

# From action-angle coordinates to geometric quantization: a round trip

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The philosophy of geometric quantization is to find and understand a “(one-way) dictionary” that “translates” classical systems into quantum systems. In this way, a quantum system is associated to a classical system in which observables (smooth functions) become operators of a Hilbert space and the classical Poisson bracket becomes the commutator of operators. In this process, the choice of additional geometric structures (polarizations) plays an important rôle. A desired property is that the quantization obtained does not depend on the polarization. Another rule in the game is that of keeping track of the symmetries on both sides. This is the deep link of geometric quantization with representation theory. The quantization commutes with reduction “principle” becomes realistic in some geometric quantization set-ups.

Our point of view in this big endeavour is very modest. We plan to construct a “representation space” in the case the polarization is given by a real polarization. For this, we follow the definition of Kostant of the representation spaces via higher cohomology groups with coefficients in the sheaf of flat sections. In this short note, we will not discuss either the (pre)Hilbert structure of this space nor the quantization rules.

## 1. QUANTIZATION VIA REAL POLARIZATIONS

Let  $(M^{2n}, \omega)$  be a symplectic manifold such that  $[\omega]$  is integral. Under these circumstances (see for instance [14] or [6]), there exists a complex line bundle  $\mathbb{L}$  with a connection  $\nabla$  over  $M$  such that  $\text{curv}(\nabla) = \omega$ . The symplectic manifold  $(M^{2n}, \omega)$  is called prequantizable and the pair  $(\mathbb{L}, \nabla)$  is called a *prequantum line bundle* of  $(M^{2n}, \omega)$ . In order to construct the representation space we need to restrict the space of sections to a subspace of sections which are flat in “privileged” directions given by a polarization. In this note we will just consider a real polarization. A real polarization  $\mathcal{P}$  is a foliation whose leaves are Lagrangian submanifolds. Integrable systems provide natural examples of real polarizations. If the manifold  $M$  is compact the “moment map”:  $F : M^{2n} \rightarrow \mathbb{R}^n$  has singularities that correspond to *equilibria*. Consider the following:

**Example 1.1.** Consider  $M = S^1 \times \mathbb{R}$  and  $\omega = dt \wedge d\theta$ . Take as  $\mathbb{L}$  the trivial bundle with connection 1-form  $\Theta = t d\theta$ . Now, let  $\mathcal{P} = \langle \frac{\partial}{\partial \theta} \rangle$  then flat sections satisfy,  $\nabla_X \sigma = X(\sigma) - i \langle \theta, X \rangle \sigma$ . Thus  $\sigma(t, \theta) = a(t) \cdot e^{it\theta}$  and Bohr-Sommerfeld leaves are given by the condition  $t = 2\pi k, k \in \mathbb{Z}$ .

This example shows that flat sections are not globally defined but they exist along a subset of leaves of the polarization. These are called Bohr-Sommerfeld leaves. The characterization of Bohr-Sommerfeld leaves for regular fibrations under some conditions is a well-known result by Guillemin and Sternberg ([4]). In particular the set of Bohr-Sommerfeld leaves is discrete and is given by “action” coordinates.

**Theorem 1.1** (Guillemin-Sternberg). *If the polarization is a regular fibration with compact leaves over a simply connected base  $B$ , then the Bohr-Sommerfeld set is discrete and assuming that the zero-fiber is a Bohr-Sommerfeld leaf, the Bohr-Sommerfeld set is given by,  $BS = \{p \in M, (f_1(p), \dots, f_n(p)) \in \mathbb{Z}^n\}$  where  $f_1, \dots, f_n$  are global action coordinates on  $B$ .*

This result connects with Arnold-Liouville-Mineur theorem for action-angle coordinates for integrable systems. When we consider a toric manifolds the base  $B$  may be identified with the image of the moment map by the toric action (Delzant polytope).

In view of the previous theorem, it would make sense to “quantize” these systems counting Bohr-Sommerfeld leaves. When the polarization is an integrable system with global action-angle coordinates, Bohr-Sommerfeld leaves are just “integral” Liouville tori. But why? Following the idea of Kostant [7], in the case there are no global sections denote by  $\mathcal{J}$  the sheaf of flat sections along the polarization, we can then define the quantization as  $\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M, \mathcal{J})$ . Then quantization is given by precisely the following theorem of Sniatycki [13]:

**Theorem 1.2** (Sniatycki). *If the leaf space  $B^n$  is a Hausdorff manifold and the natural projection  $\pi : M^{2n} \rightarrow B^n$  is a fibration with compact fibres, then all the cohomology groups vanish except for degree half of the dimension of the manifold. Furthermore,  $\mathcal{Q}(M^{2n}) = H^n(M^{2n}, \mathcal{J})$ , and the dimension of  $H^n(M^{2n}, \mathcal{J})$  is the number of Bohr-Sommerfeld leaves.*

There are two different approaches to compute this sheaf cohomology:

- (1) Using a fine resolution of the complex: Namely, we can define the sheaf:  $\Omega_{\mathcal{P}}^i(U) = \Gamma(U, \wedge^i \mathcal{P})$ . and  $\mathcal{C}$  to be the sheaf of complex-valued functions that are locally constant along  $\mathcal{P}$ . Consider the natural (fine) resolution

$$0 \rightarrow \mathcal{C} \xrightarrow{i} \Omega_{\mathcal{P}}^0 \xrightarrow{d_{\mathcal{P}}} \Omega_{\mathcal{P}}^1 \xrightarrow{d_{\mathcal{P}}} \Omega_{\mathcal{P}}^2 \xrightarrow{d_{\mathcal{P}}} \dots$$

The differential operator  $d_{\mathcal{P}}$  is the restriction of the exterior differential to the directions of the distributions (as in foliated cohomology). We can use this resolution to obtain a fine resolution of  $\mathcal{J}$  by twisting the previous resolution with the sheaf  $\mathcal{J}$ .

- (2) A different approach used in [2] and [5] is the one of Čech cohomology which turns out to be useful when we consider integrable systems with singularities.

**1.1. Applications to the general case of Lagrangian foliations.** This *fine resolution approach* can be useful to compute this geometric quantization for regular foliations (including those not coming from integrable systems like irrational slope on the torus).

In [9] we use the classification of foliations on the torus (Kneser-Denjoy-Schwartz theorem) together with basic properties of this sheaf cohomology to compute the geometric quantization of a torus. In the case of irrational slope we can compute

the quantization (see [9]) and we obtain that the quantization space is always infinite dimensional. However, if we compute the limit case of the foliated cohomology ( $\omega = 0$ ), we obtain that this foliated cohomology is finite dimensional if the irrationality measure of  $\eta$  and is infinite dimensional if the irrationality measure of  $\eta$  is infinite. The results contained in [9] seem to generalize a result of El Kacimi [8] for foliated cohomology.

Most computations in [9] rely on what we call “*geometric quantization computation kit*” (essentially a Künneth formula and a Mayer-Vietoris theorem in this context). This Künneth formula is very helpful to extend results to higher dimension by reduction to the 2-dimensional case (whenever the corresponding theorem for reduction also holds within the category of foliations considered).

## 2. QUANTIZATION USING SINGULAR ACTION-ANGLE COORDINATES

Consider the case of rotations of the sphere. There are two leaves of the polarization which are singular and correspond to fixed points of the action. What happens if we go to the edges and vertexes of Delzant’s polytope? This case and, more generally, that of toric manifolds was considered by Mark Hamilton in [2].

**Theorem 2.1** (Hamilton). *For a  $2n$ -dimensional compact toric manifold and let  $BS_r$  be the set of regular Bohr-Sommerfeld leaves,  $\mathcal{Q}(M) = H^n(M; \mathcal{J}) \cong \bigoplus_{l \in BS_r} \mathbb{C}$*

Then this geometric quantization does not see the singular elliptic points. In the example of the sphere Bohr-Sommerfeld leaves are given by integer values of height (or, equivalently) leaves which divide out the manifold in integer areas.

In order to consider more general singularities, we need to review some results for normal forms of integrable system. The theorem of Guillemin-Marle-Sternberg gives normal forms in a neighbourhood of fixed points of a toric action. This can be generalized to normal forms of integrable systems (not always toric) that we call non-degenerate. A proof of this theorem in the elliptic case can be found in [1]. For the other cases see the author’s thesis [10] where the idea of symplectic orthogonal decomposition is used and the paper [12].

**Theorem 2.2** (Eliasson-Miranda). *There exists symplectic Morse normal forms for integrable systems with non-degenerate singularities.*

The local model is given by  $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$  and  $\omega = \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$ . and the components of the moment map are:

- (1) Regular  $f_i = p_i$  for  $i = 1, \dots, k$ ;
- (2) Elliptic  $f_i = x_i^2 + y_i^2$  for  $i = k + 1, \dots, k_e$ ;
- (3) Hyperbolic  $f_i = x_i y_i$  for  $i = k_e + 1, \dots, k_e + k_h$ ;
- (4) focus-focus  $f_i = x_i y_{i+1} - x_{i+1} y_i$ ,  $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$  for  $i = k_e + k_h + 2j - 1$ ,  $j = 1, \dots, k_f$ .

We can use these models to compute geometric quantization in these cases. In the case of non-degenerate singularities in dimension 2 (only elliptic and hyperbolic singularities), we [5] obtain the following:

**Theorem 2.3** (Hamilton and Miranda). *The quantization of a compact surface endowed with an integrable system with non-degenerate singularities is given by,*

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{l \in BS_r} \mathbb{C} ,$$

where  $\mathcal{H}$  is the set of hyperbolic singularities.

In particular, this theorem shows that this quantization depends strongly on the polarization (for more details see [5]).

**2.1. New directions.** The case of general non-degenerate singularities in higher dimensions is a joint work of the author with Romero Solha and uses the above-mentioned “geometric quantization computation kit” together with the results in [11] and [10]. For these singular real polarizations a “quantization commutes with reduction” principle seems to hold.

Finally, we have learned from the symplectic case that action-angle coordinates are useful to compute geometric quantization. We can use the existence of (partial) action-angle coordinates for Poisson manifolds (recently explored in [3]) to compute geometric quantization in the Poisson context. This is a joint project with Mark Hamilton.

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