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Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions

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Abstract
In this paper, we focus on the existence and asymptotic analysis of positive solutions for a class of singular fractional differential equations subject to nonlocal boundary conditions. By constructing suitable upper and lower solutions and employing Schauder's fixed point theorem, the conditions for the existence of positive solutions are established and the asymptotic analysis for the obtained solution is carried out. In our work, the nonlinear function involved in the equation not only contains fractional derivatives of unknown functions but also has a stronger singularity at some points of the time and space variables.

Keywords: Asymptotic analysis; Nonlocal boundary conditions; Upper and lower solutions method; Fractional differential equation

1 Introduction

The purpose of this paper is to establish some new results on existence and asymptotic analysis of positive solutions for the following singular fractional differential equation with nonlocal boundary condition:

$$\begin{cases} -\mathscr{D}_t^{\alpha} x(t) = f(t, x(t), \mathscr{D}_t^{\gamma} x(t)), & 0 < t < 1, \\ \mathscr{D}_t^{\gamma} x(0) = \mathscr{D}_t^{\gamma+1} x(0) = 0, & \mathscr{D}_t^{\mu} x(1) = \int_0^1 \mathscr{D}_t^{\mu} x(s)\, d\mathcal{X}(s), \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$ with $0 < \gamma \leq \mu < \alpha - 2$, $\mathscr{D}_t^{\alpha}$ is defined as the Riemann–Liouville derivative, $\int_0^1 \mathscr{D}_t^{\mu} x(s)\, d\mathcal{X}(s)$ denotes a linear functional involving the Riemann–Stieltjes integrals, $\mathcal{X}$ is a function of bounded variation with a changing-sign measure $d\mathcal{X}$, $f : (0,1) \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ is continuous, and $f(t, x_1, x_2)$ may be singular at $t = 0, 1$ and $x_1 = x_2 = 0$.

In describing viscoelasticity, Heymans and Kitagawa [1] pointed out that the accuracy and success of the model are their abilities to describe natural phenomena including memory effects in polymers. However, in many dynamic process, the influence of memory is often persistent, even if the factors affecting the process have disappeared, such as observed in stress relaxation after a nonmonotonous loading program. Thus in order to improve the accuracy of the model, based on the non-locality of fractional order derivative, one can choose a noninteger order differential equation to describe this type of physical phenomena with memory effects. In addition, fractional calculus also has many other applications in various fields of science and engineering, such as a HIV model [2, 3] and a fluid model [4–8]. Recently, Heymans and Podlubny [9] gave some physical interpretation for the fractional spring–pot model, the Zener model, the Maxwell model and the Voigt model. In [10], Abdon introduced a new concept of differentiation and integration combining fractal differentiation and fractional differentiation, which can explain the memory effect of heterogeneity, and elasco-viscosity of the medium and also the fractal geometry of the dynamic system. Using the time-scale fractional calculus, Nadia et al. [11] gave some applications of the fractional derivatives with arbitrary time scales in white noise from signal processing.

In the aspect of mathematical theory and application, to obtain further information of the relative natural phenomena, many authors are interested in the existence and properties of solutions for fractional differential models [12–27] and many analytical techniques and methods have been developed to solve various differential equations, such as iterative methods [28–37], the Mawhin continuation theorem for resonance [38–40], the topological degree method [41, 42], the fixed point theorem [43–55], the variational method [56–73] and the upper and lower solution method [74, 75].

Inspired by the above work, in this paper, we mainly focus on the analytic results for Eq. (1.1). Our strategy is firstly introducing an accurate cone of Banach space and then constructing a couple of suitable upper and lower solutions, and finally establishing some new results on existence and asymptotic behavior of positive solutions for the equation by using the fixed point theorem. It is noteworthy that our approach and technique can solve the singularity of nonlinear term f at the space variables without the need of the complicated supremum and limit condition such as

(A) $f \in C((0,1)\times(0,+\infty)\times(0,+\infty),[0,+\infty))$ and for any $0<r<R<+\infty$,

$$\lim_{n\to+\infty}\sup_{\substack{x\in K_{\frac{R}{\Gamma(\beta+1)}}\\ y\in\overline{K}_R\setminus K_r}}\int_{e(n)}\omega(s)f\big(s,x(s),y(s)\big)\,ds=0,$$

where $e(n)=[0,\frac{1}{n}]\cup[\frac{n-1}{n},1]$.

This is applied by Zhang et al. [14] for the spectral and singularity analysis for a fractional differential equation with signed measure. The main contributions of this work are as follows:

(i) We present exact cone and suitable growth condition to overcome the difficulty due to the singularity of the nonlinear term f at the space variables.

(ii) We establish a sufficient condition for the existence of positive solutions and give the estimation of the positive solution and asymptotic behavior of the derivative of positive solutions at the

(iii) Nonsingular cases for the nonlinear term f at the time and space variables are discussed and some new results are established.

The rest of this paper is organized as follows. In Sect. 2, some preliminaries and lemmas are presented for subsequent developments. The main results are presented in Sect. 3.

2 Preliminaries and lemmas

For the convenience of the reader, we only present here some necessary properties from fractional calculus theory in the sense of Riemann–Liouville, and the corresponding definitions can be found in [76] or [12–25].

Proposition 2.1 ([76])

(1) *If* $x, y : (0, +\infty) \to \mathbb{R}$ *with order* $\alpha > 0$, *then*

$$\mathscr{D}_t^{\alpha}\big(x(t) + y(t)\big) = \mathscr{D}_t^{\alpha} x(t) + \mathscr{D}_t^{\alpha} y(t).$$

(2) *If* $x \in L^1(0,1)$, $\nu > \gamma > 0$ *and* m *is a positive integer, then*

$$I^{\nu} I^{\gamma} x(t) = I^{\nu+\gamma} x(t), \qquad \mathscr{D}_t^{\gamma} I^{\nu} x(t) = I^{\nu-\gamma} x(t),$$
$$\mathscr{D}_t^{\gamma} I^{\gamma} x(t) = x(t), \qquad \mathscr{D}_t^{m}\big(\mathscr{D}_t^{\gamma} x(t)\big) = \mathscr{D}_t^{\gamma+m} x(t).$$

(3) *If* $\alpha > 0$, $\gamma > 0$, *then*

$$\mathscr{D}_t^{\alpha} t^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} t^{\gamma-\alpha-1}.$$

(4) *Suppose* $\gamma > 0$, *and* $g(x)$ *is integrable, then*

$$I^{\gamma} \mathscr{D}_t^{\gamma} g(x) = g(x) + c_1 x^{\gamma-1} + c_2 x^{\gamma-2} + \cdots + c_n x^{\gamma-n},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, n *is the smallest integer greater than or equal to* α.

In the rest of this paper, all discussions are based on the assumption $2 < \alpha - \gamma \leq 3$. We first give the following lemma.

Lemma 2.1 *Let* $x(t) = I^{\gamma} z(t)$, $z(t) \in C[0,1]$, *then Eq.* (1.1) *is equivalent to the following boundary value problem*:

$$\begin{cases} -\mathscr{D}_t^{\alpha-\gamma} z(t) = f(t, I^{\gamma} z(t), z(t)), \\ z(0) = z'(0) = 0, \quad \mathscr{D}_t^{\mu-\gamma} z(1) = \int_0^1 \mathscr{D}_t^{\mu-\gamma} z(s)\, d\mathcal{X}(s). \end{cases} \tag{2.1}$$

Proof Firstly, let $x(t) = I^{\gamma} z(t)$ and $z(t) \in C[0,1]$. It follows from Proposition 2.1(2) that

$$\mathscr{D}_t^{\gamma} x(t) = \mathscr{D}_t^{\gamma} I^{\gamma} z(t) = z(t). \tag{2.2}$$

On the other hand, $2 < \alpha \leq 3$ and $0 < \gamma \leq \mu < \alpha - 2$ yield $\alpha - \gamma, \alpha - \mu \in (2,3)$. Consequently, by the definition of the Riemann–Liouville derivative and integral and Proposition 2.1(2), one has

$$\begin{aligned} &\mathscr{D}_t^{\gamma+1} x(t) = \mathscr{D}_t^{\gamma+1} I^{\gamma} z(t) = z'(t), \\ &\mathscr{D}_t^{\alpha} x(t) = \frac{d^3}{dt^3}\big(I^{3-\alpha} x(t)\big) = \frac{d^3}{dt^3}\big(I^{3-\alpha} I^{\gamma} z(t)\big) = \frac{d^3}{dt^3}\big(I^{3-\alpha+\gamma} z(t)\big) \\ &\quad = \mathscr{D}_t^{\alpha-\gamma} z(t). \end{aligned} \tag{2.3}$$

It follows from (1.1), (2.2) and (2.3) that $-\mathscr{D}_t^{\alpha-\gamma}z(t) = f(t, I^\gamma z(t), z(t))$ with boundary conditions

$$z(0) = \mathscr{D}_t^{\gamma}x(0) = 0, \qquad z'(0) = \mathscr{D}_t^{\gamma+1}x(0) = 0, \qquad \mathscr{D}_t^{\mu-\gamma}z(1) = \int_0^1 \mathscr{D}_t^{\mu-\gamma}z(s)\,d\mathcal{X}(s).$$

Thus, Eq. (1.1) is turned into the boundary value problem (2.1).

Conversely, if $z \in C([0,1],[0,+\infty))$ is a solution for the problem (2.1). Then letting $x(t) = I^\gamma z(t)$ and using (2.2) and (2.3), we get

$$-\mathscr{D}_t^{\alpha}x(t) = -\mathscr{D}_t^{\alpha-\gamma}z(t) = f\bigl(t, I^\gamma z(t), z(t)\bigr) = f\bigl(t, x(t), \mathscr{D}_t^{\gamma}x(t)\bigr), \quad 0 < t < 1,$$

with boundary conditions

$$\mathscr{D}_t^{\gamma}x(0) = z(0) = 0, \qquad \mathscr{D}_t^{\gamma+1}x(0) = z'(0) = 0, \qquad \mathscr{D}_t^{\mu}x(1) = \int_0^1 \mathscr{D}_t^{\mu}x(s)\,d\mathcal{X}(s).$$

Consequently, the boundary value problem (2.1) is turned into Eq. (1.1). □

The following lemma is standard according to Proposition 2.1, and we omit the proof.

Lemma 2.2 *Given $h \in L^1(0,1)$, then the boundary value problem*

$$\begin{cases} \mathscr{D}_t^{\alpha-\gamma}z(t) + h(t) = 0, & 0 < t < 1, \\ z(0) = z'(0) = 0, & \mathscr{D}_t^{\mu-\gamma}z(1) = 0, \end{cases} \tag{2.4}$$

has the unique solution

$$z(t) = \int_0^1 G(t,s)h(s)\,ds,$$

where $G(t,s)$ is the Green function of the boundary value problem (2.4) and

$$G(t,s) = \begin{cases} \frac{t^{\alpha-\gamma-1}(1-s)^{\alpha-\mu-1}-(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-\gamma-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\gamma)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

On the other hand, by Proposition 2.1, we know that the unique solution of the boundary value problem

$$\begin{cases} \mathscr{D}_t^{\alpha-\gamma}z(t) = 0, & 0 < t < 1, \\ z(0) = z'(0) = 0, & \mathscr{D}_t^{\mu-\gamma}z(1) = 1, \end{cases}$$

is $\frac{\Gamma(\alpha-\mu)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}$. Let

$$\mathcal{C} = \int_0^1 \frac{\Gamma(\alpha-\mu)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}\,d\mathcal{X}(t), \qquad \mathcal{B} = \frac{\Gamma(\alpha-\mu)}{(1-\mathcal{C})\Gamma(\alpha-\gamma)}, \tag{2.6}$$

and define

$$\mathcal{G}_{\mathcal{X}}(s) = \int_0^1 G(t,s)\,d\mathcal{X}(t).$$

Following the strategy in [24], the Green function for the boundary value problem (2.1) is

$$W(t,s) = \mathcal{B}t^{\alpha-\gamma-1}\mathcal{G}_{\mathcal{X}}(s) + G(t,s). \tag{2.7}$$

Thus we have the following lemma.

Lemma 2.3 *Let $p \in L^1[0,1]$ and $2 < \alpha \leq 3$ with $0 < \gamma \leq \mu < \alpha - 2$. Then the fractional differential equation*

$$\begin{cases} -\mathscr{D}_{\boldsymbol{t}}{}^{\alpha-\gamma} z(t) = p(t), \\ z(0) = z'(0) = 0, \quad \mathscr{D}_{\boldsymbol{t}}{}^{\mu-\gamma} z(1) = \int_0^1 \mathscr{D}_{\boldsymbol{t}}{}^{\mu-\gamma} z(s)\,d\mathcal{X}(s), \end{cases}$$

has the unique solution

$$z(t) = \int_0^1 W(t,s)p(s)\,ds,$$

where $W(t,s)$ is defined by (2.7).

In order to guarantee the nonnegativity of the Green function, the following condition is necessary.

(F0) $\mathcal{X}$ is a function of bounded variation satisfying $\mathcal{G}_{\mathcal{X}}(s) \geq 0$, $s \in [0,1]$ and $\mathcal{C} \in [0,1)$.

Lemma 2.4 *Assume* (F0) *is satisfied, then for the Green function in* (2.7) *one has the following estimation*:

(1) $W(t,s) > 0$ *for all* $0 < t, s < 1$.

(2)

$$\mathcal{B}t^{\alpha-\gamma-1}\mathcal{G}_{\mathcal{X}}(s) \leq W(t,s) \leq c(s)t^{\alpha-\gamma-1}, \tag{2.8}$$

where

$$c(s) = \frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\gamma)} + \mathcal{B}\mathcal{G}_{\mathcal{X}}(s).$$

Proof The conclusion of (1) is clear. In what follows, we prove the conclusion (2). Using (2.5) and (2.7), one gets

$$\begin{aligned} W(t,s) &= \mathcal{B}t^{\alpha-\gamma-1}\mathcal{G}_{\mathcal{X}}(s) + G(t,s) \leq \mathcal{B}\mathcal{G}_{\mathcal{X}}(s) + G(t,s) \\ &\leq \frac{t^{\alpha-\gamma-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\gamma)} + \mathcal{B}t^{\alpha-\gamma-1}\mathcal{G}_{\mathcal{X}}(s) = c(s)t^{\alpha-\gamma-1} \end{aligned}$$

and

$$W(t,s) = \mathcal{B}t^{\alpha-\gamma-1}\mathcal{G}_{\mathcal{X}}(s) + G(t,s) \geq \mathcal{B}t^{\alpha-\gamma-1}\mathcal{G}_{\mathcal{X}}(s). \qquad \square$$

It follows from Lemma 2.3 that we have the following.

Lemma 2.5 *If* $z \in C([0,1],\mathbb{R})$ *satisfies*

$$z(0) = z'(0) = 0, \quad \mathscr{D}_t{}^{\mu-\gamma} z(1) = \int_0^1 \mathscr{D}_t{}^{\mu-\gamma} z(s)\, d\mathcal{X}(s),$$

and $\mathscr{D}_t{}^{\mu-\gamma} z(t) \le 0$ *for all* $t \in (0,1)$, *then* $z(t) \ge 0$, $t \in [0,1]$.

3 Singular cases

In this section, we first give the definition of upper and lower solution on the boundary value problem (2.1), and then introduce some theories of function space and give our main results.

Definition 3.1 We call a continuous function $\psi(t)$ as a lower solution for the boundary value problem (2.1), if

$$\begin{cases} -\mathscr{D}_t{}^{\alpha-\gamma} \psi(t) \le f(t, I^\gamma \psi(t), \psi(t)), \\ \psi(0) \ge 0, \psi'(0) \ge 0, \quad \mathscr{D}_t{}^{\mu-\gamma} \psi(1) \ge \int_0^1 \mathscr{D}_t{}^{\mu-\gamma} \psi(s)\, d\mathcal{X}(s). \end{cases}$$

Definition 3.2 We call a continuous function $\phi(t)$ as a upper solution for the boundary value problem (2.1), if

$$\begin{cases} -\mathscr{D}_t{}^{\alpha-\gamma} \phi(t) \ge f(t, I^\gamma \phi(t), \phi(t)), \\ \phi(0) \le 0, \phi'(0) \le 0, \quad \mathscr{D}_t{}^{\mu-\gamma} \phi(1) \le \int_0^1 \mathscr{D}_t{}^{\mu-\gamma} \phi(s)\, d\mathcal{X}(s). \end{cases}$$

Let

$$e(t) = t^{\alpha-\gamma-1}, \qquad \kappa(t) = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} t^{\alpha-1}, \quad t \in [0,1],$$

and define our work space $E = C[0,1]$ and a subset P_e of E,

$$\begin{aligned} P_e = \bigl\{ z \in E :\ & \text{there exist two positive numbers } 0 < l_z < 1 < L_z \text{ such that} \\ & l_z e(t) \le z(t) \le L_z e(t), t \in [0,1] \bigr\}. \end{aligned} \tag{3.1}$$

Clearly, P_e is nonempty since $e(t) \in P_e$. For any $z \in P_e$, define an operator B by

$$(Bz)(t) = \int_0^1 W(t,s) f\bigl(s, I^\gamma z(s), z(s)\bigr)\, ds. \tag{3.2}$$

To overcome the difficulties of the singularity at the space variables, we introduce the following growth conditions for f:

(F1) $f \in C((0,1) \times (0,\infty) \times (0,\infty), [0,+\infty))$, and $f(t, x_1, x_2)$ is decreasing in $x_i > 0$ for $i = 1,2$.

(F2) For any $\tau > 0$, $f(t, \frac{\tau}{\Gamma(\gamma+1)} t^\gamma, \tau) \not\equiv 0$, and

$$0 < \int_0^1 c(s) f\bigl(s, \tau\kappa(s), \tau e(s)\bigr)\, ds < +\infty.$$

Lemma 3.1 *Assume* (F0) (F1) *and* (F2) *are satisfied, then* $B(P_e) \subset P_e$ *and* B *is well defined.*

Proof For any $z \in P_e$, it follows from the definition of P_e that there exist two numbers $0 < l_z < 1 < L_z$ such that $l_z e(t) \leq z(t) \leq L_z e(t)$ for any $t \in [0,1]$. Notice that $\kappa(t) = I^\gamma e(t)$, then by (2.8) and (F1)–(F2), one gets

$$\begin{aligned}(Bz)(t) &= \int_0^1 W(t,s) f\bigl(s, I^\gamma z(s), z(s)\bigr)\,ds \\ &\leq \int_0^1 c(s) f\bigl(s, I^\gamma\bigl(l_z e(s)\bigr), l_z e(s)\bigr)\,ds \\ &\leq \int_0^1 c(s) f\bigl(s, l_z\kappa(s), l_z e(s)\bigr)\,ds \\ &< +\infty. \end{aligned} \tag{3.3}$$

Take $\tau = \max_{t\in[0,1]} z(t)$, it follows from (F2) that, for any $s \in [0,1]$,

$$\mathcal{G}_{\mathcal{X}}(s) f\left(s, \frac{\tau s^\gamma}{\Gamma(\gamma+1)}, \tau\right) \not\equiv 0.$$

Consequently, from the continuity of f, one has

$$\int_0^1 \mathcal{G}_{\mathcal{X}}(s) f\left(s, \frac{\tau s^\gamma}{\Gamma(\gamma+1)}, \tau\right) ds > 0.$$

This yields

$$\int_0^1 \mathcal{G}_{\mathcal{X}}(s) f\bigl(s, I^\gamma \tau, \tau\bigr)\,ds = \int_0^1 \mathcal{G}_{\mathcal{X}}(s) f\left(s, \frac{\tau s^\gamma}{\Gamma(\gamma+1)}, \tau\right) ds > 0. \tag{3.4}$$

By (2.8), (3.3) and (3.4), we have

$$(Bz)(t) = \int_0^1 W(t,s) f\bigl(s, I^\gamma z(s), z(s)\bigr)\,ds \geq \mathcal{B} e(t) \int_0^1 \mathcal{G}_{\mathcal{X}}(s) f\bigl(s, I^\gamma\tau, \tau\bigr)\,ds \geq l'_z e(t), \tag{3.5}$$

where

$$l'_z = \min\left\{\frac{1}{2}, \mathcal{B}\int_0^1 \mathcal{G}_{\mathcal{X}}(s) f\bigl(s, I^\gamma\tau, \tau\bigr)\,ds\right\}.$$

On the other hand, in view of (2.8), we also have

$$(Bz)(t) = \int_0^1 W(t,s) f\bigl(s, I^\gamma z(s), z(s)\bigr)\,ds \leq e(t) \int_0^1 c(s) f\bigl(s, l_z\kappa(s), l_z e(s)\bigr)\,ds \leq L'_z e(t), \tag{3.6}$$

where

$$L'_z = \max\left\{2, \int_0^1 c(s) f\bigl(s, l_z\kappa(s), l_z e(s)\bigr)\,ds\right\}.$$

Thus it follows from (3.3)–(3.6) that $B(P_e) \subset P_e$ and B is well defined. $\square$

Theorem 3.1 (Existence) *Suppose* (F0)–(F2) *hold. Then Eq.* (1.1) *has at least one positive solution.*

Proof Firstly by Lemma 2.3 and (3.2), we have

$$\begin{cases} -\mathscr{D}_t^{\alpha-\gamma}(Bz)(t) = f(t, I^\gamma z(t), z(t)), \\ (Bz)(0) = (Bz)'(0) = 0, \quad \mathscr{D}_t^{\mu-\gamma}(Bz)(1) = \int_0^1 \mathscr{D}_t^{\mu-\gamma}(Bz)(s)\,d\mathcal{X}(s). \end{cases} \tag{3.7}$$

Next we seek for a couple of lower and upper solutions of the boundary value problem (2.1). To do this, take

$$\eta(t) = \min\{e(t), Be(t)\}, \qquad \xi(t) = \max\{e(t), Be(t)\}. \tag{3.8}$$

Obviously, if $e(t) = Be(t)$, then $e(t)$ is a positive solution of Eq. (1.1). If $e(t) \neq Be(t)$, then we have $\xi(t), \eta(t) \in P_e$ and

$$\eta(t) \le e(t) \le \xi(t). \tag{3.9}$$

Letting

$$\psi(t) = B\xi(t), \qquad \phi(t) = B\eta(t),$$

we claim that the functions $\psi(t), \phi(t)$ shall be the lower solution and upper solution of the boundary value problem (2.1), respectively.

In fact, it follows from (F1) that B is nonincreasing relative to z. By (3.8)–(3.9), we have

$$\begin{aligned} &\psi(t) = B\xi(t) \le B\eta(t) = \phi(t), \\ &\psi(t) = B\xi(t) \le Be(t) \le \xi(t), \\ &\phi(t) = B\eta(t) \ge Be(t) \ge \eta(t), \end{aligned} \tag{3.10}$$

and $\psi(t), \phi(t) \in P_e$. Thus (3.7) and (3.10) yield

$$\begin{aligned} &\mathscr{D}_t^{\alpha-\gamma}\psi(t) + f\bigl(t, I^\gamma\psi(t), \psi(t)\bigr) \\ &\quad = \mathscr{D}_t^{\alpha-\gamma}(B\xi)(t) + f\bigl(t, I^\gamma(B\xi)(t), (B\xi)(t)\bigr) \\ &\quad \ge \mathscr{D}_t^{\alpha-\gamma}(B\xi)(t) + f\bigl(t, I^\gamma\xi(t), \xi(t)\bigr) = 0, \quad t \in (0,1), \\ &\psi(0) = \psi'(0) = 0, \quad \mathscr{D}_t^{\mu-\gamma}\psi(1) = \int_0^1 \mathscr{D}_t^{\mu-\gamma}\psi(s)\,d\mathcal{X}(s), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\mathscr{D}_t^{\alpha-\gamma}\phi(t) + f\bigl(t, I^\gamma\phi(t), \phi(t)\bigr) \\ &\quad = \mathscr{D}_t^{\alpha-\gamma}(B\eta)(t) + f\bigl(t, I^\gamma(B\eta)(t), (B\eta)(t)\bigr) \\ &\quad \le \mathscr{D}_t^{\alpha-\gamma}(B\eta)(t) + f\bigl(t, I^\gamma\eta(t), \eta(t)\bigr) = 0, \quad t \in (0,1), \\ &\phi(0) = \phi'(0) = 0, \quad \mathscr{D}_t^{\mu-\gamma}\phi(1) = \int_0^1 \mathscr{D}_t^{\mu-\gamma}\phi(s)\,d\mathcal{X}(s). \end{aligned} \tag{3.12}$$

Thus (3.10)–(3.12) show that $\phi(t)$ and $\psi(t)$ are the lower and upper solutions of the boundary value problem (2.1), respectively, and $\psi(t), \phi(t) \in P_e$.

Now define the function

$$F(t,z) = \begin{cases} f(t, I^{\gamma}\psi(t), \psi(t)), & z < \psi(t), \\ f(t, I^{\gamma}z(t), z(t)), & \psi(t) \leq z \leq \phi(t), \\ f(t, I^{\gamma}\phi(t), \phi(t)), & z > \phi(t), \end{cases} \tag{3.13}$$

then from (3.13), $F[0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function.

Next let us consider the following auxiliary boundary value problem:

$$\begin{cases} -\mathscr{D}_{\boldsymbol{t}}{}^{\alpha-\gamma}z(t) = F(t,z), & 0 < t < 1, \\ z(0) = z'(0) = 0, & \mathscr{D}_{\boldsymbol{t}}{}^{\mu-\gamma}z(1) = \int_0^1 \mathscr{D}_{\boldsymbol{t}}{}^{\mu-\gamma}z(s)\,d\mathcal{X}(s). \end{cases} \tag{3.14}$$

Define an operator A in E by

$$(Az)(t) = \int_0^1 W(t,s)F\bigl(s,z(s)\bigr)\,ds, \quad \forall z \in E.$$

Obviously, from Lemma 2.3, a fixed point of A is a solution of the boundary value problem (3.14).

For all $z \in E$, as $\psi \in P_e$, there exists a constant $0 < l_\psi < 1$ such that $\psi(t) \geq l_\psi e(t)$, $t \in [0,1]$. Thus by Lemma 2.4, we have

$$\begin{aligned} (Az)(t) &\leq \int_0^1 c(s)F\bigl(s,z(s)\bigr)\,ds \leq \int_0^1 c(s)f\bigl(s, I^{\gamma}\psi(s), \psi(s)\bigr)\,ds \\ &\leq \int_0^1 c(s)f(s, f\bigl(s, l_\psi I^{\gamma}e(s), l_\psi e(s)\bigr)\,ds \\ &= \int_0^1 c(s)f\bigl(s, l_\psi \kappa(s), l_\psi e(s)\bigr)\,ds \\ &< +\infty. \end{aligned}$$

So A is bounded. In addition, according to the continuity of F and K, we find that $A: E \to E$ is continuous.

Let Ω be a bounded subset of E, then we have $\|z\| \leq N$ for some positive constant $N > 0$ and all $z \in \Omega$. Let $L = \max_{0\leq t\leq 1, 0\leq z\leq N} |F(t,z)| + 1$. It follows from the uniform continuity of $W(t,s)$ that, for any $\epsilon > 0$ and $s \in [0,1]$, there exists $\sigma > 0$ such that

$$\bigl|W(t_1,s) - W(t_2,s)\bigr| < \frac{\epsilon}{L},$$

for $|t_1 - t_2| < \sigma$. Then

$$\bigl|Az(t_1) - Az(t_2)\bigr| \leq \int_0^1 \bigl|W(t_1,s) - W(t_2,s)\bigr|\bigl|F\bigl(s,z(s)\bigr)\bigr|\,ds < \epsilon.$$

This implies that $A(\Omega)$ is equicontinuous.

Thus according to the Arzelà–Ascoli theorem, $A : E \to E$ is a completely continuous operator. Consequently it follows from the Schauder fixed point theorem that A has a fixed point w such that $w = Aw$.

In order to show that w is also a fixed point of the operator B, we only need to prove

$$\psi(t) \le w(t) \le \phi(t), \quad t \in [0,1].$$

We firstly verify that $w(t) \le \phi(t)$. Let $z(t) = \phi(t) - w(t)$, $t \in [0,1]$. Noticing that w is a fixed point of A and (3.12), we have

$$z(0) = z'(0) = 0, \quad \mathscr{D}_t{}^{\mu-\gamma} z(1) = \int_0^1 \mathscr{D}_t{}^{\mu-\gamma} z(s)\, d\mathcal{X}(s). \tag{3.15}$$

On the other hand, it follows from (3.11) and (F1) that

$$f\big(t, I^\gamma \phi(t), \phi(t)\big) \le f\big(t, I^\gamma \eta(t), \eta(t)\big). \tag{3.16}$$

Thus by the definition of F and (3.16), one gets

$$\begin{aligned} f\big(t, I^\gamma \psi(t), \psi(t)\big) &\le F\big(t, u(t)\big) \le f\big(t, I^\gamma \phi(t), \phi(t)\big) \\ &\le f\big(t, I^\gamma \eta(t), \eta(t)\big), \quad \forall u \in E, \forall t \in [0,1]. \end{aligned} \tag{3.17}$$

It follows from (3.7) and (3.17) that

$$\begin{aligned} \mathscr{D}_t{}^{\alpha-\gamma} z(t) &= \mathscr{D}_t{}^{\alpha-\gamma} \phi(t) - \mathscr{D}_t{}^{\alpha-\gamma} w(t) = \mathscr{D}_t{}^{\alpha-\gamma} \big(B\eta(t)\big) + F\big(w(t)\big) \\ &= -f\big(t, I^\gamma \eta(t), \eta(t)\big) + F(w(t) \le 0. \end{aligned} \tag{3.18}$$

Thus Lemma 2.5, (3.15) and (3.18) imply that

$$z(t) = \phi(t) - w(t) \ge 0, \quad t \in [0,1].$$

Similarly, we also have $w(t) - \psi(t) \ge 0$ on $[0,1]$. Thus the following estimation is valid:

$$\phi(t) \ge w(t) \ge \psi(t), \quad t \in [0,1], \tag{3.19}$$

which also implies $F(t, w(t)) = f(t, I^\gamma w(t), w(t))$, $t \in [0,1]$.

Combined with the above facts, we get that the fixed point of A is also the fixed point of B. So $w(t)$ is a positive solution of the boundary value problem (2.1), and consequently $x(t) = I^\gamma w(t)$ is a positive solution of Eq. (1.1). □

Theorem 3.2 (Estimation and asymptotic behavior) *Assume* (F0)–(F2) *are satisfied. Then there exist two positive constants m, n such that the solution $x(t)$ to Eq.* (1.1) *satisfies*

$$mt^{\alpha-1} \le x(t) \le nt^{\alpha-1} \quad \text{and} \quad \mathscr{D}_t{}^\gamma x(t) = o\big(t^{\alpha-2}\big).$$

Proof It follows from $\psi \in P_e$ and (3.19) that there exists $0 < l_\psi < 1$ such that

$$w(t) \geq \psi(t) \geq l_\psi e(t). \tag{3.20}$$

Thus, from (3.20) and (2.8), we have

$$\begin{aligned} w(t) &= \int_0^1 W(t,s)f\big(s, I^\gamma w(s), w(s)\big)\,ds \\ &\leq e(t)\int_0^1 c(s)f\big(s, l_\psi I^\gamma e(s), l_\psi e(s)\big)\,ds \\ &= e(t)\int_0^1 c(s)f\big(s, l_\psi \kappa(s), l_\psi e(s)\big)\,ds \\ &\leq Le(t). \end{aligned} \tag{3.21}$$

Consequently, one gets

$$l_\psi e(t) \leq w(t) \leq Le(t).$$

We have

$$I^\gamma e(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}e(s)\,ds = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1}. \tag{3.22}$$

By (3.22), we have

$$mt^{\alpha-1} = l_\psi \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1} \leq I^\gamma w(t) = x(t) \leq L\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1} = nt^{\alpha-1}.$$

In the end, by the l'Hospital rule,

$$\lim_{t\to 0+}\frac{\mathscr{D}_t{}^\gamma x(t)}{t^{\alpha-2}} = \lim_{t\to 0+}\frac{\mathscr{D}_t{}^{\gamma+1}x(t)}{(\alpha-2)t^{\alpha-3}} = 0,$$

that is, $\mathscr{D}_t{}^\gamma x(t) = o(t^{\alpha-2})$. □

4 Nonsingular cases

In this section, we are interested in some nonsingular cases of the nonlinear term f at time and space variables.

Case 1: f may be singular at $t = 0$ and (or) $t = 1$, but f is nonsingular at $x_1 = x_2 = 0$:

Theorem 4.1 *Suppose* (F0) *and the following assumptions are satisfied*:

(B1) $f \in C((0,1) \times [0,\infty) \times [0,\infty), [0,+\infty))$, *and* $f(t,x_1,x_2)$ *is decreasing in* $x_i > 0$ *for* $i = 1, 2$.

(B2) $f(t,0,0) \not\equiv 0$ *for any* $t \in (0,1)$, *and*

$$0 < \int_0^1 c(s)f(s,0,0)\,ds < +\infty.$$

Then Eq. (1.1) *has at least one positive solution* $x(t)$ *satisfying*

$$0 \le x(t) \le \mathcal{M}^* t^\gamma$$

for some constant $\mathcal{M}^* > 0$. *Moreover, the positive solution* $x(t)$ *has boundary asymptotic behavior*

$$\mathscr{D}_t^{\gamma} x(t) = o\big(t^{\alpha-2}\big).$$

Proof In fact, we only replace the set P_e in Theorem 3.1 by using

$$P_1 = \big\{x \in E : x(t) \ge 0, t \in [0,1]\big\}.$$

Let

$$\eta(t) = \min\{0, B0\} = 0, \qquad \xi(t) = \max\{0, B0\} = B0,$$

and set

$$\psi(t) = B\xi(t) = B(B0), \qquad \phi(t) = B\eta(t) = B0.$$

Then we have $\phi(t), \psi(t) \in P_1$ and

$$0 \le \phi(t) = B0 \quad \text{and} \quad 0 \le \psi(t) = (B\phi)(t) \le B0 = \phi(t). \tag{4.1}$$

On the other hand, we also have

$$\begin{aligned}
&\mathscr{D}_t^{\alpha-\gamma} \psi(t) + f\big(t, I^\gamma \psi(t), \psi(t)\big) \\
&\quad = \mathscr{D}_t^{\alpha-\gamma} (B\xi)(t) + f\big(t, I^\gamma \psi(t), \psi(t)\big) \\
&\quad = -f\big(t, I^\gamma \xi(t), \xi(t)\big) + f\big(t, I^\gamma \psi(t), \psi(t)\big) \\
&\quad = -f\big(t, I^\gamma \phi(t), \phi(t)\big) + f\big(t, I^\gamma \psi(t), \psi(t)\big) \\
&\quad \ge 0, \quad t \in (0,1),
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\mathscr{D}_t^{\alpha-\gamma} \phi(t) + f\big(t, I^\gamma \phi(t), \phi(t)\big) \\
&\quad = \mathscr{D}_t^{\alpha-\gamma} (B\eta)(t) + f\big(t, I^\gamma \phi(t), \phi(t)\big) \\
&\quad = -f\big(t, I^\gamma \eta(t), \eta(t)\big) + f\big(t, I^\gamma \phi(t), \phi(t)\big) \\
&\quad = -f(t, 0, 0) + f\big(t, I^\gamma \phi(t), \phi(t)\big) \\
&\quad \le 0, \quad t \in (0,1).
\end{aligned} \tag{4.3}$$

Thus from (4.1)–(4.3), $\phi(t)$ and $\psi(t)$ are still the lower and upper solutions of the boundary value problem (2.1), respectively.

Finally, it follows from Lemma 2.3 that

$$\phi(t) = B0 = \int_0^1 W(t,s)f(s,0,0)\,ds \le \int_0^1 c(s)f(s,0,0)\,ds = \mathcal{N}^*,$$
$$I^\gamma \mathcal{N}^* = \frac{\mathcal{N}^*}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\,ds = \mathcal{M}^* t^\gamma.$$

Thus according to the proofs of Theorems 3.1–3.2, the conclusion of Theorem 4.1 is true. □

Case 2: $f(t,x_1,x_2)$ is nonsingular at both $t = 0, 1$ and $x_i = 0$, $i = 1, 2$. Then, by Theorem 4.1, the following conclusion is valid.

Theorem 4.2 *Assume that* $f(t,x_1,x_2) : [0,1] \times [0,\infty) \times [0,\infty) \to [0,+\infty)$ *is a continuous and decreasing function in* x_i, $i = 1, 2$ *with* $f(t,0,0) \not\equiv 0$ *for any* $t \in [0,1]$. *If* (F0) *holds, then Eq.* (1.1) *has at least one positive solution* $x(t)$ *with the estimation*

$$0 \le x(t) \le \mathcal{M}^* t^\gamma$$

for some constant $\mathcal{M}^* > 0$ *and boundary asymptotic behavior* $\mathscr{D}_t^{\gamma} x(t) = o(t^{\alpha-2})$.

Proof In fact, if $f(t,x_1,x_2) : [0,1] \times [0,\infty) \times [0,\infty) \to [0,+\infty)$ is continuous and $f(t,0,0) \not\equiv 0$, then the condition (B2) holds naturally. □

5 Numerical examples

Example 1 Consider the existence of positive solutions for the following singular fractional differential equation with nonlocal boundary condition:

$$\begin{cases} -\mathscr{D}_t^{\frac{5}{2}} x(t) = 10t^{-\frac{1}{4}}[(\mathscr{D}_t^{\frac{1}{2}} x(t))^{-\frac{1}{8}} + x^{-\frac{1}{3}}(t)], & 0 < t < 1, \\ \mathscr{D}_t^{\frac{1}{2}} x(0) = \mathscr{D}_t^{\frac{3}{2}} x(0) = 0, \qquad \mathscr{D}_t^{\frac{1}{2}} x(1) = \int_0^1 \mathscr{D}_t^{\frac{1}{2}} x(s)\,d\mathcal{X}(s), \end{cases} \tag{5.1}$$

where $\mathcal{X}$ is a function of bounded variation such that

$$\mathcal{X}(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}), \\ \frac{3}{2}, & t \in [\frac{1}{2}, \frac{3}{4}), \\ 1, & t \in [\frac{3}{4}, 1]. \end{cases} \tag{5.2}$$

By simple calculation, Eq. (5.1) can be transformed to the following 4-point boundary value problem with coefficients of both signs in the boundary condition:

$$\begin{cases} -\mathscr{D}_t^{\frac{5}{2}} x(t) = 10t^{-\frac{1}{4}}[(\mathscr{D}_t^{\frac{1}{4}} x(t))^{-\frac{1}{8}} + x^{-\frac{1}{3}}(t)], & 0 < t < 1, \\ \mathscr{D}_t^{\frac{1}{4}} x(0) = \mathscr{D}_t^{\frac{5}{4}} x(0) = 0, \qquad \mathscr{D}_t^{\frac{1}{3}} x(1) = \frac{3}{2}\mathscr{D}_t^{\frac{1}{3}} x(\frac{1}{2}) - \frac{1}{2}\mathscr{D}_t^{\frac{1}{3}} x(\frac{3}{4}). \end{cases} \tag{5.3}$$

Conclusion: The BVP (5.1) has at least one positive solution $x(t)$, and there exist two positive constants m, n such that

$$mt^{\frac{3}{2}} \le x(t) \le nt^{\frac{3}{2}}$$

with boundary asymptotic behavior $\mathscr{D}_t^{\frac{1}{4}} x(t) = o(t^{\frac{1}{2}})$.

Proof Let $\alpha = \frac{5}{2}$, $\gamma = \frac{1}{4}$, $\mu = \frac{1}{3}$, $f(t,x_1,x_2) = 10t^{-\frac{1}{4}}[x_1^{-\frac{1}{3}} + x_2^{-\frac{1}{8}}]$. Then $2 < \alpha \le 3$ satisfying $0 < \gamma \le \mu < \alpha - 2$ and f is singular at $t = 0$ and $x_1 = x_2 = 0$.

Clearly,

$$G(t,s) = \frac{1}{\Gamma(\frac{9}{4})}\begin{cases} t^{\frac{5}{4}}(1-s)^{\frac{7}{6}} - (t-s)^{\frac{5}{4}} =: G_1(t,s), & 0 \le s \le t \le 1, \\ t^{\frac{5}{4}}(1-s)^{\frac{7}{6}} =: G_2(t,s), & 0 \le t \le s \le 1, \end{cases}$$

and

$$\mathcal{G}_{\mathcal{X}}(s) = \frac{1}{\Gamma(\frac{9}{4})}\begin{cases} \frac{3}{2}G_1(\frac{1}{2},s) - \frac{1}{2}G_1(\frac{3}{4},s) = (\frac{1}{2})^{\frac{9}{4}}(3-(\frac{3}{2})^{\frac{5}{4}})(1-s)^{\frac{7}{6}} - \frac{3}{2}(\frac{1}{2}-s)^{\frac{5}{4}} + \frac{1}{2}(\frac{3}{4}-s)^{\frac{5}{4}}, \\ \quad 0 \le s < \frac{1}{2}, \\ \frac{3}{2}G_2(\frac{1}{2},s) - \frac{1}{2}G_1(\frac{3}{4},s) = (\frac{1}{2})^{\frac{9}{4}}(3-(\frac{3}{2})^{\frac{5}{4}})(1-s)^{\frac{7}{6}} + \frac{1}{2}(\frac{3}{4}-s)^{\frac{5}{4}}, \\ \quad \frac{1}{2} \le s < \frac{3}{4}, \\ \frac{3}{2}G_2(\frac{1}{2},s) - \frac{1}{2}G_2(\frac{3}{4},s) = (\frac{1}{2})^{\frac{9}{4}}(3-(\frac{3}{2})^{\frac{5}{4}})(1-s)^{\frac{7}{6}}, \\ \quad \frac{3}{4} \le s \le 1. \end{cases}$$

Thus

$$\mathcal{G}_{\mathcal{X}}(s) \ge 0, \qquad \mathcal{C} = \frac{\Gamma(\frac{13}{6})}{\Gamma(\frac{9}{4})}\int_0^1 t^{\frac{5}{4}}\,d\mathcal{X}(t) = \frac{\Gamma(\frac{13}{6})}{\Gamma(\frac{9}{4})}\left(1 - \int_0^1 \mathcal{X}(t)\,dt^{\frac{5}{4}}\right) = 0.2691 < 1,$$
$$c(s) = \frac{1}{\Gamma(\frac{9}{4})}(1-s)^{\frac{7}{6}} + \frac{1}{1.3070}\mathcal{G}_{\mathcal{X}}(s).$$

Consequently, (F0) and (F1) hold.

Since

$$e(t) = t^{\frac{5}{4}}, \kappa(t) = \frac{\Gamma(\frac{9}{4})}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}},$$

for any $\tau > 0$ and $t \in (0,1)$, we have $f(t, \frac{\tau}{\Gamma(\gamma+1)}t^{\gamma}, \tau) = 10t^{-\frac{1}{4}}[(\frac{\tau}{\Gamma(\frac{5}{4})})^{-\frac{1}{3}}t^{-\frac{1}{12}} + \tau^{-\frac{1}{8}}] \not\equiv 0$ and

$$\begin{aligned} 0 &< \int_0^1 c(s)f\bigl(s, \tau\kappa(s), \tau e(s)\bigr)\,ds \\ &= 10\int_0^1 s^{-\frac{1}{4}}\left[\frac{1}{\Gamma(\frac{9}{4})}(1-s)^{\frac{7}{6}} + \frac{1}{1.3070}\mathcal{G}_{\mathcal{X}}(s)\right]\left[\left(\frac{\Gamma(\frac{9}{4})\tau}{\Gamma(\frac{5}{2})}\right)^{-\frac{1}{3}}s^{-\frac{1}{2}} + \tau^{-\frac{1}{8}}s^{-\frac{5}{32}}\right]ds < +\infty. \end{aligned}$$

Thus (F2) holds.

It follows from Theorem 3.1 that Eq. (5.1) has at least a positive solution $x(t)$ satisfying the estimation

$$mt^{\frac{3}{2}} \le x(t) \le nt^{\frac{3}{2}}$$

for some positive constants m, n and boundary asymptotic behavior $\mathscr{D}_t^{\frac{1}{4}}x(t) = o(t^{\frac{1}{2}})$. □

Remark 5.1 In [14], Zhang et al. use the condition (A) to overcome the singularity of the equation. Obviously, Example 1 indicates that (F1) and (F2) are easier to check than (A), thus the growth condition in this paper is more popular in handling a singularity in the space variable.

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References

1. Heymans, N., Kitagawa, M.: Modelling "unusual" behaviour after strain reversal with hierarchical fractional models. Rheol. Acta **43**, 383–389 (2004)
2. Wang, Y., Liu, L., Zhang, X., Wu, Y.: Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection. Appl. Math. Comput. **258**, 312–324 (2015)
3. Zhang, X., Mao, C., Liu, L., Wu, Y.: Exact iterative solution for an abstract fractional dynamic system model for bioprocess. Qual. Theory Dyn. Syst. **16**, 205–222 (2017)
4. Zhang, X., Liu, L., Wu, Y.: The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium. Appl. Math. Lett. **37**, 26–33 (2014)
5. Zhang, X., Liu, L., Wu, Y., Wiwatanapataphee, B.: Nontrivial solutions for a fractional advection dispersion equation in anomalous diffusion. Appl. Math. Lett. **66**, 1–8 (2017)
6. Zhang, X., Liu, L., Wu, Y.: Variational structure and multiple solutions for a fractional advection-dispersion equation. Comput. Math. Appl. **68**, 1794–1805 (2014)
7. Zhu, B., Liu, L., Wu, Y.: Local and global existence of mild solutions for a class of nonlinear fractional reaction–diffusion equations with delay. Appl. Math. Lett. **61**, 73–79 (2016)
8. Zhu, B., Liu, L., Wu, Y.: Local and global existence of mild solutions for a class of semilinear fractional integro-differential equations. Fract. Calc. Appl. Anal. **20**(6), 1338–1355 (2017)
9. Heymans, N., Podlubny, I.: Physical interpretation of initial conditions for fractional differential equations with Riemann–Liouville fractional derivatives. Rheol. Acta **45**, 765–771 (2006)
10. Atangana, A.: Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. Chaos Solitons Fractals **102**, 396–406 (2017)
11. Benkhettou, N., Cruz, A., Torres, D.: A fractional calculus on arbitrary time scales: fractional differentiation and fractional integration. Signal Process. **107**, 230–237 (2015)
12. Qin, H., Zuo, X., Liu, J., Liu, L.: Approximate controllability and optimal controls of fractional dynamical systems of order $1 < q < 2$ in Banach spaces. Adv. Differ. Equ. **2015**, 73 (2015)
13. Zhang, X., Liu, L., Wu, Y.: Existence results for multiple positive solutions of nonlinear higher order perturbed fractional differential equations with derivatives. Appl. Math. Comput. **19**, 1420–1433 (2012)
14. Zhang, X., Liu, L., Wu, Y., Wiwatanapataphee, B.: The spectral analysis for a singular fractional differential equation with a signed measure. Appl. Math. Comput. **257**, 252–263 (2015)
15. Wang, Y., Liu, L., Wu, Y.: Extremal solutions for p-Laplacian fractional integro-differential equation with integral conditions on infinite intervals via iterative computation. Adv. Differ. Equ. **2015**, 24 (2015)
16. Guan, Y., Zhao, Z., Lin, X.: On the existence of solutions for impulsive fractional differential equations. Adv. Math. Phys. **2017**, Article ID 1207456 (2017)
17. Jiang, J., Liu, L., Wu, Y.: Positive solutions to singular fractional differential system with coupled boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **18**, 3061–3074 (2013)
18. Hao, X., Wang, H.: Positive solutions of semipositone singular fractional differential systems with a parameter and integral boundary conditions. Open Math. **16**(1), 581–596 (2018)
19. Zhang, X., Liu, L., Wu, Y., Cui, Y.: New result on the critical exponent for solution of an ordinary fractional differential problem. J. Funct. Spaces **2017**, Article ID 3976469 (2017)

20. Wang, F., Yang, Y.: Quasi-synchronization for fractional-order delayed dynamical networks with heterogeneous nodes. Appl. Math. Comput. **339**, 1–14 (2018)
21. Wu, J., Zhang, X., Liu, L., Wu, Y., Cui, Y.: The convergence analysis and error estimation for unique solution of a *p*-Laplacian fractional differential equation with singular decreasing nonlinearity. Bound. Value Probl. **2018**, 82 (2018)
22. Zhang, X., Liu, L., Wu, Y., Lu, Y.: The iterative solutions of nonlinear fractional differential equations. Appl. Math. Comput. **219**, 4680–4691 (2013)
23. Zhang, X., Liu, L., Wu, Y.: Multiple positive solutions of a singular fractional differential equation with negatively perturbed term, Math. Comput. Model. **55**, 1263–1274 (2012)
24. Zhang, X., Liu, L., Wiwatanapataphee, B., Wu, Y.: The eigenvalue for a class of singular *p*-Laplacian fractional differential equations involving the Riemann–Stieltjes integral boundary condition. Appl. Math. Comput. **235**, 412–422 (2014)
25. Zhang, X., Liu, L., Wu, Y.: The eigenvalue problem for a singular higher fractional differential equation involving fractional derivatives. Appl. Math. Comput. **218**, 8526–8536 (2012)
26. Li, M., Wang, J.: Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. Appl. Math. Comput. **324**, 254–265 (2018)
27. Feng, Q., Meng, F.: Traveling wave solutions for fractional partial differential equations arising in mathematical physics by an improved fractional Jacobi elliptic equation method. Math. Methods Appl. Sci. **40**, 3676–3686 (2017)
28. Ren, T., Li, S., Zhang, X., Liu, L.: Maximum and minimum solutions for a nonlocal *p* -Laplacian fractional differential system from eco-economical processes. Bound. Value Probl. **2017**(1), 118 (2017)
29. Zhang, X., Wu, Y., Cui, Y.: Existence and nonexistence of blow-up solutions for a Schrödinger equation involving a nonlinear operator. Appl. Math. Lett. **82**, 85–91 (2018)
30. Lin, X., Zhao, Z.: Existence and uniqueness of symmetric positive solutions of 2n-order nonlinear singular boundary value problems. Appl. Math. Lett. **26**(7), 692–698 (2013)
31. Zhang, X., Liu, L., Wu, Y., Cui, Y.: The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach. J. Math. Anal. Appl. **464**(2), 1089–1106 (2018)
32. Lin, X., Zhao, Z.: Iterative technique for a third-order differential equation with three-point nonlinear boundary value conditions. Electron. J. Qual. Theory Differ. Equ. **2016**, 12 (2016)
33. Zhang, X., Liu, L., Wu, Y., Caccetta, L.: Entire large solutions for a class of Schrödinger systems with a nonlinear random operator. J. Math. Anal. Appl. **423**, 1650–1659 (2015)
34. Zhang, X., Liu, L., Wu, Y.: The uniqueness of positive solution for a singular fractional differential system involving derivatives. Commun. Nonlinear Sci. Numer. Simul. **18**, 1400–1409 (2013)
35. Wu, J., Zhang, X., Liu, L., Wu, Y., Cui, Y.: Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation. Math. Model. Anal. **23**, 611–626 (2018)
36. Mao, J., Zhao, Z., Wang, C.: The exact iterative solution of fractional differential equation with nonlocal boundary value conditions. J. Funct. Spaces **2018**, Article ID 8346398 (2018)
37. Chen, H., Chen, H., Li, M.: A new simultaneous iterative method with a parameter for solving the extended split equality problem and the extended split equality fixed point problem. Numer. Algorithms **79**(4), 1231–1256 (2018)
38. Li, J., Liu, B., Liu, L.: Solutions for a boundary value problem at resonance on $[0,\infty)$. Math. Comput. Model. **58**, 1769–1776 (2013)
39. Wang, Y., Liu, L.: Positive solutions for a class of fractional 3-point boundary value problems at resonance. Adv. Differ. Equ. **2017**, 7 (2017)
40. Liu, B., Li, J., Liu, L.: Existence and uniqueness for an *m*-point boundary value problem at resonance on infinite intervals. Comput. Math. Appl. **64**, 1677–1690 (2012)
41. Sun, F., Liu, L., Zhang, X., Wu, Y.: Spectral analysis for a singular differential system with integral boundary conditions. Mediterr. J. Math. **13**, 4763–4782 (2016)
42. Liu, L., Sun, F., Zhang, X., Wu, Y.: Bifurcation analysis for a singular differential system with two parameters via to topological degree theory. Nonlinear Anal., Model. Control **22**, 31–50 (2017)
43. Hao, X., Zuo, M., Liu, L.: Multiple positive solutions for a system of impulsive integral boundary value problems with sign-changing nonlinearities. Appl. Math. Lett. **82**, 24–31 (2018)
44. Wu, J., Zhang, X., Liu, L., Wu, Y.: Positive solutions of higher-order nonlinear fractional differential equations with changing-sign measure. Adv. Differ. Equ. **2012**, 71 (2012)
45. Liu, L., Hao, X., Wu, Y.: Positive solutions for singular second order differential equations with integral boundary conditions. Math. Comput. Model. **57**, 836–847 (2013)
46. Hao, X., Liu, L.: Mild solution of semilinear impulsive integro-differential evolution equation in Banach spaces. Math. Methods Appl. Sci. **40**(13), 4832–4841 (2017)
47. Jiang, J., Liu, L., Wu, Y.: Symmetric positive solutions to singular system with multi-point coupled boundary conditions. Appl. Math. Comput. **220**, 536–548 (2013)
48. Hao, X., Wang, H., Liu, L., Cui, Y.: Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and *p*-Laplacian operator. Bound. Value Probl. **2017**, 182 (2017)
49. Zuo, M., Hao, X., Liu, L., Cui, Y.: Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions. Bound. Value Probl. **2017**, 161 (2017)
50. He, J., Zhang, X., Liu, L., Wu, Y.: Existence and nonexistence of radial solutions of the Dirichlet problem for a class of general *k*-Hessian equations. Nonlinear Anal., Model. Control **23**(4), 475–492 (2018)
51. Xu, Y., Zhang, H.: Positive solutions of an infinite boundary value problem for *n*th-order nonlinear impulsive singular integro-differential equations in Banach spaces. Appl. Math. Comput. **218**, 5806–5818 (2012)
52. Zhang, X., Shao, Z., Zhong, Q., Zhao, Z.: Triple positive solutions for semipositone fractional differential equations *m*-point boundary value problems with singularities and *p*-*q*-order derivatives. Nonlinear Anal., Model. Control **23**(4), 889–903 (2018)
53. Zhang, X., Jiang, J., Wu, Y., Cui, Y.: Existence and asymptotic properties of solutions for a nonlinear Schrödinger elliptic equation from geophysical fluid flows. Appl. Math. Lett. (2018). https://doi.org/10.1016/j.aml.2018.11.011
54. Jiang, J., Liu, W., Wang, H.: Positive solutions to singular Dirichlet-type boundary value problems of nonlinear fractional differential equations. Adv. Differ. Equ. **2018**, 169 (2018)
55. Hao, X., Hao, H., Liu, L.: Existence results for fractional integral boundary value problem involving fractional derivatives on an infinite interval. Math. Methods Appl. Sci. **41**(16), 6984–6996 (2018)

56. Zhang, X., Liu, L., Wu, Y., Cui, Y.: Entire blow-up solutions for a quasilinear p-Laplacian Schrödinger equation with a non-square diffusion term. Appl. Math. Lett. **74**, 85–93 (2017)
57. Zhang, X., Liu, L., Wu, Y.: The entire large solutions for a quasilinear Schrödinger elliptic equation by the dual approach. Appl. Math. Lett. **55**, 1–9 (2016)
58. Liu, J., Zhao, Z.: Multiple solutions for impulsive problems with non-autonomous perturbations. Appl. Math. Lett. **64**, 143–149 (2017)
59. Liu, J., Zhao, Z.: An application of variational methods to second-order impulsive differential equation with derivative dependence. Electron. J. Differ. Equ. **2014**, 62 (2014)
60. Liu, J., Zhao, Z.: Existence of positive solutions to a singular boundary-value problem using variational methods. Electron. J. Differ. Equ. **2014**, 135 (2014)
61. He, X., Qian, A., Zou, W.: Existence and concentration of positive solutions for quasi-linear Schrödinger equations with critical growth. Nonlinearity **26**, 3137–3168 (2013)
62. Mao, A., Jing, R., Luan, S., Chu, J., Kong, Y.: Some nonlocal elliptic problem involving positive parameter. Topol. Methods Nonlinear Anal. **42**, 207–220 (2013)
63. Qian, A.: Sign solutions for nonlinear problems with strong resonance. Electron. J. Differ. Equ. **2012**, 17 (2012)
64. Qian, A.: Infinitely many sign-changing solutions for a Schrödinger equation. Adv. Differ. Equ. **2011**, 39 (2011)
65. Mao, A., Zhu, X.: Existence and multiplicity results for Kirchhoff problems. Mediterr. J. Math. **14**(2), 58 (2017)
66. Zhang, J., Lou, Z., Ji, Y., Shao, W.: Ground state of Kirchhoff type fractional Schrodinger equations with critical growth. J. Math. Anal. Appl. **462**, 57–83 (2018)
67. Mao, A., Wang, W.: Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in R^3. J. Math. Anal. Appl. **459**, 556–563 (2018)
68. Shao, M., Mao, A.: Multiplicity of solutions to Schrödinger–Poisson system with concave-convex nonlinearities. Appl. Math. Lett. **83**, 212–218 (2018)
69. Sun, F., Liu, L., Wu, Y.: Finite time blow-up for a class of parabolic or pseudo-parabolic equations. Comput. Math. Appl. **75**, 3685–3701 (2018)
70. Sun, F., Liu, L., Wu, Y.: Finite time blow-up for a thin-film equation with initial data at arbitrary energy level. J. Math. Anal. Appl. **458**, 9–20 (2018)
71. Zhang, J., Lou, Z., Ji, Y., Shao, W.: Multiplicity of solutions of the bi-harmonic Schrödinger equation with critical growth. Z. Angew. Math. Phys. **69**, 42 (2018)
72. Liu, J., Qian, A.: Ground state solution for a Schrödinger–Poisson equation with critical growth. Nonlinear Anal., Real World Appl. **40**, 428–443 (2018)
73. Zhang, X., Liu, L., Wu, Y., Cui, Y.: Existence of infinitely solutions for a modified nonlinear Schrödinger equation via dual approach. Electron. J. Differ. Equ. **2018**, 147 (2018)
74. Cui, Y., Zou, Y.: An existence and uniqueness theorem for a second order nonlinear system with coupled integral boundary value conditions. Appl. Math. Comput. **256**, 438–444 (2015)
75. Wu, J., Zhang, X., Liu, L., Wu, Y.: Positive solution of singular fractional differential system with nonlocal boundary conditions. Adv. Differ. Equ. **2014**, 323 (2014)
76. Podlubny, I.: Fractional Differential Equations, Mathematics in Science and Engineering. Academic Press, New York (1999)