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Fixed point theorems for the sum of three classes of mixed monotone operators and applications

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Abstract

In this paper we develop various new fixed point theorems for a class of operator equations with three general mixed monotone operators,namely $A(x,x) + B(x,x) + C(x,x) = x$ on ordered Banach spaces, where A, B, C are the mixed monotone operators. A is such that for any $t \in (0,1)$, there exists $\varphi(t) \in (t,1]$ such that for all $x, y \in P$, $A(tx, t^{-1}y) \geq \varphi(t)A(x,y)$; B is hypo-homogeneous, *i.e.* B satisfies that for any $t \in (0,1)$, $x, y \in P$, $B(tx, t^{-1}y) \geq tB(x,y)$; C is concave-convex, *i.e.* C satisfies that for fixed y, $C(\cdot, y) : P \to P$ is concave; for fixed x, $C(x,\cdot) : P \to P$ is convex. Also we study the solution of the nonlinear eigenvalue equation $A(x,x) + B(x,x) + C(x,x) = \lambda x$ and discuss its dependency to the parameter. Our work extends many existing results in the field of study. As an application, we utilize the results obtained in this paper for the operator equation to study the existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions.

MSC: 47H07; 47H14; 26A33; 34A08

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1 Introduction

Mixed monotone operators are an important class of operators which are used intensively in engineering, nuclear physics, biology, chemistry, technology, *etc.* They were first introduced by Guo and Lakshmikantham in [1]. Thereafter, many authors have investigated this kind of operators in Banach spaces and obtained a lot of interesting and important results (see [2–10]).

In [11], Chen discussed the conditions which guarantee the existence of an asymptotically attractive fixed point for $T = A + H : P \to P$, where P is a cone of a Banach space E. $A, H : P \to P$ are two monotone operators in E and there exist $\eta > 0$ and $\alpha \in (0,1)$ such that

$$H(tx) \geq tHx, \quad \forall t \in (\eta, 1), x \in P,$$

and

$$A(tx) \geq t^{\alpha}Ax, \quad \forall t \in (\eta, 1), x \in P,$$

where $\alpha \in (\eta, 1)$. If there is an $x_0 \in P$ such that $Ax_0 \in C_{x_0}$ and $Hx_0 \in C_{x_0}$, then there exists $\lambda : (\eta, 1) \times C_{x_0} \to (\alpha, 1)$ such that $T(tx) \geq t^{\lambda(t,x)} Tx$, $t \in (\eta, 1)$, $x \in P$. Furthermore, if $\lambda_m(t) = \sup_{x \in C_{x_0}} \lambda(t,x) < 1$, then there exists $x_* \in C_{x_0}$ such that $\lim_{n\to\infty} T^n x_0 = x_*$ and $Tx_* = x_*$.

In [12], Zhang used the partial order theory to study mixed monotone operators which have different types of concave-convex properties (for example: $A(x,y)$ is concave in x, and (-α)-convex in y). The author also assumes that there exist $u_0, v_0 \in \mathring{P}$, $\varepsilon > 0$, $\varepsilon \geq \alpha$ such that $0 \ll u_0 \leq v_0$, $u_0 \leq A(u_0, v_0)$, $A(v_0, u_0) \leq v_0$; and $A(\theta, v_0) \geq \varepsilon A(v_0, u_0)$), and he establishes the existence and uniqueness of a fixed point without assuming the operator to be compact or continuous.

In [13], Zhai and Hao considered the existence and uniqueness of positive solutions to the operator equation $A(x,x) + Bx = x$ on ordered Banach spaces, where A is a mixed monotone operator, B is an increasing sub-homogeneous operator or α-concave operator, and assume that:

(1) there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$, $Bh_0 \in P_h$;

(2) there exists a constant $\delta_0 > 0$, such that $A(x,y) \geq \delta_0 B(x,y)$, $\forall x, y \in P$.

In [14], Zhai and Anderson studied an operator equation $Ax + Bx + Cx = x$ on ordered Banach spaces, where A is an increasing α-concave operator, B is an increasing sub-homogeneous operator and C is a homogeneous operator and satisfy:

(1) there is $h > \theta$ such that $Ah \in P_h$, $Bh \in P_h$, $Ch \in P_h$;

(2) there exist constants $\delta_1, \delta_2 > 0$, such that $Ax \geq \delta_1 Bx + \delta_2 Cx$, $\forall x \in P$.

The existence and uniqueness of positive solutions of the operator equation is obtained by using the properties of cones and a fixed point theorem for increasing general β-concave operators.

Motivated by the above work, this paper studies the existence and uniqueness of positive solutions to the following operator equation on ordered Banach spaces:

$$A(x,x) + B(x,x) + C(x,x) = x, \tag{1.1}$$

where $A, B, C : P \times P \to P$ are mixed monotone operators and satisfy the following conditions, respectively:

(1) for any $t \in (0,1)$, there exists $\varphi(t) \in (t, 1]$ such that

$$A\bigl(tx, t^{-1}y\bigr) \geq \varphi(t) A(x,y), \quad \forall x, y \in P;$$

(2) for any $t \in (0,1)$,

$$B\bigl(tx, t^{-1}y\bigr) \geq tB(x,y), \quad \forall x, y \in P;$$

(3) for any fixed y, $C(\cdot, y) : P \to P$ is concave; for any fixed x, $C(x, \cdot) : P \to P$ is convex.

Also we study the solution of the nonlinear eigenvalue equation $A(x,x) + B(x,x) + C(x,x) = \lambda x$ and discuss its properties. As an application, we utilize the results obtained for the operator equations to study the existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions.

To our knowledge, so far no fixed point results have been achieved for the operator of equation (1.1) with the operators A, B, C satisfying the above listed conditions (1), (2), and

(3). Our work presented in this paper has various new features. First,by using the properties of cones, we obtain some existence and uniqueness results of positive solutions for the operator of equation (1.1) without the need of assuming the operators to be continuous or compact, which extends the corresponding results in [11–23]. Most importantly our work is about the sum of three classes of mixed monotone operators. We also discusses the solution of the nonlinear eigenvalue equation $A(x,x) + B(x,x) + C(x,x) = \lambda x$ and investigate its dependency to the parameter. To demonstrate the applicability of our abstract results, we give, in the last section of the paper, some applications to nonlinear fractional differential equations with integral boundary conditions, and also, we give some specific examples.

2 Preliminaries and lemmas

For convenience in the discussion of the following sections, we briefly present here some definitions, notations and known results. For more details, we refer the reader to [1, 16, 19, 24–26] and the references therein.

Suppose that $(E, \|\cdot\|)$ is a Banach space and θ is the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (1) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (2) $x \in P$, $-x \in P \Rightarrow x = \theta$. The Banach space E can be partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$.

A cone P is said to be solid if its interior $\overset{\circ}{P}$ is non-empty. If $x - y \in \overset{\circ}{P}$, then we denote $x \gg y$. Moreover, P is called normal if there exists a constant $N > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where the smallest N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ ($Ay \leq Ax$). An operator $A : P \to P$ is said to be α-concave if there exists $\alpha \in (0,1)$ such that for all $t \in (0,1)$, $x \in P$, $A(tx) \geq t^{\alpha} Ax$.

For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > 0$, we denote by P_h the set $P_h = \{x \in P \mid x \sim h\}$. It is easy to see that $P_h \subset P$. If P is a solid cone, take any $h \in \overset{\circ}{P}$, then $P_h = \overset{\circ}{P}$. For $x, y \in P_h$, we can define

$$M(x/y) = \inf\{\lambda \in R : x \leq \lambda y\}. \tag{2.1}$$

Definition 2.1 [1, 16] $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x, and decreasing in y, *i.e.*, u_i, v_i $(i = 1,2) \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

Definition 2.2 [22] $A : D(A) \subset E \to E$ is said to be convex if for any $x, y \in D(A)$ with $x \leq y$ and every $t \in (0,1)$, we have $A(tx + (1-t)y) \leq tAx + (1-t)Ay$. A is said to be concave if $-A$ is convex.

3 Main results

In this section we consider the existence and uniqueness of positive solutions for the operator of equation (1.1). Throughout the paper, we assume that E is a real Banach space with a partial order introduced by a normal cone P of E. Take $h \in E$, $h > \theta$, P_h is given as in the introduction.

Theorem 3.1 *Let P be a normal cone in E. Assume that $A, B, C : P \times P \to P$ are three mixed monotone operators and satisfy the following conditions*:

(1) *for any $t \in (0,1)$, there exists $\varphi(t) \in (t,1]$ such that*

$$A\bigl(tx, t^{-1}y\bigr) \geq \varphi(t)A(x,y), \quad \forall x, y \in P; \tag{3.1}$$

(2) *for any $t \in (0,1)$, $x, y \in P$,*

$$B\bigl(tx, t^{-1}y\bigr) \geq tB(x,y); \tag{3.2}$$

(3) *for any fixed $y \in P$, $C(\cdot, y) : P \to P$ is concave; for any fixed $x \in P$, $C(x, \cdot) : P \to P$ is convex*;

(4) *there is $h \in P$, $h > \theta$ such that $A(h,h) \in P_h$, $B(h,h) \in P_h$, and $C(h,h) \in P_h$*;

(5) *there exists $\frac{1}{2} \leq c \leq 1$, such that $C(\theta, lh) \geq cC(lh, \theta)$, for any $l \geq 1$*;

(6) *there exists a constant $\delta_0 > 0$, such that $B(x,y) + C(x,y) \leq \delta_0 A(x,y)$, $\forall x, y \in P_h$.*

Then the operator of equation (1.1) *has a unique positive solution x^* in P, which satisfies $\mu h \leq x^* \leq \lambda h$, where $\lambda > 0$, $\mu > 0$ are two real numbers. And for any initial values $x_0, y_0 \in P_h$, by constructing successively the sequences as follows*:

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}) + C(x_{n-1}, y_{n-1}), \\ y_n &= A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}) + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \end{aligned}$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E, as $n \to \infty$.

Proof From (3.1) and (3.2), for any $t \in (0,1)$, we have

$$A\bigl(t^{-1}x, ty\bigr) \leq \frac{1}{\varphi(t)}A(x,y), \quad \forall x, y \in P$$

and

$$B\bigl(t^{-1}x, ty\bigr) \leq t^{-1}B(x,y), \quad \forall x, y \in P. \tag{3.3}$$

Since $A(h,h) \in P_h$, $B(h,h) \in P_h$, $C(h,h) \in P_h$, there exist constants $a_i > 0$, $b_i > 0$ ($i = 1,2,3$) such that

$$a_1 h \leq A(h,h) \leq b_1 h, \qquad a_2 h \leq B(h,h) \leq b_2 h, \qquad a_3 h \leq C(h,h) \leq b_3 h. \tag{3.4}$$

First of all, we show $A : P_h \times P_h \to P_h$. For any $x, y \in P_h$, we can choose two sufficiently small numbers $\alpha_1, \alpha_2 \in (0,1)$ such that

$$\alpha_1 h \leq x \leq \frac{1}{\alpha_1}h, \qquad \alpha_2 h \leq y \leq \frac{1}{\alpha_2}h.$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, then $\alpha \in (0,1)$, by (3.1), (3.3), and (3.4), we have

$$A(x,y) \leq A\left(\frac{1}{\alpha}h, \alpha h\right) \leq \frac{1}{\varphi(\alpha)}A(h,h) \leq \frac{1}{\varphi(\alpha)}b_1 h,$$

$$A(x,y) \geq A\left(\alpha h, \frac{1}{\alpha}h\right) \geq \varphi(\alpha)A(h,h) \geq \varphi(\alpha)a_1 h.$$

Evidently, $\frac{1}{\varphi(\alpha)}b_1, \varphi(\alpha)a_1 > 0$. Thus $A(x,y) \in P_h$; that is, $A : P_h \times P_h \to P_h$.

Second, we show $B : P_h \times P_h \to P_h$. For any $x, y \in P_h$, we can choose two sufficiently small numbers $\beta_1, \beta_2 \in (0,1)$ such that

$$\beta_1 h \leq x \leq \frac{1}{\beta_1}h, \qquad \beta_2 h \leq y \leq \frac{1}{\beta_2}h.$$

Let $\beta = \min\{\beta_1, \beta_2\}$, then $\beta \in (0,1)$, by (3.2), (3.3), and (3.4), we have

$$B(x,y) \leq B\left(\frac{1}{\beta}h, \beta h\right) \leq \frac{1}{\beta}B(h,h) \leq \frac{1}{\beta}b_2 h,$$
$$B(x,y) \geq B\left(\beta h, \frac{1}{\beta}h\right) \geq \beta B(h,h) \geq \beta a_2 h.$$

Evidently, $\frac{1}{\beta}b_2, \beta a_2 > 0$. Thus $B(x,y) \in P_h$; that is, $B : P_h \times P_h \to P_h$.

Thirdly, we show $C : P_h \times P_h \to P_h$. For any $t \in (0,1)$, $x, y \in P_h$, we have

$$C(x,y) = C\bigl(x, tt^{-1}y + (1-t)\theta\bigr) \leq tC\bigl(x, t^{-1}y\bigr) + (1-t)C(x,\theta),$$

thus

$$tC\bigl(x, t^{-1}y\bigr) \geq C(x,y) - (1-t)C(x,\theta). \tag{3.5}$$

Also, we can find a sufficiently large l such that $x, y, t^{-1}y \leq lh$ and satisfies (5), so from (5), (3.5), and the concavity and convexity as well as the monotone property of operator C, we have

$$\begin{aligned}
C\bigl(tx, t^{-1}y\bigr) &\geq tC\bigl(x, t^{-1}y\bigr) + (1-t)C\bigl(\theta, t^{-1}y\bigr) \\
&\geq C(x,y) - (1-t)C(x,\theta) + (1-t)C(\theta, lh) \\
&\geq C(x,y) + (1-t)\bigl[C(\theta, lh) - C(lh, \theta)\bigr] \\
&\geq C(x,y) + (1-t)\left[C(\theta, lh) - \frac{1}{c}C(\theta, lh)\right] \\
&= C(x,y) + (1-t)\left(1 - \frac{1}{c}\right)C(\theta, lh) \\
&\geq \left[1 + (1-t)\left(1 - \frac{1}{c}\right)\right]C(x,y) \\
&= \left[\left(2 - \frac{1}{c}\right) + \left(\frac{1}{c} - 1\right)t\right]C(x,y) \\
&\geq tC(x,y).
\end{aligned}$$

That is,

$$C\bigl(tx, t^{-1}y\bigr) \geq tC(x,y), \quad \forall t \in (0,1), \forall x, y \in P_h. \tag{3.6}$$

And then it is obvious that

$$C\bigl(t^{-1}x, ty\bigr) \leq \frac{1}{t}C(x,y), \quad \forall t \in (0,1), \forall x,y \in P_h. \tag{3.7}$$

Since $x,y \in P_h$, we can choose two sufficiently small numbers $\gamma_1, \gamma_2 \in (0,1)$ such that

$$\gamma_1 h \leq x \leq \frac{1}{\gamma_1}h, \qquad \gamma_2 h \leq y \leq \frac{1}{\gamma_2}h.$$

Let $\gamma = \min\{\gamma_1, \gamma_2\}$, then $\gamma \in (0,1)$, by (3.4), (3.6), and (3.7), we have

$$\begin{aligned} C(x,y) &\leq C\left(\frac{1}{\gamma}h, \gamma h\right) \leq \frac{1}{\gamma}C(h,h) \leq \frac{1}{\gamma}b_3 h, \\ C(x,y) &\geq C\left(\gamma h, \frac{1}{\gamma}h\right) \geq \gamma C(h,h) \geq \gamma a_3 h. \end{aligned}$$

Evidently, $\frac{1}{\gamma}b_3, \gamma a_3 > 0$. Thus $C(x,y) \in P_h$; that is, $C : P_h \times P_h \to P_h$.

Now we define an operator $T = A + B + C : P_h \times P_h \to P_h$ by

$$T(x,y) = A(x,y) + B(x,y) + C(x,y), \quad x,y \in P_h.$$

Then $T : P_h \times P_h \to P_h$ is a mixed monotone operator and $T(h,h) \in P_h$.

In the following, we show that for any $x,y \in P_h$, $t \in (0,1)$, there exists $\psi(t,x,y) \in (t,1]$ such that $T(tx, t^{-1}y) \geq \psi(t,x,y)T(x,y)$. From (3.1), (3.2), and (3.6), for any $t \in (0,1)$ and $x,y \in P_h$, we have

$$\begin{aligned} T\bigl(tx, t^{-1}y\bigr) &= A\bigl(tx, t^{-1}y\bigr) + B\bigl(tx, t^{-1}y\bigr) + C\bigl(tx, t^{-1}y\bigr) \\ &\geq \varphi(t)A(x,y) + tB(x,y) + tC(x,y) \\ &\geq \varphi(t)A(x,y) + t\bigl[B(x,y) + C(x,y)\bigr]. \end{aligned}$$

For any $x,y \in P_h$, since $A,B,C : P_h \times P_h \to P_h$, we have $A(x,y) \in P_h$, $B(x,y) \in P_h$, $C(x,y) \in P_h$, and thus we get $B(x,y) + C(x,y) \sim A(x,y)$. Denote

$$F(x,y) = M\left(\frac{B(x,y) + C(x,y)}{A(x,y)}\right).$$

Then from (6) we have $F(x,y) \leq \delta_0$. For any $t \in (0,1)$, $x,y \in P_h$, consider

$$g(s) = \frac{\varphi(t) + F(x,y)t}{(F(x,y)+1)s}, \quad s \in \bigl[t, \varphi(t)\bigr].$$

It is clear that g is continuous and strictly decreasing with respect to s. Since

$$g\left(\frac{\delta_0 t + \varphi(t)}{\delta_0 + 1}\right) = \frac{\varphi(t) + F(x,y)t}{(F(x,y)+1)\frac{\delta_0 t + \varphi(t)}{\delta_0 + 1}} > 1$$

and

$$g\bigl(\varphi(t)\bigr) = \frac{\varphi(t) + F(x,y)t}{(F(x,y)+1)\varphi(t)} < 1,$$

there exists a unique $\psi(t,x,y) \in (\frac{\delta_0 t+\varphi(t)}{\delta_0+1}, \varphi(t))$ such that

$$g\bigl(\psi(t,x,y)\bigr) = \frac{\varphi(t) + F(x,y)t}{(F(x,y)+1)\psi(t,x,y)} = 1.$$

Solving for $F(x,y)$ in the above inequality leads to

$$F(x,y) = \frac{\varphi(t) - \psi(t,x,y)}{\psi(t,x,y) - t}.$$

From (2.1) we get $M(x/y) = \inf\{\lambda \in R : x \leq \lambda y\}$. So from the definitions of $F(x,y)$ and $M(x/y)$, we can get

$$B(x,y) + C(x,y) \leq \frac{\varphi(t) - \psi(t,x,y)}{\psi(t,x,y) - t} A(x,y).$$

This inequality can be rewritten as

$$\varphi(t)A(x,y) + t\bigl[B(x,y) + C(x,y)\bigr] \geq \psi(t,x,y)\bigl[A(x,y) + B(x,y) + C(x,y)\bigr].$$

Hence

$$\begin{aligned} T\bigl(tx,t^{-1}y\bigr) &= A\bigl(tx,t^{-1}y\bigr) + B\bigl(tx,t^{-1}y\bigr) + C\bigl(tx,t^{-1}y\bigr) \\ &\geq \varphi(t)A(x,y) + tB(x,y) + tC(x,y) \\ &= \varphi(t)A(x,y) + t\bigl[B(x,y) + C(x,y)\bigr] \\ &\geq \psi(t,x,y)\bigl[A(x,y) + B(x,y) + C(x,y)\bigr] \\ &= \psi(t,x,y)T(x,y). \end{aligned}$$

That is, for any $x, y \in P_h$ and $t \in (0,1)$, there exists $\psi(t,x,y) \in (\frac{\delta_0 t+\varphi(t)}{\delta_0+1}, \varphi(t)) \subseteq (t,1]$ such that

$$T\bigl(tx,t^{-1}y\bigr) \geq \psi(t,x,y)T(x,y). \tag{3.8}$$

Since $T(h,h) \in P_h$, we can choose a sufficiently small number $s_0 \in (0,1)$ such that

$$s_0 h \leq T(h,h) \leq \frac{1}{s_0} h. \tag{3.9}$$

Noting that $s_0 < \psi(s_0,x,y) \leq 1$, we can get $1 < \frac{\psi(s_0,x,y)}{s_0} \leq \frac{1}{s_0}$. By the Archimedes principle, we can take a positive integer k such that

$$\left(\frac{\psi(s_0,x,y)}{s_0}\right)^k \geq \frac{1}{s_0},$$

that is,

$$\frac{\psi(s_0,x,y)}{s_0} \geq \left(\frac{1}{s_0}\right)^{\frac{1}{k}}. \tag{3.10}$$

Put $u_0 = s_0^k h$, $v_0 = s_0^{-k} h$. Evidently, $u_0, v_0 \in P_h$ and $u_0 = s_0^{2k} v_0 < v_0$. Take any $r \in (0, s_0^{2k}]$, then $r \in (0,1)$ and $u_0 \geq r v_0$. By the mixed monotone properties of T, we have $T(u_0, v_0) \leq T(v_0, u_0)$. Further, by combining condition (3.8) with (3.9) and (3.10), we have

$$\begin{aligned}
T(u_0,v_0) &= T\left(s_0^k h, \frac{1}{s_0^k} h\right) \\
&= T\left(s_0 s_0^{k-1} h, \frac{1}{s_0}\frac{1}{s_0^{k-1}} h\right) \\
&\geq \psi\left(s_0, s_0^{k-1} h, \frac{1}{s_0^{k-1}} h\right) T\left(s_0^{k-1} h, \frac{1}{s_0^{k-1}} h\right) \\
&= \psi\left(s_0, s_0^{k-1} h, \frac{1}{s_0^{k-1}} h\right) T\left(s_0 s_0^{k-2} h, \frac{1}{s_0}\frac{1}{s_0^{k-2}} h\right) \\
&= \psi\left(s_0, s_0^{k-1} h, \frac{1}{s_0^{k-1}} h\right) \psi\left(s_0, s_0^{k-2} h, \frac{1}{s_0^{k-2}} h\right) T\left(s_0^{k-2} h, \frac{1}{s_0^{k-2}} h\right) \geq \cdots \\
&\geq \left(\left(\frac{1}{s_0}\right)^{\frac{1}{k}} s_0\right)^k T(h,h) \\
&\geq \frac{1}{s_0} s_0^k s_0 h \\
&\geq s_0^k h \\
&= u_0.
\end{aligned}$$

From (3.8), we can get, for all $t \in (0,1)$, $x, y \in P$, $T(t^{-1}x, ty) \leq \frac{1}{\psi(t,x,y)} T(x,y)$. So

$$\begin{aligned}
T(v_0,u_0) &= T\left(\frac{1}{s_0^k} h, s_0^k h\right) \\
&= T\left(\frac{1}{s_0}\frac{1}{s_0^{k-1}} h, s_0 s_0^{k-1} h\right) \\
&\leq \frac{1}{\psi(s_0, \frac{1}{s_0^{k-1}} h, s_0^{k-1} h)} T\left(\frac{1}{s_0^{k-1}} h, s_0^{k-1} h\right) \\
&= \frac{1}{\psi(s_0, \frac{1}{s_0^{k-1}} h, s_0^{k-1} h)} T\left(\frac{1}{s_0}\frac{1}{s_0^{k-2}} h, s_0 s_0^{k-2} h\right) \\
&= \frac{1}{\psi(s_0, \frac{1}{s_0^{k-1}} h, s_0^{k-1} h)} \frac{1}{\psi(s_0, \frac{1}{s_0^{k-2}} h, s_0^{k-2} h)} T\left(\frac{1}{s_0^{k-2}} h, s_0^{k-2} h\right) \leq \cdots \\
&\leq \left(\frac{1}{s_0} s_0^{\frac{1}{k}}\right)^k T(h,h) \\
&\leq \frac{1}{s_0^k} s_0 \frac{1}{s_0} h \\
&\leq \frac{1}{s_0^k} h \\
&= v_0.
\end{aligned}$$

Thus we have

$$u_0 \le T(u_0, v_0) \le T(v_0, u_0) \le v_0.$$

For u_0, v_0, construct successively the sequences as follows:

$$u_n = T(u_{n-1}, v_{n-1}), \qquad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots.$$

Evidently, $u_1 \le v_1$. By the mixed monotone properties of T, we obtain $u_n \le v_n$ ($n = 1, 2, \ldots$) and

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{3.11}$$

Noting that $u_0 \ge rv_0$, we can get $u_n \ge u_0 \ge rv_0 \ge rv_n$ ($n = 1, 2, \ldots$). Let

$$t_n = \sup\{t > 0 \mid u_n \ge tv_n, n = 1, 2, \ldots\}.$$

Then we have

$$u_n \ge t_n v_n, \quad n = 1, 2, \ldots, \tag{3.12}$$

and then by (3.11) we have

$$u_{n+1} \ge u_n \ge t_n v_n \ge t_n v_{n+1}, \quad n = 1, 2, \ldots.$$

Therefore, $t_{n+1} \ge t_n$, *i.e.*, $\{t_n\}$ is increasing with $\{t_n\} \subset (0, 1]$. Suppose $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. Otherwise, $0 < t^* < 1$. Since $t_n \le t^*$ and $\psi(t, x, y) > t$, by the mixed monotone properties of T and (3.8) as well as (3.12), we have

$$\begin{aligned}
u_{n+1} &= T(u_n, v_n) \\
&\ge T\left(t_n v_n, \frac{1}{t_n} u_n\right) \\
&= T\left(\frac{t_n}{t^*} t^* v_n, \frac{t^*}{t_n} \frac{1}{t^*} u_n\right) \\
&\ge \frac{t_n}{t^*} T\left(t^* v_n, \frac{1}{t^*} u_n\right) \\
&\ge \frac{t_n}{t^*} \psi\left(t^*, v_n, u_n\right) T(v_n, u_n) \\
&= \frac{t_n}{t^*} \psi\left(t^*, v_n, u_n\right) v_{n+1}.
\end{aligned}$$

By the definition of t_n, we have $t_{n+1} \ge \frac{t_n}{t^*} \psi(t^*, v_n, u_n)$, that is, $\psi(t^*, v_n, u_n) \le \frac{t^*}{t_n} t_{n+1}$. So we get

$$t^* < \frac{\delta_0 t^* + \varphi(t^*)}{\delta_0 + 1} < \psi\left(t^*, v_n, u_n\right) \le \frac{t^*}{t_n} t_{n+1}, \quad n = 1, 2, \ldots.$$

Since $\lim_{n\to\infty}\frac{t^*}{t_n}t_{n+1}=t^*$, we get $t^*<\frac{\delta_0 t^*+\varphi(t^*)}{\delta_0+1}\leq t^*$, which is a contradiction. Thus, $\lim_{n\to\infty}t_n=1$. For any natural number p we have

$$\theta\leq u_{n+p}-u_n\leq v_n-u_n\leq v_n-t_nv_n=(1-t_n)v_n\leq(1-t_n)v_0,$$
$$\theta\leq v_n-v_{n+p}\leq v_n-u_n\leq(1-t_n)v_0,\quad n=1,2,\ldots.$$

Since the cone P is normal, we have

$$\|u_{n+p}-u_n\|\leq N(1-t_n)\|v_0\|,\qquad\|v_n-v_{n+p}\|\leq N(1-t_n)\|v_0\|,\quad n,p=1,2,\ldots,$$

where N is the normality constant of P. So we can claim that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because E is complete, there exist u^*, v^* such that $u_n\to u^*$, $v_n\to v^*$ as $n\to\infty$. By (3.11), we know that $u_n\leq u^*\leq v^*\leq v_n$ with $u^*,v^*\in P_h$, and

$$\theta\leq v^*-u^*\leq v_n-u_n\leq(1-t_n)v_0.$$

Further, by the normality of cone P, we have

$$\left\|v^*-u^*\right\|\leq N(1-t_n)\|v_0\|\to 0,\quad n\to\infty,$$

and thus $u^*=v^*$. Let $x^*:=u^*=v^*$ and then by the mixed monotone properties of T, we obtain

$$u_{n+1}=T(u_n,v_n)\leq T\bigl(x^*,x^*\bigr)\leq T(v_n,u_n)=v_{n+1},\quad n=1,2,3,\ldots.$$

Let $n\to\infty$, then we get $x^*=T(x^*,x^*)$. That is, x^* is a fixed point of T in P_h.

In the following, we prove that x^* is the unique fixed point of T in P_h. In fact, suppose $\overline{x}$ is another fixed point of T in P_h and $\overline{x}\neq x^*$. Since $x^*,\overline{x}\in P_h$, there exist positive numbers $\mu_1,\mu_2,\lambda_1,\lambda_2>0$ such that

$$\mu_1h\leq x^*\leq\lambda_1h,\qquad\mu_2h\leq\overline{x}\leq\lambda_2h.$$

Then we obtain

$$\overline{x}\leq\lambda_2h=\frac{\lambda_2}{\mu_1}\mu_1h\leq\frac{\lambda_2}{\mu_1}x^*,\qquad\overline{x}\geq\overline{\mu}_2h=\frac{\mu_2}{\lambda_1}\lambda_1h\geq\frac{\mu_2}{\lambda_1}x^*.$$

Let

$$e_1=\sup\bigl\{t>0\mid tx^*\leq\overline{x}\leq t^{-1}x^*\bigr\}.$$

Evidently, $0<e_1\leq 1$, $e_1x^*\leq\overline{x}\leq\frac{1}{e_1}x^*$. Next we prove $e_1=1$. If $0<e_1<1$, then by the mixed monotone properties of T and (3.8), we would get

$$\overline{x}=T(\overline{x},\overline{x})\geq T\left(e_1x^*,\frac{1}{e_1}x^*\right)\geq\psi\bigl(e_1,x^*,x^*\bigr)T\bigl(x^*,x^*\bigr)=\psi\bigl(e_1,x^*,x^*\bigr)x^*.$$

Since $\psi(e_1,x^*,x^*)>e_1$, this contradicts the definition of e_1, so we get $e_1=1$. Thus $\overline{x}=x^*$. Therefore, A has a unique fixed point x^* in P_h.

Now we construct successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

for any initial points $x_0, y_0 \in P_h$. Since $x_0, y_0 \in P_h$, we can choose small numbers $e_2, e_3 \in (0,1)$ such that

$$e_2 h \leq x_0 \leq \frac{1}{e_2} h, \qquad e_3 h \leq y_0 \leq \frac{1}{e_3} h.$$

Let $e_* = \min\{e_2, e_3\}$. Then $e_* \in (0,1)$ and

$$e_* h \leq x_0, y_0 \leq \frac{1}{e_*} h.$$

Since $e_* < \psi(e_*, x, y) \leq 1$, we can get $1 < \frac{\psi(e_*,x,y)}{e_*} \leq \frac{1}{e_*}$. By the Archimedes principle, we can choose a sufficiently large positive integer m such that

$$\frac{\psi(e_*, x, y)}{e_*} \geq \left(\frac{1}{e_*}\right)^{\frac{1}{m}}.$$

Put $\overline{u}_0 = e_*^m h$, $\overline{v}_0 = \frac{1}{e_*^m} h$. It is easy to see that $\overline{u}_0, \overline{v}_0 \in P_h$, and $\overline{u}_0 < x_0, y_0 < \overline{v}_0$. Let

$$\overline{u}_n = T(\overline{u}_{n-1}, \overline{v}_{n-1}), \qquad \overline{v}_n = T(\overline{v}_{n-1}, \overline{u}_{n-1}), \quad n = 1, 2, \ldots.$$

Similarly, it follows that there exists $y^* \in P_h$ such that

$$T(y^*, y^*) = y^*, \qquad \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} \overline{v}_n = y^*.$$

By the uniqueness of the fixed points of the operator T in P_h, we get $x^* = y^*$. And by induction, $\overline{u}_n \leq x_n$, $y_n \leq \overline{v}_n$ $(n = 1, 2, \ldots)$. Since the cone P is normal, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$. □

Remark 3.1 Theorem 3.1 is a fixed point theorem for the sum of three classes of mixed monotone operators, which extends the results in [11–14].

Taking $B, C = \theta$ in Theorem 3.1, we get the following corollary.

Corollary 3.1 *Let P be a normal cone in E. Assume that $T : P \times P \to P$ is a mixed monotone operator and satisfies the following conditions*:

(A_1) *There exists $h \in P$ with $h > \theta$ such that $T(h,h) \in P_h$.*
(A_2) *For any $t \in (0,1)$, there exists $\varphi(t) \in (t,1]$ such that*

$$T(tx, t^{-1}y) \geq \varphi(t) T(x,y), \quad \forall x, y \in P.$$

Then the operator $T(x,x) = x$ has a unique solution x^* in P, which satisfies $\mu h \leq x^* \leq \lambda h$, where $\lambda > 0$, $\mu > 0$ are two real numbers. Moreover, for any initial values $x_0, y_0 \in P_h$, by *constructing successively the sequences as follows*:

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *in* E, *as* $n \to \infty$.

Proof Since X is a uniformly convex Banach space and A is bounded, we see that A_0 is non-empty and $\{(PT)^n(x)\}$ is bounded for any $x \in A_0$. By Theorem 3.1, T has at least one best proximity point. □

Remark 3.2 Under the conditions (A_1), (A_2), this corollary not only guarantees the existence of upper-lower solutions for the operator T and the existence of a unique fixed point, but also it constructs successively some sequences for approximating the fixed point.

Taking $A, B = \theta$ and $\frac{1}{2} < c \leq 1$ in Theorem 3.1, from the proof of Theorem 3.1 we get the following corollary.

Corollary 3.2 *Let* P *be a normal cone in* E. *Assume that* $C : P \times P \to P$ *is a mixed monotone operator and satisfies the following conditions*:

(1) *for any fixed* $y \in P$, $C(\cdot, y) : P \to P$ *is concave*; *for any fixed* $x \in P$, $C(x, \cdot) : P \to P$ *is convex*;
(2) *there is* $h \in P$, $h > \theta$ *such that* $C(h, h) \in P_h$;
(3) *there exists* $\frac{1}{2} < c \leq 1$, *such that* $C(\theta, lh) \geq cC(lh, \theta)$, $l \geq 1$.

Then the operator $C(x, x) = x$ *has a unique solution* x^* *in* P, *which satisfies* $\mu h \leq x^* \leq \lambda h$, *where* $\lambda > 0$, $\mu > 0$ *are two real numbers*. *Furthermore*, *for any initial values* $x_0, y_0 \in P_h$, *by constructing successively the sequences as follows*:

$$x_n = C(x_{n-1}, y_{n-1}), \qquad y_n = C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *in* E, *as* $n \to \infty$.

Remark 3.3 In the above corollary we do not need to require the operator C to satisfy $0 < C(\theta, v) \leq v$, which extends the result in [12].

Taking $\varphi(t) = t^\alpha$, $\alpha \in (0, 1)$, we get the following corollary, which generalizes and improves Theorem 2.1 in [21].

Corollary 3.3 *Let* P *be a normal cone in* E, $\alpha \in (0, 1)$. *Assume that* $A, B, C : P \times P \to P$ *are three mixed monotone operators and satisfy the following conditions*:

(1) *for any* $t \in (0, 1)$, $x, y \in P$, *we have* $A(tx, t^{-1}y) \geq t^\alpha A(x, y)$;
(2) *for any* $t \in (0, 1)$, $x, y \in P$, *we have* $B(tx, t^{-1}y) \geq tB(x, y)$;
(3) *for any fixed* $y \in P$, $C(\cdot, y) : P \to P$ *is concave*; *for any fixed* $x \in P$, $C(x, \cdot) : P \to P$ *is convex*;
(4) *there is* $h \in P$, $h > \theta$ *such that* $A(h, h) \in P_h$, $B(h, h) \in P_h$, *and* $C(h, h) \in P_h$;
(5) *there exists* $\frac{1}{2} \leq c \leq 1$, *such that* $C(\theta, lh) \geq cC(lh, \theta)$, *for any* $l \geq 1$;
(6) *there exists a constant* $\delta_0 > 0$, *such that* $B(x, y) + C(x, y) \leq \delta_0 A(x, y)$, $\forall x, y \in P_h$.

Then the operator equation (1.1) *has a unique solution* x^* *in* P, *which satisfies* $\mu h \leq x^* \leq \lambda h$, *where* $\lambda > 0$, $\mu > 0$ *are two real numbers*. *Furthermore*, *for any initial values* $x_0, y_0 \in P_h$, *by constructing successively the sequences as follows*:

$$x_n = A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}) + C(x_{n-1}, y_{n-1}),$$
$$y_n = A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}) + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Corollary 3.4 *Let* P *be a normal cone in* E. *Let* $h > 0$ *and* $A, B, C : P_h \times P_h \to P_h$ *are three mixed monotone operators and satisfy the following conditions*:

(1) *for any* $t \in (0,1)$, *there exists* $\varphi(t) \in (t, 1]$ *such that*

$$A\bigl(tx, t^{-1}y\bigr) \geq \varphi(t)A(x,y), \quad \forall x, y \in P_h;$$

(2) *for any* $t \in (0,1)$, $x, y \in P_h$, $B(tx, t^{-1}y) \geq tB(x,y)$;
(3) *for any fixed* $y \in P_h$, $C(\cdot, y) : P_h \to P_h$ *is concave*; *for any fixed* $x \in P_h$, $C(x, \cdot) : P_h \to P_h$ *is convex*;
(4) *there exists* $\frac{1}{2} \leq c \leq 1$, *such that* $C(\theta, lh) \geq cC(lh, \theta)$, $l \geq 1$;
(5) *there exists a constant* $\delta_0 > 0$, *such that* $B(x,y) + C(x,y) \leq \delta_0 A(x,y)$, $\forall x, y \in P_h$.

Then the operator of equation (1.1) *has a unique solution* x^* *in* P, *which satisfies* $\mu h \leq x^* \leq \lambda h$, *where* $\lambda > 0$, $\mu > 0$ *are two real numbers*. *And for any initial values* $x_0, y_0 \in P_h$, *by constructing successively the sequences as follows*:

$$x_n = A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}) + C(x_{n-1}, y_{n-1}),$$
$$y_n = A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}) + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Remark 3.4 If P is a solid cone, $h \in \mathring{P}$. If we suppose that the operators $A, B, C : P_h \times P_h \to P_h$ or $A, B, C : \mathring{P} \times \mathring{P} \to \mathring{P}$, then $A(h,h) \in P_h$, $B(h,h) \in P_h$, and $C(h,h) \in P_h$ are automatically satisfied in Corollary 3.4.

Theorem 3.2 *Let* P *be a normal cone in* E. $A, B, C : P \times P \to P$ *are three mixed monotone operators and satisfy the following conditions*:

(1) *for any* $t \in (0,1)$, *there exists* $\varphi(t) \in (t, 1]$ *such that*

$$A\bigl(tx, t^{-1}y\bigr) \geq \varphi(t)A(x,y), \quad \forall x, y \in P;$$

(2) *for any* $t \in (0,1)$, $x, y \in P$, $B(tx, t^{-1}y) \geq tB(x,y)$;
(3) *for any fixed* $y \in P$, $A(\cdot, y) : P \to P$ *is concave*; *for any fixed* $x \in P$, $A(x, \cdot) : P \to P$ *is convex*;
(4) *there is* $h \in P$, $h > \theta$ *such that* $A(h,h) \in P_h$, $B(h,h) \in P_h$, *and* $C(h,h) \in P_h$;
(5) *there exists* $\frac{1}{2} \leq c \leq 1$, *such that* $C(\theta, lh) \geq cC(lh, \theta)$, $l \geq 1$;
(6) *there exists a constant* $\delta_0 > 0$, *such that* $B(x,y) + C(x,y) \leq \delta_0 A(x,y)$, $\forall x, y \in P_h$.

Then for any given $\lambda > 0$, *the operator equation*

$$A(x,x) + B(x,x) + C(x,x) = \lambda x$$

has a unique solution x_λ in P, which satisfies $\mu h \leq x_\lambda \leq \lambda h$, where $\lambda > 0$, $\mu > 0$ are two real numbers. Furthermore, we have the following conclusions:

(R1) *if $\varphi(t) > t^{\frac{1}{2}}(\delta_0 + 1) - \delta_0 t$ for $t \in (0,1)$, then x_λ is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;*

(R2) *if there exists $\beta \in (0,1)$ such that $\varphi(t) \geq t^\beta(\delta_0 + 1) - \delta_0 t$ for $t \in (0,1)$, then x_λ is continuous in λ, that is, $\lambda \to \lambda_0$ ($\lambda_0 > 0$) implies $\|x_\lambda - x_{\lambda_0}\| \to 0$;*

(R3) *if there exists $\beta \in (0,\frac{1}{2})$ such that $\varphi(t) \geq t^\beta(\delta_0 + 1) - \delta_0 t$ for $t \in (0,1)$, then $\lim_{\lambda\to\infty} \|x_\lambda\| = 0$, $\lim_{\lambda\to 0^+} \|x_\lambda\| = \infty$.*

Proof For fixed $\lambda > 0$, by Theorem 3.1, $\frac{1}{\lambda}T : P_h \times P_h \to P_h$ is mixed monotone and satisfies

$$\left(\frac{1}{\lambda}T\right)\left(tx, t^{-1}y\right) = \frac{1}{\lambda}T\left(tx, t^{-1}y\right) \geq \frac{1}{\lambda}\psi(t,x,y)T(x,y) = \psi(t,x,y)\left(\frac{1}{\lambda}T\right)(x,y),$$

for any $x, y \in P_h$, $t \in (0,1)$. So it follows from Theorem 3.1 that $\frac{1}{\lambda}T$ has a unique fixed point x_λ in P_h. That is, $T(x_\lambda, x_\lambda) = \lambda x_\lambda$. For convenience of proof, we let

$$\alpha(t,x,y) = \frac{\ln\psi(t,x,y)}{\ln t}, \quad \forall t \in (0,1).$$

Then $\alpha(t,x,y) \in [0,1)$ and $\psi(t,x,y) = t^{\alpha(t,x,y)}$. Thus $T(tx, t^{-1}y) \geq t^{\alpha(t,x,y)}T(x,y)$, for any $x, y \in P_h$, $t \in (0,1)$.

(1) Proof of (R1). Suppose $0 < \lambda_1 < \lambda_2$ and let

$$t_0 = \sup\{t > 0 \mid x_{\lambda_1} \geq tx_{\lambda_2}, x_{\lambda_2} \geq tx_{\lambda_1}\},$$

then we have $0 < t_0 < 1$ and

$$x_{\lambda_1} \geq t_0 x_{\lambda_2}, \qquad x_{\lambda_2} \geq t_0 x_{\lambda_1}. \tag{3.13}$$

By the mixed monotone properties of T,

$$\begin{aligned} \lambda_1 x_{\lambda_1} &= T(x_{\lambda_1}, x_{\lambda_1}) \geq T\left(t_0 x_{\lambda_2}, t_0^{-1} x_{\lambda_2}\right) \geq t_0^{\alpha(t_0, x_{\lambda_2}, x_{\lambda_2})} T(x_{\lambda_2}, x_{\lambda_2}) = t_0^{\alpha(t_0, x_{\lambda_2}, x_{\lambda_2})} \lambda_2 x_{\lambda_2}, \\ \lambda_2 x_{\lambda_2} &= T(x_{\lambda_2}, x_{\lambda_2}) \geq T\left(t_0 x_{\lambda_1}, t_0^{-1} x_{\lambda_1}\right) \geq t_0^{\alpha(t_0, x_{\lambda_1}, x_{\lambda_1})} T(x_{\lambda_1}, x_{\lambda_1}) = t_0^{\alpha(t_0, x_{\lambda_1}, x_{\lambda_1})} \lambda_1 x_{\lambda_1}. \end{aligned}$$

Further

$$x_{\lambda_1} \geq \lambda_1^{-1}\lambda_2 t_0^{\alpha(t_0, x_{\lambda_2}, x_{\lambda_2})} x_{\lambda_2}, \qquad x_{\lambda_2} \geq \lambda_2^{-1}\lambda_1 t_0^{\alpha(t_0, x_{\lambda_1}, x_{\lambda_1})} x_{\lambda_1}. \tag{3.14}$$

Noting that $\lambda_1^{-1}\lambda_2 t_0^{\alpha(t_0, x_{\lambda_2}, x_{\lambda_2})} > t_0$, from the definition of t_0 and (3.14), we get

$$\lambda_2^{-1}\lambda_1 t_0^{\alpha(t_0, x_{\lambda_1}, x_{\lambda_1})} \leq t_0,$$

which in turn yields

$$t_0 \geq \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0, x_{\lambda_1}, x_{\lambda_1})}}. \tag{3.15}$$

Hence

$$x_{\lambda_1} \geq \lambda_1^{-1}\lambda_2\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha(t_0,x_{\lambda_2},x_{\lambda_2})}{1-\alpha(t_0,x_{\lambda_1},x_{\lambda_1})}} x_{\lambda_2} = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1-2\alpha(t_0,x_{\lambda_1},x_{\lambda_1})}{1-\alpha(t_0,x_{\lambda_2},x_{\lambda_2})}} x_{\lambda_2}. \tag{3.16}$$

Noting that $\varphi(t_0) > t_0^{\frac{1}{2}}(\delta_0+1) - \delta_0 t_0$, we have $\psi(t_0,x,y) > \frac{\varphi(t_0)+\delta_0 t_0}{\delta_0+1} > t_0^{\frac{1}{2}}$, and thus we have $\alpha(t_0,x,y) < \frac{1}{2}$ and consequently,

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1-2\alpha(t_0,x_{\lambda_1},x_{\lambda_1})}{1-\alpha(t_0,x_{\lambda_2},x_{\lambda_2})}} > 1.$$

Thus, $x_{\lambda_1} > x_{\lambda_2}$.

(2) Proof of (R2). Since $\varphi(t) \geq t^{\beta}(\delta_0+1) - \delta_0 t$ for $t \in (0,1)$, we have $\psi(t,x,y) > \frac{\varphi(t_0)+\delta_0 t_0}{\delta_0+1} \geq t^{\beta}$ for $t \in (0,1)$, and thus we get $\alpha(t,x,y) \leq \beta$ for $t \in (0,1)$. By (3.13) and (3.15), we have

$$\begin{aligned}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2} \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0,x_{\lambda_1},x_{\lambda_1})}} x_{\lambda_2} \leq x_{\lambda_1} \\ \leq \frac{1}{t_0}x_{\lambda_2} \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\alpha(t_0,x_{\lambda_1},x_{\lambda_1})}} x_{\lambda_2} \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} x_{\lambda_2},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_1} \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0,x_{\lambda_1},x_{\lambda_1})}} x_{\lambda_1} \leq x_{\lambda_2} \leq \frac{1}{t_0}x_{\lambda_1} \\ \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\alpha(t_0,x_{\lambda_1},x_{\lambda_1})}} x_{\lambda_1} \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} x_{\lambda_1}.\end{aligned} \tag{3.18}$$

Further

$$\theta \leq x_{\lambda_1} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2} \leq \left[\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}}\right] x_{\lambda_2}.$$

Consequently, from the normality of cone P and (3.17), we get

$$\begin{aligned}\|x_{\lambda_1} - x_{\lambda_2}\| &\leq \left\| x_{\lambda_1} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2}\right\| + \left\|\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2} - x_{\lambda_2}\right\| \\ &\leq N\left[\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}}\right]\|x_{\lambda_2}\| + \left|\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} - 1\right| \|x_{\lambda_2}\|,\end{aligned}$$

where N is the normality constant. Let $\lambda_1 \to \lambda_2^-$, we have $\|x_{\lambda_1} - x_{\lambda_2}\| \to 0$. Similarly, let $\lambda_2 \to \lambda_1^+$, from (3.18) we can also prove $\|x_{\lambda_2} - x_{\lambda_1}\| \to 0$. So the conclusion (R2) holds.

(3) Proof of (R3). Since $\varphi(t) \geq t^{\beta}(\delta_0+1) - \delta_0 t$ for $t \in (0,1)$, we have $\psi(t,x,y) > \frac{\varphi(t_0)+\delta_0 t_0}{\delta_0+1} \geq t^{\beta}$ for $t \in (0,1)$, and thus we have $\alpha(t,x,y) \leq \beta < \frac{1}{2}$ for $t \in (0,1)$. Let $\lambda_1 = 1$, $\lambda_2 = \lambda$ in (3.16), then we have

$$x_1 \geq \lambda^{\frac{1-2\alpha(t_0,x_1,x_1)}{1-\alpha(t_0,x_\lambda,x_\lambda)}} x_\lambda \geq \lambda^{\frac{1-2\beta}{1-\beta}} x_\lambda, \quad \forall \lambda > 1.$$

Thus we can easily obtain

$$\|x_\lambda\| \le \frac{N}{\lambda^{\frac{1-2\beta}{1-\beta}}}\|x_1\|, \quad \forall\lambda > 1,$$

where N is the normality constant. Let $\lambda \to \infty$, then $\|x_\lambda\| \to 0$. Similarly, let $\lambda_1 = \lambda$, $\lambda_2 = 1$ in (3.16), then

$$x_\lambda \ge \lambda^{-\frac{1-2\alpha(t_0,x_\lambda,x_\lambda)}{1-\alpha(t_0,x_1,x_1)}} x_1 \ge \lambda^{-\frac{1-2\beta}{1-\beta}} x_1, \quad \forall 0 < \lambda < 1.$$

Thus

$$\|x_\lambda\| \ge N^{-1}\lambda^{-\frac{1-2\beta}{1-\beta}}\|x_1\|, \quad \forall 0 < \lambda < 1,$$

where N is the normality constant. Let $\lambda \to 0^+$, then we have $\|x_\lambda\| \to \infty$. □

4 Applications

Many problems in various areas, such as differential equations, integral equations, boundary value problems and nonlinear matrix equations, can be converted to the operator equation (1.1). We refer the reader to [27–34] and the references therein. In this section, we apply the results in Section 3 to study a class of nonlinear fractional differential equations with integral boundary conditions. We focus on the existence and uniqueness of positive solutions for the following nonlinear fractional differential equations with integral boundary conditions:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t),u(t)) + g(t,u(t),u(t)) + k(t,u(t),u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \int_0^{\eta} u(s)\,ds, \end{cases} \tag{4.1}$$

where $3 < \alpha \le 4$, $0 < \eta < 1$, and D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $\alpha > 0$, defined by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1}u(\tau)\,d\tau.$$

In the following, for the sake of convenience, set $E = C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the norm $\|y\| = \max\{|y(t)| : t \in [0,1]\}$. $P = \{y \in C[0,1] \mid y(t) \ge 0, t \in [0,1]\}$. It is clear that P is a normal cone of which the normality constant is 1. The Green function of problem (4.1) is as follows:

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - \frac{(\eta-s)^{\alpha}}{\alpha}t^{\alpha-1} - (1-\frac{\eta^{\alpha}}{\alpha})(t-s)^{\alpha-1}, \\ \quad 0 \le s \le t \le 1, s \le \eta; \\ t^{\alpha-1}(1-s)^{\alpha-1} - (1-\frac{\eta^{\alpha}}{\alpha})(t-s)^{\alpha-1}, \quad 0 \le \eta \le s \le t \le 1; \\ t^{\alpha-1}(1-s)^{\alpha-1} - \frac{(\eta-s)^{\alpha}}{\alpha}t^{\alpha-1}, \quad 0 \le t \le s \le \eta \le 1; \\ t^{\alpha-1}(1-s)^{\alpha-1}, \quad 0 \le t \le s \le 1, \eta \le s, \end{cases} \tag{4.2}$$

where $p(s) = 1 - \frac{\eta^{\alpha}}{\alpha}(1-s)$. Obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$.

Lemma 4.1 [35] *Let $3 < \alpha \le 4$. Then the Green function $G(t,s)$ defined by* (4.2) *has the following properties*:

(1) $G(t,s) \ge 0$, $(t,s) \in [0,1] \times [0,1]$;
(2) $\frac{\eta^\alpha}{\Gamma(\alpha)(\alpha-\eta^\alpha)} s(1-s)^{\alpha-1}h(t) \le G(t,s) \le \frac{(\alpha-1)(\alpha-\eta^\alpha)+4\eta^{\alpha-1}}{\Gamma(\alpha)(\alpha-\eta^\alpha)}(1-s)^{\alpha-1}h(t)$, $t,s \in [0,1]$, *where* $h(t) = t^{\alpha-1}$.

Theorem 4.2 *Assume that the following conditions* (H_1)-(H_5) *hold*:

(H_1) $f,g,k : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *are continuous, with* $f(t,0,1) \not\equiv 0$, $g(t,0,1) \not\equiv 0$, $k(t,0,1) \not\equiv 0$, $t \in [0,1]$;

(H_2) *for any fixed* $t \in [0,1]$ *and* $v \in [0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$, $k(t,u,v)$ *are increasing in* $u \in [0,+\infty)$; *for any fixed* $t \in [0,1]$ *and* $u \in [0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$, $k(t,u,v)$ *are decreasing in* $v \in [0,+\infty)$;

(H_3) *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u,v \in [0,+\infty)$, $g(t,\lambda u,\lambda^{-1}v) \ge \lambda g(t,u,v)$; *for any* $\lambda \in (0,1)$, $t \in [0,1]$, $u,v \in [0,+\infty)$, *there exists* $\varphi(\lambda) \in (\lambda,1]$ *such that* $f(t,\lambda u,\lambda^{-1}v) \ge \varphi(\lambda)f(t,u,v)$; *and for fixed* $t \in [0,1]$, $v \in [0,+\infty)$, $k(t,\cdot,v)$ *is concave*; *for fixed* $t \in [0,1]$, $u \in [0,+\infty)$, $k(t,u,\cdot)$ *is convex*;

(H_4) *there exists* $\frac{1}{2} \le c \le 1$ *such that* $k(s,\theta,lh(t)) \ge ck(s,lh(t),\theta)$, $l \ge 1$;

(H_5) *there exists a constant* $\delta_0 > 0$, *such that* $g(t,u,v) + k(t,u,v) \le \delta_0 f(t,u,v)$, $\forall t \in [0,1]$, $u,v \in [0,+\infty)$.

Then the problem (4.1) *has a unique positive solution* u^* *in* P, *which satisfies* $\mu t^{\alpha-1} \le u^*(t) \le \lambda t^{\alpha-1}$, *where* $\lambda > 0$, $\mu > 0$ *are two real numbers*, $t \in [0,1]$. *Furthermore for any* $x_0, y_0 \in P_h$, *by constructing successively the sequences*

$$\begin{aligned} x_{n+1}(t) &= \int_0^1 G(t,s)f\bigl(s,x_n(s),y_n(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,x_n(s),y_n(s)\bigr)\,ds \\ &\quad + \int_0^1 G(t,s)k\bigl(s,x_n(s),y_n(s)\bigr)\,ds, \quad n = 0,1,2,\ldots, \\ y_{n+1}(t) &= \int_0^1 G(t,s)f\bigl(s,y_n(s),x_n(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,y_n(s),x_n(s)\bigr)\,ds \\ &\quad + \int_0^1 G(t,s)k\bigl(s,y_n(s),x_n(s)\bigr)\,ds, \quad n = 0,1,2,\ldots, \end{aligned}$$

we have $x_n(t) \rightrightarrows u^*$, $t \in [0,1]$ *and* $y_n(t) \rightrightarrows u^*(t)$, $t \in [0,1]$.

Proof From [29], the problem (4.1) has an integral formulation given by

$$u(t) = \int G(t,s)\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s),u(s)\bigr) + k\bigl(s,u(s),u(s)\bigr)\bigr]\,ds,$$

where G(t,s) is given as in (4.2).

Define three operators $A,B,C : P \times P \to E$ by

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds, \\ B(u,v)(t) &= \int_0^1 G(t,s)g\bigl(s,u(s),v(s)\bigr)\,ds, \end{aligned}$$

$$C(u,v)(t) = \int_0^1 G(t,s)k\bigl(s,u(s),v(s)\bigr)\,ds.$$

It is easy to prove that u is the solution of the problem (4.1) if and only if $u = A(u,u) + B(u,u) + C(u,u)$. From ($H_1$), we know that $A, B, C : P \times P \to P$. Further, it follows from (H_2) that A, B, C are mixed monotone. For any $\lambda \in (0,1)$ and $u, v \in P$, by (H_3) we obtain

$$\begin{aligned} A\bigl(\lambda u, \lambda^{-1}v\bigr) &= \int_0^1 G(t,s)f\bigl(s,\lambda u(s),\lambda^{-1}v(s)\bigr)\,ds \\ &\geq \varphi(\lambda)\int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds \\ &= \varphi(\lambda)A(u,v)(t). \end{aligned}$$

That is, $A(\lambda u, \lambda^{-1}v) \geq \varphi(\lambda)A(u,v)$ for $\lambda \in (0,1)$, $u, v \in P$. So the operator A satisfies (3.1). Also, for any $\lambda \in (0,1)$ and $u, v \in P$, from (H_3) we know that

$$\begin{aligned} B\bigl(\lambda u, \lambda^{-1}v\bigr) &= \int_0^1 G(t,s)g\bigl(s,\lambda u(s),\lambda^{-1}v(s)\bigr)\,ds \\ &\geq \lambda\int_0^1 G(t,s)g\bigl(s,u(s),v(s)\bigr)\,ds \\ &= \lambda B(u,v)(t). \end{aligned}$$

That is, $B(\lambda u, \lambda^{-1}v) \geq \lambda B(u,v)$ for $\lambda \in (0,1)$, $u, v \in P$. So the operator A satisfies (3.2).

Now we prove that for fixed $v \in [0,+\infty)$, $C(t,\cdot,v) : P \to P$ is concave; for fixed $u \in [0,+\infty)$, $C(t,u,\cdot) : P \to P$ is convex. For fixed $t \in (0,1)$, $v \in [0,+\infty)$, for any $a \in (0,1)$, $u_1, u_2 \in P$,

$$\begin{aligned} C\bigl(au_1 + (1-a)u_2, v\bigr) &= \int_0^1 G(t,s)k\bigl(s, au_1(s) + (1-a)u_2(s), v(s)\bigr)\,ds \\ &\geq \int_0^1 G(t,s)\bigl(ak\bigl(s,u_1(s),v(s)\bigr) + (1-a)k\bigl(s,u_2(s),v(s)\bigr)\bigr)\,ds \\ &= a\int_0^1 G(t,s)k\bigl(s,u_1(s),v(s)\bigr)\,ds \\ &\quad + (1-a)\int_0^1 G(t,s)k\bigl(s,u_2(s),v(s)\bigr)\,ds \\ &= aC\bigl(u_1(s),v(s)\bigr) + (1-a)C\bigl(u_2(s),v(s)\bigr), \end{aligned}$$

so for fixed $v \in [0,+\infty)$, $C(t,\cdot,v) : P \to P$ is concave; for fixed $t \in (0,1)$, $u \in [0,+\infty)$, for any $a \in (0,1)$, $v_1, v_2 \in P$,

$$\begin{aligned} C\bigl(u, av_1 + (1-a)v_2\bigr) &= \int_0^1 G(t,s)k\bigl(s, u(s), av_1(s) + (1-a)v_2(s)\bigr)\,ds \\ &\leq \int_0^1 G(t,s)\bigl(ak\bigl(s,u(s),v_1(s)\bigr) + (1-a)k\bigl(s,u(s),v_2(s)\bigr)\bigr)\,ds \\ &= a\int_0^1 G(t,s)k\bigl(s,u(s),v_1(s)\bigr)\,ds \end{aligned}$$

$$+ (1-a)\int_0^1 G(t,s)k\big(s,u(s),v_2(s)\big)\,ds$$
$$= aC\big(u(s),v_1(s)\big) + (1-a)C\big(u(s),v_2(s)\big),$$

so for fixed $u \in [0,+\infty)$, $C(t,u,\cdot): P \to P$ is convex.

Then we show that $A(h,h) \in P_h$, $B(h,h) \in P_h$, and $C(h,h) \in P_h$. In fact, from (4.2) and Lemma 4.1

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s)k\big(s,f(s),f(s)\big)\,ds \\ &\le \frac{(\alpha-1)(\alpha-\eta^\alpha)+4\eta^{\alpha-1}}{\Gamma(\alpha)(\alpha-\eta^\alpha)}h(t)\int_0^1 (1-s)^{\alpha-1}f(s,1,0)\,ds, \\ A(h,h)(t) &= \int_0^1 G(t,s)f\big(s,h(s),h(s)\big)\,ds \ge \frac{\eta^\alpha}{\Gamma(\alpha)(\alpha-\eta^\alpha)}h(t)\int_0^1 s(1-s)^{\alpha-1}f(s,0,1)\,ds. \end{aligned}$$

From (H_2), we have

$$f(s,1,0) \ge f(s,0,1) \not\equiv 0, \quad \forall s \in [0,1],$$

so

$$\int_0^1 f(s,1,0)\,ds \ge \int_0^1 f(s,0,1)\,ds > 0,$$

and consequently $A(h,h) \in P_h$. Similarly,

$$\begin{aligned} &\frac{\eta^\alpha}{\Gamma(\alpha)(\alpha-\eta^\alpha)}h(t)\int_0^1 s(1-s)^{\alpha-1}g(s,0,1)\,ds \\ &\quad\le B\big(h(t),h(t)\big) \le \frac{(\alpha-1)(\alpha-\eta^\alpha)+4\eta^{\alpha-1}}{\Gamma(\alpha)(\alpha-\eta^\alpha)}h(t)\int_0^1 g(s,1,0)(1-s)^{\alpha-1}\,ds, \end{aligned}$$

from $g(t,0,1) \not\equiv 0$, we have $B(h,h) \in P_h$.

$$\begin{aligned} &\frac{\eta^\alpha}{\Gamma(\alpha)(\alpha-\eta^\alpha)}h(t)\int_0^1 s(1-s)^{\alpha-1}k(s,0,1)\,ds \\ &\quad\le C\big(h(t),h(t)\big) \le \frac{(\alpha-1)(\alpha-\eta^\alpha)+4\eta^{\alpha-1}}{\Gamma(\alpha)(\alpha-\eta^\alpha)}h(t)\int_0^1 (1-s)^{\alpha-1}k(s,1,0)\,ds, \end{aligned}$$

from $k(t,0,1) \not\equiv 0$, we have $C(h,h) \in P_h$. Hence the condition (4) of Theorem 3.1 is satisfied.

In the following we show that the condition (6) of Theorem 3.1 is satisfied. From (H_4), there exists $\frac{1}{2} \le c \le 1$ such that

$$\begin{aligned} C\big(\theta,lh(t)\big) &= \int_0^1 G(t,s)k\big(s,\theta,lh(s)\big)\,ds \ge c\int_0^1 G(t,s)k\big(s,lh(s),\theta\big)\,ds \\ &= cC\big(lh(t),\theta\big), \quad l \ge 1. \end{aligned}$$

For $u, v \in P$, from (H_5),

$$B(u,v)(t) + C(u,v)(t) = \int_0^1 G(t,s)\bigl[g\bigl(s,u(s),v(s)\bigr) + k\bigl(s,u(s),v(s)\bigr)\bigr]\,ds$$
$$\leq \delta_0 \int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds = \delta_0 A(u,v)(t).$$

Then we get $B(u,v) + C(u,v) \leq \delta_0 A(u,v)$, $u, v \in P$. So the conclusions of Theorem 4.2 follow from Theorem 3.1. □

Example 4.1 Consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) = t^2 + (u+1)^{\frac{1}{2}} + (u+1)^{\frac{1}{3}} + (v+1)^{-1} + (v+1)^{-\frac{1}{5}} - \frac{1}{4}e^{-u} + \frac{1}{4}e^{-v} + 3, \\ \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \int_0^{\frac{1}{2}} u(s)\,ds. \end{cases} \tag{4.3}$$

Consider the functions $f, g, k : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, defined by

$$f(t,x,y) = t^2 + 1 + (x+1)^{\frac{1}{2}} + (y+1)^{-\frac{1}{5}}, \quad t \in [0,1], x, y \geq 0,$$
$$g(t,x,y) = t^2 + (x+1)^{\frac{1}{3}} + (y+1)^{-1}, \quad t \in [0,1], x, y \geq 0,$$
$$k(t,x,y) = -t^2 + 2 - \frac{1}{4}e^{-x} + \frac{1}{4}e^{-y}, \quad t \in [0,1], x, y \geq 0.$$

Then (4.3) is equivalent to

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) = f(t,u,u) + g(t,u,u) + k(t,u,u), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \int_0^{\frac{1}{2}} u(s)\,ds. \end{cases} \tag{4.4}$$

Let us check that all the required conditions of Theorem 4.2 are satisfied.

(1) Clearly, the functions $f, g, k : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous with $f(t,0,1) \not\equiv 0$, $g(t,0,1) \not\equiv 0$, $k(t,0,1) \not\equiv 0$.

(2) We observe easily that for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, $f(t,x,y)$, $g(t,x,y)$, $k(t,x,y)$ are increasing in $x \in [0,+\infty)$; for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, $f(t,x,y)$, $g(t,x,y)$, $k(t,x,y)$ are decreasing in $y \in [0,+\infty)$.

(3) For all $\lambda \in (0,1)$, $t \in [0,1]$, and $x \geq 0$, $y \geq 0$, taking $\varphi(\lambda) = \lambda^{\frac{1}{2}} \in (\lambda, 1)$, we have

$$\begin{aligned} f\bigl(t,\lambda x,\lambda^{-1}y\bigr) &= t^2 + 1 + (\lambda x + 1)^{\frac{1}{2}} + \bigl(\lambda^{-1}y + 1\bigr)^{-\frac{1}{5}} \\ &\geq t^2 + 1 + (\lambda x + \lambda)^{\frac{1}{2}} + \bigl(\lambda^{-1}y + \lambda^{-1}\bigr)^{-\frac{1}{5}} \\ &= t^2 + 1 + \lambda^{\frac{1}{2}}(x+1)^{\frac{1}{2}} + \lambda^{\frac{1}{5}}(y+1)^{-\frac{1}{5}} \\ &\geq \varphi(\lambda)\bigl[t^2 + 1 + (x+1)^{\frac{1}{2}} + (y+1)^{-\frac{1}{5}}\bigr] \\ &= \varphi(\lambda)f(t,x,y). \end{aligned}$$

For all $\lambda \in (0,1)$, $t \in [0,1]$, and $x \geq 0$, $y \geq 0$, we have

$$\begin{aligned} g\bigl(t,\lambda x,\lambda^{-1}y\bigr) &= t^2 + (\lambda x + 1)^{\frac{1}{2}} + \bigl(\lambda^{-1}y + 1\bigr)^{-1} \\ &\geq t^2 + (\lambda x + \lambda)^{\frac{1}{2}} + \bigl(\lambda^{-1}y + \lambda^{-1}\bigr)^{-1} \\ &\geq \lambda t^2 + \lambda^{\frac{1}{2}}(x+1)^{\frac{1}{2}} + \lambda(y+1)^{-1} \\ &\geq \lambda\bigl[t^2 + (x+1)^{\frac{1}{2}} + (y+1)^{-1}\bigr] \\ &= \lambda g(t,x,y). \end{aligned}$$

It is easy to prove that

$$k''_{uu}(t,x,y) = -\frac{1}{4}e^{-x} < 0, \qquad k''_{vv}(t,x,y) = \frac{1}{4}e^{-y} > 0.$$

So for fixed $t \in (0,1)$, $y \in [0,+\infty)$, $k(t,\cdot,y)$ is concave; for fixed $t \in (0,1)$, $x \in [0,+\infty)$, $k(t,x,\cdot)$ is convex.

(4) For all $s \in (0,1)$, $y \in [0,+\infty)$, taking $c = \frac{8}{15}$, it is easy to prove

$$k(s,\theta,y) = \bigl(1-s^2\bigr) + \left(\frac{3}{4} + \frac{1}{4}e^{-y}\right) \geq c\left[\bigl(1-s^2\bigr) + \left(\frac{5}{4} - \frac{1}{4}e^{-y}\right)\right] = ck(s,y,\theta).$$

(5) Taking $\delta_0 = 3$, then

$$\begin{aligned} g(t,x,y) + k(t,x,y) &= \bigl[t^2 + (x+1)^{\frac{1}{3}} + (y+1)^{-1}\bigr] + \left[-t^2 + 2 - \frac{1}{4}e^{-x} + \frac{1}{4}e^{-y}\right] \\ &= (x+1)^{\frac{1}{3}} + (y+1)^{-\frac{1}{4}} + 2 - \frac{1}{4}e^{-x} + \frac{1}{4}e^{-y} \\ &\leq 3\bigl[t^2 + 1 + (x+1)^{\frac{1}{2}} + (y+1)^{-\frac{1}{5}}\bigr] \\ &= 3f(t,x,y) \\ &= \delta_0 f(t,x,y). \end{aligned}$$

Thus we proved that all the hypotheses of Theorem 4.2 are satisfied. Then we deduce that (4.3) has one and only one positive solution $x^* \in P_h$, where $h(t) = t^{\frac{5}{2}}$, $t \in [0,1]$.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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