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Existence of positive solutions for a singular nonlinear fractional differential equation with integral boundary conditions involving fractional derivatives

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Abstract

In this article, by using the spectral analysis of the relevant linear operator and Gelfand's formula, some properties of the first eigenvalue of a fractional differential equation are obtained. Based on these properties and through the fixed point index theory, the singular nonlinear fractional differential equations with Riemann–Stieltjes integral boundary conditions involving fractional derivatives are considered under some appropriate conditions, and the nonlinearity is allowed to be singular in regard to not only time variable but also space variable and it includes fractional derivatives. The existence of positive solutions for boundary conditions involving fractional derivatives is established. Finally, an example is given to demonstrate the validity of our main results.

MSC: 26A33; 34A08; 34B16; 34B18

Keywords: Singular fractional differential equations; Integral boundary conditions involving fractional derivatives; Positive solution; Fixe point index theory; Cone

1 Introduction

Recently, fractional differential equations have drawn more and more attention of the research community due to their numerous applications in various fields of science such as engineering, chemistry, physics, mechanics, etc. [1–4]. Boundary value problems of fractional differential equations have been investigated for many years. Now, there are many papers dealing with the problem for different kinds of boundary value conditions such as multi-point boundary condition (see [5–10]), integral boundary condition (see [11–24]), and many other boundary conditions (see [25–32]).

In this paper, we consider the existence of positive solutions for the following integral boundary value problems of singular nonlinear fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + p(t)f(t,u(t),D_{0^+}^{\beta_1}u(t),\ldots,D_{0^+}^{\beta_{n-1}}u(t)) = 0, \quad 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \\ D_{0^+}^{\beta}u(1)=\int_0^1 l(t)D_{0^+}^{\beta_{n-1}}u(t)\,dA(t), \end{cases} \tag{1.1}$$

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where $n-1 < \alpha \leq n, i-1 < \beta_i \leq i\,(i = 1, 2, \ldots, n-1), \alpha - \beta_{n-1} > \alpha - \beta > 1, p \in C((0,1), \mathbb{R}_+)$ and $p(t)$ is allowed to be singular at $t = 0$ or $t = 1$, in which $\mathbb{R}_+ = [0, +\infty), f : [0,1] \times (0, +\infty)^n \to \mathbb{R}_+$ is continuous and f may be singular at $x_0 = x_1 = \cdots = x_{n-1} = 0, l : (0,1) \to \mathbb{R}_+$ is continuous with $l \in L^1(0,1)$, $\int_0^1 l(t)u(t)\,dA(t)$ denotes the Riemann–Stieltjes integral with a signed measure, in which $A : [0,1] \to \mathbb{R} = (-\infty, +\infty)$ is a function of bounded variation.

Zhang *et al.* [15] studied the existence of positive solutions of the following singular nonlinear fractional differential equation with integral boundary value conditions:

$$\begin{cases} -D_t^{\alpha} x(t) = f(t, x(t), D_t^{\beta} x(t)), \quad 0 < t < 1, \\ D_t^{\beta} x(0) = 0, \qquad D_t^{\beta} x(1) = \int_0^1 D_t^{\beta} x(s)\,dA(s), \end{cases}$$

where $0 < \beta \leq 1 < \alpha \leq 2, f(t, x, y)$ may be singular at both $t = 0, 1$ and $x = y = 0$, $\int_0^1 x(s)\,dA(s)$ denotes the Riemann–Stieltjes integral with a signed measure, in which $A : [0,1] \to \mathbb{R}$ is a function of bounded variation. Through the spectral analysis and fixed point index theorem, the author obtained the existence of positive solutions.

By means of the fixed point index theory, Hao *et al.* [16] studied the existence of positive solutions of the following nth order differential equation:

$$\begin{cases} u^{(n)}(t) + \lambda a(t) f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \int_0^1 u(s)\,dA(s), \end{cases}$$

where $\lambda > 0$ is a parameter, $0 \leq \Gamma := \int_0^1 t^{n-1}\,dA(t) < 1, a : (0,1) \to \mathbb{R}_+$ is continuous and $a(t)$ may be singular at $t = 0$ and $t = 1, f : [0,1] \times (0, +\infty) \to \mathbb{R}_+$ is continuous and $f(t, x)$ may also have singularity at $x = 0$. $\int_0^1 u(s)\,dA(s)$ denotes the Riemann–Stieltjes integral with a signed measure, that is, A has bounded variation.

Li *et al.* [17] studied the existence of positive solutions of the following singular nonlinear fractional differential equation with integral boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + p(t) f(t, u(t)) + q(t) g(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \int_0^1 h(s) u(s)\,dA(s), \end{cases}$$

where $n - 1 < \alpha \leq n$, $p, q \in C((0,1), \mathbb{R}_+)$, $p(t)$ and $q(t)$ are allowed to be singular at $t = 0$ or $t = 1, f, g : [0,1] \times (0, +\infty) \to \mathbb{R}_+$ are continuous and $f(t, x)$, $g(t, x)$ may be singular at $x = 0$, $h : (0,1) \to \mathbb{R}_+$ is continuous with $h \in L^1(0,1)$; $\int_0^1 h(s) u(s)\,dA(s)$ denotes the Riemann–Stieltjes integral with a signed measure, in which $A : [0,1] \to \mathbb{R}$ is a function of bounded variation. Through a well-known fixed point theorem, the author obtained the existence and multiplicity of positive solutions.

Zhang [29] obtained several cases of local existence and multiplicity of positive solutions for the following infinite-point boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + q(t) f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases}$$

where $\alpha > 2$, $n-1 < \alpha \leq n$, $i \in [1, n-2]$ is fixed, $f : (0,1) \times (0, +\infty)$ is continuous and $f(t,x)$ permits singularities with $t = 0, 1$ and $x = 0$.

Motivated by the above mentioned papers, the purpose of this article is to investigate the existence of positive solutions for a more general problem. Obviously, our work is different from those in [15–17, 29]. The main new features presented in this paper are as follows. Firstly, the nonlinear term f in our question includes multiple fractional order derivatives of unknown function, and the boundary value conditions also involve the fractional derivative. Secondly, the nonlinear term f in our work can be singular at $x_0 = x_1 = \cdots = x_{n-1} = 0$, which implies all of the above work. Lastly, by using the spectral analysis and fixed point index theorem, we get the existence of positive solutions.

The rest of the paper is organized as follows. Firstly, we present some preliminaries and lemmas that are to be used to prove our main results and develop some properties of the Green function. Secondly, we prove the existence of a positive solution of BVP (1.1). Finally, we give an example to prove our main conclusion.

2 Preliminaries and lemmas

In this section, for the convenience of the reader, we present some notations and lemmas that will be used in the proof of our main results.

Definition 2.1 ([4]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([4]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $\alpha > 0$ and $n-1 < \alpha \leq n$ $(n = 1, 2, 3, \dots)$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([2]) *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad C_i \in \mathbb{R}, i = 1, 2, \dots, N,$$

as the unique solution, where $N = [\alpha] + 1$.

From the definition of the Riemann–Liouville derivative, we can obtain the statement.

Lemma 2.2 ([2]) *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

for some $C_i \in \mathbb{R}$ $(i = 1, 2, \ldots, N)$, where $N = [\alpha] + 1$.

In the following, we present the Green function of the fractional differential equation boundary value problem.

Lemma 2.3 *Take f be as in* (1.1) *and let $v(t) = D_{0^+}^{\beta_{n-1}} u(t)$. Then problem* (1.1) *is transformed to the following equation*:

$$\begin{cases} D_{0^+}^{\alpha-\beta_{n-1}} v(t) + f(t, I_{0^+}^{\beta_{n-1}} v(t), I_{0^+}^{\beta_{n-1}-\beta_1} v(t), \ldots, v(t)) = 0, & t \in (0,1), \\ I_{0^+}^{\beta_{n-1}-n+2} v(0) = 0, \qquad D_{0^+}^{\beta-\beta_{n-1}} v(1) = \int_0^1 l(t) v(t)\, dA(t). \end{cases} \tag{2.1}$$

Furthermore, assume that $0 \leq \delta \neq \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}$, then the solution of problem (2.1) *is equivalent to the solution of the following fractional integral equation*:

$$v(t) = \int_0^1 G(t,s) f\bigl(s, I_{0^+}^{\mu_{n-1}} v(s), \ldots, I_{0^+}^{\mu_{n-1}-\mu_{n-2}} v(s), v(s)\bigr), \tag{2.2}$$

where

$$G(t,s) = K(t,s) + \frac{t^{\alpha-\beta_{n-1}-1}}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} - \delta} g_A(s),$$

in which

$$\begin{aligned} K(t,s) &= \frac{1}{\Gamma(\alpha-\beta_{n-1})} \begin{cases} t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta_{n-1}-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ \delta &= \int_0^1 t^{\alpha-\beta_{n-1}-1} l(t)\, dA(t), \\ g_A(s) &= \int_0^1 K(t,s) l(t)\, dA(t). \end{aligned} \tag{2.3}$$

Moreover, if $v(t)$ is a positive solution of (2.1), *then $u(t) = I_{0^+}^{\mu_{n-1}} v(t)$ is a positive solution of problem* (1.1).

Proof Let $v(t) = D_{0^+}^{\beta_{n-1}} u(t)$, then from the boundary value conditions of (1.1) we have $u(t) = I_{0^+}^{\beta_{n-1}} v(t)$, and

$$\begin{aligned} u^{(n-2)} &= \left(\frac{d}{dt}\right)^{(n-2)} \frac{1}{\Gamma(\beta_{n-1})} \int_0^t (t-s)^{\beta_{n-1}-1} v(s)\, ds \\ &= \frac{1}{\Gamma(\beta_{n-1})} \times \beta_{n-1} \times \cdots \times (\beta_{n-1} - n + 2) \int_0^t (t-s)^{\beta_{n-1}-n+1} v(s) \\ &= I_{0^+}^{\beta_{n-1}-n+2} v(t), \end{aligned}$$

and from Lemma 2.3 we have $D_{0^+}^{\beta} u(t) = D_{0^+}^{\beta} I_{0^+}^{\beta_{n-1}} v(t) = D_{0^+}^{\beta-\beta_{n-1}} v(t)$.

Next, we may apply Lemma 2.2 to reduce (2.1) to an equivalent integral equation

$$v(t) = -I_{0^+}^{\alpha-\beta_{n-1}} y(t) + C_1 t^{\alpha-\beta_{n-1}-1} + C_2 t^{\alpha-\beta_{n-1}-2}$$

for some $C_i \in \mathbb{R}$ $(i = 1, 2)$.

$$\begin{aligned} I_{0^+}^{\beta_{n-1}-n+2} v(t) &= I_{0^+}^{\beta_{n-1}-n+2}\bigl(-I_{0^+}^{\alpha-\beta_{n-1}} y(t) + C_1 t^{\alpha-\beta_{n-1}-1} + C_2 t^{\alpha-\beta_{n-1}-2}\bigr) \\ &= -I_{0^+}^{\alpha-n+2} y(t) + C_1 I_{0^+}^{\beta_{n-1}-n+2} t^{\alpha-\beta_{n-1}-1} + C_2 I_{0^+}^{\beta_{n-1}-n+2} t^{\alpha-\beta_{n-1}-2} \\ &= -I_{0^+}^{\alpha-n+2} y(t) + C_1 \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-n+2)} t^{\alpha-n+1} + C_2 \frac{\Gamma(\alpha-\beta_{n-1}-1)}{\Gamma(\alpha-n+1)} t^{\alpha-n}. \end{aligned} \tag{2.4}$$

From (2.4) and $I_{0^+}^{\beta_{n-1}-n+2} v(0) = 0$, we know that $C_2 = 0$. Then we obtain

$$v(t) = -I_{0^+}^{\alpha-\beta_{n-1}} y(t) + C_1 t^{\alpha-\beta_{n-1}-1}. \tag{2.5}$$

From (2.5) we have

$$\begin{aligned} D_{0^+}^{\beta-\beta_{n-1}} v(t) &= -D_{0^+}^{\beta-\beta_{n-1}} I_{0^+}^{\alpha-\beta_{n-1}} y(t) + C_1 D_{0^+}^{\beta-\beta_{n-1}} t^{\alpha-\beta_{n-1}-1} \\ &= -I_{0^+}^{\alpha-\beta} y(t) + C_1 \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} \\ &= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)\,ds + C_1 \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \end{aligned}$$

So, we have

$$D_{0^+}^{\beta-\beta_{n-1}} v(1) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds + C_1 \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}. \tag{2.6}$$

And from (2.5) we also have

$$\begin{aligned} \int_0^1 l(t)v(t)\,dA(t) &= \int_0^1 l(t)\biggl[-\frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^t (t-s)^{\alpha-\beta_{n-1}-1} y(s)\,ds + C_1 t^{\alpha-\beta_{n-1}-1}\biggr]\,dA(t) \\ &= -\frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 \int_0^t l(t)(t-s)^{\alpha-\beta_{n-1}-1} y(s)\,ds\,dA(t) \\ &\quad + C_1 \int_0^1 t^{\alpha-\beta_{n-1}-1} l(t)\,dA(t). \end{aligned} \tag{2.7}$$

From (2.6), (2.7), and $D_{0^+}^{\beta-\beta_{n-1}} v(1) = \int_0^1 l(t)v(t)\,dA(t)$, we obtain that

$$\begin{aligned} \biggl[\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} - \delta\biggr] C_1 &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds \\ &\quad - \frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 \int_0^t l(t)(t-s)^{\alpha-\beta_{n-1}-1} y(s)\,ds\,dA(t), \end{aligned}$$

where $\delta = \int_0^1 t^{\alpha-\beta_{n-1}-1} l(t)\,dA(t)$, thus

$$C_1 = \frac{\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds - \frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 \int_0^t l(t)(t-s)^{\alpha-\beta_{n-1}-1} y(s)\,ds\,dA(t)}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} - \delta}.$$

Putting C_1 into equation (2.5), we obtain that

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha-\beta_{n-1})}\int_0^t (t-s)^{\alpha-\beta_{n-1}-1}y(s)\,ds + t^{\alpha-\beta_{n-1}-1}\frac{\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}y(s)\,ds}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}-\delta} \\
&\quad - t^{\alpha-\beta_{n-1}-1}\frac{\frac{1}{\Gamma(\alpha-\beta_{n-1})}\int_0^1\int_0^t l(t)(t-s)^{\alpha-\beta_{n-1}-1}y(s)\,ds\,dA(t)}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}-\delta} \\
&= \frac{\int_0^t (t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta_{n-1}-1})y(s)\,ds + \int_t^1 t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}y(s)\,ds}{\Gamma(\alpha-\beta_{n-1})} \\
&\quad + \frac{t^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})}\frac{\int_0^1\int_0^t [t^{\alpha-\beta_{n-1}-1}l(t)(1-s)^{\alpha-\beta-1}-l(t)(t-s)^{\alpha-\beta_{n-1}-1}]y(s)\,ds\,dA(t)}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}-\delta} \\
&\quad + \frac{t^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})}\frac{\int_0^1\int_t^1 t^{\alpha-\beta_{n-1}-1}l(t)(1-s)^{\alpha-\beta-1}y(s)\,ds\,dA(t)}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}-\delta}.
\end{aligned}$$

Let

$$K(t,s) = \frac{1}{\Gamma(\alpha-\beta_{n-1})}\begin{cases} t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta_{n-1}-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases}$$

$$g_A(s) = \int_0^1 l(t)K(t,s)\,dA(t).$$

From the above we obtain that

$$u(t) = \int_0^1 K(t,s)y(s)\,ds + \frac{t^{\alpha-\beta_{n-1}-1}}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}-\delta}\int_0^1 g_A(s)y(s)\,ds = \int_0^1 G(t,s)y(s)\,ds.$$

Lastly, by the computation above, we know that if $v(t)$ is a positive solution of (2.1), then $u(t) = I_{0^+}^{\mu_{n-1}}v(t)$ is a positive solution of problem (1.1). Thus we complete the proof. □

Lemma 2.4 *Let* $0 \le \delta < \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}$ *and* $g_A(s) \ge 0$, $s \in [0,1]$, *the Green function* $G(t,s)$ *defined by* (2.2) *satisfies*

(1) $G : [0,1] \times [0,1] \to \mathbb{R}_+$ *is continuous,*

(2) *For any* $t,s \in [0,1]$, *we have* $t^{\alpha-\beta_{n-1}-1}\phi(s) \le G(t,s) \le \phi(s)$, *where*

$$\phi(s) = K(1,s) + \frac{g_A(s)}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}-\delta}, \quad s \in [0,1].$$

Proof (1) holds obviously, so we only prove (2) holds. By (2.3), when $0 \le s \le t \le 1$,

$$\begin{aligned}
\frac{\partial K(t,s)}{\partial t} &= \frac{1}{\Gamma(\alpha-\beta_{n-1})}\bigl[(\alpha-\beta_{n-1}-1)t^{\alpha-\beta_{n-1}-2}(1-s)^{\alpha-\beta-1} \\
&\quad - (\alpha-\beta_{n-1}-1)(t-s)^{\alpha-\beta_{n-1}-2}\bigr] \\
&= \frac{(\alpha-\beta_{n-1}-1)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-2}\left[(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-\beta_{n-1}-2}\right] \ge 0,
\end{aligned}$$

and

$$
\begin{aligned}
K(t,s) - t^{\alpha-\beta_{n-1}-1}K(1,s) &= \frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \\
&\quad - t^{\alpha-\beta_{n-1}-1}\frac{(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \\
&= \frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta_{n-1}-1} - (t-s)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \geq 0.
\end{aligned}
$$

In the same way, when $0 \leq t \leq s \leq 1$,

$$
\frac{\partial K(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha-\beta_{n-1})}(\alpha-\beta_{n-1}-1)t^{\alpha-\beta_{n-1}-2}(1-s)^{\alpha-\beta-1} \geq 0,
$$

and

$$
\begin{aligned}
K(t,s) - t^{\alpha-1}K(1,s) &= \frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta_{n-1})} - t^{\alpha-\beta_{n-1}-1}\frac{(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \\
&= \frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \geq 0.
\end{aligned}
$$

It follows from the above that

$$
t^{\alpha-\beta_{n-1}-1}K(1,s) \leq K(t,s) \leq K(1,s), \quad t,s \in [0,1].
$$

Furthermore, by the definition of $\phi(s)$, the conclusion of (2) is proved. □

Let $E = C[0,1]$, $\|v\| = \sup_{0\leq t\leq 1}|v(t)|$. Then $(E, \|\cdot\|)$ is a Banach space. Let

$$
\begin{aligned}
&P = \bigl\{v \in E : v(t) \geq 0, t \in [0,1]\bigr\}, \\
&K = \bigl\{v \in P : v(t) \geq t^{\alpha-\beta_{n-1}-1}\|v\|, t \in [0,1]\bigr\}, \\
&K^{(i)} = \bigl\{v \in P : v(t) \geq t^{\alpha-\beta_i-1}\|v\|, t \in [0,1]\bigr\}, \quad i = 0,1,\ldots,n-2,
\end{aligned}
$$

where $\beta_0 = 0$. And for any $r > 0$, define $\Omega_r = \{v \in K : \|v\| < r\}$, $\partial\Omega_r = \{v \in K : \|v\| = r\}$, $\overline{\Omega}_r = \{v \in K : \|v\| \leq r\}$, $\Omega_r^{(i)} = \{v \in K^{(i)} : \|v\| < r\}$. It is easy to see that K and $K^{(i)}$ $(i = 0,1,\ldots,n-2)$ are cones in E and $\overline{\Omega}_R \setminus \Omega_r \subset K$ for any $0 < r < R$. Throughout the paper we need the following conditions:

(H_1) $A : [0,1] \to \mathbb{R}$ is a function of bounded variation and $g_A(s) \geq 0$ for all $s \in [0,1]$;

(H_2) $l \in C(0,1) \cap L^1(0,1)$ and

$$
0 \leq \delta = \int_0^1 t^{\alpha-\beta_{n-1}-1}l(t)\,dA(t) < \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)};
$$

(H_3) $p : (0,1) \to \mathbb{R}_+$ is continuous, and $\int_0^1 \phi(s)p(s)\,ds < +\infty$;

(H_4) $f : [0,1] \times (0,\infty)^n \to \mathbb{R}_+$ is continuous, and for any $0 < r < R < +\infty$,

$$
\lim_{m\to\infty} \sup_{\substack{x_i \in \overline{\Omega}_{R_i} \setminus \Omega_{r_i}^{(i)} (i=0,1,\ldots,n-2) \\ x_{n-1} \in \overline{\Omega}_R \setminus \Omega_r}} \int_{H(m)} \phi(s)p(s)f\bigl(s, x_0(s), x_1(s), \ldots, x_{n-1}(s)\bigr)\,ds = 0,
$$

where $R_i = \frac{R}{\Gamma(\beta_{n-1}-\beta_i+1)}$, $r_i = \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_i)r}$ $(i = 0,1,\ldots,n-2)$, and $H(m) = [0,\frac{1}{m}] \cup [\frac{m-1}{m},1]$, $\phi(s)$ is defined in Lemma 2.4.

In what follows, let us define a nonlinear operator $L : \overline{\Omega}_R \setminus \Omega_r \to E$ and a linear operator $T : E \to E$ by

$$(Lv)(t) = \int_0^1 G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s),\ldots,v(s)\bigr)\,ds \tag{2.8}$$

and

$$(Tv)(t) = \int_0^1 G(t,s)p(s)v(s)\,ds, \quad t \in [0,1], \tag{2.9}$$

respectively. And for any $\tau : 0 < \tau < \delta$, we define $T_\tau : E \to E$ by

$$T_\tau u(t) = \int_\tau^{1-\tau} G(t,s)p(s)u(s)\,ds, \quad t \in [0,1]. \tag{2.10}$$

Lemma 2.5 *Suppose that* (H$_1$)–(H$_4$) *hold. Then* $L : \overline{\Omega}_R \setminus \Omega_r \to K$ *is a completely continuous operator, and the fixed point of* L *in* $\overline{\Omega}_R \setminus \Omega_r$ *is the positive solutions to BVP* (2.1).

Proof It follows from (H$_4$) that there exists a natural number $m_1 \geq 2$ such that

$$\sup_{v\in\overline{\Omega}_R\setminus\Omega_r} \int_{H(m_1)} \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s),\ldots,v(s)\bigr)\,ds < 1.$$

It is easy to see that for each $v \in \overline{\Omega}_R \setminus \Omega_r$ there exists $r_1 \in [r,R]$ such that $\|v\| = r_1$. For $v \in \Omega$, we have

$$t^{\alpha-\beta_{n-1}-1}r \leq t^{\alpha-\beta_{n-1}-1}r_1 \leq v(t) \leq r_1 \leq R,$$

and for any $i = 0,1,\ldots,n-2$, we have

$$\begin{aligned} I_{0^+}^{\beta_{n-1}-\beta_i}v(t) &= \frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\int_0^t (t-s)^{\beta_{n-1}-\beta_i-1}v(s)\,ds \\ &\geq \frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\int_0^t (t-s)^{\beta_{n-1}-\beta_i-1}s^{\alpha-\beta_{n-1}-1}r_1\,ds \\ &\geq \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_i)}t^{\alpha-\beta_i-1}r, \end{aligned}$$

and

$$\begin{aligned} I_{0^+}^{\beta_{n-1}-\beta_i}v(t) &= \frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\int_0^t (t-s)^{\beta_{n-1}-\beta_i-1}v(s)\,ds \\ &\leq \frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\int_0^t (t-s)^{\beta_{n-1}-\beta_i-1}r_1\,ds \\ &\leq \frac{1}{\Gamma(\beta_{n-1}-\beta_i+1)}R. \end{aligned}$$

And for all $t \in [\frac{1}{m}, \frac{m-1}{m}]$, we have $\frac{1}{m^{\alpha-\beta_{n-1}-1}} r \le v(t) \le R$, and

$$\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_i)} \frac{r}{m^{\alpha-\beta_i-1}} \le I_{0^+}^{\beta_{n-1}-\beta_i} v(t) \le \frac{1}{\Gamma(\beta_{n-1}-\beta_i+1)} R.$$

Let

$$M_1 = \max\bigl\{f(t, x_0, \dots, x_{n-1}) : (t, x_0, \dots, x_{n-1}) \in I \times J_0 \times J_1 \times \cdots \times J_{n-1}\bigr\},$$

where $I = [\frac{1}{m_1}, \frac{m_1-1}{m_1}]$, $J_i = [\frac{\Gamma(\alpha-\beta_{n-1})r}{\Gamma(\alpha-\beta_i)m_1^{\alpha-\beta_i-1}}, \frac{1}{\Gamma(\beta_{n-1}-\beta_i+1)}R]$ $(i = 0, 1, 2, \dots, n-2)$, $J_{n-1} = [\frac{1}{m_1^{\alpha-\beta_{n-1}-1}} r, R]$. So, by Lemma 2.4(2), (H_3), and (H_4), we have

$$\begin{aligned}
&\sup_{v \in \overline{\Omega_R} \setminus \Omega_r} \int_0^1 G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds \\
&\quad \le \sup_{v \in \overline{\Omega_R} \setminus \Omega_r} \int_0^1 \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds \\
&\quad \le \sup_{v \in \overline{\Omega_R} \setminus \Omega_r} \int_{H(m_1)} \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds \\
&\qquad + \sup_{v \in \overline{\Omega_R} \setminus \Omega_r} \int_{\frac{1}{m_1}}^{\frac{m_1-1}{m_1}} \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds \\
&\quad \le 1 + M_1 \int_{\frac{1}{m_1}}^{\frac{m_1-1}{m_1}} \phi(s)p(s)\, ds \\
&\quad \le 1 + M_1 \int_0^1 \phi(s)p(s)\, ds < +\infty.
\end{aligned} \tag{2.11}$$

This implies that the operator L defined by (2.8) is well defined.

Next, we show that $L : \overline{\Omega}_R \setminus \Omega_r \to K$. For any $v \in \overline{\Omega}_R \setminus \Omega_r$, $t \in [0,1]$, we have

$$\begin{aligned}
(Lv)(t) &= \int_0^1 G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds \\
&\le \int_0^1 \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds.
\end{aligned}$$

Hence,

$$\|Lv\| \le \int_0^1 \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds.$$

On the other hand, by Lemma 2.4, we have

$$\begin{aligned}
(Lv)(t) &\ge t^{\alpha-\beta_{n-1}-1} \int_0^1 \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \dots, v(s)\bigr)\, ds \\
&\ge t^{\alpha-\beta_{n-1}-1} \|Lv\|, \quad t \in [0,1],
\end{aligned}$$

thus $Lv \in K$. Therefore $L : \overline{\Omega}_R \setminus \Omega_r \to K$.

Finally, we prove that $L : \overline{\Omega}_R \setminus \Omega_r \to K$ is a completely continuous map. Suppose $D \subset \overline{\Omega}_R \setminus \Omega_r$ is an arbitrary bounded set. Firstly, from the above proof, we know that $L(D)$ is uniformly bounded. Secondly, we show that $L(D)$ is equicontinuous. In fact, for any $\varepsilon > 0$, there exists a natural number $m_2 \geq 3$ such that

$$\sup_{v\in\overline{\Omega}_R\setminus\Omega_r} \int_{H(m_2)} \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \dots, v(s)\bigr)\,ds < \frac{\varepsilon}{4}. \tag{2.12}$$

Since $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for the above $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, $s \in [\frac{1}{m_2}, \frac{m_2-1}{m_2}]$,

$$\bigl|G(t_1,s) - G(t_2,s)\bigr| < \frac{\varepsilon}{2\overline{p}M_2},$$

where

$$M_2 = \max\bigl\{f(t,x_0,\dots,x_{n-1}) : (t,x_0,\dots,x_{n-1}) \in I \times J_0 \times J_1 \times \cdots \times J_{n-1}\bigr\},$$

in which

$$\begin{aligned}
&I = \left[\frac{1}{m_2}, \frac{m_2-1}{m_2}\right],\\
&J_i = \left[\frac{\Gamma(\alpha-\beta_{n-1})r}{\Gamma(\alpha-\beta_i)m_2^{\alpha-\beta_i-1}}, \frac{1}{\Gamma(\beta_{n-1}-\beta_i+1)}R\right], \quad i = 0,1,2,\dots,n-2,\\
&J_{n-1} = \left[\frac{1}{m_2^{\alpha-\beta_{n-1}-1}}r, R\right].
\end{aligned}$$

Consequently, for any $v \in D$, $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&\bigl|(Lv)(t_1) - (Lv)(t_2)\bigr|\\
&\quad= \left|\int_0^1 \bigl(G(t_1,s) - G(t_2,s)\bigr)\right| p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \dots, v(s)\bigr)\,ds\\
&\quad\leq 2\int_{H(m_2)} \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \dots, v(s)\bigr)\,ds\\
&\qquad+ \sup_{v\in D}\int_{\frac{1}{m_2}}^{\frac{m_2-1}{m_2}} \bigl|\bigl(G(t_1,s) - G(t_2,s)\bigr)\bigr| p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \dots, v(s)\bigr)\,ds\\
&\quad\leq 2 \times \frac{\varepsilon}{4} + \frac{\varepsilon}{2\overline{p}M_2}\overline{p}M_2\\
&\quad= \varepsilon,
\end{aligned}$$

where

$$\overline{p} = \max\left\{p(t) : \frac{1}{m_2} \leq t \leq \frac{m_2-1}{m_2}\right\}.$$

This shows that $L(D)$ is equicontinuous. By the Arzela–Ascoli theorem, $L : \overline{\Omega}_R \setminus \Omega_r \to K$ is compact. Thirdly, we prove that $L : \overline{\Omega}_R \setminus \Omega_r \to K$ is continuous. Assume $v_0, v_n \in \overline{\Omega}_R \setminus \Omega_r$

and $\|v_n - v_0\| \to 0$ $(n \to \infty)$. Then $r \le \|v_n\| \le R$ and $r \le \|v_0\| \le R$. From (H_4) there exists a natural number $m_3 > m_2$ such that

$$\sup_{v\in\overline{\Omega}_R\setminus\Omega_r} \int_{H(m_3)} \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \ldots, v(s)\bigr)\,ds < \frac{\varepsilon}{4}. \tag{2.13}$$

Since $f(t, x_0, \ldots, x_{n-1})$ is uniformly continuous in

$$\left[\frac{1}{m_3}, \frac{m_3-1}{m_3}\right] \times \prod_{i=0}^{n-2}\left[\frac{\Gamma(\alpha-\beta_{n-1})r}{\Gamma(\alpha-\beta_i)m^{\alpha-\beta_i-1}}, \frac{1}{\Gamma(\beta_{n-1}-\beta_i+1)}R\right] \times \left[\left(\frac{1}{m_3}\right)^{\alpha-\beta_{n-1}-1} r, R\right],$$

we have

$$\lim_{n\to\infty}\bigl|f\bigl(s, I_{0^+}^{\beta_{n-1}}v_n(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_n(s), \ldots, v_n(s)\bigr) - f\bigl(s, I_{0^+}^{\beta_{n-1}}v_0(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_0(s), \ldots, v_0(s)\bigr)\bigr| = 0$$

uniformly on $s \in [\frac{1}{m_3}, \frac{m_3-1}{m_3}]$. Then the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}&\int_{\frac{1}{m_3}}^{\frac{m_3-1}{m_3}} \phi(s)p(s)\bigl|f\bigl(s, I_{0^+}^{\beta_{n-1}}v_n(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_n(s), \ldots, v_n(s)\bigr)\\ &\quad - f\bigl(s, I_{0^+}^{\beta_{n-1}}v_0(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_0(s), \ldots, v_0(s)\bigr)\bigr|\,ds \to 0, \quad n\to\infty.\end{aligned}$$

Thus, for the above $\varepsilon > 0$, there exists a natural number N such that for $n > N$ we have

$$\begin{aligned}&\int_{\frac{1}{m_3}}^{\frac{m_3-1}{m_3}} \phi(s)p(s)\bigl|f\bigl(s, I_{0^+}^{\beta_{n-1}}v_n(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_n(s), \ldots, v_n(s)\bigr)\\ &\quad - f\bigl(s, I_{0^+}^{\beta_{n-1}}v_0(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_0(s), \ldots, v_0(s)\bigr)\bigr|\,ds < \frac{\varepsilon}{2}.\end{aligned} \tag{2.14}$$

It follows from (2.13), (2.14) that when $n > N$

$$\begin{aligned}\|Lv_n - Lv_0\| \le{}& \sup_{v\in\overline{\Omega}_R\setminus\Omega_r} \int_{H(m_3)} \phi(s)p(s)\bigl|f\bigl(s, I_{0^+}^{\beta_{n-1}}v_n(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_n(s), \ldots, v_n(s)\bigr)\\ &- f\bigl(s, I_{0^+}^{\beta_{n-1}}v_0(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_0(s), \ldots, v_0(s)\bigr)\bigr|\,ds\\ &+ \sup_{v\in\overline{\Omega}_R\setminus\Omega_r} \int_{\frac{1}{m_3}}^{\frac{m_3-1}{m_3}} \phi(s)p(s)\bigl|f\bigl(s, I_{0^+}^{\beta_{n-1}}v_n(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_n(s), \ldots, v_n(s)\bigr)\\ &- f\bigl(s, I_{0^+}^{\beta_{n-1}}v_0(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_0(s), \ldots, v_0(s)\bigr)\bigr|\,ds\\ \le{}& \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This implies that $L : \overline{\Omega}_R \setminus \Omega_r \to K$ is continuous. Thus $L : \overline{\Omega}_R \setminus \Omega_r \to K$ is completely continuous. It is clear that if v is a fixed point of L in $\overline{\Omega}_R \setminus \Omega_r$, then v satisfies (2.1) and is a positive solution of BVP (2.1). □

Lemma 2.6 *Assume that* (H_1)–(H_3) *hold, then for the linear bounded operator* T *the spectral radius* $r(T) \neq 0$ *and* T *has a positive eigenfunction* φ_1 *corresponding to its first eigenvalue* $\lambda_1 = (r(T))^{-1}$, *that is,* $\varphi_1 = \lambda_1 T\varphi_1$. *In the same way,* T_τ *has a positive eigenfunction corresponding to its first eigenvalue* $\lambda_\tau = (r(T_\tau))^{-1}$.

Proof The proof is similar to Lemma 2.5 of [12], so we omit it. □

To prove the main results, we need the following well-known fixed point index theorem.

Lemma 2.7 ([33]) *Let* K *be a cone in a real Banach space* E. *Suppose that* $L : \overline{\Omega}_r \to K$ *is a completely continuous operator. If there exists* $u_0 \in K\backslash\{\theta\}$ *such that* $u - Lu \neq \mu u_0$ *for any* $u \in \partial\Omega_r$ *and* $\mu \geq 0$, *then* $i(L, \Omega_r, K) = 0$.

Lemma 2.8 ([33]) *Let* K *be a cone in a real Banach space* E. *Suppose that* $L : \overline{\Omega}_r \to K$ *is a completely continuous operator. If* $Lu \neq \mu u$ *for any* $u \in \partial\Omega_r$ *and* $\mu \geq 1$, *then* $i(L, \Omega_r, K) = 1$.

3 Existence of positive solutions

Theorem 3.1 *Assume that* (H_1)–(H_4) *hold, and*

$$\overline{f}^{\infty} = \limsup_{x_{n-1}\to+\infty} \sup_{\substack{t\in[0,1] \\ x_i>0(i=0,1,\ldots,n-2)}} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{x_{n-1}} < \lambda_1, \tag{3.1}$$

$$\underline{f}_0 = \liminf_{\substack{x_i\to 0 \\ i=1,2,\ldots,n-1}} \min_{t\in[0,1]} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{x_0 + x_1 + \cdots + x_{n-1}} > \lambda_1. \tag{3.2}$$

Then BVP (1.1) *has at least one positive solution, where* λ_1 *is the first eigenvalue of the operator* T *defined by* (2.9).

Proof From (3.2) we can choose $\varepsilon_0 > 0$, there exists $r > 0$ such that, for any $t \in [0,1]$ and $0 \leq x_i \leq \frac{r}{\Gamma(\beta_{n-1}-\beta_i+1)}$ $(i = 0,1,\ldots,n-2)$, $0 \leq x_{n-1} \leq r$, we have

$$f(t,x_0,x_1,\ldots,x_{n-1}) \geq (\lambda_1 + \varepsilon_0)(x_0 + x_1 + \cdots + x_{n-1}). \tag{3.3}$$

For any $v \in \partial\Omega_r$, since

$$0 \leq I_{0^+}^{\beta_{n-1}-\beta_i} v(t) \leq \frac{r}{\Gamma(\beta_{n-1}-\beta_i+1)} \quad (i = 0,1,\ldots,n-2), 0 \leq v(t) \leq r,$$

thus from (3.3), we have

$$\begin{aligned}
(Lv)(t) &= \int_0^1 G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \ldots, v(s)\bigr)\,ds \\
&\geq \int_0^1 G(t,s)p(s)(\lambda_1 + \varepsilon_0)\bigl(I_{0^+}^{\beta_{n-1}} v(s) + I_{0^+}^{\beta_{n-1}-\beta_1} v(s) + \cdots + v(s)\bigr)\,ds \\
&\geq \int_0^1 G(t,s)p(s)(\lambda_1 + \varepsilon_0)v(s)\,ds \\
&\geq \lambda_1(Tv)(t), \quad t \in [0,1].
\end{aligned}$$

Let φ_1 be the positive eigenfunction corresponding to the first eigenvalue λ_1, thus $\varphi_1 = \lambda_1 T\varphi_1$. We may suppose that L has no fixed points on $\partial\Omega_r$ (otherwise, the proof is finished). Now we show that

$$v - Lv \neq \mu\varphi_1, \quad v \in \partial\Omega_r, \mu \geq 0. \tag{3.4}$$

If not, there exist $v_1 \in \partial\Omega_r$ and $\mu_1 \geq 0$ such that $v_1 - Lv_1 = \mu_1\varphi_1$, then $\mu_1 > 0$ and $v_1 = Lv_1 + \mu_1\varphi_1 \geq \mu_1\varphi_1$. Let $\widetilde{\mu} = \sup\{\mu | v_1 \geq \mu\varphi_1\}$, then $\widetilde{\mu} \geq \mu_1$, $v_1 \geq \widetilde{\mu}\varphi_1$ and $Lv_1 \geq \lambda_1 Tv_1 \geq \lambda_1\widetilde{\mu}T\varphi_1 = \widetilde{\mu}\varphi_1$. Thus,

$$v_1 = Lv_1 + \mu_1\varphi_1 \geq \widetilde{\mu}\varphi_1 + \mu_1\varphi_1 = (\widetilde{\mu} + \mu_1)\varphi_1,$$

which contradicts the definition of $\widetilde{\mu}$. So (3.4) is true, and by Lemma 2.7 we have

$$i(L, \Omega_r, K) = 0. \tag{3.5}$$

On the other hand, from (3.1) we can choose $\varepsilon_1 > 0$, $0 < \sigma < 1$ such that, for any $R > r > 0$, $t \in [0, 1]$, $x_i \geq 0$ ($i = 0, 1, \ldots, n-2$), and $x_{n-1} \geq R$, we have

$$f(t, x_0, x_1, \ldots, x_{n-1}) < \sigma(\lambda_1 - \varepsilon_1)x_{n-1}. \tag{3.6}$$

Let $\overline{T}v = \sigma\lambda_1 Tv$, then $\overline{T} : E \to E$ is a bounded linear operator and $\overline{T}(\Omega) \subset K$. Since λ_1 is the first eigenvalue of T and $0 < \sigma < 1$,

$$\bigl(r(\overline{T})\bigr)^{-1} = \bigl(\sigma\lambda_1\bigl(r(T)\bigr)\bigr)^{-1} = \sigma^{-1} > 1. \tag{3.7}$$

Let $\varepsilon_2 = \frac{1}{2}(1 - r(\overline{T}))$, then by $r(\overline{T}) = \lim_{n\to\infty} \|\overline{T}^n\|^{\frac{1}{n}}$, we know that there exists a natural number $N \geq 1$ such that $n \geq N$ implies that $\|\overline{T}^n\| \leq [r(\overline{T}) + \varepsilon_2]^n$. For any $v \in E$, define

$$\|v\|_* = \sum_{i=1}^{N} \bigl[r(\overline{T}) + \varepsilon_2\bigr]^{N-i} \bigl\|\overline{T}^{i-1}v\bigr\|,$$

where $\overline{T}^0 = I$ is the identity operator. It is easy to verify that $\|v\|_*$ is a new norm in E. Let

$$M = \sup_{v\in\partial\Omega_R} \int_0^1 \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \ldots, v(s)\bigr)\,ds,$$

by (2.11) we know that $M < +\infty$. Select $R_1 > \max\{R, 2M_*\varepsilon_2^{-1}\}$, where $M_* = \|M\|_*$.

In the following we prove that

$$Lv \neq \mu v, \quad v \in \partial\Omega_{R_1}, \mu \geq 1. \tag{3.8}$$

If otherwise, there exist $v_1 \in \partial\Omega_{R_1}$ and $\mu_1 \geq 1$ such that $Lv_1 = \mu_1 v_1$. Let $\widetilde{v}(t) = \min\{v_1(t), R\}$, $t \in [0, 1]$ and $D(v_1) = \{t \in [0, 1] : v_1(t) > R\}$. Notice that $\widetilde{v} \in C([0, 1], \mathbb{R}_+)$, $t^{\alpha-\beta_{n-1}-1}R_1 \leq v_1(t) \leq \|v_1\| = R_1$, and $R_1 > R$, thus there exists some $t_0 \in (0, 1]$ satisfying $v_1(t_0) = R$. So, by the definition of $\widetilde{v}$, we have $\widetilde{v}(t) \leq R$, $\widetilde{v}(t_0) = \min\{v_1(t_0), R\}$, $\widetilde{v}(t) \geq t^{\alpha-\beta_{n-1}-1}R$. Hence $\|\widetilde{v}\| = R$, so $\widetilde{v} \in \partial\Omega_R$. From (3.6) and the definition of M, we have

$$
\begin{aligned}
\mu_1 v_1(t) = (Lv_1)(t) &= \int_0^1 G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v_1(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_1(s), \ldots, v_1(s)\bigr)\,ds \\
&\leq \int_{D(v_1)} G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v_1(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_1(s), \ldots, v_1(s)\bigr)\,ds \\
&\quad + \int_{[0,1]\setminus D(v_1)} \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v_1(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_1(s), \ldots, v_1(s)\bigr)\,ds \\
&\leq \int_{D(v_1)} G(t,s)p(s)\sigma(\lambda_1 - \varepsilon_1)v_1(s)\,ds \\
&\quad + \int_0^1 \phi(s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v_1(s), I_{0^+}^{\beta_{n-1}-\beta_1}v_1(s), \ldots, v_1(s)\bigr)\,ds \\
&\leq (\overline{T}v_1)(t) + M, \quad t \in [0,1].
\end{aligned}
$$

Since $\overline{T}(\Omega) \subset K$, we have $0 \leq (\overline{T})^j(Lv_1)(t) \leq (\overline{T})^j(\overline{T}v_1 + M)(t)$ $(j = 0,1,2,\ldots,N-1)$, then

$$
\bigl\|(\overline{T})^j(Lv_1)\bigr\| \leq \bigl\|(\overline{T})^j(\overline{T}v_1 + M)\bigr\|, \quad j = 0,1,2,\ldots,N-1.
$$

Hence

$$
\begin{aligned}
\|Lv_1\|_* &= \sum_{i=1}^N \bigl[r(\overline{T}) + \varepsilon_2\bigr]^{N-i} \bigl\|(\overline{T})^{i-1}(Lv_1)\bigr\| \\
&\leq \sum_{i=1}^N \bigl[r(\overline{T}) + \varepsilon_2\bigr]^{N-i} \bigl\|(\overline{T})^{i-1}(\overline{T}v_1 + M)\bigr\| \\
&= \|\overline{T}v_1 + M\|_*.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mu_1\|v_1\|_* = \|Lv_1\|_* \leq \|\overline{T}v_1\|_* + M_* &= \sum_{i=1}^N \bigl[r(\overline{T}) + \varepsilon_2\bigr]^{N-i} \bigl\|\overline{T}^i v_1\bigr\| + M_* \\
&= \bigl[r(\overline{T}) + \varepsilon_2\bigr] \sum_{i=1}^{N-1} \bigl[r(\overline{T}) + \varepsilon_2\bigr]^{N-i-1} \bigl\|\overline{T}^i v_1\bigr\| + \bigl\|\overline{T}^N v_1\bigr\| + M_* \\
&\leq \bigl[r(\overline{T}) + \varepsilon_2\bigr] \sum_{i=1}^{N-1} \bigl[r(\overline{L}) + \varepsilon_2\bigr]^{N-i-1} \bigl\|\overline{T}^i v_1\bigr\| + \bigl[r(\overline{L}) + \varepsilon_2\bigr]^N \|v_1\| + M_* \\
&= \bigl[r(\overline{T}) + \varepsilon_2\bigr] \sum_{i=1}^{N} \bigl[r(\overline{T}) + \varepsilon_2\bigr]^{N-i} \bigl\|\overline{T}^{i-1} v_1\bigr\| + M_* \\
&= \bigl[r(\overline{T}) + \varepsilon_2\bigr]\|v_1\|_* + M_* \\
&\leq \bigl[r(\overline{T}) + \varepsilon_2\bigr]\|v_1\|_* + \frac{\varepsilon_2}{2}\|v_1\|_* \\
&= \left[\frac{1}{4}r(\overline{T}) + \frac{3}{4}\right]\|v_1\|_* < \|v_1\|_*,
\end{aligned}
$$

that is, $\mu_1 < 1$, which contradicts $\mu_1 \geq 1$. This implies that (3.8) holds. It follows from Lemma 2.8 that

$$i(L, \Omega_R, K) = 1. \tag{3.9}$$

By (3.4) and (3.8), we have

$$i(L, \Omega_R \backslash \overline{\Omega}_r, K) = i(L, \Omega_R, K) - i(L, \Omega_r, K) = 1.$$

Therefore, L has at least one fixed point $v^* \in \Omega_R \backslash \overline{\Omega}_r$, and v^* is a positive solution of BVP (2.1). Therefore problem (1.1) also has at least one positive solution. □

Lemma 3.1 *Suppose that* (H_1)–(H_3) *hold, then there exists an eigenvalue* $\widetilde{\lambda}_1$ *of* T *such that* $\lim_{\tau\to 0} \lambda_\tau = \widetilde{\lambda}_1$.

Proof Take $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \geq \cdots$ and $\tau_n \to 0$ ($n \to +\infty$), $\tau_n \in (0, \delta)$. For any $m > n$ and $\varphi \in E$, we have

$$\begin{aligned}
&(T_{\tau_n}\varphi)(t) \leq (T_{\tau_m}\varphi)(t) \leq (T\varphi)(t), \quad t \in [0,1], \\
&\bigl(T_{\tau_n}^k\varphi\bigr)(t) \leq \bigl(T_{\tau_m}^k\varphi\bigr)(t) \leq \bigl(T^k\varphi\bigr)(t), \quad t \in [0,1], k = 2, 3, \ldots,
\end{aligned}$$

where $T_{\tau_n}^k = T_{\tau_n}(T_{\tau_n}^{k-1})$ $(k = 2, 3, \ldots)$. Consequently, $\|T_{\tau_n}^k\| \leq \|T_{\tau_m}^k\| \leq \|T^k\|$ $(k = 1, 2, \ldots)$, by Gelfand's formula, we know that $\lambda_{\tau_n} \geq \lambda_{\tau_m} \geq \lambda_1$, where λ_1 is the first eigenvalue of T. Since λ_{τ_n} is monotonous with lower boundedness λ_1, let $\lim_{n\to\infty} \lambda_{\tau_n} = \widetilde{\lambda}_1$.

In the following we shall show that $\widetilde{\lambda}_1$ is an eigenvalue of T. Let φ_{τ_n} be the positive eigenfunction corresponding to φ_{τ_n}, i.e.,

$$\varphi_{\tau_n}(t) = \lambda_{\tau_n} \int_{\tau_n}^{1-\tau_n} G(t,s)p(s)\varphi_{\tau_n}(s)\,ds = \lambda_{\tau_n} L_{\tau_n}\varphi_{\tau_n}(t), \quad t \in [0,1], \tag{3.10}$$

with $\|\varphi_{\tau_n}\| = 1$ $(n = 1, 2, \ldots)$. From

$$\begin{aligned}
\|L_{\tau_n}\varphi_{\tau_n}\| &= \max_{t\in[0,1]} \int_{\tau_n}^{1-\tau_n} G(t,s)p(s)\varphi_{\tau_n}(s)\,ds \\
&\leq \int_0^1 \phi(s)p(s)\,ds < +\infty,
\end{aligned}$$

we know that $T_{\tau_n}\varphi_{\tau_n} \subset E$ is uniformly bounded.

On the other hand, for any n and $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
\bigl|T_{\tau_n}\varphi_{\tau_n}(t_1) - T_{\tau_n}\varphi_{\tau_n}(t_2)\bigr| &\leq \int_{\tau_n}^{1-\tau_n} \bigl|G(t_1,s) - G(t_2,s)\bigr|p(s)\varphi_{\tau_n}(s)\,ds \\
&\leq \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr|p(s)\,ds.
\end{aligned}$$

So, as $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$, we obtain that $T_{\tau_n}\varphi_{\tau_n}(t)$ is equicontinuous for $t \in [0,1]$. By the Arzela–Ascoli theorem and $\lim_{n\to\infty} \lambda_{\tau_n} = \widetilde{\lambda}_1$, we get that

$\varphi_{\tau_n} \to \varphi_0$ as $n \to \infty$. This leads to $\|\varphi_0\| = 1$, and then by (3.10) we have

$$\varphi_0(t) = \widetilde{\lambda}_1 \int_0^1 G(t,s)p(s)\varphi_0(s)\,ds, \quad t \in [0,1],$$

that is, $\varphi_0 = \widetilde{\lambda}_1 T\varphi_0$. This completes the proof. □

Theorem 3.2 *Assume that* (H_1)–(H_4) *hold, and*

$$\overline{f}^0 = \limsup_{\substack{x_i \to 0 \\ i=0,1,\ldots,n-1}} \max_{t\in[0,1]} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{x_{n-1}} < \lambda_1, \tag{3.11}$$

$$\underline{f}_{+\infty} = \liminf_{\sum_{i=0}^{n-1} x_i \to +\infty} \min_{t\in[0,1]} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{x_0 + x_1 + \cdots + x_{n-1}} > \widetilde{\lambda}_1. \tag{3.12}$$

Then BVP (1.1) *has at least one positive solution, where* λ_1 *is the first eigenvalue of* T *defined by* (2.9), *and* $\widetilde{\lambda}_1$ *is the eigenvalue of* T.

Proof Firstly, from (3.11) we know that there exist $r > 0$, $\tau_0 > 0$ such that, for any $t \in [0,1]$, and $0 \le x_i \le \frac{r}{\Gamma(\beta_{n-1}-\beta_i+1)}$ $(i = 0,1,\ldots,n-2)$, $0 \le x_{n-1} \le r$, we have

$$f(t,x_0,x_1,\ldots,x_{n-1}) \le (\lambda_1 - \tau_0)x_{n-1}. \tag{3.13}$$

For any $v \in \partial\Omega_r$, since

$$0 \le I_{0^+}^{\beta_{n-1}-\beta_i} v(t) \le \frac{r}{\Gamma(\beta_{n-1}-\beta_i+1)} \quad (i = 0,1,\ldots,n-2), 0 \le v(t) \le r,$$

thus from (3.13) we obtain that

$$\begin{aligned}(Lv)(t) &= \int_0^1 G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s),\ldots,v(s)\bigr)\,ds \\ &\le \int_0^1 G(t,s)p(s)(\lambda_1-\tau_0)v(s)\,ds \\ &\le \lambda_1(Tv)(t), \quad t \in [0,1],\end{aligned}$$

that is,

$$Lv \le \lambda_1 Tv, \quad v \in \partial\Omega_r. \tag{3.14}$$

Without loss of generality, we may suppose that T has no fixed point in $\partial\Omega_r$ (otherwise the conclusion is proved). In what follows, we will show that

$$Lv \ne \mu v, \quad \forall v \in \partial\Omega_r, \mu \ge 1. \tag{3.15}$$

As a contradiction, if there exist $v_1 \in \partial\Omega_r, \mu_1 \ge 1$ such that $Lv_1 = \mu_1 v_1$, obviously, $\mu_1 > 1$ and $\mu_1 v_1 = Lv_1 \le \lambda_1 Tv_1$. By induction we have $\mu_1^n v_1 \le \lambda_1^n T^n v_1$ $(n = 1,2,\ldots)$, so we have

$\|T^n\| \geq \frac{\|T^n v_1\|}{\|v_1\|} \geq \frac{\mu_1^n}{\lambda_1^n}$. By Gelfand's formula, we have

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \geq \frac{\mu_1}{\lambda_1} > \frac{1}{\lambda_1},$$

which contradicts $r(T) = \frac{1}{\lambda_1}$. So $Tv \neq \mu v$, $v \in \partial\Omega_r$, $\mu \geq 1$. From Lemma 2.8 we have

$$i(L, \Omega_r, K) = 1. \tag{3.16}$$

From (3.12) and $\lim_{\tau\to 0} \lambda_\tau = \widetilde{\lambda}_1$, we know there exists sufficiently small $\tau \in (0, 1)$. Taking

$$l_\tau = \sum_{i=0}^{n-2} \frac{\Gamma(\alpha - \beta_{n-1})}{\Gamma(\alpha - \beta_i)} \tau^{\alpha-\beta_i-1} + \tau^{\alpha-\beta_{n-1}-1},$$

there exist $\tau_0 > 0$, $R > r > 0$ such that, for any $x_i \geq 0$ ($i = 0, 1, \ldots, n-1$) and $x_0 + x_1 + \cdots + x_{n-1} > l_\tau R$, $t \in [0, 1]$, we have

$$f(t, x_0, x_1, \ldots, x_{n-1}) \geq (\lambda_\tau + \tau_0)(x_0 + x_1 + \cdots + x_{n-1}) \geq (\lambda_\tau + \tau_0)x_{n-1}, \tag{3.17}$$

where λ_τ is the first eigenvalue of T_τ.

Let φ_τ be the positive eigenfunction of T_τ corresponding to λ_τ, i.e., $\varphi_\tau = \lambda_\tau T_\tau \varphi_\tau$. For any $v \in \partial\Omega_R$, $s \in [\tau, 1-\tau]$, taking $\beta_0 = 0$, then

$$\begin{aligned}
&\sum_{i=0}^{n-2} I_{0^+}^{\beta_{n-1}-\beta_i} v(s) + v(s) \\
&\quad = \sum_{i=0}^{n-2} \frac{1}{\Gamma(\beta_{n-1} - \beta_i)} \int_0^s (s-\tau)^{\beta_{n-1}-\beta_i-1} v(\tau)\, d\tau + v(s) \\
&\quad \geq \sum_{i=0}^{n-2} \frac{1}{\Gamma(\beta_{n-1} - \beta_i)} \int_0^s (s-\tau)^{\beta_{n-1}-\beta_i-1} \tau^{\alpha-\beta_{n-1}-1}\, d\tau \|v\| + s^{\alpha-\beta_{n-1}-1}\|v\| \\
&\quad \geq \left(\sum_{i=0}^{n-2} \frac{\Gamma(\alpha - \beta_{n-1})}{\Gamma(\alpha - \beta_i)} s^{\alpha-\beta_i-1} + s^{\alpha-\beta_{n-1}-1} \right) R \\
&\quad \geq \left(\sum_{i=0}^{n-2} \frac{\Gamma(\alpha - \beta_{n-1})}{\Gamma(\alpha - \beta_i)} \tau^{\alpha-\beta_i-1} + \tau^{\alpha-\beta_{n-1}-1} \right) R \\
&\quad = l_\tau R.
\end{aligned} \tag{3.18}$$

So, from (3.17) and (3.18) we have

$$\begin{aligned}
(Lv)(t) &= \int_0^1 G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \ldots, v(s)\bigr)\, ds \\
&\geq \int_\tau^{1-\tau} G(t,s)p(s)f\bigl(s, I_{0^+}^{\beta_{n-1}} v(s), I_{0^+}^{\beta_{n-1}-\beta_1} v(s), \ldots, v(s)\bigr)\, ds \\
&\geq \int_\tau^{1-\tau} G(t,s)p(s)(\lambda_\tau + \tau_0)v(s)\, ds \\
&\geq \lambda_\tau (T_\tau u)(t), \quad t \in [0, 1].
\end{aligned}$$

We may suppose that L has no fixed points on $\partial\Omega_R$ (otherwise, the proof is ended). Following the procedure used in the first part of Theorem 3.1, it follows that

$$v - Lv \neq \mu\varphi_\tau, v \in \partial\Omega_R, \mu \geq 0.$$

From Lemma 2.7, we know

$$i(L, \Omega_R, K) = 0. \tag{3.19}$$

So, from (3.16) and (3.19) we have

$$i(L, \overline{\Omega}_R \backslash \Omega_r, K) = i(L, \Omega_R, K) - i(L, \Omega_r, K) = 0 - 1 = -1.$$

Therefore, L has at least one fixed point on $\overline{\Omega}_R \backslash \Omega_r$, which is a positive solution of BVP (2.1). Consequently, it is a positive solution of BVP (1.1). The proof is completed. □

4 An example

Example 4.1 We consider the singular fractional differential equation as follows:

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t) + p(t) f(t, u(t), D_{0^+}^{\frac{1}{16}} u(t), D_{0^+}^{\frac{9}{8}} u(t), D_{0^+}^{\frac{9}{4}} u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ D_{0^+}^{\frac{19}{8}} u(1) = \int_0^1 l(t) D_{0^+}^{\frac{9}{4}} u(t) \, dA(t), \end{cases} \tag{4.1}$$

where $p(t) = (1-t)^{-\frac{1}{8}}$, $l(t) = t^{-\frac{1}{20}}$, $f(t, x_0, x_1, x_2, x_3) = [x_0 + x_1 + x_2 + x_3]^{-\frac{1}{3}} + \ln x_3$, and

$$A(t) = \begin{cases} 0, & t \in [0, \frac{1}{4}), \\ \frac{1}{200}, & t \in [\frac{1}{4}, 1]. \end{cases}$$

It is obvious that $p(t)$ is singular at $t = 1$, and f is singular at $x_0 = x_1 = x_2 = x_3 = 0$. Let $u(t) = I_{0^+}^{\frac{9}{4}} v(t)$, then problem (4.1) can be transformed to the following equation:

$$\begin{cases} D_{0^+}^{\frac{5}{4}} v(t) + p(t) f(t, I_{0^+}^{\frac{9}{4}} v(t), I_{0^+}^{\frac{35}{16}} v(t), I_{0^+}^{\frac{9}{8}} v(t), v(t)) = 0, \quad 0 < t < 1, \\ I_{0^+}^{\frac{1}{4}} v(0) = 0, \\ D_{0^+}^{\frac{1}{8}} v(1) = \int_0^1 l(t) v(t) \, dA(t). \end{cases} \tag{4.2}$$

Then

$$K(t,s) = \frac{1}{\Gamma(\frac{5}{4})} \begin{cases} t^{\frac{1}{4}}(1-s)^{\frac{1}{8}} - (t-s)^{\frac{1}{4}}, & 0 \leq s \leq t \leq 1, \\ t^{\frac{1}{4}}(1-s)^{\frac{1}{8}}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{4.3}$$

and

$$g_A(s) = \begin{cases} \frac{1}{200} K_2(\frac{1}{4}, s) l(\frac{1}{4}) = \frac{1}{200} \frac{1}{\Gamma(\frac{5}{4})} (\frac{1}{4})^{\frac{1}{5}} (1-s)^{\frac{1}{8}}, & 0 \leq s \leq \frac{1}{4}, \\ \frac{1}{200} K_1(\frac{1}{4}, s) l(\frac{1}{4}) = \frac{1}{200} [\frac{1}{\Gamma(\frac{5}{4})} (\frac{1}{4})^{\frac{1}{4}} (1-s)^{\frac{1}{8}} - (\frac{1}{4} - s)^{\frac{1}{4}}] (\frac{1}{4})^{-\frac{1}{20}}, & \frac{1}{4} \leq s \leq 1. \end{cases}$$

Obviously, $0 \le g_A(s) \le (1-s)^{\frac{1}{8}}$, $\delta = \int_0^1 t^{\frac{1}{4}} l(t)\,dA(t) = \int_0^1 t^{\frac{1}{5}}\,dA(t) = (\frac{1}{4})^{\frac{1}{5}} \times \frac{1}{200} = 0.00125$, that is, $\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{9}{8})} - \delta > 0$, and

$$\phi(s) = K(1,s) + \frac{g_A(s)}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} - \delta} \le 100(1-s)^{\frac{1}{8}}.$$

Now we will check that all the conditions of Theorem 3.1 are satisfied, define a cone

$$K = \bigl\{\nu \in C[0,1] : \nu(t) \ge t^{\frac{1}{4}}\|\nu\|, t \in [0,1]\bigr\}.$$

$K^{(0)} = \{\nu \in C[0,1] : \nu(t) \ge t^{\frac{5}{2}}\|\nu\|, t \in [0,1]\}$, $K^{(1)} = \{\nu \in C[0,1] : \nu(t) \ge t^{\frac{39}{16}}\|\nu\|, t \in [0,1]\}$, $K^{(2)} = \{\nu \in C[0,1] : \nu(t) \ge t^{\frac{11}{8}}\|\nu\|, t \in [0,1]\}$.

For any $0 < r < R < +\infty$ and $\nu \in \overline{\Omega}_R \backslash \Omega_r$, we have $\nu(t) \ge t^{\alpha-\beta_{n-1}-1}\|\nu\| = t^{\frac{1}{4}}\|\nu\|$, $t \in [0,1]$, then

$$\begin{aligned} &0 \le rt^{\frac{1}{4}} \le x_3(t) = \nu(t) \le R, \quad t \in [0,1], \\ &\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{2}-\beta_i)} t^{\frac{5}{2}-\beta_i} r \le x_i(t) = I_{0^+}^{\frac{9}{4}-\beta_i}\nu(t) \le \frac{1}{\Gamma(\frac{13}{4})}R, \quad t \in [0,1], i = 0,1,2, \end{aligned}$$

where $\beta_0 = 0$, $\beta_1 = \frac{1}{16}$, $\beta_2 = \frac{9}{8}$. Since $|\ln x|$ is decreasing on $(0,1)$ and is increasing on $(1,+\infty)$, we have

$$\begin{aligned} &\bigl|\ln \nu(x)\bigr| \le \bigl|\ln rt^{\frac{1}{4}}\bigr| + |\ln R| \le |\ln r| + |\ln R| + \bigl|\ln t^{\frac{1}{4}}\bigr|, \\ &[x_0 + x_1 + x_2 + x_3]^{-\frac{1}{3}} \le \biggl[\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{2})} + \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{55}{16})} + \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{19}{8})} + 1\biggr]^{-\frac{1}{3}} r^{-\frac{1}{3}} t^{-\frac{1}{12}}, \end{aligned}$$

and

$$\int_0^1 \bigl|\ln t^{\frac{1}{4}}\bigr|\,dt + \int_0^1 t^{-\frac{1}{12}}\,dt = \frac{1}{4} + \frac{12}{11} = \frac{59}{44} < +\infty.$$

The absolute continuity of the integral yields that

$$\lim_{n\to\infty} \int_{e(n)} \bigl|\ln t^{\frac{1}{4}}\bigr|\,dt = 0, \qquad \lim_{n\to\infty} \int_{e(n)} t^{-\frac{1}{12}}\,dt = 0. \tag{4.4}$$

So,

$$\begin{aligned} &\sup_{\substack{x_i \in \overline{\Omega}_{R_i} \backslash \Omega_{r_i}^{(i)} \\ x_3 \in \overline{\Omega}_R \backslash \Omega_r}} \int_{e(n)} \phi(s)p(s)f\bigl(s, x_0(s), x_1(s), x_2(s), x_3(s)\bigr)\,ds \\ &\quad \le \sup_{\substack{x_i \in \overline{\Omega}_{R_i} \backslash \Omega_{r_i}^{(i)} \\ x_3 \in \overline{\Omega}_R \backslash \Omega_r}} \int_{e(n)} 100(1-s)^{\frac{1}{8}}(1-s)^{-\frac{1}{8}}\bigl[\bigl(x_0(s) + x_1(s) + x_2(s) + x_3(s)\bigr)^{-\frac{1}{3}} + \ln\bigl|x_3(s)\bigr|\bigr]\,ds \\ &\quad \le \sup_{\substack{x_i \in \overline{\Omega}_{R_i} \backslash \Omega_{r_i}^{(i)} \\ x_3 \in \overline{\Omega}_R \backslash \Omega_r}} \int_{e(n)} 100\biggl[|\ln r| + |\ln R| + \bigl|\ln t^{\frac{1}{4}}\bigr| \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{2})}+\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{55}{16})}+\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{19}{8})}+1\right)^{-\frac{1}{3}} r^{-\frac{1}{3}} t^{-\frac{1}{12}}\Bigg] ds \\
&= 2\bigl(\ln|r| + \ln|R|\bigr)\frac{1}{n} + \frac{1}{4}\int_{e(n)} |\ln s|\, ds \\
&+\left(\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{2})}+\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{55}{16})}+\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{19}{8})}+1\right)^{-\frac{1}{3}} r^{-\frac{1}{3}} \int_{e(n)} s^{-\frac{1}{12}}\, ds,
\end{aligned}
$$

and from (4.4) we obtain that

$$
\lim_{n\to\infty} \sup_{\substack{x_i\in\overline{\Omega}_{R_i}\setminus\Omega^{(i)}_{r_i} \\ x_3\in\overline{\Omega}_R\setminus\Omega_r}} \int_{e(n)} \phi(s)p(s)f\bigl(s,x_0(s),x_1(s),x_2(s),x_3(s)\bigr)\, ds = 0.
$$

On the other hand, by a simple calculation, we have

$$
\begin{aligned}
&\overline{f}^{\infty} = \limsup_{x_3\to+\infty} \sup_{\substack{t\in[0,1] \\ x_i>0, i=0,1,2}} \frac{f(t,x_0,x_1,x_2,x_3)}{x_3} = 0 < \lambda_1, \\
&\underline{f}_0 = \liminf_{\substack{x_i\to 0 \\ i=0,1,2,3}} \min_{t\in[0,1]} \frac{f(t,x_0,x_1,x_2,x_3)}{x_0+x_1+x_2+x_3} = +\infty > \lambda_1.
\end{aligned}
$$

Therefore, the assumptions of Theorem 3.1 are satisfied. Thus the above problem possesses at least one positive solution in K.

5 Conclusions

In this paper, we study a type of singular nonlinear fractional differential equation with integral boundary conditions involving derivatives. The biggest difference from other papers is that our nonlinear term and boundary value conditions contain several fractional derivatives, and f is singular at $x_0 = x_1 = \cdots = x_{n-1} = 0$. So our work is valued.

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