UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Departament de Matemàtiques

July 2017
December 2017

The effective resistance of extended or contracted networks

Á. Carmona, A.M. Encinas and M. Mitjana

Abstract. The effective resistance on a given a network is a distance on it, intrinsically associated with the combinatorial Laplacian. This means that to compute the effective resistance, all vertices are equally considered and the only parameters really significant are the weight on each edge, its *conductance*. It results that this distance is very sensitive to small changes in the conductances and then allows us to discriminate between networks with similar structure.
It is possible to define effective resistances that, in addition to the conductance, also take into account a weight on each vertex. These *generalized effective resistances* also determine distances on the network, one for each normalized weight on the vertex set, and coincide with the former one if the weight is constant; that is, when it does not discriminate between vertices. It is known that this family of distances are associated with linear operators on the network, more general that the combinatorial Laplacian, namely positive semidefinite Schödinger operators.
The aim of this communication is to analyze the behavior of these distances under the usual network transformations, specially the so–called *Star–Mesh* transformation. We also compute the effective resistance for an *extended network*; that is the network obtained from the former one by joining a new vertex, and then study the effect of the *contraction of this new network*; that is we apply a star-mesh transformation with center in the joined vertex.

1. Extending Networks

In the previous section, given an admissible potential $q = \mathfrak{q}(\omega,\lambda)$, we have defined the effective resistance between two vertices with respect to the potential q and an unitary weight $\sigma \in \Omega$ and have proved some of its main properties. However, except for the case in which σ is a positive multiple of ω, we have not yet analyzed when the effective resistance determines a metric on Γ, that we declared as our main objective. In fact, we only have to analyze when the triangular inequality holds, since we have proved the other properties.

In this section we show as the q–harmonicity of the chosen weight σ is essential to prove that the corresponding effective resistance is a distance on the network. This property was assured in the case $\lambda = 0$, since then σ should be a multiple of ω and hence it is q–harmonic. Therefore, in this section we assume that the admissible potential q satisfies that $q = \mathfrak{q}(\omega,\lambda)$ where $\omega \in \Omega$ and $\lambda > 0$ and consider $\mathcal{L}_q$, $\mathcal{G}_q^V$, G_q^V its corresponding Schrödinder operator, Green operator for V and Green function for V, respectively.

2000 Mathematics Subject Classification:
Keywords and Prases: Effective resistance, extended networks, contracted networks, Green functions, Neighborhood Transformation

We also denote by $\mathcal{H}_q$ the set of q–superharmonic weights. Since $\lambda > 0$, G_q^V is positive, and moreover for any $y \in V$ we can define the weight $\sigma_q^y = G_q^V(\cdot, y)$. Then, we have the following characterization of the set $\mathcal{H}_q$.

Proposition 1.1. *For any $y \in V$ we have that $\sigma_q^y \in \mathcal{H}_q$. Moreover,*

$$\mathcal{H}_q = \Big\{ \mathcal{G}_q^V(f) = \sum_{y \in V} f(y)\sigma_q^y : f \geq 0 \text{ and } f \neq 0 \Big\}.$$

In particular, $\omega = \lambda^{-1} \sum\limits_{y\in V} \omega(y)\sigma_q^y \in \mathcal{H}_q$ and moreover $1 \in \mathcal{H}_q$ iff $q \geq 0$ (and $q \neq 0$).

Given $\sigma \in \mathcal{H}_q$, we also consider $R_{q,\sigma}$ the effective resistance with respect to q and σ. To prove that $R_{q,\sigma}$ determines a distance on Γ we use a well known technique originally implemented by M. Fiedler, see [12] for nonnegative potentials and also [4] for admissible potentials and when $\sigma = \omega$. The main idea is to embed the given network Γ in other one in such a way the Green function for V appears as the Green function of a subset in the host network. The most simple way to do this, is consider a new vertex, *the grounded vertex,* $\hat{x}$ and extend the conductance to form a host network $\widehat{\Gamma} = (\hat{V}, \hat{c})$, where $\widehat{V} = V \cup \{\hat{x}\}$. In addition, given σ, a weight on V, we consider $\hat{\sigma} > 0$ on $\widehat{V}$ such that $\hat{\sigma}(x) = \sigma(x)$ for any $x \in V$. So, the extension $\hat{\sigma}$ of the weight σ is determined by its value at the grounded vertex, $\sigma(\hat{x})$. For any $a > 0$, we call the a–extension of σ, the weight $\hat{\sigma}$ on $\widehat{V}$ defined as $\hat{\sigma} = \sigma$ on V and as $\hat{\sigma}(\hat{x}) = a$.

To define the value of the conductance $\hat{c}$ at the pairs $(x, \hat{x})$, $x \in V$, consider $u \in \mathcal{C}(V)$ and the Doob transform with respect to σ. Then, for any $x \in V$ we have

$$\mathcal{L}_q(u)(x) = \mathcal{L}_{q_\sigma}(u)(x) + (q - q_\sigma)(x)u(x) = \frac{1}{\sigma(x)} \sum_{y\in V} c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)} - \frac{u(y)}{\sigma(y)}\right) + (q - q_\sigma)(x)u(x).$$

If we impose $u(\hat{x}) = 0$, that electrically means that the new vertex $\hat{x}$ is grounded, then for any $x \in V$ we have

$$(q - q_\sigma)(x)u(x) = \frac{1}{\sigma(x)}\,(q - q_\sigma)(x)\sigma(x)^2 \frac{u(x)}{\sigma(x)} = \frac{1}{\sigma(x)}\,\big[(q - q_\sigma)(x)\sigma(x)\sigma(\hat{x})^{-1}\big]\sigma(x)\sigma(\hat{x})\left(\frac{u(x)}{\sigma(x)} - \frac{u(\hat{x})}{\sigma(\hat{x})}\right).$$

Since $\sigma \in \mathcal{H}_q$, we know that $q - q_\sigma \geq 0$ on V and that $q \neq q_\sigma$. Therefore, if we define the conductance $\hat{c}(x, \hat{x}) = (q - q_\sigma)(x)\sigma(x)\sigma(\hat{x})^{-1} = \sigma(\hat{x})^{-1}\mathcal{L}_q(\sigma)(x)$ for any $x \in V$, then

$$\mathcal{L}_q(u)(x) = \frac{1}{\sigma(x)} \sum_{y\in\hat{V}} \hat{c}(x,y)\hat{\sigma}(x)\hat{\sigma}(y)\left(\frac{u(x)}{\hat{\sigma}(x)} - \frac{u(y)}{\hat{\sigma}(y)}\right), \quad x \in V.$$

Fixed $\sigma \in \mathcal{H}_q$ and $a > 0$ we consider the a–extension of σ and call the *Fiedler extension of the network* $\Gamma = (V, c)$ *with respect to σ and a*, the network $\widehat{\Gamma} = (V \cup \{\hat{x}\}, \hat{c})$ where $\hat{c} = c$ on $V \times V$ and

$$\hat{c}(x, \hat{x}) = \hat{c}(\hat{x}, x) = a^{-1}\,\big(q(x) - q_\sigma(x)\big)\sigma(x) = a^{-1}\,\mathcal{L}_q(\sigma)(x), \quad x \in V.$$

Since $\mathsf{supp}\big(\mathcal{L}_q(\sigma)\big) \neq \emptyset$, the Fiedler extension consists in joining each vertex $x \in \mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$ with the grounded vertex $\hat{x}$ through an edge whose conductance depends on the value of the weight at both extremes and on the *potential excess* at x, $q(x) - q_\sigma(x)$. In particular, the host network $\widehat{\Gamma}$ is always connected. Next, we prove that any connected host network of Γ with exactly one more vertex, is in fact a Fiedler extension of Γ.

Lemma 1.2. *Let $\hat{x} \notin V$ and $\widehat{\Gamma} = (V \cup \{\hat{x}\}, \hat{c})$ a connected host network of Γ. Then, for any $a > 0$ there exists $\sigma \in \mathcal{H}_q$ such that $\widehat{\Gamma}$ is the Fiedler extension of Γ, with respect to σ and a.*

Proof. If we define $f \in \mathcal{C}(V)$ as $f(x) = a\hat{c}(x, \hat{x})$, then $f \geq 0$ and moreover $f \neq 0$. Therefore, from Proposition 1.1, if $\sigma = \mathcal{G}_q(f)$, then $\sigma \in \mathcal{H}_q$ and moreover $\mathcal{L}_q(\sigma) = f$. This last identity implies that $\hat{c}(x, \hat{x}) = a^{-1}\mathcal{L}_q(\sigma)$ and hence, $\widehat{\Gamma}$ is the Fiedler extension of Γ, with respect to σ and a. □

The combinatorial Laplacian corresponding to the Fiedler extension is denoted by $\widehat{\mathcal{L}}$. The next result establishes the relationship between the original Schrödinger operator $\mathcal{L}_q$ and a new singular and positive semi-definite Schrödinger operator on $\widehat{\Gamma}$.

Proposition 1.3. *If we consider $\hat{q}=\mathsf{q}(\hat{\sigma},0)$, then*

$$\hat{q}=q-a^{-1}\mathcal{L}_q(\sigma)\ \ \textit{on } V \quad \textit{and} \quad \hat{q}(\hat{x})=a^{-2}\big(\langle\mathcal{L}_q(\sigma),\sigma\rangle u(\hat{x})-a\langle\mathcal{L}_q(\sigma),u\rangle\big).$$

Moreover, for any $u\in\mathcal{C}(\widehat{V})$ we get that

$$\widehat{\mathcal{L}}_{\hat{q}}(u)=\mathcal{L}_q(u)-a^{-1}\mathcal{L}_q(\sigma)\,u(\hat{x})\ \ \textit{on } V \quad \textit{and} \quad \widehat{\mathcal{L}}_{\hat{q}}(u)(\hat{x})=a^{-2}\big(\langle\mathcal{L}_q(\sigma),\sigma\rangle u(\hat{x})-a\langle\mathcal{L}_q(\sigma),u\rangle\big).$$

Proof. Given $u\in\mathcal{C}(\hat{V})$, then for any $x\in V$ we get that

$$\begin{aligned}\widehat{\mathcal{L}}(u)(x)&=\mathcal{L}(u_{|_V})(x)+a^{-1}\mathcal{L}_q(\sigma)(x)u(x)-a^{-1}\mathcal{L}_q(\sigma)(x)u(\hat{x}),\quad x\in V,\\ \widehat{\mathcal{L}}(u)(\hat{x})&=a^{-1}\big(u(\hat{x})\langle q,\sigma\rangle-\langle\mathcal{L}_q(\sigma),u\rangle\big),\end{aligned}$$

since $\langle\mathcal{L}_q(\sigma),1\rangle=\langle q,\sigma\rangle$. In particular, taking $u=\hat{\sigma}$ we obtain

$$\begin{aligned}\hat{q}&=-\hat{\sigma}^{-1}\widehat{\mathcal{L}}(\hat{\sigma})=q_\sigma-a^{-1}\mathcal{L}_q(\sigma)+\sigma^{-1}\mathcal{L}_q(\sigma)=q-a^{-1}\mathcal{L}_q(\sigma),\quad \text{on } V,\\ \hat{q}(\hat{x})&=a^{-2}\big(\langle\mathcal{L}_q(\sigma),\sigma\rangle-a\langle q,\sigma\rangle\big)\end{aligned}$$

and hence

$$\begin{aligned}\widehat{\mathcal{L}}_{\hat{q}}(u)&=\mathcal{L}_q(u)-a^{-1}\mathcal{L}_q(\sigma)\,u(\hat{x}), &&\text{on } V,\\ \widehat{\mathcal{L}}_{\hat{q}}(u)(\hat{x})&=a^{-2}\big(\langle\mathcal{L}_q(\sigma),\sigma\rangle u(\hat{x})-a\langle\mathcal{L}_q(\sigma),u\rangle\big).\end{aligned}$$

□

Corollary 1.4. *Given $f\in\mathcal{C}(V)$ consider $u\in\mathcal{C}(V)$, the unique solution of the Poisson equation $\mathcal{L}_q(u)=f$ on V. Then, $u=v_{|_V}$ where v is the unique solution of the Dirichlet Problem on $\widehat{\Gamma}$*

$$\widehat{\mathcal{L}}_{\hat{q}}(v)=f\ \ \textit{on } V\ \ \textit{and}\ \ v(\hat{x})=0.$$

In particular, G_q^V, the Green function for V is the bottleneck function at $\hat{x}$ for $\widehat{\Gamma}$.

Proof. We know that both, the Poisson equation on Γ and the Dirichlet problem have a unique solution, say $u,v\in\mathcal{C}(V)$, respectively. From the above Proposition we have that

$$f=\widehat{\mathcal{L}}_{\hat{q}}(v)=\mathcal{L}_q(v_{|_V})-a^{-1}\mathcal{L}_q(\sigma)\,v(\hat{x})=\mathcal{L}_q(v_{|_V})\ \ \text{on } V$$

which implies that $u=v_{|_V}$. □

Now we have all ingredients to prove the result we are looking for.

Theorem 1.5. *Given $\sigma\in\mathcal{H}_q$, then $R_{q,\sigma}$, the effective resistance with respect to q and σ, determines a distance on Γ. Moreover given $x,y,z\in V$, $R_{q,\sigma}(x,z)+R_{q,\sigma}(z,y)=R_{q,\sigma}(x,y)$ iff z separates x,y in Γ and moreover either $x\notin\mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$ or $y\notin\mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$.*

Proof. Consider the potential $\hat{q}=\mathsf{q}(\hat{\sigma},0)$ in the host network $\widehat{\Gamma}$ and $\widehat{R}_{\hat{q},\hat{\sigma}}$ its associated effective resistance. Then $\widehat{R}_{\hat{q},\hat{\sigma}}=d_{\hat{q}}$ and hence determines a distance on $\widehat{\Gamma}$. Therefore, its restriction ot $V\times V$ also determines a distance on Γ.

On the other hand, applying the Identity (**??**), or equivalently the Identity (**??**), and taking into account that G_q^V is the bottleneck function for $\widehat{\Gamma}$ at $\hat{x}$, we have that

$$d_{\hat{q}}(x,y)=R_{\hat{q},\hat{\sigma}}(x,y)=\frac{G_q^V(x,x)}{\sigma(x)^2}+\frac{G_q^V(y,y)}{\sigma(y)^2}-\frac{2G_q^V(x,y)}{\sigma(y)\sigma(x)}=R_{q,\sigma}(x,y),\quad x,y\in V.$$

□

We remark that, applying the identity (**??**), in the host network we also have the following identity

$$\widehat{R}_{\hat{q},\hat{\sigma}}(x,\hat{x}) = \frac{G_q^V(x,x)}{\sigma(x)^2}, \quad \text{for any } x \in V.$$

As a by–product, we have the following relation between the Kirchhoff indexes

$$\mathsf{K}(\hat{q},\hat{\sigma}) = \mathsf{K}(q,\sigma) + a^2\mathsf{tr}(\mathcal{G}_q^V) = \big(a^2 + ||\sigma||_2^2\big)\mathsf{tr}(G_q^V) - \langle \mathcal{G}_q^V(\sigma),\sigma\rangle \tag{1}$$

In addition, we can newly apply the part (iii) of Corollary **??** to obtain the Green function for $\widehat{\Gamma}$

$$\widehat{G}_{\hat{q}}(x,y) = \begin{cases} \sigma(x)\sigma(y)\Big[\dfrac{G_q^V(x,y)}{\sigma(x)\sigma(y)} - \dfrac{\mathcal{G}_q^V(\sigma)(y)}{(a^2+||\sigma||_2^2)\sigma(y)} - \dfrac{\mathcal{G}_q^V(\sigma)(x)}{(a^2+||\sigma||_2^2)\sigma(x)} + \dfrac{\langle \mathcal{G}_q^V(\sigma),\sigma\rangle}{(a^2+||\sigma||_2^2)^2}\Big], & x,y \in V,\\[2ex] a\sigma(x)\Big[\dfrac{\langle \mathcal{G}_q^V(\sigma),\sigma\rangle}{(a^2+||\sigma||_2^2)^2} - \dfrac{\mathcal{G}_q^V(\sigma)(x)}{(a^2+||\sigma||_2^2)\sigma(x)}\Big] & x \in V,\ y = \hat{x},\\[2ex] \dfrac{a^2\langle \mathcal{G}_q^V(\sigma),\sigma\rangle}{(a^2+||\sigma||_2^2)^2} & x = y = \hat{x}. \end{cases}$$

Moreover applying the part (iv) of the Corollary **??**, we have that the bottleneck function for Γ and any vertex $z \in V$ is given by

$$\widehat{G}^z_{\hat{q}}(x,y) = \begin{cases} \sigma(x)\sigma(y)\Big[\dfrac{G_q^V(x,y)}{\sigma(x)\sigma(y)} - \dfrac{G_q^V(y,z)}{\sigma(y)\sigma(z)} - \dfrac{G_q^V(x,z)}{\sigma(x)\sigma(z)} + \dfrac{G_q^V(z,z)}{\sigma(z)^2}\Big], & x,y \in V,\\[2ex] \sigma(x)a\Big[\dfrac{G_q^V(z,z)}{\sigma(z)^2} - \dfrac{G_q^V(x,z)}{\sigma(x)\sigma(z)}\Big], & x \in V,\ y = \hat{x},\\[2ex] \dfrac{a^2 G_q^V(z,z)}{\sigma(z)^2}, & x = y = \hat{x}. \end{cases}$$

If given $z \in V$ we consider the weight σ_q^z, we have the following result.

Corollary 1.6. *For any $z \in V$, the function*

$$d_z(x,y) = \frac{G_q^V(x,x)}{G_q^V(x,z)^2} + \frac{G_q^V(y,y)}{G_q^V(y,z)^2} - \frac{2G_q^V(x,y)}{G_q^V(x,z)G_q^V(y,z)}, \quad x,y \in V,$$

determines a cutpoint additive distance on Γ.

Notice that

$$d_z(z,y) = \frac{G_q^V(z,z)}{G_q^V(z,z)^2} + \frac{G_q^V(y,y)}{G_q^V(y,z)^2} - \frac{2G_q^V(z,y)}{G_q^V(z,z)G_q^V(y,z)} = \frac{G_q^V(y,y)}{G_q^V(y,z)^2} - \frac{1}{G_q^V(z,z)}$$

and hence the positiveness of $d_z(z,y)$ is equivalent to the Cauchy-Schwarz inequality

$$G_q^V(z,y)^2 = \langle \mathcal{G}_q^V(\varepsilon_z),\varepsilon_y\rangle^2 \le \langle \mathcal{G}_q^V(\varepsilon_z),\varepsilon_z\rangle\langle \mathcal{G}_q^V(\varepsilon_y),\varepsilon_y\rangle = G_q^V(z,z)G_q^V(y,y)$$

with equality iff $z = y$, since $\mathcal{G}_q$ is positive definite.

We end this section by observing that the effective resistance on $\widehat{\Gamma}$ with respect to $\hat{q} = \mathsf{q}(\hat{\sigma},0)$ in fact does not depend of a the value of the extended weight $\hat{\sigma}$ at the grounded vertex $\hat{x}$.

2. Contracting networks: The Neighborhood Transformation

In this section we introduce a transformation on the network Γ by deleting a given vertex $x\in V$ but maintaining the connectedness. In what follows, we fix $x_0\in V$ and $F=V\setminus\{x_0\}$. Therefore, $\mathcal{C}(F)$ is the set of real functions on V vanishing at x_0.

Let us consider a potential $q\in\mathcal{C}(F)$ and hence such that $q(x_0)=0$, and the Poisson equation $\mathcal{L}_q(u)=f$, where $f\in\mathcal{C}(V)$. Then, Identity (**??**) implies that

$$\kappa(x_0)u(x_0)=f(x_0)+\sum_{y\in V}c(x_0,y)\,u(y)$$

and hence, we get

$$u(x_0)=\frac{1}{\kappa(x_0)}\Big[f(x_0)+\sum_{z\in F}c(x_0,z)\,u(z)\Big]. \tag{2}$$

Lemma 2.1. *If $q\in\mathcal{C}(F)$, then for any $f\in\mathcal{C}(V)$, $u\in\mathcal{C}(V)$ is a solution of the Poisson equation $\mathcal{L}_q(u)=f$ iff for any $x\in F$ we have*

$$f(x)+\frac{c(x_0,x)}{\kappa(x_0)}\,f(x_0)=\sum_{y\in F}\left[\frac{c(x,x_0)c(x_0,y)}{\kappa(x_0)}+c(x,y)\right]\big(u(x)-u(y)\big)++q(x)\,u(x),$$

and, in addition, $u(x_0)=\dfrac{1}{\kappa(x_0)}\Big[f(x_0)+\sum_{y\in F}c(x_0,y)\,u(y)\Big].$

Proof. Given $x\in F$, then Identity (2) implies that

$$u(x)-u(x_0)=\frac{1}{\kappa(x_0)}\sum_{y\in F}c(x_0,y)\big(u(x)-u(y)\big)-\frac{f(x_0)}{\kappa(x_0)}$$

and hence

$$\begin{aligned}f(x)&=c(x,x_0)\big(u(x)-u(x_0)\big)+\sum_{y\in F}c(x,y)\big(u(x)-u(y)\big)+q(x)u(x)\\&=-\frac{c(x_0,x)}{\kappa(x_0)}\,f(x_0)+\sum_{y\in F}\left[\frac{c(x,x_0)c(x_0,y)}{\kappa(x_0)}+c(x,y)\right]\big(u(x)-u(y)\big)+q(x)u(x).\end{aligned}$$

□

The identity given in the above Lemma motivates the definition of the function $c^{x_0}\colon F\times F\longrightarrow[0,+\infty)$, given by $c^{x_0}(x,x)=0$ for any $x\in F$ and by

$$c^{x_0}(x,x)=0,\quad x\in F\quad\text{and}\quad c^{x_0}(x,y)=c(x,y)+\frac{c(x,x_0)c(x_0,y)}{\kappa(x_0)},\quad x,y\in F,\ x\neq y. \tag{3}$$

If we define in $F\times F$ the adjacency relation $x\overset{x_0}{\sim}y$ iff $c^{x_0}(x,y)>0$ and denote the new graph as Γ^{x_0}, clearly c^{x_0} is a conductance on Γ^{x_0} and then (Γ^{x_0},c^{x_0}) is a new network that, in the sequel, we denote simply by Γ^{x_0}. Moreover, its corresponding combinatorial Laplacian is denoted by $\mathcal{L}^{x_0}$.

We say that the network Γ^{x_0} has been obtained from Γ after the *Neighborhood Transformation at vertex* x_0. Observe that any pair of vertices that are adjacent to x_0 in Γ are adjacent in the network Γ^{x_0}, because if $x,y\sim x_0$ in Γ, then $c(x,x_0)c(x_0,y)>0$. In other words, the subnetwork in Γ^{x_0} consisting in the neighborhood of x_0 in Γ is complete. For this reason, the Neighborhood Transformation is also named *Star–Mesh transformation.* Notice that , when $x\not\sim x_0$ in Γ, then $c^{x_0}(x,y)=c(x,y)$ for all $y\in F$.

Lemma 2.2. *The network Γ^{x_0} is connected.*

Proof. Let $x,y\in V^x$ and consider $x=z_0\sim z_1\sim\cdots\sim z_n\sim z_{n+1}=y$ a path joining x and y in Γ. If $z_j\neq x_0$ for any $j=1,\dots,n$, then $c^{x_0}(z_i,z_{i+1})\geq c(z_i,z_{i+1})>0$ and hence $xz_1\cdots z_ny$ is a path in Γ^{x_0}.

On the other hand, if $z_i = x$ for some $i = 1, \dots, n$, then $z_{i-1}, z_{i+1} \sim x_0$ and hence $z_{i-1} \sim z_{i+1}$ in Γ^{x_0}. So, we can delete vertex z_i in the above path and continue having a path on Γ^{x_0}. □

Given $u \in \mathcal{C}(F)$, *the harmonic extension of* u *at* x_0 is $u_h \in \mathcal{C}(V)$ defined as

$$u_h(y) = u(y) \quad \text{for any} \quad y \in F \quad \text{and} \quad u_h(x_0) = \frac{1}{\kappa(x_0)} \sum_{y \in F} c(x_0, y)\, u(y). \tag{4}$$

Observe that if $u \geq 0$ on the neighborhood of x_0, then $u_h(x_0) \geq 0$ and the equality holds iff $u(x) = 0$ for any $x \sim x_0$.

Lemma 2.3. *Given $q \in \mathcal{C}(F)$, if $u \in \mathcal{C}(F)$ then $\mathcal{L}_q(u_h)(x_0) = 0$ and conversely if $u \in \mathcal{C}(V)$ satisfied that $\mathcal{L}_q(u)(x_0) = 0$, then $u = v_h$ where $v = u_{|_F}$.*

After the above definitions, (3) and (4), the result of Lemma 2.1 can straightforwardly be re-written as follows.

Proposition 2.4. *Given $q, f \in \mathcal{C}(F)$, then $u \in \mathcal{C}(F)$ is a solution of the Poisson equation $\mathcal{L}_q^{x_0}(u) = f$ on F iff $u_h \in C(V)$ is a solution of the Poisson equation $\mathcal{L}_q(u_h) = f$ on V. In particular, if $\omega \in \mathcal{C}(F)$ satisfies that $\omega > 0$ on F, then ω_h is a weight, $\mathcal{L}(\omega_h) \in \mathcal{C}(F)$ and moreover $\mathcal{L}^{x_0}(\omega) = \mathcal{L}(\omega_h)$ on F.*

In the sequel we consider the set $\Omega(F) = \{\omega \in \mathcal{C}(F) : \omega > 0 \text{ on } F\}$ and denote by Ω_{x_0}, the set of weights on Γ that are harmonic at x_0. It is clear that $q_\omega \in \mathcal{C}(F)$ for all $\omega \in \Omega_{x_0}$. Moreover it is clearly satisfied that

$$\Omega(F) = \left\{\omega_{|_F} : \omega \in \Omega_{x_0}\right\} \quad \text{and} \quad \Omega_{x_0} = \left\{\omega_h : \omega \in \Omega(F)\right\} \tag{5}$$

Lemma 2.5. *Given $\omega \in \Omega(F)$, then $\omega^{-1}\mathcal{L}^{x_0}(\omega) = \omega_h^{-1}\mathcal{L}(\omega_h)$ on F. Conversely, given $\omega \in \Omega_{x_0}(V)$, then $\omega^{-1}\mathcal{L}^{x_0}(\omega) = \omega^{-1}\mathcal{L}(\omega)$ on F.*

Given $\omega \in \Omega(F)$, the above Lemma permits to identify the potential on Γ^{x_0} associated with ω with the potential on Γ associated with ω_h; that is with $\omega_h^{-1}\mathcal{L}(\omega_h)$. In the sequel we systematically use this identification and then both will be denoted by q_ω.

Corollary 2.6. *Given $\omega \in \Omega(F)$ and $f \in \mathcal{C}(F)$ such that $f \in \omega^\perp$, then $u \in \mathcal{C}(F)$ is a solution of the Poisson equation $\mathcal{L}_{q_\omega}^{x_0}(u) = f$ on F iff $u_h \in C(V)$ is a solution of the Poisson equation $\mathcal{L}_{q_\omega}(u_h) = f$ on V.*

Consider now fixed $\omega \in \Omega(F)$ and $G_{q_\omega}^{x_0}$ the Green function for Γ^{x_0} and G_{q_ω} the Green function for Γ. In addition, $R_{q_\omega}^{x_0}$ and R_{q_ω} denote their corresponding effective resistances, respectively.

Theorem 2.7. *For any $x, y \in F$, we get*

$$G_{q_\omega}^{x_0}(x, y) = G_{q_\omega}(x, y) + \omega_h(x_0)\Big[\omega(y)G_{q_\omega}(x, x_0) + \omega(x)G_{q_\omega}(x_0, y)\Big] + \omega(x)\omega(y)\omega_h^2(x_0)G_{q_\omega}(x_0, x_0),$$

where $\omega_h(x_0) = \dfrac{1}{\kappa(x_0)} \sum\limits_{y \in F} c(x_0, y)\, \omega(y).$

Proof. Given $y \in F$, consider $f = \varepsilon_y - \omega(y)\omega$ and u the unique solution of the Poisson equation $\mathcal{L}_{q_\omega}^{x_0}(u) = f$ such that $u \in \omega^\perp$. According with Corollary 2.6,

$$u = G_{q_\omega}(\cdot, y) - \omega(y)\mathcal{G}_{q_\omega}(\omega_h - \omega_h(x_0)\varepsilon_{x_0}) + \alpha\omega = G_{q_\omega}(\cdot, y) + \omega(y)\omega_h(x_0)G_{q_\omega}(\cdot, x_0) + \alpha\omega$$

where

$$\begin{aligned} 0 = \langle u, \omega \rangle &= \sum_{x \in F} \omega(x)G_{q_\omega}(x, y) + \omega(y)\omega_h(x_0) \sum_{x \in F} \omega(x)G_{q_\omega}(x, x_0) + \alpha \\ &= -\omega_h(x_0)G_{q_\omega}(x_0, y) - \omega(y)\omega_h^2(x_0)G_{q_\omega}(x_0, x_0) + \alpha \end{aligned}$$

that is, $\alpha = \omega_h(x_0)\Big[G(x_0, y) + \omega(y)\omega_h(x_0)G(x_0, x_0)\Big]$. □

Corollary 2.8. *$R_{q_\omega}^{x_0}$ is the restriction of R_{q_ω} to $F \times F$.*

Proof. Applying the Identity (**??**), for any $x, y \in F$,

$$
\begin{aligned}
R^{x_0}_{\omega}(x,y) &= \frac{G^{x_0}_{q_\omega}(x,x)}{\omega^2(x)} + \frac{G^{x_0}_{q_\omega}(y,y)}{\omega^2(y)} - \frac{2G^{x_0}_{q_\omega}(x,y)}{\omega(x)\omega(y)} \\
&= \frac{G_{q_\omega}(x,x)}{\omega^2(x)} + 2\omega_h(x_0)\frac{G_{q_\omega}(x,x_0)}{\omega(x)} + \omega_h^2(x_0)G_{q_\omega}(x_0,x_0) \\
&+ \frac{G_{q_\omega}(y,y)}{\omega^2(y)} + 2\omega_h(x_0)\frac{G_{q_\omega}(y,x_0)}{\omega(y)} + \omega_h^2(x_0)G_{q_\omega}(x_0,x_0) \\
&- \frac{2G_{q_\omega}(x,y)}{\omega(x)\omega(y)} - 2\omega_h(x_0)\Big[\frac{G_{q_\omega}(x,x_0)}{\omega(x)} + \frac{G_{q_\omega}(x_0,y)}{\omega(y)}\Big] - 2\omega_h^2(x_0)G_{q_\omega}(x_0,x_0) = R_{q_\omega}(x,y).
\end{aligned}
$$

□

We end this section considering newly the connected network $\Gamma = (V, c)$ and the admissible potential $q = \mathfrak{q}(\omega, \lambda)$, where $\omega \in \Omega$. Therefore, $\mathcal{L}_q$ is the corresponding Schrödinger operator, $\mathcal{G}_q$ and G_q the Green operator for the network, and R_q the associated effective resistance. In addition we also consider fixed a new vertex $\hat{x} \notin V$.

Fixed $\sigma \in \mathcal{H}_q$ and $a > 0$, we consider the a–extension of σ and $\widehat{\Gamma} = (V \cup \{\hat{x}\}, \hat{c})$, the Fiedler extension of the network Γ with respect to σ and a, where $\hat{c} = c$ on $V \times V$ and $\hat{c}(x, \hat{x}) = a^{-1}\mathcal{L}_q(\sigma)(x)$, for any $x \in V$. Observe that $\hat{c}(x, \hat{x}) > 0$ only when $x \in \mathsf{supp}\big(\mathcal{L}_q(\sigma)\big)$. Then $\widehat{\kappa}(\hat{x}) = a^{-1}\langle q, \sigma\rangle$ and hence, if σ_h is the harmonic extension of σ to $\widehat{\Gamma}$, then

$$
\sigma_h(\hat{x}) = \frac{1}{\widehat{\kappa}(\hat{x})}\sum_{y \in F}\hat{c}(\hat{x}, y)\,\sigma(y) = \langle q, \sigma\rangle^{-1}\langle \mathcal{L}_q(\sigma), \sigma\rangle.
$$

Therefore, $\hat{\sigma} = \sigma_h$ iff $a = \langle q, \sigma\rangle^{-1}\langle \mathcal{L}_q(\sigma), \sigma\rangle$. In particular, when $\sigma = \omega$, then $\hat{c}(x, \hat{x}) = a^{-1}\lambda\,\omega(x)$, for any $x \in V$ and $\hat{\omega} = \omega_h$ iff $a = ||\omega||_1^{-1}$.

Now, consider the Neighborhood Transformation at vertex $\hat{x}$. Then for any $x, y \in V$, $x \neq y$, we have that

$$
c^{\hat{x}}(x,y) = \hat{c}(x,y) + \frac{\hat{c}(x,\hat{x})\hat{c}(\hat{x},y)}{\hat{\kappa}(\hat{x})} = c(x,y) + a^{-1}\langle q,\sigma\rangle^{-1}\mathcal{L}_q(\sigma)(x)\mathcal{L}_q(\sigma)(y).
$$

As a consequence of the Corollary 2.8, we have the following relation between the effective resistance of the Fiedler extension and the effective resistance after a Neighborhood Transformation.

Theorem 2.9. *Given $\sigma \in \mathcal{H}_q$, consider the conductance $c^\sigma \colon V \times V \longrightarrow \mathbb{R}$ defined as*

$$
c^\sigma(x,y) = c(x,y) + \langle \mathcal{L}_q(\sigma), \sigma\rangle^{-1}\mathcal{L}_q(\sigma)(x)\mathcal{L}_q(\sigma)(y), \quad \textit{for any } x, y \in V,\ x \neq y.
$$

Then $R_{q,\sigma} = R^\sigma$, where R^σ is the effective resistance of the network $\Gamma^\sigma = (V, c^\sigma)$ with respect to q_σ.

Observe that $\Gamma^\sigma = (V, c^\sigma)$ appears as the perturbation of the initial network Γ that consists in to take the perturbation $\varepsilon(x,y) = \langle \mathcal{L}_q(\sigma), \sigma\rangle^{-1}\mathcal{L}_q(\sigma)(x)\mathcal{L}_q(\sigma)(y)$, $x \neq y$. In particular, for any $z \in V$, we have that $c^{\sigma_z} = c$ and hence

$$
R_{q_\sigma} = R_{q,\sigma}
$$

Acknowledgments. This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología,) under project MTM2014-60450-R.

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