks

$$\mathsf{R}_k = \mathbb{I}^{\mathrm{iso}}_{/\underline{k}} \times_{\mathbb{I}^{\mathrm{iso}}_{/\underline{1}}} \mathbb{R}^{\mathrm{iso}}$$

along the functor $\mathbb{R}^{\mathrm{iso}} \to \mathbb{I}^{\mathrm{iso}} = \mathbb{I}^{\mathrm{iso}}_{/\underline{1}}$. Thus R_k is the groupoid of R-structures with a k-layering of the underlying sets. This defines a diagram of shape $\underline{\mathbb{\Delta}} = \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}}$:

$$R_{\mathrm{act}} : \underline{\mathbb{\Delta}} \to \boldsymbol{Grpd}.$$

The pullback construction also shows that forgetting the R-structure and retaining only the layering of the underlying set provides a cartesian natural transformation (of $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}}$-diagrams)

$$R_{\mathrm{act}} \to I_{\mathrm{act}}.$$

So far the construction works for any species, not necessarily restriction species. To define also the inert maps (i.e. outer face maps) we need the restriction structure on R, which allows us to lift the outer face maps we constructed for I. Recall that the outer face map $d_\perp : \mathbb{I}^{\mathrm{iso}}_{/\underline{k}} \to \mathbb{I}^{\mathrm{iso}}_{/\underline{k-1}}$ is defined by sending $A {\to} \underline{k}$ to the pullback

$$\begin{array}{ccc} A' & \xrightarrow{\ \subset\ } & A \\ \big\downarrow & \lrcorner & \big\downarrow \\ \underline{k-1} & \xrightarrow[d_\perp]{} & \underline{k}. \end{array}$$

Since $A' \hookrightarrow A$ is an injection, we can use functoriality of R (the fact that R is a restriction species) to get also the face map for R_k: for example,

$$d_\perp : \mathsf{R}_k \to \mathsf{R}_{k-1}$$

is defined as

$$\big(A {\to} \underline{k},\ X\big) \mapsto \big(\underline{d}^*_\perp A {\to} \underline{k-1},\ X|\underline{d}^*_\perp A\big).$$

We see that the point is to be covariantly functorial in all maps in $\underline{\mathbb{\Delta}}$ and to be contravariantly functorial in convex maps. To establish the simplicial identities is to exhibit a certain compatibility between these two functorialities. These conditions are precisely condensed in the notion of sesquicartesian fibration which we introduce in §9 below.

Theorem 5.8. *Given a restriction species R, the above construction defines a simplicial groupoid R, which is a decomposition space. Furthermore, a morphism of restriction* *species $R' \to R$ induces a CULF functor $\mathsf{R}' \to \mathsf{R}$. These assignments define a functor from the category of restriction species to that of decomposition spaces and CULF functors.*

Proof. The simplicial identities in R can be checked by hand, arguing along the lines of the proof of Proposition 4.3. (Later we will give a more elegant proof using the machinery introduced in Sections 8–10 and there will be no need for ad hoc arguments). Since by construction the simplicial groupoid R is CULF over a decomposition space $\mathbf{\mathsf{I}}$, it is itself a decomposition space (by Lemma 1.12).

A morphism $f : R' \to R$ amounts to a morphism of right fibrations

$$\begin{array}{ccc} \mathbb{R}' & \longrightarrow & \mathbb{R} \\ & \searrow \quad \swarrow & \\ & \mathbb{I} & \end{array}$$

inducing simplicial maps

$$\begin{array}{ccc} \mathsf{R}' & \xrightarrow{\mathsf{f}} & \mathsf{R} \\ & \searrow \quad \swarrow & \\ & \mathbf{\mathsf{I}}. & \end{array}$$

Indeed, at level n, the morphism of groupoids $\mathsf{R}'_n \to \mathsf{R}_n$ is induced from $\mathbb{R}' \to \mathbb{R}$, since the layering only affects the underlying set which does not change. Finally, since the projection maps to $\mathbf{\mathsf{I}}$ are CULF, also f is CULF, by Lemma 1.10. □

5.9. Decalage. The decomposition space R constructed from the restriction species $\mathbb{R}$ can be seen as an 'un-decking': we have

$$\mathrm{Dec}_\bot \mathsf{R} \simeq \mathbf{N}\mathbb{R}, \qquad \mathrm{Dec}_\top \mathsf{R} \simeq \mathbf{N}\mathbb{R}^{\mathrm{op}}.$$

We postpone the proof until 11.3.

Lemma 5.10. *The groupoid $\mathsf{R}_1 = \mathbb{R}^{\mathrm{iso}}$ is locally finite.*

Proof. For each $n \in \mathbb{I}^{\mathrm{iso}}$ we have a fibre sequence (homotopy pullback)

$$\begin{array}{ccc} R[n] & \longrightarrow & \mathbb{R}^{\mathrm{iso}} \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow[\ulcorner n \urcorner]{} & \mathbb{I}^{\mathrm{iso}}. \end{array}$$

Since $\mathbb{I}^{\mathrm{iso}}$ is locally finite, and since $R[n]$ is locally finite by our standing assumption (see 5.4), also $\mathbb{R}^{\mathrm{iso}}$ is locally finite. □

Proposition 5.11. *The decomposition space R is complete, locally finite, locally discrete, and locally of finite length.*

Proof. R_1 is locally finite by Lemma 5.10. The remaining finiteness properties and the discreteness property follow from Lemmas 1.11 and 4.4 since R is CULF over $\mathbf{\mathsf{I}}$. □

5.12. Coalgebras. (See [**19**] and [**20**].) For any decomposition space X, there is induced a coalgebra structure on the slice category $\boldsymbol{Grpd}_{/\mathsf{X}_1}$, as stated in Theorem 1.6 (for details, see [**19**, Theorem 7.4]). The comultiplication is the linear functor

$$\Delta : \boldsymbol{Grpd}_{/\mathsf{X}_1} \xrightarrow{(d_2,d_0)_! \circ d_1{}^*} \boldsymbol{Grpd}_{/\mathsf{X}_1 \times \mathsf{X}_1},$$

defined by the span

$$\mathsf{X}_1 \xleftarrow{d_1} \mathsf{X}_2 \xrightarrow{(d_2,d_0)} \mathsf{X}_1 \times \mathsf{X}_1 \tag{4}$$

(and the counit is the linear functor $\varepsilon : \boldsymbol{Grpd}_{/\mathsf{X}_1} \longrightarrow \boldsymbol{Grpd}$ defined by $\mathsf{X}_1 \xleftarrow{s_0} \mathsf{X}_0 \longrightarrow 1$.)

From an abstract viewpoint, 'linear' means to preserve homotopy sums [**18**]. From a practical viewpoint, it means to be given by a span, thought of as a matrix. The upshot is that under suitable finiteness conditions, one can take homotopy cardinality of these linear functors to obtain linear maps between vector spaces, to obtain a coalgebra structure on the vector space $\mathbb{Q}_{\pi_0 \mathsf{X}_1}$, the free vector space spanned by iso-classes of 1-cells of X. We refer to [**20**, §7] for the details.

Similarly a CULF functor $\mathsf{f} : \mathsf{X}' \to \mathsf{X}$ induces a coalgebra homomorphism $\boldsymbol{Grpd}_{/\mathsf{X}'_1} \to \boldsymbol{Grpd}_{/\mathsf{X}_1}$ ([**19**, Lemma 8.2]), given simply by post-composition with f_1.

The following proposition is the motivation for channelling the Schmitt construction through decomposition spaces.

Proposition 5.13. *For R a restriction species, the Schmitt coalgebra of R is the homotopy cardinality of the incidence coalgebra of the associated decomposition space* R. *For a morphism of restriction species $f : R' \to R$, Schmitt's coalgebra homomorphism is the homotopy cardinality of the associated CULF functor* $\mathsf{f} : \mathsf{R}' \to \mathsf{R}$.

Proof. At the objective level, the comultiplication is given by pullback along $d_1 : \mathsf{R}_2 \to \mathsf{R}_1$, followed by composing with (d_2, d_0), with reference to the span (4) for R. For a given R-structure X, viewed as a morphism $\ulcorner X \urcorner : 1 \to \mathsf{R}_1$, the pullback is the d_1-fibre over X, that is the groupoid $(\mathsf{R}_2)_X$ of all 2-layerings on the R-structure X. This is a groupoid over $\mathsf{R}_1 \times \mathsf{R}_1$ by composing with (d_2, d_0), which amounts to returning the restriction of X to each of the two layers. To recover the formula in 5.2, it remains to take homotopy cardinality of this groupoid, relative to $\mathsf{R}_1 \times \mathsf{R}_1$. This is meaningful since R is locally finite by Proposition 5.11. There are general formulae for this in [**22**], but in the present case it is straightforward: since R is locally discrete by Proposition 5.11, the groupoid $(\mathsf{R}_2)_X$ is discrete, and hence homotopy cardinality amounts to counting isomorphism classes, yielding Schmitt's formula in 5.2.

If $f : R' \Rightarrow R$ is a morphism of restriction species, the effect of Schmitt's coalgebra homomorphism (see 5.2) on an R-structure X with underlying set A is $X \mapsto f_A(X)$. On the other hand, Theorem 5.8 gives a CULF functor $\mathsf{f} : \mathsf{R}' \to \mathsf{R}$, inducing the objective-level coalgebra homomorphism $\boldsymbol{Grpd}_{/\mathsf{R}'_1} \to \boldsymbol{Grpd}_{/\mathsf{R}_1}$ that takes $1 \xrightarrow{\ulcorner X' \urcorner} \mathsf{R}'_1$ to $1 \xrightarrow{\ulcorner X' \urcorner} \mathsf{R}'_1 \xrightarrow{\mathsf{f}_1} \mathsf{R}_1$, which is $\ulcorner f_A X' \urcorner$. The cardinality of this linear functor is precisely Schmitt's coalgebra homomorphism. □

5.14. Monoidal restriction species. We introduce the notion of monoidal restriction species. The idea is simply that many restriction species are 'closed under disjoint union', in a way compatible with restrictions. This compatibility with restrictions ensures that the resulting algebra structure is compatible with the coalgebra structure to result altogether in a bialgebra. This bialgebra is always graded (by the number of elements in the underlying set), and most often connected (this happens when there is only one possible structure on the empty set), and hence a Hopf algebra. Schmitt [**49**] arrives at Hopf algebras through a notion of *coherent exponential restriction species*. Our notion is a bit more general, and conceptually simpler.

The category $\mathbb{I}$ has a symmetric monoidal structure given by disjoint union, as already exploited to make I a monoidal decomposition space (Lemma 4.6). We define a *monoidal restriction species* to be a right fibration $\mathbb{R} \to \mathbb{I}$ for which the total space $\mathbb{R}$ has a monoidal structure $\sqcup$ and the projection to $\mathbb{I}$ is strong monoidal.

If X_1 is an R-structure with underlying set S_1, and X_2 is an R-structure with underlying set S_2, and if $K_1 \subset S_1$ and $K_2 \subset S_2$ are subsets (or injective maps), then there is a canonical isomorphism

$$(X_1 \sqcup X_2) \mid (K_1 + K_2) \simeq (X_1 \mid K_1) \sqcup (X_2 \mid K_2).$$

This follows from unique comparison between cartesian lifts and the fact that the projection is strong monoidal. This isomorphism expresses the desired compatibility between the monoidal structure and restrictions.

A *morphism of monoidal restriction species* is a strong monoidal functor which is also a morphism of right fibrations.

Proposition 5.15. *The functor of Theorem 5.8 extends to a functor from the category of monoidal restriction species and their morphisms to that of monoidal decomposition spaces and CULF monoidal functors.*

Proof. If $\mathbb{R}$ is a monoidal restriction species, then the associated decomposition space R is monoidal: in degree n, this is simply given by the monoidal structure $\sqcup : \mathbb{R}_{/\underline{n}} \times \mathbb{R}_{/\underline{n}} \to \mathbb{R}_{/\underline{n}}$. This is well defined because the projection functor is strong monoidal. Furthermore, this monoidal structure is CULF thanks to the above compatibility: to give a pair of R-structures with a layering of each is the same as giving a pair of R-structures with a layering of its disjoint union. This is to say that this square is a pullback:

$$\begin{array}{ccc}
\mathbb{R}^{\text{iso}}_{/\underline{1}} \times \mathbb{R}^{\text{iso}}_{/\underline{1}} & \xleftarrow{g\times g} & \mathbb{R}^{\text{iso}}_{/\underline{k}} \times \mathbb{R}^{\text{iso}}_{/\underline{k}} \\
\downarrow & & \downarrow \\
\mathbb{R}^{\text{iso}}_{/\underline{1}} & \xleftarrow[g]{} & \mathbb{R}^{\text{iso}}_{/\underline{k}},
\end{array}$$

where g is the unique active map (and k could be 0). □

It follows that every monoidal restriction species defines a bialgebra, and a morphism of monoidal restriction species defines a bialgebra homomorphism. Furthermore, the decomposition space of a restriction species and its associated incidence coalgebra are canonically graded, by the number of elements in the underlying set, and for monoidal restriction species, this grading is also compatible with the multiplication. (At the objective level, this is an instance of the general notion of *length filtration* [**20**, §6].) In the connected case, i.e. when there is a unique R-structure on $\varnothing$, we therefore get a Hopf algebra. (The antipode exists at the vector-space level, but not on the objective level, as it always involves minus signs.)

5.16. Remark. There is a kind of converse to the construction $R \rightsquigarrow \mathsf{R}$. Namely, starting from a decomposition space R CULF over I (and with R_1 locally finite), we can take lower dec of both and obtain a Segal space which by Lemma 1.17 is a right fibration over $\text{Dec}_\bot \mathsf{I} = \mathbf{N}\mathbb{I}$ (Proposition 4.5). In fact $\text{Dec}_\bot \mathsf{R}$ is a Rezk-complete Segal space. Indeed, since R is CULF over I, it is complete, locally finite, locally discrete and of locally finite length, by Lemma 1.11. But also the dec map $\text{Dec}_\bot \mathsf{R} \to \mathsf{R}$ is CULF, so $\text{Dec}_\bot \mathsf{R}$ also has all these properties. Since it is furthermore a Segal space, it follows from [**20**, Corollary 8.7] that it is Rezk complete. Hence $\mathrm{Dec}_\perp \mathsf{R}$ is essentially the fat nerve of a category $\mathbb{R}$ (with a right fibration over $\mathbb{I}$).

6. The decomposition space $\mathbf{C}$ of layered finite posets

We define and study the monoidal decomposition space $\mathbf{C}$ of finite posets and their 'admissible cuts', which will play the same role for directed restriction species as $\mathbf{I}$ does for plain restriction species. An important difference is that while the simplicial groupoid $\mathbf{I}$ is a Segal space, $\mathbf{C}$ is only a decomposition space, not a Segal space.

6.1. Convex maps of posets. A subposet K of a poset P is *convex* if it is full and if $a \leq x \leq b$ in P and $a, b \in K$ imply $x \in K$. A map of posets $f : K \to P$ is *convex* if for all $a, b \in K$ and $fa \leq x \leq fb$ in P there is a unique $k \in K$ with $a \leq k \leq b$ and $fk = x$. In other words, f is injective and $f(K) \subset P$ is a convex subposet. We denote by $\mathbb{C}$ the category of finite posets and convex maps.

Lemma 6.2. *In the category of posets, convex maps are stable under pullback.*

Lemma 6.3. *For a subposet $K \subset P$ the following are equivalent.*

(1) *K is convex*
(2) *K is the middle fibre of some monotone map $P \to \underline{3}$*
(3) *$K \subset P$ is a fully faithful ULF functor of categories.*

6.4. Layered posets. An *n-layering* of a finite poset P is a monotone map $\ell : P \to \underline{n}$. We refer to the fibres $P_i = \ell^{-1}(i)$, $i \in \underline{n}$, as *layers*. Layers are convex subposets, by the previous lemma, and may be empty.

For sets, considered as discrete posets, the notion of set layering from 4.1 agrees with the notion of poset layering. Poset layering is more subtle, however, as it contains more information than just the list of layers.

6.5. The groupoid of n-layered finite posets. Consider the groupoid $\mathbb{C}^{\mathrm{iso}}_{/\underline{n}}$ of n-layerings of finite posets. That is, the objects of $\mathbb{C}^{\mathrm{iso}}_{/\underline{n}}$ are monotone maps $\ell : P \to \underline{n}$, and the morphisms are commutative triangles

$$\begin{array}{ccc} P & \xrightarrow{\ \simeq\ } & P' \\ & \searrow \quad \swarrow & \\ & \underline{n}, & \end{array}$$

where $P \xrightarrow{\simeq} P'$ is a monotone bijection (a poset isomorphism).

6.6. The simplicial groupoid of layered finite posets. We can define face and degeneracy maps between the groupoids of layered finite posets to assemble them into a simplicial groupoid $\mathbf{C}$, in the same way as for layered finite sets in 4.2:

The degeneracy and the inner face maps are defined using the correspondence $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}} \simeq \underline{\mathbb{\Delta}}$: if $g : [n] \to [m]$ is an active map in $\mathbb{\Delta}$ then $g^* : \mathbb{C}^{\mathrm{iso}}_{/\underline{m}} \to \mathbb{C}^{\mathrm{iso}}_{/\underline{n}}$ is given by postcomposition with the corresponding map $\underline{g} : \underline{m} \to \underline{n}$ in $\underline{\mathbb{\Delta}}$,

$$P \to \underline{m} \qquad \mapsto \qquad P \to \underline{m} \to \underline{n}.$$

The definition for inert maps (composites of outer face maps) is by pullback: for example, $d_\top : \mathbb{C}^{\text{iso}}_{/\underline{n}} \to \mathbb{C}^{\text{iso}}_{/\underline{n-1}}$ is given by taking $P' \to \underline{n}$ to $P \to \underline{n-1}$ in the pullback square

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \big\downarrow & \lrcorner & \big\downarrow \\ \underline{n-1} & \underset{\underline{d}_\top}{\longrightarrow} & \underline{n}. \end{array}$$

Since $\underline{d}_\top : \underline{n-1} \to \underline{n}$ is a convex map of posets, so is $P \to P'$. To be explicit, we can take this convex map to be an actual subset inclusion.

Proposition 6.7. *The groupoids $\mathbb{C}^{\text{iso}}_{/\underline{n}}$ and the maps between them, defined above, form a simplicial groupoid $\mathbf{C}$.*

Proof. The check may be performed in precisely the same way as done for $\mathbf{I}$ in Proposition 4.3: one checks the constructions above are covariantly functorial in all maps in $\underline{\mathbb{\Delta}}$ (giving the active part), contravariantly functorial in the convex maps of $\underline{\mathbb{\Delta}}$ (giving the inert part), and that these two functorialities are compatible. We will formalise this later in the notions of $\mathbb{V}$-spaces and sesquicartesian fibrations (Sections 8–9). □

6.8. Lower-set inclusions. Let P be a poset. A full subposet $L \subset P$ is a *lower set* if $x \leq b$ in P and $b \in L$ imply $x \in L$. A map of posets $L \to P$ is a *lower-set inclusion* if it is injective, full, and its image is a lower set in P. Clearly lower-set inclusions are convex. Let $\mathbb{C}^{\text{lower}}$ denote the category of finite posets and lower-set inclusions. Note that $L \to P$ is a lower-set inclusion if and only if it is a right fibration of categories. Upper sets are defined analogously.

Lemma 6.9. *In the category of posets, lower-set inclusions are stable under pullback.*

Proposition 6.10. *The map $\underline{d}_\top : \underline{1} \to \underline{2}$ classifies lower-set inclusions. That is, if P is a poset, pullback along $\underline{d}_\top$ defines a bijection*

$$\{\textit{monotone maps } P \to \underline{2}\} \;\cong\; \{\textit{isoclasses of lower-set inclusions } L \subseteq P\}.$$

Proposition 6.11. *There are natural (levelwise) equivalences*

$$\mathrm{Dec}_\bot \mathbf{C} \simeq \mathbf{N}\mathbb{C}^{\text{lower}} \qquad \mathrm{Dec}_\top \mathbf{C} \simeq \mathbf{N}(\mathbb{C}^{\text{upper}})^{\text{op}}$$

Proof. There is a natural equivalence

$$\mathbb{C}^{\text{iso}}_{/\underline{n}} \;\simeq\; \mathrm{Map}([n{-}1], \mathbb{C}^{\text{lower}}).$$

Indeed, given an n-layering of a poset P (i.e. a monotone map $P \to \underline{n}$), let first $P_{\underline{n}} = P$, and define then, by downward induction, $P_{\underline{k}} \to \underline{k}$ as the pullback of $P_{\underline{k+1}} \to \underline{k+1}$ along the lower-set inclusion $\underline{d}_\top : \underline{k} \to \underline{k+1}$. By Lemma 6.9, we obtain lower-set inclusions $P_{\underline{k}} \to P_{\underline{k+1}}$. Then the equivalence assigns to $P \to \underline{n}$ the sequence of lower-set inclusions

$$(P_{\underline{1}} \hookrightarrow P_{\underline{2}} \hookrightarrow \cdots \hookrightarrow P_{\underline{n-1}} \hookrightarrow P) \;\in\; \mathrm{Map}([n{-}1], \mathbb{C}^{\text{lower}}).$$

This assignment is fully faithful since each automorphism of such a sequence corresponds to a unique automorphism of P over $\underline{n}$. Finally, given such a sequence of lower-set inclusions, we recover a monotone map $P \to \underline{n}$, sending x to the least k for which $x \in P_{\underline{k}}$. It is straightforward to check that the face maps match up as required, so as to assemble these equivalences into a levelwise equivalence of simplicial groupoids.

The result for the upper dec is analogous. The 'op' appears in that case because the smallest subset in the chain is the last one, not the first as above. □

Proposition 6.12. **C** *is a decomposition space (but not a Segal space).*

Proof. We apply the decalage criterion [**19**, Theorem 4.10 (4)]. We already proved that the two Decs are Segal spaces. It remains to check that the following two squares are pullbacks:

$$\begin{array}{ccc} \mathbb{C}^{\text{iso}}_{/\underline{0}} & \xrightarrow{s_0} & \mathbb{C}^{\text{iso}}_{/\underline{1}} \\ {\scriptstyle d_\perp}\uparrow & \lrcorner & \uparrow{\scriptstyle d_\perp} \\ \mathbb{C}^{\text{iso}}_{/\underline{1}} & \xrightarrow[s_1]{} & \mathbb{C}^{\text{iso}}_{/\underline{2}} \end{array} \qquad \begin{array}{ccc} \mathbb{C}^{\text{iso}}_{/\underline{0}} & \xrightarrow{s_0} & \mathbb{C}^{\text{iso}}_{/\underline{1}} \\ {\scriptstyle d_\top}\uparrow & \lrcorner & \uparrow{\scriptstyle d_\top} \\ \mathbb{C}^{\text{iso}}_{/\underline{1}} & \xrightarrow[s_0]{} & \mathbb{C}^{\text{iso}}_{/\underline{2}} \end{array}$$

But it is clear they are strict pullbacks: this amounts to saying that if a 2-layered poset has one layer empty, it is determined by the other layer. Since the inert face maps are iso-fibrations, the squares are also (homotopy) pullbacks. Clearly **C** is not a Segal space as $\mathbb{C}^{\text{iso}}_{/\underline{m+n}} \not\simeq \mathbb{C}^{\text{iso}}_{/\underline{m}} \times \mathbb{C}^{\text{iso}}_{/\underline{n}}$. □

Lemma 6.13. *The decomposition space* **C** *is complete, locally finite and locally discrete, and of locally finite length.*

Proof. Since $\mathbb{C}^{\text{iso}}_{/\underline{0}}$ is contractible, consisting of the empty poset with no non-trivial automorphisms, we know $s_0 : \mathbb{C}^{\text{iso}}_{/\underline{0}} \to \mathbb{C}^{\text{iso}}_{/\underline{1}}$ is mono, so **C** is complete. Now observe that $\mathbb{C}^{\text{iso}}_{/\underline{1}}$ is locally finite as each finite poset has only finitely many automorphisms. We have just seen that $s_0 : \mathbb{C}^{\text{iso}}_{/\underline{0}} \to \mathbb{C}^{\text{iso}}_{/\underline{1}}$ is finite and discrete, and for $d_1 : \mathbb{C}^{\text{iso}}_{/\underline{2}} \to \mathbb{C}^{\text{iso}}_{/\underline{1}}$ the fibre over each finite poset P is the finite discrete groupoid $\{P \to \underline{2}\}$ of all monotone maps. Lastly, **C** is of locally finite length: the degenerate simplices are precisely the layerings with an empty layer. The fibre of $g : \mathbb{C}^{\text{iso}}_{/\underline{n}} \to \mathbb{C}^{\text{iso}}_{/\underline{1}}$ over P has no non-degenerate simplices if n is greater than the number of elements of the finite poset P. □

Lemma 6.14. **C** *is a monoidal decomposition space under disjoint union.*

Proof. For fixed k, we have $\mathbb{C}^{\text{iso}}_{/\underline{k}} \times \mathbb{C}^{\text{iso}}_{/\underline{k}} \to \mathbb{C}^{\text{iso}}_{/\underline{k}}$ given by disjoint union. It is clear that these maps assemble into a simplicial map $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$. CULFness of this simplicial map follows because to give a pair of posets, each with a k-layering, is the same as giving a pair of posets, together with a k-layering of their disjoint union. In other words, disjoint union of layered posets are computed layer-wise. Diagrammatically, this square is a pullback:

$$\begin{array}{ccc} \mathbb{C}^{\text{iso}}_{/\underline{1}} \times \mathbb{C}^{\text{iso}}_{/\underline{1}} & \xleftarrow{g\times g} & \mathbb{C}^{\text{iso}}_{/\underline{k}} \times \mathbb{C}^{\text{iso}}_{/\underline{k}} \\ \downarrow & & \downarrow \\ \mathbb{C}^{\text{iso}}_{/\underline{1}} & \xleftarrow[g]{} & \mathbb{C}^{\text{iso}}_{/\underline{k}}, \end{array}$$

where g is the unique active map (and k could be 0). □

7. Directed restriction species

We introduce the new notion of directed restriction species, with associated incidence coalgebras generalising well-known constructions with rooted forests [**9, 7**], acyclic directed graphs [**42, 44**], posets and distributive lattices [**50, 12**], and double posets [**41**].

7.1. Directed restriction species. A *directed restriction species* is by definition a (pseudo)-functor

$$R : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Grpd},$$

or equivalently, by the Grothendieck construction, a right fibration $\mathbb{R} \to \mathbb{C}$; we shall always assume that all values are locally finite groupoids. The idea is that the value on a poset P is the groupoid of all possible R-structures that have P as underlying poset.

A morphism of directed restriction species is just a (pseudo)-natural transformation. This defines the category of directed restriction species $\boldsymbol{DRSp}$, equivalent to the categories of groupoid-valued presheaves on $\mathbb{C}$, and of right fibrations over $\mathbb{C}$:

$$\boldsymbol{DRSp} \simeq \boldsymbol{Grpd}^{\mathbb{C}^{\mathrm{op}}} \simeq \boldsymbol{RFib}_{/\mathbb{C}}.$$

7.2. Coalgebras from directed restriction species. Let R be any directed restriction species. An *admissible cut* of an object $X \in R[P]$ is by definition a 2-layering of the underlying poset. In other words, the cut separates P into a lower-set and an upper-set. This agrees with the notion of admissible cut in Butcher–Connes–Kreimer (cf. 2.2), and in related examples.

A coalgebra is defined by the rule

$$\Delta(X) = \sum_{c \in \mathrm{cut}(P)} X|D_c \otimes X|U_c, \qquad X \in R[P], \tag{5}$$

where the sum is over all admissible cuts $c = (D_c, U_c)$.

Note that the incidence coalgebra of a directed restriction species is generally noncocommutative. It is cocommutative if and only if it is actually supported on discrete posets, so that in reality it is an ordinary restriction species, as we explain next.

7.3. Sets as discrete posets. Any finite set can be regarded as a discrete poset, and any injective map of sets is then a convex map. Hence there is a natural functor $\mathbb{I} \to \mathbb{C}$. This functor is easily seen to be a right fibration. Hence every restriction species is also a directed restriction species. This is to say that there is a natural functor

$$\boldsymbol{RSp} \to \boldsymbol{DRSp}$$

from restriction species to directed restriction species, clearly fully faithful.

7.4. Directed restriction species as decomposition spaces. If $\mathbb{R} \to \mathbb{C}$ is a directed restriction species, let R_k be the groupoid of R-structures on posets P with a k-layering. (In other words, R_2 is the groupoid of R-structures with an admissible cut, and R_k is the groupoid of R-structures with $k-1$ compatible admissible cuts.)

Theorem 7.5. *The R_k form a simplicial groupoid R, which is a decomposition space. Morphisms of directed restriction species induce CULF functors between decomposition spaces. The construction defines a functor from the category of directed restriction species and their morphisms to that of decomposition spaces and CULF maps.*

Proof. This can be proved in the same way as Theorem 5.8 for ordinary restriction species. A more elegant proof will be given in Theorem 10.9, after setting up fancier machinery. □

Since we assume directed restriction species $R : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Grpd}$ take locally finite groupoids as values, it follows by Lemma 5.10 that R_1 is a locally finite groupoid. Now by Lemmas 1.11 and 6.13 we have the necessary finiteness conditions to obtain classical incidence coalgebras by taking homotopy cardinality:

Lemma 7.6. *The decomposition space* R *is complete, locally finite, locally discrete, and of locally finite length.*

Lemma 7.7. *The incidence coalgebra obtained by taking homotopy cardinality coincides with formula* (5)*.*

Proof. The main point here is that since R is locally discrete by Lemma 7.6, the homotopy sum resulting from the decomposition space is just an ordinary sum, as in (5). □

7.8. Monoidal directed restriction species. The category $\mathbb{C}$ is symmetric monoidal under disjoint union. We define a *monoidal directed restriction species* to be a directed restriction species $\mathbb{R} \to \mathbb{C}$ for which the total space $\mathbb{R}$ has a monoidal structure and the right fibration is also a strong monoidal functor. This extends the notion of ordinary monoidal restriction species introduced in 5.14, as $\mathbb{I} \to \mathbb{C}$ is easily seen to be a monoidal directed restriction species. Since strong monoidal right fibrations compose, every monoidal restriction species is also a monoidal directed restriction species. We have:

Proposition 7.9. *The functor of Theorem 7.5 extends to a functor from monoidal directed restriction species and their morphisms, to monoidal decomposition spaces and CULF monoidal functors.*

If a restriction species is monoidal, the associated incidence coalgebra becomes a bialgebra. The projection $\mathsf{R} \to \mathbf{C}$ is monoidal, and so the incidence bialgebra of $\mathbb{R}$ comes with a bialgebra homomorphism to the incidence bialgebra of $\mathbb{C}$.

Except when explicitly mentioned otherwise, all the following examples are in fact monoidal directed restriction species and hence induce bialgebras.

7.10. First examples. Just as for ordinary restriction species, it is sometimes useful to describe a directed restriction species by describing the associated right fibration $\mathbb{R} \to \mathbb{C}$, where the restriction structure is encoded in the arrows.

(1) *Posets.* The category $\mathbb{C}$ of finite posets and convex maps is the terminal directed restriction species. The resulting coalgebra comultiplies a poset by splitting it along 'admissible cuts' into lower-sets and upper-sets (cf. Example 2.5).

(2) *One-way categories and Möbius categories.* For a finite category $\mathscr{C}$ to have an underlying poset, it is required that for any two objects $x, y \in \mathscr{C}$ at least one of the hom sets $\mathrm{Hom}_{\mathscr{C}}(x, y)$ and $\mathrm{Hom}_{\mathscr{C}}(y, x)$ is empty. (This implies that $\mathscr{C}$ is skeletal.) The underlying poset $\underline{\mathscr{C}}$ is then given by declaring $x \leq y$ to mean that $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is nonempty. Such categories form a directed restriction species U: for a convex map of posets $K \subset \underline{\mathscr{C}}$, the restriction of $\mathscr{C}$ to K is given as the full subcategory spanned by the objects in K. For the corresponding right fibration $\mathbb{U} \to \mathbb{C}$, the arrows in $\mathbb{U}$ are the fully faithful CULF functors (automatically injective on objects since the categories are skeletal).

With the further condition imposed that the only endomorphisms are the identities, we arrive at the notion of *finite delta*, in the terminology of Mitchell [**45**], now more commonly called finite *one-way categories.* This is equivalent (cf. [**38**]) to the notion of finite *Möbius category* of Leroux [**39**]. Möbius categories play an important role as a generalisation of locally finite posets, and in particular admit Möbius inversion. It is clear that we also have a directed restriction subspecies of finite Möbius categories.

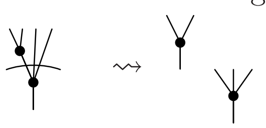
7.11. Convex-closed classes of posets. Ordinary (restriction) species are mostly about structure, not property, since the only property that can be assigned to a finite set is its cardinality. For directed restriction species, property plays a more important role, since posets can have many properties. Any class of posets closed under taking convex subposets and closed under isomorphisms defines a (fully faithful) right fibration, and hence a directed restriction species. Such a class may or may not be monoidal under disjoint union. (Note that this notion, which could reasonably be called convex-closed classes of posets, is different from the classical closure property in incidence coalgebras, where a class of *intervals* is required to be closed under subintervals [**50**].)

For example, forests (cf. 7.12 below), linear orders, and discrete posets (cf. 7.3) are convex-closed classes of posets, and form (monoidal) directed restriction species. Considering linear orders leads to $\mathbb{L}$-species, in the sense of [**4**].

Just as in the case of ordinary restriction species, the minimal such 'ideals' are defined by picking any single poset P, and considering the 'principal ideal generated by P', more precisely the slice category $\mathbb{C}_{/P}$. Note that $\mathbb{C}_{/P}$ cannot be monoidal in the sense of 7.8. Since the morphisms in $\mathbb{C}$ are just the convex maps, $\mathbb{C}_{/P}$ is equivalent to the full subcategory of $\mathbb{C}$ consisting of P and all its convex subposets. This reflects the standard fact that any element in a coalgebra generates a subcoalgebra.

7.12. Examples: various flavours of trees (actually forests). (1) *Combinatorial trees.* Consider the directed restriction species of rooted forests: a rooted forest has an underlying poset, whose convex subposets inherit each a rooted-forest structure. Regarded as a right fibration $\mathbb{H} \to \mathbb{C}$, the category $\mathbb{H}$ has objects rooted forests and morphisms subforest inclusions (not required to preserve the root). The resulting bialgebra is the Butcher–Connes–Kreimer Hopf algebra [**9, 7**] already treated in 2.2. As explained, this is not a Segal groupoid: a tree cannot be reconstructed from its layers. An important non-commutative variation comes from planar forests [**13**].

(2) *Operadic trees (with nodes).* Consider the combinatorial structure of rooted forests allowing open-ended edges (leaves and root) as in [**31, 16**], but disallowing isolated edges, i.e. edges not adjacent to any node. As before, each such forest has an underlying poset of nodes, and for each convex subset of the node set, there is induced a forest again. These are full forest inclusions, meaning that for each node, all incoming edges as well as the outgoing edge must be included (see [**31**] for details). It is an important feature that the local structure at the nodes is always preserved under taking such subforests. This means that one can consider trees whose nodes are decorated with 'operation symbols' of matching arity (more precisely P-trees for P a polynomial endofunctor [**31, 32**]) and that subtrees inherit such decorations. This is not possible for combinatorial trees, where the cuts destroy the local structure of nodes (such as for example being a binary node). Operadic forests (with nodes) form a directed restriction species. Note that in contrast to what happens for combinatorial trees, cuts do not delete inner edges, they cut them in two (as a consequence of the fullness of subforest inclusions). But if an isolated edge results from a cut, it is deleted, as illustrate in this figure:



(3) *Non-example: operadic trees, including nodeless ones.* If one allows the nodeless tree, the resulting notion of forest does not form a directed restriction species. Indeed,

with all the nodeless forests being different structures on the empty set of nodes, and since there exist non-invertible maps between such node-less forests, the functor to $\mathbb{C}$ cannot be a right fibration (it has non-invertible arrows in its fibres). (It is only over the empty set that this problem arises: between trees with nodes, every non-invertible map can be detected on nodes.)

This variation, which is subsumed in the class of decomposition spaces coming from operads [**22, 37**], has some different features which have been exploited to good effect in various contexts [**16, 33, 34, 36**]. In particular it is important that the cut locus expresses a type match between the roots of the crown forest and the leaves of the bottom tree, and that there is a grading [**20**] given by number of leaves minus number of roots. The incidence bialgebra is not connected: the zeroth graded piece is spanned by the node-less forests. These are all group-like, and the connected quotient (dividing out by this coideal) is precisely the incidence Hopf algebra of the directed restriction species of forests without isolated edges. One can then further take *core* [**33, 36**], which means shave off leaves and root (and forget the P-decoration). This is a monoidal CULF functor, and altogether there is a monoidal CULF functor from the decomposition space of P-trees to the decomposition space of combinatorial trees. This is an interesting example of a relative 2-Segal space in the sense of Young [**56**] and Walde [**53**].

7.13. Examples: various flavours of acyclic directed graphs. (1) *Acyclic directed graphs.* These have underlying posets, where $x \leq y$ if there is a directed path from x to y. Any convex subposet of the poset of vertices induces a subgraph S, which is convex in the usual sense of directed graphs, meaning that any directed path from $x \in S$ to $y \in S$ in the whole graph must be entirely contained in S. There is now induced a natural notion of admissible cut, similar to Butcher–Connes–Kreimer, and a Hopf algebra results (see Manchon [**42**, §5]).

(2) *Acyclic directed open graphs.* Now we allow open-ended edges, thought of as input edges and output edges (see [**35**]), but we do not allow graphs containing isolated edges. This situation and the resulting bialgebra have been studied by Manchon [**42**, §4]. Interesting decorated versions have been studied by Manin [**43, 44**] in the theory of computation. His graphs are decorated by operations on partial recursive functions and switches.

(3) *Non-example: Acyclic directed open graphs, allowing isolated edges.* Again, if one allows isolated edges, it is not a restriction species. In contrast it is a Segal groupoid, and the comultiplication resulting from it enjoys a nice grading (by number of input edges minus number of output edges).

7.14. Examples: double posets and related structures. A *double poset* [**41**] is a poset $(P, \leq)$ with an additional poset structure $\preccurlyeq$, not required to have any compatibility with $\leq$. Let $\mathbb{D}$ denote the category of finite double posets $(P, \leq, \preccurlyeq)$ and inclusions that are convex for $\leq$. For every $\leq$-convex subset $(K, \leq) \subset (P, \leq)$, there is induced a $\preccurlyeq$ structure on K, simply by the fact that posets form an ordinary restriction species (cf. 5.5 (3)). It follows that $\mathbb{D} \to \mathbb{C}$ is a right fibration, and hence a directed restriction species. The associated incidence coalgebra was first studied by Malvenuto and Reutenauer [**41**]; see [**14**] and [**15**] for more recent developments.

The case where the second poset structure is a linear order is called *special double poset* or just *special poset*, and is equivalent to Stanley's notion of labelled poset [**51**].

Double posets and special posets are just two instances of the following general construction: for any ordinary restriction species R, consider the directed restriction species consisting of having simultaneously a poset structure and an R-structure, without compatibility conditions. Let the morphisms be inclusions that are convex for the poset structure.

7.15. Decalage. While for ordinary restriction species $\mathbb{R} \to \mathbb{I}$ we have $\mathrm{Dec}_\bot \mathsf{R} \simeq \mathbf{N}\mathbb{R}$ and $\mathrm{Dec}_\top \mathsf{R} \simeq \mathbf{N}\mathbb{R}^{\mathrm{op}}$, the situation is slightly more complicated for directed restriction species. The result is (as we shall see in Proposition 11.1):

$$\mathrm{Dec}_\bot \mathsf{R} \simeq \mathbf{N}\mathbb{R}^{\mathrm{lower}} \qquad\qquad \mathrm{Dec}_\top \mathsf{R} \simeq \mathbf{N}(\mathbb{R}^{\mathrm{upper}})^{\mathrm{op}}$$

where $\mathbb{R}^{\mathrm{lower}} \subset \mathbb{R}$ denotes the subcategory of R-structures with all the objects, but only the maps whose underlying poset map is a lower-set inclusion. (Similarly, $\mathbb{R}^{\mathrm{upper}}$ has only upper-set inclusion.) (Note that this result does not contradict 5.9: if an ordinary restriction species $\mathbb{R}$ is considered a directed restriction species (as in 7.3) supported on discrete posets, then all inclusion maps are both lower-set inclusions and upper-set inclusions.)

7.16. Example: rooted trees (continued). An interesting example of the decalage result is provided by the Butcher–Connes–Kreimer Hopf algebra of rooted trees 2.2. Although it cannot be realised directly as the incidence coalgebra of a category, Dür [**9**] (Ch.IV, §3) constructed it as the *reduced* incidence coalgebra of a category. In our language, he starts with the category $\mathscr{C}$ of forests and root-preserving inclusions, and takes the incidence coalgebra of the fat nerve of $\mathscr{C}$; then he imposes the equivalence relation that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests. To be precise, this yields the opposite of the Butcher–Connes–Kreimer coalgebra, in the sense that the factors P_c and R_c are interchanged; to remedy this, one should just use $\mathscr{C}^{\mathrm{op}}$ instead of $\mathscr{C}$. Note also that since the underlying poset of a forest is oriented from leaves to roots, the root-preserving inclusions are the upper-set inclusions. The relationship with Dür's construction is now clear: the 'raw' decomposition space $\mathbf{N}\mathscr{C}^{\mathrm{op}}$ is the decalage of H:

$$\mathrm{Dec}_\top \mathsf{H} \simeq \mathbf{N}\mathscr{C}^{\mathrm{op}}.$$

Furthermore, the dec map $\mathrm{Dec}_\top \mathsf{H} \to \mathsf{H}$ realises precisely Dür's reduction.

8. Convex correspondences and 'nabla spaces'

8.1. Convex correspondences. Consider the category ∇ of *convex correspondences* in $\underline{\mathbb{\Delta}}$, a subcategory of the category of spans in $\underline{\mathbb{\Delta}}$. Objects are those of $\underline{\mathbb{\Delta}}$, and morphisms are spans

$$\underline{n}' \xleftarrow{\;j\;} \underline{n} \xrightarrow{\;f\;} \underline{k}$$

where j is convex. Composition of such spans is given by pullback, which exist by Lemma 3.11. By construction, ∇ has a factorisation system in which the left-hand class (called *backward convex* maps) consists of spans of the form $\cdot \leftarrow \cdot \xrightarrow{=} \cdot$, and the right-hand class (called *ordinalic* maps) consists of spans of the form $\cdot \xleftarrow{=} \cdot \rightarrow \cdot$. Composition of an ordinalic map followed by backward convex map is defined by

$$(\cdot \xleftarrow{\;i\;} \cdot \xrightarrow{\;=\;} \cdot) \circ (\cdot \xleftarrow{\;=\;} \cdot \xrightarrow{\;g\;} \cdot) \;=\; (\cdot \xleftarrow{\;j\;} \cdot \xrightarrow{\;f\;} \cdot) \tag{6}$$

with reference to the pullback square

$$\begin{array}{ccc} \cdot & \overset{j}{\leftarrowtail} & \cdot \\ {\scriptstyle g}\big\downarrow & \llcorner & \big\downarrow{\scriptstyle f} \\ \cdot & \underset{i}{\leftarrowtail} & \cdot \end{array}$$

Lemma 8.2. *There is a canonical functor*

$$\gamma : \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathbb{\nabla}, \qquad [n] \longmapsto \underline{n},$$

restricting to isomorphisms

(7) $$\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}} \cong \underline{\mathbb{\Delta}} \cong \mathbb{\nabla}_{\mathrm{ordinalic}}, \qquad (\mathbb{\Delta}^{\geq 1}_{\mathrm{inert}})^{\mathrm{op}} \cong (\underline{\mathbb{\Delta}}^{\geq 1}_{\mathrm{convex}})^{\mathrm{op}} \cong \mathbb{\nabla}^{\geq 1}_{\mathrm{back.conv.}},$$

and sending all maps $[0] \to [n]$ in $\mathbb{\Delta}$ to the zero map $\underline{n} \leftarrowtail \underline{0} \to \underline{0}$ in $\mathbb{\nabla}$. In particular, γ is bijective on objects and full.

In summary, the categories $\mathbb{\Delta}^{\mathrm{op}}$ and $\mathbb{\nabla}$ differ only in the fact that $\underline{0} \in \mathbb{\nabla}$ is initial *and terminal*, whereas $\mathrm{Hom}_{\mathbb{\Delta}^{\mathrm{op}}}([n], [0])$ contains $n + 1$ maps.

Proof. The first isomorphism is Lemma 3.3 and the second is Lemma 3.10. □

Proposition 8.3. *Precomposing with the canonical functor $\gamma : \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\nabla}$ of Lemma 8.2 induces a fully faithful functor*

$$\gamma^* : \mathrm{Fun}(\mathbb{\nabla}, \boldsymbol{Grpd}) \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \boldsymbol{Grpd})$$

whose essential image is the full subcategory consisting of simplicial groupoids X with $d_\bot = d_\top : \mathsf{X}_1 \to \mathsf{X}_0$.

Proof. Any functor which is bijective on objects and full induces a fully faithful functor of the presheaf categories. The main point is to characterise the essential image. Note that every simplicial groupoid X in the image will have all maps $\mathsf{X}_n \to \mathsf{X}_0$ equal, since the functor γ sends all maps $[0] \to [n]$ to the same image. Given a simplicial groupoid X with all $\mathsf{X}_n \to \mathsf{X}_0$ equal, we define a $\mathbb{\nabla}$-diagram by sending each object $\underline{n}$ to X_n and sending each convex correspondence $\underline{n}' \overset{j}{\leftarrowtail} \underline{n} \overset{f}{\to} \underline{k}$ to the composite

$$\mathsf{X}_{n'} \xrightarrow{X(\gamma^{-1}(j))} \mathsf{X}_n \xrightarrow{\mathsf{X}(\gamma^{-1}(f))} \mathsf{X}_k,$$

assuming $n > 0$ so as to invoke the bijections (7) separately on backward convex and ordinalic maps. For $n = 0$, $\gamma^{-1}(j)$ is not well defined in $\mathbb{\Delta}$, but taking X on it *is* well defined, since we have assumed all the maps $\mathsf{X}_n \to \mathsf{X}_0$ coincide. To check functoriality of the assignment, it is enough to treat the situation of an ordinalic map followed by a backward convex map. These compose by pullback in $\underline{\mathbb{\Delta}}$, and by Proposition 3.12 these pullback squares correspond to commutative squares in $\mathbb{\Delta}$, in a way compatible with the assignments on arrows, so as to ensure that composition is respected. It is clear that this nabla space induces X as required. □

8.4. Iesq condition on functors. For a functor $X : \mathbb{\nabla} \to \boldsymbol{Grpd}$, the image of a backward convex map is denoted by upperstar: if the backward convex map corresponds to $i : \underline{k} \rightarrowtail \underline{k}'$ in $\underline{\mathbb{\Delta}}$, we denote its image by $i^* : X_{k'} \to X_k$. Similarly, the image of an ordinalic map, corresponding to $f : \underline{n} \to \underline{k}$ in $\underline{\mathbb{\Delta}}$ is denoted $f_! : X_n \to X_k$. As observed in 3.13, any identity-extension square in $\underline{\mathbb{\Delta}}$

$$\begin{array}{ccc} \underline{a}+\underline{n}+\underline{b} & \overset{j}{\leftarrowtail} & \underline{n} \\ {\scriptstyle \mathrm{id}_a+f+\mathrm{id}_b=g}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{a}+\underline{k}+\underline{b} & \underset{i}{\leftarrowtail} & \underline{k} \end{array} \tag{8}$$

is a pullback and hence a commutative square in $\mathbb{\nabla}$ between maps from $\underline{a}+\underline{n}+\underline{b}$ to $\underline{k}$. The corresponding square of groupoids

$$\begin{array}{ccc} X_{a+n+b} & \xrightarrow{j^*} & X_n \\ {\scriptstyle g_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ X_{a+k+b} & \xrightarrow[i^*]{} & X_k. \end{array} \tag{9}$$

therefore commutes by functoriality (this is the 'Beck–Chevalley condition' (BC).)

We say that X satisfies the *iesq condition* if (9) not only commutes but is furthermore a pullback for every identity-extension square (8).

If a nabla space $M : \mathbb{\nabla} \to \textbf{\textit{Grpd}}$ sends identity-extension squares to pullbacks then the composite $\mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\nabla} \to \textbf{\textit{Grpd}}$ is a decomposition space. This follows from the correspondence between iesq in $\underline{\mathbb{\Delta}}$ and active-inert squares in $\mathbb{\Delta}$ (Lemma 3.14).

A morphism of nabla spaces is called *CULF* if it is cartesian on (forward) ordinalic maps, i.e. on arrows in $\underline{\mathbb{\Delta}} \subset \mathbb{\nabla}$. If $u : M' \Rightarrow M : \mathbb{\nabla} \to \textbf{\textit{Grpd}}$ is a CULF natural transformation between functors that send identity-extension squares to pullbacks, then it induces a CULF functor between decomposition spaces. Altogether:

Proposition 8.5. *Precomposition with $\mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\nabla}$ defines a canonical functor*

$$\mathrm{Fun}^{\mathrm{culf}}_{\mathrm{iesq}}(\mathbb{\nabla}, \textbf{\textit{Grpd}}) \to \textbf{\textit{Decomp}}^{\mathrm{culf}}$$

from iesq (pseudo)-functors (and CULF (pseudo)-natural transformations) to decomposition spaces and CULF functors.

9. Sesquicartesian fibrations

9.1. Functors out of $\mathbb{\nabla}$. In view of Proposition 8.5, we are interested in defining functors out of $\mathbb{\nabla}$. By its construction as a category of spans, this amounts to defining a covariant functor on $\underline{\mathbb{\Delta}}$ and a contravariant functor on $\underline{\mathbb{\Delta}}_{\mathrm{convex}}$ which agree on objects, and such that for every pullback along a convex map the Beck–Chevalley condition holds. Better still, we can describe these as certain fibrations over $\underline{\mathbb{\Delta}}$, called sesquicartesian fibrations, which we now introduce.

9.2. Sesquicartesian fibrations. A *sesquicartesian fibration* is a cocartesian fibration $X \to \underline{\mathbb{\Delta}}$ that is also cartesian over $\underline{\mathbb{\Delta}}_{\mathrm{convex}}$, and in addition satisfies the Beck–Chevalley condition: for each pullback in $\underline{\mathbb{\Delta}}$ of a convex map τ,

$$\begin{array}{ccc} \cdot & \xrightarrow{\sigma'} & \cdot \\ {\scriptstyle \tau'}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ \cdot & \xrightarrow[\sigma]{} & \cdot \end{array}$$

the comparison map $\sigma'_! \tau'^* \to \tau^* \sigma_!$ is an isomorphism.

Let ***Sesq*** be the category that has as objects the sesquicartesian fibrations and as arrows the functors of sesquicartesian fibrations (required to preserve cocartesian arrows and cartesian arrows over convex maps).

Proposition 9.3. *There is a canonical functor*

$$\boldsymbol{Sesq} \longrightarrow \mathrm{Fun}(\mathbb{\nabla}, \boldsymbol{Cat}).$$

Recall that Fun denotes the category of pseudo-functors and pseudo-natural transformations.

Proof. Given a sesquicartesian fibration $p : X \to \underline{\mathbb{\Delta}}$, we can define a pseudo-functor $P : \mathbb{\nabla} \to \boldsymbol{Cat}$ as follows. On objects, send $\underline{n}$ to the category X_n. Send a convex correspondence $\underline{n}' \overset{j}{\leftarrowtail} \underline{n} \overset{f}{\to} \underline{k}$ to the composite functor $X_{n'} \overset{j^*}{\to} X_n \overset{f_!}{\to} X_k$. Individually, the covariant and contravariant reindexing functors compose up to coherent isomorphisms because that's how cocartesian and cartesian fibrations work. The Beck–Chevalley isomorphisms provide the coherence isomorphisms for general composition.

On arrows: given a morphism $c : p' \to p$ of sesquicartesian fibrations, assign a pseudo-natural transformation $u : P' \Rightarrow P$: its component on $\underline{n}$ is $c_n : X'_n \to X_n$, its pseudo-naturality square on a backward convex map $\underline{n}' \overset{j}{\leftarrowtail} \underline{n}$ is given (at an object $x' \in X'_{n'}$) by the isomorphisms $c(j^*(x')) \simeq j^*(c(x'))$ expressing that c preserves cartesian arrows (but not chosen cartesian). Similarly with the forward maps and cocartesian lifts. Again BC is invoked to ensure these are really pseudo-natural. $\square$

9.4. Remark. From work of Hermida [**25**] and Dawson–Paré–Pronk [**8**], it can be expected that this functor is actually an equivalence, but we do not need this result and do not pursue the question further here.

9.5. The iesq property. A sesquicartesian fibration $p : X \to \underline{\mathbb{\Delta}}$ is said to have the *iesq property* if for every identity-extension square

$$\begin{array}{ccc} \underline{a} + \underline{n} + \underline{b} & \overset{j}{\longleftarrowtail} & \underline{n} \\ {\scriptstyle \mathrm{id}_a + f + \mathrm{id}_b = g} \downarrow & & \downarrow {\scriptstyle f} \\ \underline{a} + \underline{k} + \underline{b} & \underset{i}{\longleftarrowtail} & \underline{k} \end{array}$$

the diagram of categories

$$\begin{array}{ccc} X_{a+n+b} & \overset{j^*}{\longrightarrow} & X_n \\ {\scriptstyle g_!} \downarrow & & \downarrow {\scriptstyle f_!} \\ X_{a+k+b} & \underset{i^*}{\longrightarrow} & X_k \end{array}$$

not only commutes up to natural isomorphism (the BC condition), but is furthermore a homotopy pullback of categories (i.e. it is equivalent to a iso-comma square).

Let ***IesqSesq*** be the category whose objects are the sesquicartesian fibrations $p : X \to \underline{\mathbb{\Delta}}$ having the iesq property, and whose arrows are functors over $\underline{\mathbb{\Delta}}$

$$\begin{array}{ccc} X & \xrightarrow{c} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \underline{\mathbb{\Delta}} & \end{array}$$

that preserve cocartesian arrows and cartesian arrows (over convex maps), and satisfying the condition that for every arrow $f : \underline{n} \to \underline{k}$ in $\underline{\mathbb{\Delta}}$, the following square is a homotopy pullback:

$$\begin{array}{ccc} X_n & \xrightarrow{f_!} & X_k \\ {\scriptstyle c}\downarrow & \lrcorner & \downarrow{\scriptstyle c} \\ Y_n & \xrightarrow[f_!]{} & Y_k. \end{array} \tag{10}$$

This condition on arrows $c : X \to Y$ is equivalent to saying that the associated (pseudo)-natural transformation of pseudo-functors $\underline{\mathbb{\Delta}} \to \boldsymbol{Cat}$ is homotopy cartesian, i.e. all its (pseudo)-naturality squares are homotopy pullbacks.

Proposition 9.6. *The functor of Proposition 9.3 restricts to a functor*

$$\boldsymbol{IesqSesq} \longrightarrow \mathrm{Fun}^{\mathrm{culf}}_{\mathrm{iesq}}(\mathbb{\nabla}, \boldsymbol{Cat})$$

Here $\mathrm{Fun}^{\mathrm{culf}}_{\mathrm{iesq}}(\mathbb{\nabla}, \boldsymbol{Cat})$ is the subcategory of $\mathrm{Fun}(\mathbb{\nabla}, \boldsymbol{Cat})$ whose objects are those $X : \mathbb{\nabla} \to \boldsymbol{Cat}$ such that for every identity extension square the corresponding Beck–Chevalley square is a homotopy pullback in $\boldsymbol{Cat}$, and whose morphisms are those pseudo-natural transformations $X \to Y$ that are homotopy cartesian on (forward) ordinalic maps, i.e. on arrows in $\underline{\mathbb{\Delta}} \subset \mathbb{\nabla}$. Compare 8.4 for corresponding notions in $\mathrm{Fun}(\mathbb{\nabla}, \boldsymbol{Grpd})$.

Taking maximal subgroupoids to get a functor $\mathrm{Fun}^{\mathrm{culf}}_{\mathrm{iesq}}(\mathbb{\nabla}, \boldsymbol{Cat}) \to \mathrm{Fun}^{\mathrm{culf}}_{\mathrm{iesq}}(\mathbb{\nabla}, \boldsymbol{Grpd})$, and combining Propositions 9.6 and 8.5, we obtain:

Theorem 9.7. *The constructions so far define a functor*

$$\boldsymbol{IesqSesq} \to \boldsymbol{Decomp}^{\mathrm{culf}}.$$

9.8. Example: monoids. A monoid viewed as a monoidal functor $X : (\underline{\mathbb{\Delta}}, +, 0) \to (\boldsymbol{Grpd}, \times, 1)$ defines a iesq sesquicartesian fibration. The contravariant functoriality on the convex maps is given as follows. The cartesian lift of a convex map $\underline{a} + \underline{n} + \underline{b} \leftarrowtail \underline{n}$ is simply the projection

$$X_{a+n+b} \simeq X_a \times X_n \times X_b \longrightarrow X_n,$$

where the first equivalence expresses that X is monoidal. For any identity-extension square (8), it is clear that the corresponding diagram

$$\begin{array}{ccc} X_{a+n+b} & \xrightarrow{j^*} & X_n \\ {\scriptstyle g_!}\downarrow & \lrcorner & \downarrow{\scriptstyle f_!} \\ X_{a+k+b} & \xrightarrow[i^*]{} & X_k \end{array}$$

is a pullback, since the upperstar functors are just projections. The associated decomposition space is the classifying space of the monoid.

10. From restriction species to iesq-sesqui

In order to construct nabla spaces satisfying the iesq property, we can construct sesquicartesian fibrations satisfying iesq, and then take maximal sub-groupoid.

All our examples originate as the left leg of a two-sided fibration, as we proceed to explain.

10.1. Two-sided fibrations. Classically (the notion is due to Street [**52**]), a *two-sided fibration* is a diagram of categories and functors

$$\begin{array}{ccc} X & \xrightarrow{q} & T \\ {\scriptstyle p}\downarrow & & \\ S & & \end{array}$$

such that

— p is a cocartesian fibration whose p-cocartesian arrows are precisely the q-vertical arrows,

— q is a cartesian fibration whose q-cartesian arrows are precisely the p-vertical arrows,

— for $x \in X$, an arrow $f : px \to s$ in S and $g : t \to qx$ in T, the canonical map $f_! g^* x \to g^* f_! x$ is an isomorphism.

In the setting of ∞-categories, Lurie [**40**, §2.4.7] (using the terminology 'bifibration') characterises two-sided fibrations as functors $X \to S \times T$ subject to a certain horn-filling condition, which among other technical advantages makes it clear that the notion is stable under pullback along functors $S' \times T' \to S \times T$. The classical axioms are derived from the horn-filling condition.

10.2. Comma categories. For I a category, $\mathrm{Ar}(I) \xrightarrow{(\mathrm{codom},\mathrm{dom})} I \times I$ is a two-sided fibration. Given categories and functors

$$\begin{array}{ccc} & & S \\ & & \downarrow{\scriptstyle G} \\ T & \xrightarrow[F]{} & I \end{array}$$

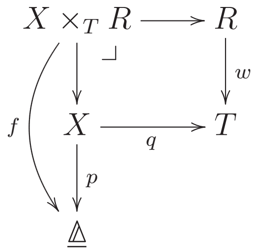
the *comma category* $T{\downarrow}S$ is the category whose objects are triples (t, s, ϕ), where $t \in T$, $s \in S$, and $\phi : Ft \to Gs$. More formally it is defined as the pullback two-sided fibration

$$\begin{array}{ccc} T{\downarrow}S & \longrightarrow & \mathrm{Ar}(I) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{codom},\mathrm{dom})} \\ S \times T & \xrightarrow[G \times F]{} & I \times I. \end{array}$$

Note that the factors come in the opposite order: $T{\downarrow}S \to S$ is the cocartesian fibration, and $T{\downarrow}S \to T$ the cartesian fibration. The left leg cocartesian fibration comes with a canonical splitting. The two-sided fibration sits in a comma square which we depict like this:

$$\begin{array}{ccc} T{\downarrow}S & \longrightarrow & T \\ \downarrow & \Leftarrow & \downarrow \\ S & \longrightarrow & I \end{array}$$

Lemma 10.3. *In a diagram*



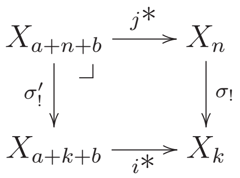
where

— $(p, q) : X \to \underline{\Delta} \times T$ *is a two-sided fibration;*
— $p : X \to \underline{\Delta}$ *is a iesq sesquicartesian fibration; and*
— $w : R \to T$ *is a cartesian fibration;*

we have

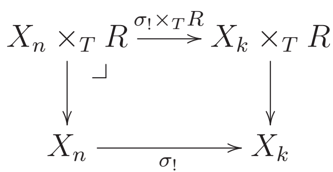
(1) f *is a iesq sesquicartesian fibration.*
(2) *the map* $X \times_T R \to X$ *is a morphism of iesq sesquicartesian fibrations from* f *to* p *(in the sense of 9.5).*

Proof. (1) f is a cocartesian fibration because it is the left leg of the pullback two-sided fibration of $X \to \underline{\Delta} \times T$ along $\underline{\Delta} \times R \to \underline{\Delta} \times T$. The f-cartesian lift of a given convex arrow has components (ℓ, c) where ℓ is a p-cartesian lift to X, and c is a w-cartesian lift of $q(\ell)$. Given the pullback square



expressing that $X \to \underline{\Delta}$ has the iesq property, the corresponding square for $X \times_T R \to \underline{\Delta}$ is simply obtained applying $- \times_T R$ to it, hence is again a pullback, so f has the iesq property.

(2) By construction $X \times_T R \to X$ preserves cocartesian arrows and cartesian arrows over convex maps, so it is indeed a morphism of sesquicartesian fibrations. For each arrow $\sigma : n \to k$ in $\underline{\Delta}$, the square required to be a pullback is



which is clear. □

10.4. Restriction species and directed restriction species. Recall that $\mathbb{I}$ denotes the category of finite sets and injections, and that a restriction species is a functor $R : \mathbb{I}^{\text{op}} \to \boldsymbol{Grpd}$, or equivalently, a right fibration $\mathbb{R} \to \mathbb{I}$. Recall also that $\mathbb{C}$ denotes the category of finite posets and convex maps, and that a *directed restriction species* is a functor $R : \mathbb{C}^{\text{op}} \to \boldsymbol{Grpd}$, or equivalently, a right fibration $\mathbb{R} \to \mathbb{C}$.

We are going to establish that every ordinary restriction species and every directed restriction species defines naturally a iesq sesquicartesian fibration. We will do the proofs for directed restriction species, and then exploit the fact that ordinary restriction species are a special kind of directed restriction species to deduce the results also for ordinary restriction species.

Proposition 10.5. *The projection* $\mathbb{C}{\downarrow}\underline{\mathbb{\Delta}} \to \underline{\mathbb{\Delta}}$ *is a iesq sesquicartesian fibration.*

Proof. The comma category is taken over ***Poset***. The objects of $\mathbb{C}{\downarrow}\underline{\mathbb{\Delta}}$ are poset maps $P \to \underline{k}$, and the arrows are squares in ***Poset***

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ \underline{n} & \longrightarrow & \underline{k} \end{array}$$

with $Q \to P$ a convex map and $\underline{n} \to \underline{k}$ a monotone map. Just from being a comma category projection, $\mathbb{C}{\downarrow}\underline{\mathbb{\Delta}} \to \underline{\mathbb{\Delta}}$ is a (split) cocartesian fibration. The chosen cocartesian arrows are squares in ***Poset*** of the form

$$\begin{array}{ccc} P & \xrightarrow{=} & P \\ \downarrow & & \downarrow \\ \underline{n} & \longrightarrow & \underline{k}. \end{array}$$

Over $\underline{\mathbb{\Delta}}_{\text{convex}}$ it is also a (split) cartesian fibration, as follows readily from Lemma 6.2 on pullback stability of convex maps in ***Poset***: the cartesian arrows over a convex map are squares in ***Poset*** of the form

$$\begin{array}{ccc} P & \rightarrowtail & P' \\ {\scriptstyle i^*\beta}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ \underline{k} & \underset{i}{\rightarrowtail} & \underline{k}'. \end{array}$$

The chosen cartesian arrows are the squares in which the map $P \to P'$ is an actual inclusion.

Finally for the iesq property, we need to check that given

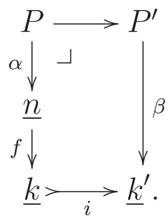
$$\begin{array}{ccc} \underline{a}+\underline{n}+\underline{b} = \underline{n}' & \overset{j}{\leftarrowtail} & \underline{n} \\ {\scriptstyle \mathrm{id}_a+f+\mathrm{id}_b}\downarrow{\scriptstyle =g} & & \downarrow{\scriptstyle f} \\ \underline{a}+\underline{k}+\underline{b} = \underline{k}' & \underset{i}{\leftarrowtail} & \underline{k} \end{array}$$

the resulting strictly commutative square

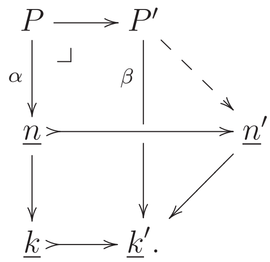
$$\begin{array}{ccc} \mathbb{C}_{/\underline{n}'} & \xrightarrow{j^*} & \mathbb{C}_{/\underline{n}} \\ {\scriptstyle g_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ \mathbb{C}_{/\underline{k}'} & \underset{i^*}{\longrightarrow} & \mathbb{C}_{/\underline{k}} \end{array}$$

is a pullback. To this end, note first that lowershriek functors between slices are cartesian fibrations, so it is enough to show that this square is a strict pullback. We first compute the strict pullback at the level of objects. A pair $(P' \xrightarrow{\beta} \underline{k}', P \xrightarrow{\alpha} \underline{n})$ lies in the pullback

$\mathbb{C}_{/\underline{k}'} \times_{\mathbb{C}_{/\underline{k}}} \mathbb{C}_{/\underline{n}}$ if $i^*\beta = f_!\alpha$, that is, P is an actual subposet of P' and this diagram is a pullback:



The claim is then that there is a unique way to complete this diagram to



Indeed, at the level of elements, P' is constituted by three subsets, namely the inverse images $P'_{\underline{a}}$, $P'_{\underline{k}}$ and $P'_{\underline{b}}$. (We don't need to worry about the poset structure, since we already know all of P'. The point is that the covariant functoriality does not change the total space.) We now define $P' \to \underline{n}' = \underline{a}+\underline{n}+\underline{b}$ as follows: we use β to define $P'_{\underline{a}} \to \underline{a}$ and $P'_{\underline{b}} \to \underline{b}$ on the outer subsets, and on the middle subset we use α to define $P'_{\underline{k}} = P \to \underline{n}$. Conversely, an element in $\mathbb{C}_{/\underline{n}'}$ defines a element in the pullback, and it is clear that the two constructions are inverse to each other. Having established that the two groupoids have the same objects, it remains to check that their automorphism groups agree. An automorphism of a pair $(P' \stackrel{\beta}{\to} \underline{k}', P \stackrel{\alpha}{\to} \underline{n})$ is an automorphism of P' compatible with the k'-layering and whose restriction to $\underline{k}$ is furthermore compatible with the refined layering here, given by $P \to \underline{n}$. But this is precisely to say that it is an automorphism of P' that is compatible with the layering $P' \to \underline{n}'$ constructed. □

Proposition 10.6. *There is a natural functor*

$$\boldsymbol{DRSp} \simeq \boldsymbol{RFib}_{/\mathbb{C}} \to \boldsymbol{IesqSesq},$$

*which takes a directed restriction species $R : \mathbb{C}^{\mathrm{op}} \to$ **Grpd** with associated right fibration $\mathbb{R} \to \mathbb{C}$ to the comma category projection $\mathbb{R}{\downarrow}\underline{\mathbb{\Delta}} \to \underline{\mathbb{\Delta}}$.*

Proof. Just note that stacking pullbacks on top of a comma square yields again comma squares:

$$
\begin{array}{ccc}
\mathbb{R}'{\downarrow}\underline{\mathbb{\Delta}} & \longrightarrow & \mathbb{R}' \\
\downarrow & \lrcorner & \downarrow \\
\mathbb{R}{\downarrow}\underline{\mathbb{\Delta}} & \longrightarrow & \mathbb{R} \\
\downarrow & \lrcorner & \downarrow \\
\mathbb{C}{\downarrow}\underline{\mathbb{\Delta}} & \longrightarrow & \mathbb{C} \\
\downarrow & \Leftarrow & \downarrow \\
\underline{\mathbb{\Delta}} & \longrightarrow & \boldsymbol{Poset}.
\end{array}
$$

Now $\mathbb{C}{\downarrow}\underline{\mathbb{\Delta}} \to \underline{\mathbb{\Delta}}$ is a iesq sesquicartesian fibration by Proposition 10.5, so the statement about objects follows from Lemma 10.3 (1) and the statement about morphisms from Lemma 10.3 (2). ☐

From these results for directed restriction species, the analogous results for ordinary restriction species can be deduced, remembering from 7.3 that $\mathbb{I} \to \mathbb{C}$ is a right fibration.

Corollary 10.7. *The projection* $\mathbb{I}{\downarrow}\underline{\mathbb{\Delta}} \to \underline{\mathbb{\Delta}}$ *is a iesq sesquicartesian fibration.*

Corollary 10.8. *For any ordinary restriction species* $R : \mathbb{I}^{\mathrm{op}} \to \boldsymbol{Grpd}$ *with associated right fibration* $\mathbb{R} \to \mathbb{I}$*, the comma category projection* $\mathbb{R}{\downarrow}\underline{\mathbb{\Delta}} \to \underline{\mathbb{\Delta}}$ *is a iesq sesquicartesian fibration.*

Proposition 10.6, together with Theorem 9.7 (that is, Propositions 8.5 and 9.6), gives the following result, summarising our constructions so far.

Theorem 10.9. *The preceding constructions define functors*

$$\boldsymbol{RSp} \xrightarrow{7.3} \boldsymbol{DRSp} \xrightarrow{10.6} \boldsymbol{IesqSesq} \xrightarrow{9.6} \mathrm{Fun}^{\mathrm{culf}}_{\mathrm{iesq}}(\mathbb{\nabla}, \boldsymbol{Grpd}) \xrightarrow{8.5} \boldsymbol{Decomp}^{\mathrm{culf}}.$$

These functors are not exactly fully faithful but we shall see in the next section that they become fully faithful when suitably sliced.

10.10. Unpacking, and comparison with the discussion in §7. Given a directed restriction species $R : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Grpd}$, we may consider the associated right fibration $p : \mathbb{R} \to \mathbb{C}$ as a morphism in $\boldsymbol{RFib}_{/\mathbb{C}}$ from p to the terminal object $\mathbb{C} \to \mathbb{C}$. Theorem 10.9 then associates to this a decomposition space $\mathsf{R} : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$ with a CULF functor $\Psi(p) : \mathsf{R} \to \mathbf{C}$, constructed via iesq-sesqui and nabla spaces.

Indeed, we have a functor

$$\Psi : \boldsymbol{RFib}_{/\mathbb{C}} \to \boldsymbol{Decomp}^{\mathrm{culf}}_{/\mathbf{C}}$$

to the category of decomposition spaces which are CULF over $\mathbf{C}$.

Let us unpack the constructions. Consider the pullback of p to the comma categories

$$\begin{array}{ccccc}
\mathbb{R}_{/\underline{n}} & \longrightarrow & \mathbb{R}{\downarrow}\underline{\mathbb{\Delta}} & \longrightarrow & \mathbb{R} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
\mathbb{C}_{/\underline{n}} & \longrightarrow & \mathbb{C}{\downarrow}\underline{\mathbb{\Delta}} & \longrightarrow & \mathbb{C} \\
\downarrow & \lrcorner & \downarrow & \Leftarrow & \downarrow \\
1 & \xrightarrow[\ulcorner \underline{n} \urcorner]{} & \underline{\mathbb{\Delta}} & \longrightarrow & \boldsymbol{Poset}.
\end{array}$$

The values of the simplicial groupoids R and $\mathbf{C}$ at $[n]$, are groupoid interiors of the fibres over $\underline{n} \in \underline{\mathbb{\Delta}}$,

$$\mathbf{C}_n = (\mathbb{C}{\downarrow}\underline{\mathbb{\Delta}})^{\text{iso}}_{\underline{n}} = \mathbb{C}^{\text{iso}}_{/\underline{n}}, \qquad \mathsf{R}_n = (\mathbb{R}{\downarrow}\underline{\mathbb{\Delta}})^{\text{iso}}_{\underline{n}} = \mathbb{R}^{\text{iso}}_{/\underline{n}} = \mathbb{C}^{\text{iso}}_{/\underline{n}} \times_{\mathbb{C}^{\text{iso}}} \mathbb{R}^{\text{iso}},$$

and $\Psi(p)_n : \mathsf{R}_n \to \mathbf{C}_n$ is the canonical projection. The simplicial structure is given as follows:

- An active map $g : [n] \rightarrow\!\!\!\shortmid [k]$ in $\mathbb{\Delta}$ and the corresponding $\underline{g} : \underline{k} \to \underline{n}$ in $\underline{\mathbb{\Delta}}$ induce, by postcomposition, the map of groupoids

  $$\mathbf{C}_k \to \mathbf{C}_n, \qquad (P \to \underline{k}) \mapsto \underline{g}_!(P \to \underline{k}) = (P \to \underline{k} \to \underline{n}).$$

  This in turn induces the map $\mathsf{R}_k \to \mathsf{R}_n$,

  $$\begin{array}{ccccc}
  \mathsf{R}_k & \longrightarrow & \mathsf{R}_n & \longrightarrow & \mathsf{R}_1 \\
  \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
  \mathbf{C}_k & \longrightarrow & \mathbf{C}_n & \longrightarrow & \mathbf{C}_1.
  \end{array}$$

  and hence the projection $\mathsf{R} \to \mathbf{C}$ is cartesian on active maps.
- An inert map $f : [n] \rightarrowtail [k]$ in $\mathbb{\Delta}$ and the associated convex map $\underline{f} : \underline{n} \rightarrowtail \underline{k}$ in $\underline{\mathbb{\Delta}}$ induce, by pullback, the homomorphism

  $$\mathbf{C}_k \to \mathbf{C}_n, \qquad (P \to \underline{k}) \mapsto (\underline{f}^* P \to \underline{n}).$$

  The definition of $\mathsf{R}_k \to \mathsf{R}_n$ uses the directed restriction species structure,

  $$\mathbf{C}_k \times_{\mathbf{C}_1} \mathsf{R}_1 \longrightarrow \mathbf{C}_n \times_{\mathbf{C}_1} \mathsf{R}_1, \qquad (P \to \underline{k},\ S) \longmapsto (f^* P \to \underline{n},\ (S|f^*P)\,).$$

11. Decalage and fully faithfulness

We have already exploited (Proposition 6.11) the decalage formulae

$$\mathrm{Dec}_{\bot}\mathbf{C} \simeq \mathbf{N}\mathbb{C}^{\text{lower}} \qquad \mathrm{Dec}_{\top}\mathbf{C} \simeq \mathbf{N}(\mathbb{C}^{\text{upper}})^{\text{op}}$$

which we now generalise as follows. For each directed restriction species R, we can pull back the corresponding right fibration $\mathbb{R} \to \mathbb{C}$ to these subcategories of lower- and upper-set inclusions, giving

$$\mathbb{R}^{\text{lower}} := \mathbb{C}^{\text{lower}} \times_{\mathbb{C}} \mathbb{R}, \qquad \mathbb{R}^{\text{upper}} := \mathbb{C}^{\text{upper}} \times_{\mathbb{C}} \mathbb{R},$$

the categories of R-structures and their lower-set and upper-set inclusions. Thus we have pullback functors

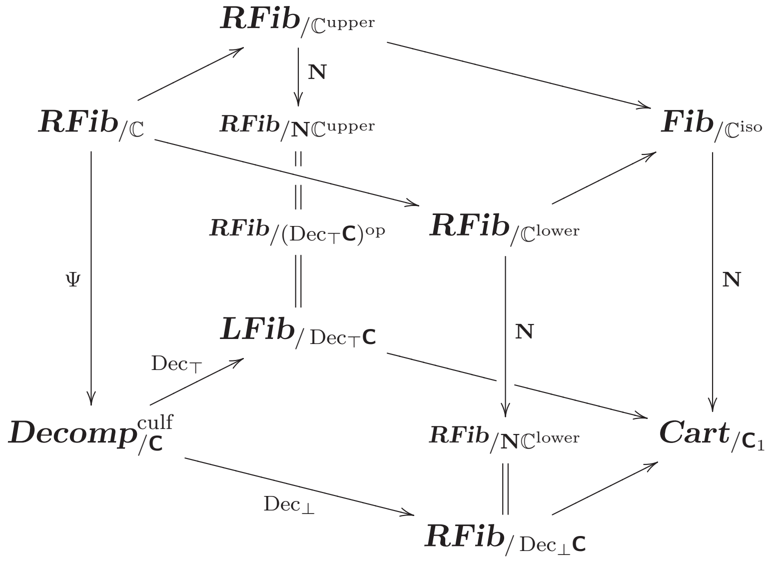
$$\boldsymbol{RFib}_{/\mathbb{C}^{\text{lower}}} \xleftarrow{\text{pbk}} \boldsymbol{RFib}_{/\mathbb{C}} \xrightarrow{\text{pbk}} \boldsymbol{RFib}_{/\mathbb{C}^{\text{upper}}}.$$

Proposition 11.1. *We have the following natural (levelwise) equivalences of simplicial groupoids:*

$$\mathrm{Dec}_\perp \mathsf{R} \simeq \mathbf{N}\mathbb{R}^{\mathrm{lower}} \qquad\qquad \mathrm{Dec}_\top \mathsf{R} \simeq \mathbf{N}(\mathbb{R}^{\mathrm{upper}})^{\mathrm{op}}.$$

Proof. The equivalences are expressed by commutativity of the left-hand faces (incident with the edge labelled by the functor $\Psi : \mathbb{R} \mapsto \mathsf{R}$) of the cube in the following lemma. $\square$

Lemma 11.2. *We have the commutative diagram*



Proof. We first prove that the left-hand faces commute. In simplicial degree zero the images of $p : \mathbb{R} \to \mathbb{C}$ clearly coincide: they are $p^{\mathrm{iso}} : \mathsf{R}_1 = \mathbb{R}^{\mathrm{iso}} \to \mathbf{C}_1 = \mathbb{C}^{\mathrm{iso}}$. Analogously to 10.10 we can write

$$(\mathrm{Dec}_\perp \mathsf{R})_k = \mathsf{R}_{k+1} = \mathbf{C}_{k+1} \times_{\mathbf{C}_1} \mathsf{R}_1 = (\mathrm{Dec}_\perp \mathbf{C})_k \times_{\mathbf{C}_1} \mathsf{R}_1$$
$$(\mathbf{N}\mathbb{R}^{\mathrm{lower}})_k = (\mathbf{N}\mathbb{C}^{\mathrm{lower}})_k \times_{(\mathbf{N}\mathbb{C}^{\mathrm{lower}})_0} (\mathbf{N}\mathbb{R}^{\mathrm{lower}})_0 = (\mathbf{N}\mathbb{C}^{\mathrm{lower}})_k \times_{\mathbf{C}_1} \mathsf{R}_1,$$

and similarly for $\mathrm{Dec}_\top$ and the categories of upper-set inclusions. From Proposition 6.11 we have canonical equivalences of simplicial groupoids $\mathrm{Dec}_\perp \mathbf{C} = \mathbf{N}\mathbb{C}^{\mathrm{lower}}$ and $\mathrm{Dec}_\top \mathbf{C} = \mathbf{N}(\mathbb{C}^{\mathrm{upper}})^{\mathrm{op}}$. We also have commuting diagrams for active or bottom face maps

$$\begin{array}{ccccccc}
(\mathrm{Dec}_\perp \mathsf{R})_k & \xrightarrow{=} & (\mathrm{Dec}_\perp \mathbf{C})_k \times_{\mathbf{C}_1} \mathsf{R}_1 & \xrightarrow{\cong} & (\mathbf{N}\mathbb{C}^{\mathrm{lower}})_k \times_{\mathbf{C}_1} \mathsf{R}_1 & \xrightarrow{=} & (\mathbf{N}\mathbb{R}^{\mathrm{lower}})_k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\mathrm{Dec}_\perp \mathsf{R})_n & \xrightarrow{=} & (\mathrm{Dec}_\perp \mathbf{C})_n \times_{\mathbf{C}_1} \mathsf{R}_1 & \xrightarrow{\cong} & (\mathbf{N}\mathbb{C}^{\mathrm{lower}})_n \times_{\mathbf{C}_1} \mathsf{R}_1 & \xrightarrow{=} & (\mathbf{N}\mathbb{R}^{\mathrm{lower}})_n
\end{array}$$

The diagram for $d_\top : [k-1] \to [k]$ also commutes:

$$\begin{array}{ccc}
\left(\begin{smallmatrix} P \\ \downarrow \\ \underline{k+1} \end{smallmatrix} , S \right) & \longleftrightarrow & (P_{\underline{1}} \subseteq P_{\underline{2}} \subseteq \cdots \subseteq P, S) \\
\Big\downarrow & & \Big\downarrow \\
\left(\begin{smallmatrix} \underline{d}^{\top *} P \\ \downarrow \\ \underline{k} \end{smallmatrix} , (S|\underline{d}^{\top *} P) \right) & \longleftrightarrow & (P_{\underline{1}} \subseteq \cdots \subseteq P_{\underline{k}},\ (S|P_{\underline{k}})) \,.
\end{array}$$

This shows that the two left-hand faces commute.

The top face is just pullback to $\mathbb{C}^{\text{iso}}$ taken in two steps in two ways. For the bottom face, observe first that C_1 is the constant simplicial groupoid with value $\mathsf{C}_1 = \mathbb{C}^{\text{iso}}$. The bottom face commutes because both ways around send a CULF map $\mathsf{R} \to \mathsf{C}$ to the (obviously cartesian) simplicial map of constant simplicial groupoids $\mathsf{R}_1 \to \mathsf{C}_1$. The right-hand faces are easier to understand with $\boldsymbol{RFib}_{/\mathbf{N}\mathbb{C}^{\text{upper}}}$ instead of $\boldsymbol{LFib}_{/\operatorname{Dec}_\top \mathsf{C}}$ and $\boldsymbol{RFib}_{/\mathbf{N}\mathbb{C}^{\text{lower}}}$ instead of $\boldsymbol{RFib}_{/\operatorname{Dec}_\perp \mathsf{C}}$: commutativity of the two squares then just amounts to the fact that the fat nerve commutes with pullbacks. $\square$

Since ordinary restriction species are just directed restriction species supported on discrete posets, Proposition 11.1 implies the following result, remembering that for discrete posets, every inclusion is both a lower-set and an upper-set inclusion:

Corollary 11.3. *For an ordinary restriction species $\mathbb{R} \to \mathbb{I}$ with associated decomposition space R, we have*

$$\operatorname{Dec}_\perp \mathsf{R} \simeq \mathbf{N}\mathbb{R} \qquad \operatorname{Dec}_\top \mathsf{R} \simeq \mathbf{N}\mathbb{R}^{\text{op}}.$$

Theorem 11.4. *The functor*

$$\Psi : \boldsymbol{DRSp} \longrightarrow \boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{C}}$$

is fully faithful.

Proof. From the cube diagram in Lemma 11.2 we get the commutative square

$$\begin{array}{ccc}
\boldsymbol{RFib}_{/\mathbb{C}} & \xrightarrow{(\text{pbk},\text{pbk})} & \boldsymbol{RFib}_{/\mathbb{C}^{\text{lower}}} \underset{\boldsymbol{Fib}_{/\mathbb{C}^{\text{iso}}}}{\times} \boldsymbol{RFib}_{/\mathbb{C}^{\text{upper}}} \\
{\scriptstyle \Psi}\downarrow & & \downarrow {\scriptstyle \mathbf{N} \underset{\mathbf{N}}{\times} \mathbf{N}} \\
\boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{C}} & \xrightarrow[(\operatorname{Dec}_\perp,\operatorname{Dec}_\top)]{} & \boldsymbol{RFib}_{/\operatorname{Dec}_\perp \mathsf{C}} \underset{\boldsymbol{Cart}_{/\mathsf{C}_1}}{\times} \boldsymbol{LFib}_{/\operatorname{Dec}_\top \mathsf{C}} .
\end{array}$$

Now the main point is that the pair of pullback functors is jointly fully faithful. Indeed, a transformation is natural in all convex maps if and only if it is natural in both lower-set inclusions and upper-set inclusions, since every convex inclusion factors (non-uniquely) as a lower-set inclusion followed by an upper-set inclusion. Since also the fibre product of fat nerves is fully faithful, and since the pair of Decs is faithful, we conclude that Ψ is fully faithful. $\square$

Theorem 11.5. *The functor*

$$\boldsymbol{RSp} \longrightarrow \boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{I}}$$

is fully faithful.

Proof. In the commutative diagram

$$\begin{array}{ccc}
\boldsymbol{RSp} & \longrightarrow & \boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{I}} \\
{\scriptstyle \text{f.f.}}\downarrow & & \downarrow{\scriptstyle \text{f.f.}} \\
\boldsymbol{DRSp} & \xrightarrow[\text{f.f.}]{} & \boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{C}}
\end{array}$$

$\boldsymbol{RSp} \subset \boldsymbol{DRSp}$ is clearly fully faithful; $\boldsymbol{DRSp} \to \boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{C}}$ is fully faithful by Theorem 11.4, and $\boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{I}} \to \boldsymbol{Decomp}^{\text{culf}}_{/\mathsf{C}}$ is fully faithful since $\mathsf{I} \to \mathsf{C}$ is a monomorphism in $\boldsymbol{Decomp}^{\text{culf}}$. □

12. Remarks on strictness

Since our general philosophy is that the homotopy content is the essence—and in the end we want to take homotopy cardinality anyway—we have worked in this paper with groupoids up to homotopy: when we say simplicial groupoid, we mean pseudo-functor $\mathbb{\Delta}^{\text{op}} \to \boldsymbol{Grpd}$, and all pullbacks mentioned are homotopy pullbacks.

Nevertheless, one may rightly feel that it is nicer to work with strict simplicial objects. In the present situation one can actually have a strict version of everything, if just restriction species and directed restriction species are assumed to be *strict* groupoid-valued functors, not pseudo-functors (and their morphisms *strict* natural transformations rather than pseudo-natural transformations). It is doable to trace through all the construction with sufficient care to ensure that the resulting decomposition spaces are again strict.

We finish the paper by outlining the arguments going into this. First of all:

12.1. Strict decomposition spaces. We define *strict decomposition spaces* to be strict functors $\mathbb{\Delta}^{\text{op}} \to \boldsymbol{Grpd}$ such that the active-inert squares are simultaneously strict pullbacks and homotopy pullbacks.

Note that the squares in question are already strictly commutative since they are strict simplicial identities, so in practice the pullback condition happens because it is a strict pullback in which one of the legs is an iso-fibration.

For example, the fat nerve of a small category is a strict decomposition space: it is clearly a strict functor, the Segal squares are readily seen to be strict pullbacks, and the face maps are iso-fibrations because the coface maps in $\mathbb{\Delta}$ are injective on objects.

12.2. Strict CULF functors. We define a *strict CULF functor* to be a strictly simplicial map, whose naturality squares on active maps are simultaneously strict pullbacks and homotopy pullbacks.

Again, this typically happens when the simplicial map is degree-wise an iso-fibration.

Theorem 12.3. *The functors* $\boldsymbol{RSp} \to \boldsymbol{DRSp} \to \boldsymbol{Decomp}^{\text{culf}}$ *of Theorem 10.9 take strict (directed) restriction species and their strict morphisms to strict decomposition spaces and strict CULF functors.*

Let us explain the main intermediate step.

12.4. Strictly iesq sesquicartesian fibrations. A sesquicartesian fibration is *split* when there are specified functorial cocartesian lifts for all maps and specified functorial cartesian lifts for convex maps, and such that the Beck–Chevalley isomorphisms are strict identities. A split sesquicartesian fibration is *strictly iesq* when the strictly commutative Beck–Chevalley squares are both strict pullbacks and homotopy pullbacks. A strict morphism of strictly iesq sesquicartesian fibrations is by definition a functor that preserves the specified lifts, both cocartesian and cartesian, and for which the square (10) is both a strict pullback and a homotopy pullback.

Lemma 12.5. *The functors* $\boldsymbol{RSp} \to \boldsymbol{DRSp} \to \boldsymbol{IesqSesq}$ *of Proposition 10.6 take strict (directed) restriction species and their strict morphisms to strictly iesq sesquicartesian fibrations and strict morphisms.*

The main ingredient in checking this is the fact that the base case $\mathbb{C}{\downarrow}\underline{\mathbb{\Delta}} \to \underline{\mathbb{\Delta}}$ is a strictly iesq sesquicartesian fibration. This follows from inspection of the proof of Proposition 10.5, where in fact the crucial pullback square was established as a strict pullback along an isofibration. For this we exploited in particular that the pullbacks of convex maps can be taken to be actual subset inclusions.

For the general strict directed restriction species (which includes $\mathbb{I}$), the proof follows from niceness of comma categories, including the fact that comma-category projections are always split cartesian and cocartesian fibrations, and therefore the top squares in the proof of Proposition 10.6 can be taken to be strict pullbacks.

Finally, it is straightforward to verify that all the strictnesses are preserved by the functor of Proposition 9.6 to (suitably strict) nabla spaces, and from there to strict decomposition spaces via Proposition 8.5.

We stress that for the sake of taking homotopy cardinality to obtain incidence coalgebras, the strictness is irrelevant.

References

[1] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile. *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*. Compos. Math. **142** (2006), 1–30. arXiv:math/0310016.

[2] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, vol. 29 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown, Stephen Chase, and André Joyal.

[3] John C. Baez and James Dolan. *From finite sets to Feynman diagrams*. In *Mathematics unlimited—2001 and beyond*, pp. 29–50. Springer, Berlin, 2001.

[4] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial species and tree-like structures*, vol. 67 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, with a foreword by Gian-Carlo Rota.

[5] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer. *2-Segal sets and the Waldhausen construction*. Topol. Appl. **235** (2018), 445–484. arXiv:1609.02853.

[6] Pierre Cartier and Dominique Foata. *Problèmes combinatoires de commutation et réarrangements*. No. 85 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, New York, 1969. Republished in the "books" section of the Séminaire Lotharingien de Combinatoire.

[7] Alain Connes and Dirk Kreimer. *Hopf algebras, renormalization and noncommutative geometry*. Comm. Math. Phys. **199** (1998), 203–242. arXiv:hep-th/9808042.

[8] Robert J. MacG. Dawson, Robert Paré, and Dorette A. Pronk. *Universal properties of Span*. Theory Appl. Categ. **13** (2004), 61–85.

[9] Arne Dür. *Möbius functions, incidence algebras and power series representations*, vol. 1202 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986.

[10] Tobias Dyckerhoff. *Higher categorical aspects of Hall Algebras*. In *Advanced Course on (Re)emerging Methods in Commutative Algebra and Representation Theory*, vol. 70 of Quaderns. CRM, Barcelona, 2015. arXiv:1505.06940.

[11] Tobias Dyckerhoff and Mikhail Kapranov. *Higher Segal spaces I*. Preprint, arXiv:1212.3563, to appear in Lecture Notes in Mathematics.

[12] Héctor Figueroa and José M. Gracia-Bondía. *Combinatorial Hopf algebras in quantum field theory. I*. Rev. Math. Phys. **17** (2005), 881–976. arXiv:hep-th/0408145.

[13] Loïc Foissy. *Les algèbres de Hopf des arbres enracinés décorés. I.* Bull. Sci. Math. **126** (2002), 193–239.

[14] Loïc Foissy. *Algebraic structures on double and plane posets.* J. Algebraic Combin. **37** (2013), 39–66. arXiv:1101.5231.

[15] Loïc Foissy. *Plane posets, special posets, and permutations.* Adv. Math. **240** (2013), 24–60. arXiv:1109.1101.

[16] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Groupoids and Faà di Bruno formulae for Green functions in bialgebras of trees.* Adv. Math. **254** (2014), 79–117. arXiv:1207.6404.

[17] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition Spaces, Incidence Algebras and Möbius Inversion.* (Old omnibus version, not intended for publication.) Preprint, arXiv:1404.3202.

[18] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Homotopy linear algebra.* Proc. Royal Soc. Edinburgh A. **148** (2018) 293–325. arXiv:1602.05082.

[19] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces, incidence algebras and Möbius inversion I: basic theory.* Adv. Math. **331** (2018) 952–1015. arXiv:1512.07573.

[20] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces, incidence algebras and Möbius inversion II: completeness, length filtration, and finiteness.* Adv. Math. **333** (2018) 1242–1292. arXiv:1512.07577.

[21] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces, incidence algebras and Möbius inversion III: the decomposition space of Möbius intervals.* Adv. Math. **334** (2018) 544–584. arXiv:1512.07580.

[22] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces in combinatorics.* Preprint, arXiv:1612.09225.

[23] David Gepner, Rune Haugseng and Joachim Kock. *∞-operads as analytic monads.* Preprint, arXiv:1712.06469.

[24] Ira M. Gessel. *Multipartite P-partitions and inner products of skew Schur functions.* In *Combinatorics and algebra (Boulder, CO, 1983)*, vol. 34 of Contemp. Math., pp. 289–317. Amer. Math. Soc., Providence, RI, 1984.

[25] Claudio Hermida. *Representable multicategories.* Adv. Math. **151** (2000), 164–225.

[26] Brandon Humpert and Jeremy L. Martin. *The incidence Hopf algebra of graphs.* SIAM J. Discrete Math. **26** (2012), 555–570. arXiv:1012.4786.

[27] Luc Illusie. *Complexe cotangent et déformations. II.* No. 283 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1972.

[28] Saj-nicole A. Joni and Gian-Carlo Rota. *Coalgebras and bialgebras in combinatorics.* Stud. Appl. Math. **61** (1979), 93–139.

[29] André Joyal. *Une théorie combinatoire des séries formelles.* Adv. Math. **42** (1981), 1–82.

[30] André Joyal. *Disks, duality and Θ-categories*, September 1997.

[31] Joachim Kock. *Polynomial functors and trees.* Int. Math. Res. Notices **2011** (2011), 609–673. arXiv:0807.2874.

[32] Joachim Kock. *Data types with symmetries and polynomial functors over groupoids.* In *Proceedings of the 28th Conference on the Mathematical Foundations of Programming Semantics (Bath, 2012)*, vol. 286 of Electr. Notes Theoret. Comput. Sci., pp. 351–365, 2012. arXiv:1210.0828.

[33] Joachim Kock. *Categorification of Hopf algebras of rooted trees.* Cent. Eur. J. Math. **11** (2013), 401–422. arXiv:1109.5785.

[34] Joachim Kock. *Perturbative renormalisation for not-quite-connected bialgebras.* Lett. Math. Phys. **105** (2015), 1413–1425. arXiv:1411.3098.

[35] Joachim Kock. *Graphs, hypergraphs, and properads.* Collect. Math. **67** (2016), 155–190. arXiv:1407.3744.

[36] Joachim Kock. *Polynomial functors and combinatorial Dyson–Schwinger equations.* J. Math. Phys. **58** (2017), 041703, 36pp. arXiv:1512.03027.

[37] Joachim Kock and Mark Weber. *Faà di Bruno for operads and internal algebras.* Preprint, arXiv:1609.03276.

[38] F. William Lawvere and Matías Menni. *The Hopf algebra of Möbius intervals.* Theory Appl. Categ. **24** (2010), 221–265.

[39] Pierre Leroux. *Les catégories de Möbius.* Cahiers Topol. Géom. Diff. **16** (1976), 280–282.
[40] Jacob Lurie. *Higher Algebra.* Available from http://www.math.harvard.edu/~lurie/, 2013.
[41] Claudia Malvenuto and Christophe Reutenauer. *A self paired Hopf algebra on double posets and a Littlewood-Richardson rule.* J. Combin. Theory Ser. A **118** (2011), 1322–1333. arXiv:0905.3508.
[42] Dominique Manchon. *On bialgebras and Hopf algebras of oriented graphs.* Confluentes Math. **4** (2012), 1240003, 10pp. arXiv:1011.3032.
[43] Yuri I. Manin. *A course in mathematical logic for mathematicians*, vol. 53 of Graduate Texts in Mathematics. Springer, New York, second edition, 2010. Chapters I–VIII translated from the Russian by Neal Koblitz, with new chapters by Boris Zilber and the author.
[44] Yuri I. Manin. *Renormalization and computation I: motivation and background.* In *OPERADS 2009*, vol. 26 of Sémin. Congr., pp. 181–222. Soc. Math. France, Paris, 2013. arXiv:0904.4921.
[45] Barry Mitchell. *Rings with several objects.* Adv. Math. **8** (1972), 1–161.
[46] James G. Oxley. *Matroid Theory.* Oxford Graduate Texts in Mathematics. Oxford University Press, 1997.
[47] Mark D. Penney. *The universal Hall bialgebra of a double* 2*-Segal space.* Preprint, arXiv:1711.10194.
[48] Gian-Carlo Rota. *On the foundations of combinatorial theory. I. Theory of Möbius functions.* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368.
[49] William R. Schmitt. *Hopf algebras of combinatorial structures.* Canad. J. Math. **45** (1993), 412–428.
[50] William R. Schmitt. *Incidence Hopf algebras.* J. Pure Appl. Algebra **96** (1994), 299–330.
[51] Richard P. Stanley. *Ordered structures and partitions.* Memoirs of the American Mathematical Society, No. 119. American Mathematical Society, Providence, R.I., 1972.
[52] Ross Street. *Fibrations and Yoneda's lemma in a* 2*-category.* In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pp. 104–133. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.
[53] Tashi Walde. *Hall monoidal categories and categorical modules.* Preprint, arXiv:1611.08241.
[54] Mark Weber. *Generic morphisms, parametric representations and weakly Cartesian monads.* Theory Appl. Categ. **13** (2004), 191–234.
[55] Mark Weber. *Familial 2-functors and parametric right adjoints.* Theory Appl. Categ. **18** (2007), 665–732.
[56] Matthew B. Young. *Relative 2-Segal spaces.* Preprint, arXiv:1611.09234.

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