JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION

doi:10.3934/jimo.2016078

INERTIAL ACCELERATED ALGORITHMS FOR SOLVING A SPLIT FEASIBILITY PROBLEM

Yazheng Dang*

School of Management
University of Shanghai for Science and Technology
Shanghai 200093, China

Jie Sun

Department of Mathematics and Statistics
Curtin University, Perth, WA 6102, Australia

Honglei Xu

Department of Mathematics and Statistics
Curtin University, Perth, WA 6102, Australia

Abstract. Inspired by the inertial proximal algorithms for finding a zero of a maximal monotone operator, in this paper, we propose two inertial accelerated algorithms to solve the split feasibility problem. One is an inertial relaxed-CQ algorithm constructed by applying inertial technique to a relaxed-CQ algorithm, the other is a modified inertial relaxed-CQ algorithm which combines the KM method with the inertial relaxed-CQ algorithm. We prove their asymptotical convergence under some suitable conditions. Numerical results are reported to show the effectiveness of the proposed algorithms.

1. **Introduction.** We are concerned with the following split feasibility problem (SFP): Find a point x satisfying

$$x \in C,\ Ax \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H^1 and H^2, respectively, and $A : H^1 \to H^2$ is a bounded linear operator. The SFP was introduced in [9], which has broad applications in many fields such as image reconstruction problem [8, 18], approximation theory [13], control [17], and so on. Many projection methods have been developed for solving the SFP, see [6, 10, 23, 25]. Denote by P_C the orthogonal projection onto C; that is, $P_C(x) = \arg\min_{y\in C} \|x-y\|$, over all $x \in C$. In [5], Byrne introduced the so-called CQ algorithm, taking an initial point x^0 arbitrarily, and defining the iterative step as:

$$x^{k+1} = P_C[(I - \gamma A^T(I - P_Q)A)(x^k)],$$

where $0 < \gamma < 2/\rho(A^TA)$ and $\rho(A^TA)$ is the spectral radius of A^TA. Another approach is the so-called Krasnoselski-Mann (KM) algorithm, which was proposed

2010 *Mathematics Subject Classification.* Primary: 65K05, 65K10; Secondary: 49J52.

Key words and phrases. Asymptotical convergence, inertial technique, nonsmooth analysis, split feasibility problem, subdifferential.

* Corresponding author: Yazheng Dang. The reviewing process of the paper was handled by Changzhi Wu as a Guest Editor.

originally for solving fixed point problem [11]. Byrne [6] first applied KM iterations to a CQ algorithm for solving the SFP. Subsequently, Zhao [26] applied KM iteration to a perturbed CQ algorithm, Dang and Gao [14] combined the KM iterative method with a modified CQ algorithm to construct a KM-CQ-Like algorithm for solving the SFP. The implementation of the algorithms mentioned above is under the condition that the orthogonal projections onto C and Q are easily calculated. However, in most cases, it is impossible or needs too much work to compute the exact orthogonal projection. Similar to the inexact techniques used in optimization (e.g., [15, 16]), an extension of the CQ algorithm that incorporate an inexact technique has been proposed by Qu and Xiu [22]. In [24], by using the inexact projection technique, Yang developed a relaxed CQ algorithm for solving the SFP, where he used two halfspaces C_k and Q_k in place of C and Q, respectively, at the $k-$th iteration such that the orthogonal projections onto C_k and Q_k are easily done.

In nonlinear analysis, as a powerful unified framework for solving problems such as minimization problem, complementarity problem and variational inequality problem, the problem of finding a zero of a maximal monotone operator G on a real Hilbert space H is formulated as

$$\text{finding } x \in H \text{ such that } 0 \in G(x). \tag{1.2}$$

A classical method for solving this problem is the proximal method, in which the next iteration x^{k+1} is generated by solving the subproblem

$$0 \in \lambda_k G(x) + (x - x^k), \tag{1.3}$$

where x^k is the current iteration and λ_k is a regularization parameter. Attouch and Alvarez [2] applied an inertial technique to (1.3) to develop an inertial proximal method for solving(1.2). It works as follows. Given $x^{k-1}, x^k \in H$ and two parameters $\theta_k \in [0, 1), \lambda_k > 0$, find $x^{k+1} \in H$such that

$$0 \in \lambda_k G(x^{k+1}) + x^{k+1} - x^k - \theta_k(x^k - x^{k-1}). \tag{1.4}$$

Here, the inertia is induced by the term $\theta_k(x^k - x^{k-1})$.

It is well known that the proximal iteration (1.3) may be interpreted as an implicit one-step discretization method for the evolution differential inclusion

$$0 \in \frac{dx}{dt}(t) + G(x(t)) \ a.e. \ t \geq 0. \tag{1.5}$$

Similarly, (1.4) may be thought of as coming from the implicit discretization of the second-order differential system

$$0 \in \frac{d^2x}{dt^2}(t) + \rho\frac{dx}{dt}(t) + G(x(t)) \ a.e. \ t \geq 0. \tag{1.6}$$

where $\rho > 0$ is a damping or a friction parameter. This point of view inspired various numerical methods related to the inertial terminology, all those methods had nice convergence properties [1-3, 19, 20] by incorporating second order information.

In this paper, we apply the inertial technique to the relaxed CQ method [24] to propose inertial relaxed CQ algorithms for the SFP. Under certain suitable conditions, the asymptotical convergence are proved. Preliminary numerical results are reported to show the effectiveness of the algorithms.

The paper is organized as follows. In Section 2, we state some basic definitions and lemmas. In Section 3, we present an inertial relaxed CQ algorithm and show its convergence. In Section 4, a modified inertial relaxed CQ algorithm is presented and its convergence is also proved. In Section 5, numerical experiment results are given.

2. **Preliminaries.** Throughout the rest of the paper, I denotes the identity operator, $\mathrm{Fix}(T)$ denotes the set of the fixed points of an operator T i.e., $\mathrm{Fix}(T) := \{x \mid x = T(x)\}$.

Recall that an operator T is called nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|, \tag{2.1}$$

firmly nonexpansive if

$$\|T(x) - T(y)\|^2 \leq \langle x - y, T(x) - T(y)\rangle. \tag{2.2}$$

It is well known that the orthogonal projection operator P_C for any x, y, is characterized by the inequalities

$$\langle x - P_C(x), c - P_C(x)\rangle \leq 0,\ c \in C$$

and

$$\langle P_C(y) - P_C(x), y - x\rangle \geq \|P_C(y) - P_C(x)\|^2.$$

Therefore, the operator P_C is firmly nonexpansive. From Cauchy inequality we conclude that

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|,$$

that is, the operator P_C is nonexpansive.

Recall the notion of the subdifferential for an appropriate convex function.

Definition 2.1 [12]. Let $f : H \to R$ be appropriate convex. The subdifferential of f at x is defined as

$$\partial f(x) = \{\xi \in H |\ f(y) \geq f(x) + \langle \xi, y - x\rangle, \quad \forall\ y \in H\}.$$

The lemmas below are necessary for the convergence analysis in the next section.

Lemma 2.1 [20]. *Assume $\varphi_k \in [0, \infty)$ and $\delta_k \in [0, \infty)$ satisfy:*

(1) $\varphi_{k+1} - \varphi_k \leq \theta_k(\varphi_k - \varphi_{k-1}) + \delta_k$,
(2) $\sum_{k=1}^{+\infty} \delta_k < \infty$,
(3) $\{\theta_k\} \subset [0, \theta]$, *where* $\theta \in [0, 1)$.

Then, the sequence $\{\varphi_k\}$ *is convergent with* $\sum_{k=1}^{+\infty}[\varphi_{k+1} - \varphi_k]_+ < \infty$, where $[t]_+ := \max\{t, 0\}$ (for any $t \in R$).

Lemma 2.2 (Opial [21]). *Let H be a Hilbert space and $\{x^k\}$ a sequence such that there exists a nonempty set $S \subset H$ verifying:*
(1) For every $z \in S, \lim_{k\to\infty} \|x^k - z\|$ exists.
(2) If $x^{k_j} \rightharpoonup x^*$ weakly in H for a sequence $k_j \to \infty$ then $x^* \in S$.

Then, there exists $x \in S$ such that $x^k \rightharpoonup x$ weakly in H as $k \to \infty$.

3. The inertial relaxed CQ algorithm and its asymptotic convergence.

3.1. **The inertial relaxed CQ algorithm.** As in [24], we make the following blanket assumptions.

(1) The solution set of the SFP is nonempty.
(2) The set C is denoted as

$$C = \{x \in H^1 \mid c(x) \leq 0\}, \tag{3.1}$$

where $c : H^1 \to R$ is appropriate convex and C is nonempty.

The set Q is denoted by

$$Q = \{h \in H^2 \mid q(h) \le 0\}, \tag{3.2}$$

where $q : H^2 \to R$ is appropriate convex and Q is nonempty.

(3) For any $x \in H^1$, at least one subgradient $\xi \in \partial c(x)$ can be calculated; and for any $h \in H^2$, at least one subgradient $\eta \in \partial q(h)$ can be computed.

Now, we define two sets at point x^k,

$$C_k = \{x \in H^1 \mid c(x^k) + \langle \xi^k, x - x^k\rangle \le 0\}, \tag{3.3}$$

where ξ^k is an element in $\partial c(x^k)$, and

$$Q_k = \{h \in H^2 \mid q(Ax^k) + \langle \eta^k, h - Ax^k\rangle \le 0\}, \tag{3.4}$$

where η^k is an element in $\partial q(Ax^k)$.

By the definition of subgradient, it is clear that the halfspaces C_k and Q_k contain C and Q, respectively. Due to the specific form of C_k and Q_k, the orthogonal projections onto C_k and Q_k may be computed directly, see [4].

The following lemma provides an important boundedness property for the subdifferential.

Lemma 3.1 [12]. *Suppose that $f : H \to R$ is convex. Then its subdifferential are uniformly bounded on any bounded subsets of finite dimensional space H.*

Next, we state our inertial relaxed CQ algorithm.

Algorithm 3.1

Initialization: Take $x^0,\ x^1$ in H^1.

Iterative step: For $k \ge 0$, given the points x^k, x^{k-1}, the next iterative point x^{k+1} is generated by

$$x^{k+1} = P_{C_k}[U_k(x^k + \theta_k(x^k - x^{k-1}))], \tag{3.5}$$

where $U_k = I - \gamma F_k, F_k = A^T(I - P_{Q_k})A,\ \ \gamma \in (0, 2/L),\ L$ denotes the spectral radius of A^TA, $\theta_k \in [0, 1)$, C_k and Q_k are given by (3.3) and (3.4), respectively.

Evidently, when $\theta_k \equiv 0$, (3.5) happens to be the standard relaxed CQ method.

3.2. **Asymptotic convergence of the inertial relaxed CQ algorithm.** In this subsection, we establish the asymptotic convergence of Algorithm 3.1.

Define $f(x) = \frac{1}{2}\|Ax - P_QAx\|^2, x \in H^1$. It is not hard to see that x solves the SFP (1.1) if and only if x solves the minimization $f_{min} := \min_{x\in C} f(x)$ with $f_{min} = 0$. It is well known that the gradient of $f : F = \nabla f = A^T(I - P_Q)A$ is L-Lipschitz continuous with $L = \rho(A^TA)$, and thus it is $\frac{1}{L}-$ cocoercive (see [6]), that is $\langle F(x) - F(y), x - y\rangle \ge \frac{1}{L}\|F(x) - F(y)\|^2$. The same is true for the operators $F_k = A^T(I - P_{Q_k})A$ for $k = 0, 1, \cdots$.

Theorem 3.1. *Choose parameter $\theta_k \in [0, \bar{\theta}_k]$ with $\bar{\theta}_k := \min\{\theta, (\max\{k^2\|x^k - x^{k-1}\|^2, k^2\|x^k - x^{k-1}\|\})^{-1}\}$, $\theta \in [0, 1)$, then the sequence $\{x^k\}$ generated by (3.5) converges weakly to a point x^* as $k \to \infty$, where x^* is a solution of (1.1).*

Proof. The case for $\theta_k \equiv 0$, we can see the detailed proof in [23].

Now we see the case for $\theta_k > 0$ for some $k \in N$. Let z be a solution of the SFP. Since $C \subset C_k, Q \subset Q_k$, then $z = P_C(z) = P_{C_k}(z)$ and $F_k(z) = F(z) = 0$. Define the auxiliary real sequence $\varphi_k := \frac{1}{2}\|x^k - z\|^2$. Let $y^k = x^k + \theta_k(x^k - x^{k-1})$.

Therefore, from the nonexpansive of the operator P_C and (3.5), we have

$$\begin{aligned}\varphi_{k+1} &= \frac{1}{2}\|x^{k+1}-z\|^2\\ &= \frac{1}{2}\|P_{C_k}[U_k(x^k+\theta_k(x^k-x^{k-1}))]-z\|^2\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2}\|P_{C_k}[y^k-\gamma F_k(y^k)]-P_{C_k}(z)\|^2\\ &\le \frac{1}{2}\|(y^k-z)-\gamma F_k(y^k)\|^2\\ &= \frac{1}{2}\|y^k-z\|^2+\frac{\gamma^2}{2}\|F_k(y^k)\|^2-\gamma\langle y^k-z, F_k(y^k)-F_k(z)\rangle.\end{aligned}$$

Furthermore, by the cocoercivity of the operator F_k, we have

$$\varphi_{k+1} \le \frac{1}{2}\|y^k-z\|^2+\frac{\gamma^2}{2}\|F_k(y^k)\|^2-\gamma\frac{1}{L}\|F_k(y^k)\|^2$$

namely,

$$\varphi_{k+1} \le \frac{1}{2}\|y^k-z\|^2-(\frac{\gamma}{L}-\frac{\gamma^2}{2})\|A^T(P_{Q_k}-I)Ay^k\|^2. \tag{3.6}$$

Note that we have

$$\begin{aligned}\frac{1}{2}\|y^k-z\|^2 &= \frac{1}{2}\|x^k+\theta_k(x^k-x^{k-1})-z\|^2\\ &= \frac{1}{2}\|x^k-z\|^2+\theta_k\langle x^k-z, x^k-x^{k-1}\rangle+\frac{\theta_k^2}{2}\|x^k-x^{k-1}\|^2\\ &= \varphi_k+\theta_k\langle x^k-z, x^k-x^{k-1}\rangle+\frac{\theta_k^2}{2}\|x^k-x^{k-1}\|^2.\end{aligned}$$

It is easy to check that $\varphi_k = \varphi_{k-1}+\langle x^k-z, x^k-x^{k-1}\rangle-\frac{1}{2}\|x^k-x^{k-1}\|^2$. Hence

$$\frac{1}{2}\|y^k-z\|^2 = \varphi_k+\theta_k(\varphi_k-\varphi_{k-1})+\frac{\theta_k+\theta_k^2}{2}\|x^k-x^{k-1}\|^2. \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$\varphi_{k+1} \le \varphi_k+\theta_k(\varphi_k-\varphi_{k-1})+\frac{\theta_k+\theta_k^2}{2}\|x^k-x^{k-1}\|^2-(\frac{\gamma}{L}-\frac{\gamma^2}{2})\|A^T(P_{Q_k}-I)Ay^k\|^2. \tag{3.8}$$

Since $0<\gamma<2/L$, we have $(\frac{\gamma}{L}-\frac{\gamma^2}{2})>0$. According to $\theta_k^2 \le \theta_k$ and (3.8), we derive

$$\varphi_{k+1} \le \varphi_k+\theta_k(\varphi_k-\varphi_{k-1})+\theta_k\|x^k-x^{k-1}\|^2. \tag{3.9}$$

Evidently,

$$\sum_{k=1}^{+\infty}\theta_k\|x^k-x^{k-1}\|^2<\infty, \tag{3.10}$$

due to $\theta_k\|x^k-x^{k-1}\|^2 \le \frac{1}{k^2}$. Let $\delta_k := \theta_k\|x^k-x^{k-1}\|^2$ in Lemma 2.1. We deduce that the sequence $\{\|x^k-z\|\}$ is convergent (hence $\{x^k\}$ is bounded). By (3.9) and Lemma 2.1, we obtain $\sum_{k=1}^{+\infty}[\|x^k-z\|^2-\|x^{k-1}-z\|^2]_+<\infty$. In view of (3.6), we have

$$(\frac{\gamma}{L}-\frac{\gamma^2}{2})\|A^T(P_{Q_k}-I)Ay^k\|^2 \le \varphi_k-\varphi_{k+1}+\theta_k(\varphi_k-\varphi_{k-1})+\theta_k\|x^k-x^{k-1}\|^2.$$

Therefore

$$\sum_{k=1}^{+\infty}(\frac{\gamma}{L}-\frac{\gamma^2}{2})\|A^T(P_{Q_k}-I)Ay^k\|^2<\infty.$$

By $0<\gamma<2/L$, we get

$$\|A^T(P_{Q_k}-I)Ay^k\|^2\to 0 \tag{3.11}$$

and

$$\|(P_{Q_k}-I)Ay^k\|^2\to 0. \tag{3.12}$$

Obviously, $F_k(y^k)\to 0$. We next show that

$$\|x^{k+1}-x^k\|\to 0. \tag{3.13}$$

To do this, we proceed as follows.

$$\begin{aligned}\|x^{k+1}-z\|^2 &= \|(x^{k+1}-y^k)+(y^k-z)\|^2\\ &= \|x^{k+1}-y^k\|^2+\|y^k-z\|^2+2\langle x^{k+1}-y^k,y^k-z\rangle\\ &= \|x^{k+1}-y^k\|^2+\|y^k-z\|^2\\ &\quad +2\langle x^{k+1}-y^k,y^k-x^{k+1}\rangle+2\langle x^{k+1}-y^k,x^{k+1}-z\rangle.\end{aligned}$$

Hence

$$\|x^{k+1}-y^k\|^2=\|y^k-z\|^2-\|x^{k+1}-z\|^2+2\langle x^{k+1}-y^k,x^{k+1}-z\rangle. \tag{3.14}$$

Substituting (3.7) into (3.14), we have

$$\begin{aligned}\|x^{k+1}-y^k\|^2 &= 2\varphi_k+2\theta_k(\varphi_k-\varphi_{k-1})-\|x^{k+1}-z\|^2+(\theta_k+\theta_k^2)\|x^k-x^{k-1}\|^2\\ &\quad +2\langle x^{k+1}-y^k,x^{k+1}-z\rangle\\ &= \|x^k-z\|^2-\|x^{k+1}-z\|^2+\theta_k(\|x^k-z\|^2-\|x^{k-1}-z\|^2)\\ &\quad +(\theta_k+\theta_k^2)\|x^k-x^{k-1}\|^2+2\langle x^{k+1}-y^k,x^{k+1}-z\rangle.\end{aligned}$$

By $\theta_k\in[0,\bar{\theta}_k]$, $\bar{\theta}_k:=\min\{\theta,\frac{1}{\max\{k^2\|x^k-x^{k-1}\|^2,k^2\|x^k-x^{k-1}\|\}}\},\theta\in[0,1)$, we get

$$\begin{aligned}\|x^{k+1}-y^k\|^2 &\le (\|x^k-z\|^2-\|x^{k+1}-z\|^2)_+ +\theta(\|x^k-z\|^2-\|x^{k-1}-z\|^2)_+\\ &\quad +2\theta_k\|x^k-x^{k-1}\|^2+2\langle x^{k+1}-y^k,x^{k+1}-z\rangle,\end{aligned} \tag{3.15}$$

where $t_+=\max\{t,0\}$. On the other hand, $x^{k+1}=P_{C_k}(x^k-\gamma F_k(y^k))$ implies

$$\langle (y^k-\gamma F_k(y^k))-x^{k+1},z-x^{k+1}\rangle\le 0.$$

Then,

$$\langle x^{k+1}-y^k,x^{k+1}-z\rangle\le\gamma\langle F_k(y^k),z-x^{k+1}\rangle\to 0. \tag{3.16}$$

From (3.10) , we obtain

$$\lim_{k\ge 1}\theta_k\|x^k-x^{k-1}\|^2=0 \tag{3.17}$$

Combining (3.15) with (3.16) and (3.17) leads to

$$\|x^{k+1}-y^k\|\to 0. \tag{3.18}$$

By the triangle inequality, we get

$$\|x^{k+1}-x^k\|\le\|x^{k+1}-y^k\|+\|y^k-x^k\|=\|x^{k+1}-y^k\|+\theta_k\|x^k-x^{k-1}\|. \tag{3.19}$$

Since $\theta_k\in[0,\bar{\theta}_k]$ with $\bar{\theta}_k:=\min\{\theta,\frac{1}{\max\{k^2\|x^k-x^{k-1}\|^2,k^2\|x^k-x^{k-1}\|\}}\},\theta\in[0,1)$, we have

$$\sum_{k=1}^{+\infty}\theta_k\|x^k-x^{k-1}\|<\infty, \tag{3.20}$$

which implies

$$\theta_k\|x^k - x^{k-1}\| \to 0. \tag{3.21}$$

Using (3.18) and (3.21), from (3.19), we derive that

$$\|x^{k+1} - x^k\| \to 0.$$

We have known that $\{x^k\}$ is bounded, which implies that ξ_k is bounded. Assume that x^* is an accumulation point of $\{x^k\}$ and $x^{k_l} \to x^*$, where $\{x^{k_l}\}$ is a subsequence of $\{x^k\}$. Since $\theta_k\|x^k - x^{k-1}\| \to 0$, we have $y^{k_l} \to x^*$. Then, from (3.11), it follows

$$P_{Q_{k_l}}(Ax^{k_l}) \to Ax^*, \quad k_l \to +\infty. \tag{3.22}$$

Finally, we show that x^* is a solution of the SFP. Since $x^{k_l+1} \in C_{k_l}$, We obtain

$$c(x^{k_l}) + \langle \xi^{k_l}, x^{k_l+1} - x^{k_l}\rangle \leq 0.$$

Thus

$$c(x^{k_l}) \leq -\langle \xi^{k_l}, x^{k_l+1} - x^{k_l}\rangle \leq \xi\|x^{k_l+1} - x^{k_l}\|.$$

where ξ satisfies $\|\xi^k\| \leq \xi$ for all k. By virtue of the continuity of function c and $\|x^{k_l+1} - x^k\| \to 0$, we get that

$$c(x^*) = \lim_{l\to\infty} c(x^{k_l}) \leq 0.$$

Therefore, $x^* \in C$.

Now we show that $Ax^* \in Q$. To do this, let $z^k = Ay^k - P_{Q_k}(Ay^k) \to 0$ and let η be such that $\|\eta_k\| \leq \eta$ for all k. Since $Ay^{k_l} - z^{k_l} = P_{Q_{k_l}}(Ay^{k_l}) \in Q_{k_l}$, we have

$$q(Ax^{k_l}) + \langle \eta^{k_l}, (Ay^{k_l} - z^{k_l}) - Ax^{k_l}\rangle \leq 0.$$

Hence,

$$q(Ax^{k_l}) \leq \langle \eta^{k_l}, Ax^{k_l} - Ay^{k_l}\rangle + \langle \eta^{k_l}, z^{k_l}\rangle \leq \eta\|A\|\theta_{k_l}\|x^{k_l} - x^{k_l-1}\| + \eta\|z^{k_l}\| \to 0.$$

By the continuity of q and $A(x^{k_l}) \to Ax^*$, we arrive at the conclusion

$$q(Ax^*) = \lim_{l\to\infty} q(Ax^{k_l}) \leq 0,$$

namely $Ax^* \in Q$. □

Remark 3.1. Since the current value of $\|x^k - x^{k-1}\|$ is known before choosing the parameter θ_k, then θ_k is well-defined in Theorem 3.1. In fact, from the process of proof Theorem 3.1, we can claim that the convergence result of Theorem 3.1 always holds provided that we take $\theta_k \in [0, \theta]$, $\theta \in [0, 1), \forall k \geq 0$, with

$$\sum_{k=1}^{+\infty} \theta_k\|x^k - x^{k-1}\|^2 < \infty$$

and

$$\sum_{k=1}^{+\infty} \theta_k\|x^k - x^{k-1}\| < \infty.$$

4. **A modified inertial relaxed CQ algorithm and its asymptotic convergence.** In this section, a modified inertial relaxed CQ algorithm is presented, the asymptotic convergence is shown under some conditions.

Algorithm 4.1. Take $x^0, x^1 \in H^1$, the sequence $\{x^k\}_{k\geq 0}$ is generated by the iterative process

$$x^{k+1} = (1-\alpha_k)y^k + \alpha_k P_{C_k}[U_k(y^k)], \tag{4.1}$$

where $U_k = I - \gamma F_k, F_k = A^T(I - P_{Q_k})A,\ y^k = x^k + \theta_k(x^k - x^{k-1}),\ \alpha_k \in (0,1), \theta_k \in [0,1),\ \gamma, C_k, Q_k$ are given as in Algorithm 3.1.

Now, we establish the asymptotic convergence of the algorithm 4.1.

Theorem 4.1. *Suppose (1.1) is consistent, $\theta_k \in [0,\theta]$, $\theta \in [0,1), \forall k \geq 0$. If (3.12), (3.21) and the following condition holds*

$$1 > R_1 = \inf_{k\geq 0} \alpha_k > 0, \tag{4.2}$$

then the sequence $\{x^k\}$ generated by (4.1) converges weakly to a point x^* contained in the set of solution of (1.1).

Proof. Let z be a solution of the SFP. $C \subset C_k$ implies $z = P_C(z) = P_{C_k}(z)$. Define the auxiliary real sequence $\varphi_k := \frac{1}{2}\|x^k - z\|^2$. From (4.1), we have

$$\begin{aligned}
\varphi_{k+1} &= \frac{1}{2}\|x^{k+1} - z\|^2 \\
&= \frac{1}{2}\|(1-\alpha_k)y^k + \alpha_k P_{C_k} U_k(x^k + \theta_k(x^k - x^{k-1})) - z\|^2 \\
&= \frac{1}{2}\|(1-\alpha_k)y^k + \alpha_k P_{C_k}[y^k - \gamma F_k(y^k)] - z\|^2 \\
&\leq \frac{1}{2}(1-\alpha_k)\|y^k - z\|^2 + \frac{1}{2}\alpha_k\|P_{C_k}[y^k - \gamma F_k(y^k)] - P_{C_k} z\|^2 \\
&\leq \frac{1}{2}(1-\alpha_k)\|y^k - z\|^2 + \frac{1}{2}\alpha_k\|y^k - \gamma F_k(y^k) - z\|^2 \\
&= \frac{1}{2}(1-\alpha_k)\|y^k - z\|^2 + \frac{1}{2}\alpha_k\|y^k - z\|^2 + \alpha_k\frac{\gamma^2}{2}\|F_k(y^k)\|^2 \\
&\quad -\alpha_k\gamma\langle y^k - z, F_k(y^k) - F_k(z)\rangle.
\end{aligned}$$

So we get

$$\varphi_{k+1} \leq \frac{1}{2}\|y^k - z\|^2 + \alpha_k\frac{\gamma^2}{2}\|F_k(y^k)\|^2 - \alpha_k\gamma\frac{1}{L}\|F_k(y^k)\|^2, \tag{4.3}$$

therefore

$$\varphi_{k+1} \leq \frac{1}{2}\|y^k - z\|^2 - \alpha_k(\frac{\gamma}{L} - \frac{\gamma^2}{2})\|F_k(y^k)\|^2. \tag{4.4}$$

Substituting (3.10) into (4.4), we get

$$\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2 - \alpha_k(\frac{\gamma}{L} - \frac{\gamma^2}{2})\|A^T(P_{Q_k} - I)Ay^k\|^2. \tag{4.5}$$

Since $0 < \gamma < 2/L$ and $0 < \inf \alpha_k < 1$, we have

$$\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2 - R_1(\frac{\gamma}{L} - \frac{\gamma^2}{2})\|A^T(P_{Q_k} - I)Ay^k\|^2, \tag{4.6}$$

moreover

$$\varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2. \tag{4.7}$$

Suppose $\sum_{k=1}^{+\infty}\theta_k\|x^k-x^{k-1}\|^2<\infty$, choose $\delta_k:=\theta_k\|x^k-x^{k-1}\|^2$ in Lemma 2.1, we deduce that the sequence $\{\|x^k-z\|\}$ is convergent (hence $\{x^k\}$ is bounded). From (4.7) and Lemma 2.1 we obtain $\sum_{k=1}^{+\infty}[\|x^k-z\|^2-\|x^{k-1}-z\|^2]_+<\infty$, while from (4.6) we have

$$R_1(\frac{\gamma}{L}-\frac{\gamma^2}{2})\|A^T(P_{Q_k}-I)Ay^k\|^2\leq\varphi_k-\varphi_{k+1}+\theta_k(\varphi_k-\varphi_{k-1})+\theta_k\|x^k-x^{k-1}\|^2.$$

Obviously,

$$\sum_{k=1}^{+\infty}R_1(\frac{\gamma}{L}-\frac{\gamma^2}{2})\|A^T(P_{Q_k}-I)Ay^k\|^2<\infty.$$

Since $0<\gamma<2/L$ and $0<R_1<1$, we have

$$\|A^T(P_{Q_k}-I)Ay^k\|^2\to 0. \tag{4.8}$$

The rest part of the proof is similar to that of Theorem 3.1, and hence it is omitted. $\square$

5. **Numerical results.** In this section, we will test three numerical examples. Throughout the computational experiments, we set $\varepsilon=10^{-4}$. In the algorithms, we take $\gamma=1/L, L$ denotes the spectral radius of A^TA, $\theta=0.8$. If $\theta\leq\left[\max\{k^2\|x^k-x^{k-1}\|^2,k^2\|x^k-x^{k-1}\|\}\right]^{-1}$, we take $\theta_k=\frac{\theta}{2}=0.4$; Otherwise, we take $\theta_k=\left[\max\{(k+1)^2\|x^k-x^{k-1}\|^2,(k+1)^2\|x^k-x^{k-1}\|\}\right]^{-1}$, $k=1,2\cdots$.

Example 5.1. Let

$$A=\begin{bmatrix}2&1&0\\0&1&0\\0&0&1\end{bmatrix}$$

$C=\{x\in R^3|\ x_1^2+x_2^2-9\leq 0\}$; $Q=\{x\in R^3|\ x_1+x_3^2-3\leq 0\}$. Find $x\in C$ with $Ax\in Q$.

Example 5.2. Let

$$A=\begin{bmatrix}2&-1&3&2&3\\1&2&5&2&1\\2&0&2&1&-2\\2&-1&0&-3&5\end{bmatrix}$$

$C=\{x\in R^5|\ x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-0.25\leq 0\}$; $Q=\{x\in R^4|\ x_1+x_2+x_3+x_4+0.6\leq 0\}$. Find $x\in C$ with $Ax\in Q$.

Example 5.3. Let $A=(a_{ij})_{M\times N}$, $a_{ij}\in(0,1)$ be a random matrix, M,N are two positive integers. $C=\{x\in R^N|\ \sum_{l=1}^N x_l^2\leq r^2\}$; $Q=\{x\in R^M|\ x\leq b\}$. To ensure the existence of the solution of the problem, the vector b is generated by using the following way: Given a random $N-$dimensional vector $x^*\in C, r=\|x^*\|$, taking $b=Ax^*$. Find $x\in C$ with $Ax\in Q$.

The numerical results of Examples 5.1-5.3 can be seen from Tables 1-5. In Tables 1-5, "R-Iter", "Iner-R-Iter" and "Iner-KM-R-Iter" denote the relaxed CQ algorithm, the inertial relaxed CQ algorithm and the modified inertial relaxed CQ algorithm, respectively. "k" ," s" and "x^*" denote the number of iterations, cpu time in seconds and the solution, respectively. To compare conveniently, we take the initial point x^1 in the latter two algorithms as in the R-iter, that is, the point x^1 is generated by the R-iter.

TABLE 1. The numerical results of example 5.1

Initiative point	R-Iter	Iner-R-Iter
$x^0 = (3.2, 4.2, 5.2)$ $x^1 = (-0.5843, 2.3078, 3.3435)$	$k = 74; s = 0.068$ $x^* = (-0.6200, 1.6180, 1.6216)$	$k = 5; s = 0.016$ $x^* = (-1.1281, 1.0720, 1.9694)$
$x^0 = (10, 0, 10)$ $x^1 = (2.0825, -2.5275, 6.4589)$	$k = 93; s = 0.090$ $x^* = (0.9000, -1.7152, 1.7074)$	$k = 84; s = 0.085$ $x^* = (-0.1061, -1.4514, 2.1596)$
$x^0 = (2, -5, 2)$ $x^1 = (1.3327, -3.2657, 1.9328)$	$k = 73; s = 0.075$ $x^* = (1.1512, -2.7679; 1.8616)$	$k = 35; s = 0.035$ $x^* = (0.9010, -2.1029, 1.8169)$

TABLE 2. The numerical results of example 5.1

Initiative point	α_k	Iner-KM-R-Iter
$x^0 = (3.2, 4.2, 5.2)$ $x^1 = (-0.5843, 2.3078, 3.3435)$	0.4	$k = 3; s = 0.016$ $x^* = (-2.6931, 1.2534, 2.2937)$
	0.8	$k = 3; s = 0.013$ $x^* = (-2.6828, 1.2585, 2.2835)$
$x^0 = (10, 0, 10)$ $x^1 = (2.0825, -2.5275, 6.4589)$	0.4	$k = 76; s = 0.086$ $x^* = (-0.1346, -2.6392, 2.3046)$
	0.8	$k = 74; s = 0.085$ $x^* = (-0.0799, -2.6190, 2.3611)$
$x^0 = (2, -5, 2)$ $x^1 = (1.3327, -3.2657, 1.9328)$	0.6	$k = 62; s = 0.056$ $x^* = (0.9006, -2.1031, 1.8171)$
	0.8	$k = 45; s = 0.046$ $x^* = (0.9008, -2.1030, 1.8170)$

TABLE 3. The numerical results of example 5.2

Initiative point	R-Iter	Iner-R-Iter
$x^0 = (0, 0, 0, 0, 0)$ $x^1 = (-0.0092, 0, -0.0132, -0.0026, -0.0092)$	$k = 15$; s = 0.675 $x^* = (-0.0208, 0, -0.0297, -0.0059, -0.0208)$	$k = 5$; s = 0.018 $x^* = (0.0015, 0, -0.0412, -0.0082, -0.0288)$
$x^0 = (1, 1, 1, 1, 1)$ $x^1 = (0.3237, 0.5471, 0.2280, 0.4833, 0.3237)$	$k = 20$; s = 0.083 $x^* = (0.0171, 0.3822, -0.1394, 0.2779, 0.0171)$	$k = 3$; s = 0.0272 $x^* = (-0.0784, 0.2935, -0.2378, 0.1873, -0.0784)$
$x^0 = (20, 10, 20, 10, 20)$ $x^1 = (6.1605, 5.0023, 4.5130, 3.9040, 6.1605)$	$k = 22$; s = 0.090 $x^* = (0.0837, 0.3910, -0.2155, 0.1915, 0.0837)$	$k = 6$; s = 0.067 $x^* = (-0.2490, -0.2117, -0.1742, -0.1619, -0.2490)$

TABLE 4. The numerical results of example 5.2

Initiative point	α_k	Iner-KM-R-Iter
$x^0 = (0, 0, 0, 0, 0)$ $x^1 = (-0.0092, 0, -0.0132, -0.0026, -0.0092)$	0.6	$k = 6$; s = 0.020 $x^* = (-0.0209, 0, -0.0299, -0.0059, -0.0209)$
	0.8	k=5; s=0.018 $x^* = (-0.0212, 0, -0.0304, -0.0060, -0.0212)$
$x^0 = (1, 1, 1, 1, 1)$ $x^1 = (0.3237, 0.5471, 0.2280, 0.4833, 0.3237)$	0.4	$k = 3$; s = 0.034 $x^* = (-0.0644, 0.2935, -0.2177, 0.1913, -0.0644)$
	0.6	k=3; s=0.034 $x^* = (-0.0691, 0.2935, -0.2244, 0.1899, -0.0691)$
$x^0 = (20, 10, 20, 10, 20)$ $x^1 = (6.1605, 5.0023, 4.5130, 3.9040, 6.1605)$	0.6	$k = 9$; s = 0.072 $x^* = (-0.2263, -0.1045, -0.2337, -0.1094, -0.2263)$
	0.8	k= 7; s=0.071 $x^* = (-0.2283, -0.1610, -0.1881, -0.1342, -0.2283)$

TABLE 5. The numerical results of example 5.3

M, N	R-Iter	Iner-R-Iter	Iner-KM-R-Iter
$M = 20, N = 10$	$k = 436, s = 0.970$	$k = 174, s = 0.500$	$k = 210, s = 0.270$
$M = 100, N = 90$	$k = 3788, s = 0.130$	$k = 602, s = 0.680$	$k = 534, s = 0.690$

Table 1 and Table 2 give the numerical results of Example 5.1 with the R-Iter, the Iner-R-Iter, and the Iner-KM-R-Iter for different α_k, respectively. Table 3 and Table 4 show the numerical results of Example 5.2 with the R-Iter, the Iner-R-Iter, and the Iner-KM-R-Iter for different α_k, respectively. Table 5 gives the numerical results of Example 5.3 with $\alpha_k = 0.6, k = 1, 2, \cdots$.

From Tables 1-5, we can see that our algorithms are effective. It appears that they converge more quickly than the relaxed CQ algorithm in [23]. It also appears to suggest greater values of α_k because the convergence of the Iner-KM-R-Iter took less number of iterations in the experiment.

Acknowledgments. We would like to thank two anonymous referees for their constructive comments which helped us revise this paper. This work is partially supported by Natural Science Foundation of Shanghai (14ZR1429200), Innovation Program of Shanghai Municipal Education Commission(15ZZ074) and Australian Research Council (DP160102819).

REFERENCES

[1] F. Alvarez, On the minimizing property of a second order dissipative dynamical system in Hilbert spaces, *SIAM Journal on Control and Optimization*, **39** (2000), 1102–1119.

[2] F. Alvarez and H. Attouch, An inertial proximal method for maximal monotone operators via Discretization of a nonlinear oscillator with damping, *Set-Valued Analysis*, **9** (2001), 3–11.

[3] F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *SIAM Journal on Optimization*, **14** (2003), 773–782.

[4] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Review*, **38** (1996), 367–426.

[5] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, *Inverse Problems,* **18** (2002), 441–453.

[6] C. Byrne, An unified treatment of some iterative algorithm algorithms in signal processing and image reconstruction, *Inverse Problems,* **20** (2004), 103–120.

[7] J. W. Chinneck, The constraint consensus method for finding approximately feasible points in nonlinear programs, *INFORMS Journal on Computing*, **16** (2004), 255–265.

[8] Y. Censor, Parallel application of block iterative methods in medical imaging and radiation therapy, *Mathematical Programming*, **42** (1998), 307–325.

[9] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numerical Algorithms*, **8** (1994), 221–239.

[10] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets solit feasibility problem and its applications for inverse problems, *Inverse Problems*, **21** (2005), 2071–2084.

[11] G. Crombez, A geometrical look at iterative methods for operators with fixed points, *Numerical Functional Analysis and Optimization*, **26** (2005), 137–175.

[12] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983.

[13] F. Deutsch, The method of alternating orthogonal projections, *Approximation Theory, Spline Functions and Applications*, 105–121, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 356, Kluwer Acad. Publ., Dordrecht, 1992.

[14] Y. Dang and Y. Gao, The strong convergence of a KM-CQ-Like algorithm for split feasibility problem, *Inverse Problems*, **27** (2011), 015007, 9 pp.

[15] M. Fukushima, On the convergence of a class of outer approximation algorithms for convex programs, *Journal of Computational and Applied Mathematics*, **10** (1984), 147–156.

[16] M. Fukushima, A relaxed projection method for variational inequalities, *Mathematics Programming*, **35** (1986), 58–70.

[17] Y. Gao, Piecewise smooth Lyapunov function for a nonlinear dynamical system, *Journal of Convex Analysis,* **19** (2012), 1009–1015.

[18] G. T. Herman, *Image Reconstruction From Projections: The Fundamentals of Computerized Tomography*, Academic Press, New York, 1980.

[19] P. E. Mainge, Inertial iterative process for fixed points of certain quasi-nonexpansive mappings, *Set-valued Analysis*, **15** (2007), 67–79.

[20] P. E. Mainge, Convergence theorem for inertial KM-type algorithms, *Journal of Computational and Applied Mathematics*, **219** (2008), 223–236.

[21] Z. Opial, Weak convergence of the sequence of successive approximations for non-expansive mappings, *Bull. American Mathematical Society*, **73** (1967), 591–597.

[22] B. Qu and N. Xiu, A new halfspace-relaxation projection method for the split feasibility problem, *Linear Algebra and Its Application*, **428** (2008), 1218–1229.

[23] H. Xu, A variabe Krasnoselski-Mann algorithm and the multiple-set split feasibility problem, *Inverse Problems*, **22** (2006), 2021–2034.

[24] Q. Yang, The relaxed CQ algorithm solving the split feasibility problem, *Inverse Problems*, **20** (2004), 1261–1266.

[25] A. L. Yan, G. Y. Wang and N. H. Xiu, Robust solutions of split feasibility problem with uncertain linear operator, *Journal of Industrial and Management Optimization*, **3** (2007), 749–761.

[26] J. Zhao and Q. Yang, Several solution methods for the split feasibility problem, *Inverse Problems*, **21** (2005), 1791–1799.

Received February 2015; 1st revision January 2016; final revision August 2016.

E-mail address: jgdyz@163.com
E-mail address: jie.sun@curtin.edu.au
E-mail address: H.Xu@curtin.edu.au