A new Z-eigenvalue inclusion theorem for tensors[☆]

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Abstract

A new Z-eigenvalue inclusion theorem for tensors is given and proved to be tighter than those in [G. Wang, G.L. Zhou, L. Caccetta, Z-eigenvalue inclusion theorems for tensors, Discrete and Continuous Dynamical Systems Series B, 22(1) (2017) 187–198]. Based on this set, a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is obtained. Finally, numerical examples are given to show the effectiveness of the proposed bound.

Keywords: Z-eigenvalue; Inclusion theorem; Nonnegative tensors; Spectral radius; Weakly symmetric
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1. Introduction

For a positive integer n, $n \geq 2$, N denotes the set $\{1, 2, \cdots, n\}$. $\mathbb{C}$ ($\mathbb{R}$) denotes the set of all complex (real) numbers. We call $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ a real tensor of order m dimension n, denoted by $\mathbb{R}^{[m,n]}$, if

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{R},$$

where $i_j \in N$ for $j = 1, 2, \cdots, m$. $\mathcal{A}$ is called nonnegative if $a_{i_1 i_2 \cdots i_m} \geq 0$. $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called symmetric [1] if

$$a_{i_1 \cdots i_m} = a_{\pi(i_1 \cdots i_m)}, \ \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called weakly symmetric [2] if the associated homogeneous polynomial

$$\mathcal{A}x^m = \sum_{i_1, \cdots, i_m \in N} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

satisfies $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$. It is shown in [2] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

Given a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x = (x_1, x_2 \cdots, x_n)^T \in \mathbb{C} \backslash \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x \text{ and } x^T x = 1,$$

then λ is called an E-eigenvalue of $\mathcal{A}$ and x an E-eigenvector of $\mathcal{A}$ associated with λ, where $\mathcal{A}x^{m-1}$ is an n dimension vector whose ith component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \cdots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

If λ and x are all real, then λ is called a Z-eigenvalue of $\mathcal{A}$ and x a Z-eigenvector of $\mathcal{A}$ associated with λ; for details, see [1, 3].

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We define the Z-spectrum of $\mathcal{A}$, denoted $\sigma(\mathcal{A})$ to be the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq 0$, then the Z-spectral radius [2] of $\mathcal{A}$, denoted $\varrho(\mathcal{A})$, is defined as

$$\varrho(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Recently, much literature has focused on locating all Z-eigenvalues of tensors and bounding the Z-spectral radius of nonnegative tensors in [4–10]. It is well known that one can use eigenvalue inclusion sets to obtain the lower and upper bounds of the spectral radius of nonnegative tensors; for details, see [4, 11–14]. Therefore, the main aim of this paper is to give a tighter Z-eigenvalue inclusion set for tensors, and use it to obtain a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors.

In 2017, Wang *et al.* [4] established the following Geršgorin-type Z-eigenvalue inclusion theorem for tensors.

Theorem 1.1. [4, Theorem 3.1] *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i\in N} \mathcal{K}_i(\mathcal{A}),$$

where

$$\mathcal{K}_i(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq R_i(\mathcal{A})\},\ \ R_i(\mathcal{A}) = \sum_{i_2,\cdots,i_m\in N} |a_{ii_2\cdots i_m}|.$$

To get tighter Z-eigenvalue inclusion sets than $\mathcal{K}(\mathcal{A})$, Wang *et al.* [4] also gave a Brauer-type Z-eigenvalue inclusion theorem for tensors.

Theorem 1.2. [4, Theorem 3.3] *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) = \bigcup_{i,j\in N, i\neq j} \left(\mathcal{M}_{i,j}(\mathcal{A}) \bigcup \mathcal{H}_{i,j}(\mathcal{A})\right),$$

where

$$\mathcal{M}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : \left[|z| - (R_i(\mathcal{A}) - |a_{ij\cdots j}|\right](|z| - P_j^i(\mathcal{A})) \leq |a_{ij\cdots j}|(R_j(\mathcal{A}) - P_j^i(\mathcal{A}))\right\},$$
$$\mathcal{H}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : |z| < R_i(\mathcal{A}) - |a_{ij\cdots j}|, |z| < P_j^i(\mathcal{A})\right\},$$

and

$$P_j^i(\mathcal{A}) = \sum_{\substack{i_2,\cdots,i_m\in N,\\ i\notin\{i_2,\cdots,i_m\}}} |a_{ji_2\cdots i_m}|.$$

In this paper, we continue this research on the Z-eigenvalue localization problem for tensors and its applications. We give a new Z-eigenvalue inclusion set for tensors and prove that the new set is tighter than those in Theorem 1.1 and Theorem 1.2. As an application of this set, we obtain a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors, which is sharper than existing bounds in some cases.

2. A new Z-eigenvalue inclusion theorem

In this section, we give a new Z-eigenvalue inclusion theorem for tensors, and establish the comparison between this set with those in Theorem 1.1 and Theorem 1.2.

Theorem 2.1. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \bigcup_{i,j\in N, j\neq i} \left(\hat{\Omega}_{i,j}(\mathcal{A}) \bigcup \left(\tilde{\Omega}_{i,j}(\mathcal{A}) \bigcap \mathcal{K}_i(\mathcal{A})\right)\right),$$

where

$$\hat{\Omega}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : |z| < P_i^j(\mathcal{A}), |z| < P_j^i(\mathcal{A})\right\}$$

and

$$\tilde{\Omega}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : \left(|z| - P_i^j(\mathcal{A})\right)\left(|z| - P_j^i(\mathcal{A})\right) \leq \left(R_i(\mathcal{A}) - P_i^j(\mathcal{A})\right)\left(R_j(\mathcal{A}) - P_j^i(\mathcal{A})\right)\right\}.$$

Proof. Let λ be a Z-eigenvalue of $\mathcal{A}$ with corresponding Z-eigenvector $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x, \text{ and } ||x||_2 = 1. \tag{1}$$

Let $|x_t| \geq |x_s| \geq \max\limits_{i\in N, i\neq t,s} |x_i|$. Obviously, $0 < |x_t|^{m-1} \leq |x_t| \leq 1$. From (1), we have

$$\lambda x_t = \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\in\{i_2,\cdots i_m\}}} a_{ti_2\cdots i_m} x_{i_2}\cdots x_{i_m} + \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\notin\{i_2,\cdots i_m\}}} a_{ti_2\cdots i_m} x_{i_2}\cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality gives

$$\begin{aligned} |\lambda||x_t| &\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\in\{i_2,\cdots i_m\}}} |a_{ti_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\notin\{i_2,\cdots i_m\}}} |a_{ti_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\in\{i_2,\cdots i_m\}}} |a_{ti_2\cdots i_m}||x_s| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ s\notin\{i_2,\cdots i_m\}}} |a_{ti_2\cdots i_m}||x_t| \\ &= (R_t(\mathcal{A}) - P_t^s(\mathcal{A}))|x_s| + P_t^s(\mathcal{A})|x_t|, \end{aligned}$$

i.e.,

$$\big(|\lambda| - P_t^s(\mathcal{A})\big)|x_t| \leq (R_t(\mathcal{A}) - P_t^s(\mathcal{A}))|x_s|. \tag{2}$$

If $|x_s| = 0$, then $|\lambda| - P_t^s(\mathcal{A}) \leq 0$ as $|x_t| > 0$. When $|\lambda| \geq P_s^t(\mathcal{A})$, we have

$$\big(|\lambda| - P_t^s(\mathcal{A})\big)\big(|\lambda| - P_s^t(\mathcal{A})\big) \leq 0 \leq (R_t(\mathcal{A}) - P_t^s(\mathcal{A}))\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big),$$

which implies $\lambda \in \tilde{\Omega}_{t,s}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$. When $|\lambda| < P_s^t(\mathcal{A})$, we have $\lambda \in \hat{\Omega}_{t,s}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$.

Otherwise, $|x_s| > 0$. By (1), we can get

$$\begin{aligned} |\lambda||x_s| &\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\in\{i_2,\cdots i_m\}}} |a_{si_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\notin\{i_2,\cdots i_m\}}} |a_{si_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\in\{i_2,\cdots i_m\}}} |a_{si_2\cdots i_m}||x_t|^{m-1} + \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\notin(i_2,\cdots i_m)}} |a_{si_2\cdots i_m}||x_s|^{m-1}, \\ &\leq \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\in\{i_2,\cdots i_m\}}} |a_{si_2\cdots i_m}||x_t| + \sum_{\substack{i_2,\cdots,i_m\in N,\\ t\notin(i_2,\cdots i_m)}} |a_{si_2\cdots i_m}||x_s|, \end{aligned}$$

i.e.,

$$\big(|\lambda| - P_s^t(\mathcal{A})\big)|x_s| \leq (R_s(\mathcal{A}) - P_s^t(\mathcal{A}))|x_t|. \tag{3}$$

By (2), it is not difficult to see $|\lambda| \leq R_t(\mathcal{A})$, that is, $\lambda \in \mathcal{K}_t(\mathcal{A})$. When $|\lambda| \geq P_t^s(\mathcal{A})$ or $|\lambda| \geq P_s^t(\mathcal{A})$ holds, multiplying (2) with (3) and noting that $|x_t||x_s| > 0$, we have

$$\big(|\lambda| - P_t^s(\mathcal{A})\big)\big(|\lambda| - P_s^t(\mathcal{A})\big) \leq \big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big),$$

which implies $\lambda \in \big(\tilde{\Omega}_{t,s}(\mathcal{A}) \bigcap \mathcal{K}_t(\mathcal{A})\big) \subseteq \Omega(\mathcal{A})$.

And when $|\lambda| < P_t^s(\mathcal{A})$ and $|\lambda| < P_s^t(\mathcal{A})$ hold, we have $\lambda \in \hat{\Omega}_{t,s}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$. Hence, the conclusion $\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A})$ follows immediately from what we have proved. ■

Next, a comparison theorem is given for Theorem 1.1, Theorem 1.2 and Theorem 2.1.

Theorem 2.2. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then*

$$\Omega(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

Proof. By Corollary 3.2 in [4], $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ holds. Hence, we only prove $\Omega(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. Let $z \in \Omega(\mathcal{A})$. Then there are $t, s \in N$ and $t \neq s$ such that $z \in \hat{\Omega}_{t,s}(\mathcal{A})$ or $z \in \big(\tilde{\Omega}_{t,s}(\mathcal{A}) \bigcap \mathcal{K}_t(\mathcal{A})\big)$. We divide the proof into two parts.

Case I: If $z \in \hat{\Omega}_{t,s}(\mathcal{A})$, that is, $|z| < P_t^s(\mathcal{A})$ and $|z| < P_s^t(\mathcal{A})$. Then, it is easily to see that

$$|z| < P_t^s(\mathcal{A}) \leq R_t(\mathcal{A}) - |a_{ts\cdots s}|,$$

which implies that $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$, consequently, $\Omega(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

Case II: If $z \notin \hat{\Omega}_{t,s}(\mathcal{A})$, that is,

$$|z| \geq P_s^t(\mathcal{A}) \tag{4}$$

or

$$|z| \geq P_t^s(\mathcal{A}), \tag{5}$$

then $z \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \bigcap \mathcal{K}_t(\mathcal{A})\right)$, i.e.,

$$|z| \leq R_t(\mathcal{A}) \tag{6}$$

and

$$\big(|z| - P_t^s(\mathcal{A})\big)\big(|z| - P_s^t(\mathcal{A})\big) \leq \big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big). \tag{7}$$

(i) Assume $\big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big) = 0$. When (4) holds, we have

$$\begin{aligned}\big[|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\big]\big(|z| - P_s^t(\mathcal{A})\big) &\leq \big(|z| - P_t^s(\mathcal{A})\big)\big(|z| - P_s^t(\mathcal{A})\big)\\ &\leq \big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big)\\ &= 0 \leq |a_{ts\cdots s}|\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big),\end{aligned}$$

which implies that $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

On the other hand, when (5) holds and $|z| < P_s^t(\mathcal{A})$, we have $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ if

$$P_t^s(\mathcal{A}) \leq |z| < R_t(\mathcal{A}) - |a_{ts\cdots s}|,$$

and $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ from

$$\big[|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\big]\big(|z| - P_s^t(\mathcal{A})\big) \leq 0 \leq |a_{ts\cdots s}|\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big)$$

if

$$R_t(\mathcal{A}) - |a_{ts\cdots s}| \leq |z| \leq R_t(\mathcal{A}).$$

(ii) Assume $\big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big) > 0$. Then dividing both sides by $\big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big)$ in (7), we have

$$\frac{|z| - P_t^s(\mathcal{A})}{R_t(\mathcal{A}) - P_t^s(\mathcal{A})}\frac{|z| - P_s^t(\mathcal{A})}{R_s(\mathcal{A}) - P_s^t(\mathcal{A})} \leq 1. \tag{8}$$

Let $a = |z|, b = P_t^s(\mathcal{A}), c = R_t(\mathcal{A}) - |a_{ts\cdots s}| - P_t^s(\mathcal{A})$ and $d = |a_{ts\cdots s}|$. If $|a_{ts\cdots s}| > 0$, by (6) and Lemma 2.2 in [11], we have

$$\frac{|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)}{|a_{ts\cdots s}|} = \frac{a - (b+c)}{d} \leq \frac{a-b}{c+d} = \frac{|z| - P_t^s(\mathcal{A})}{R_t(\mathcal{A}) - P_t^s(\mathcal{A})}. \tag{9}$$

When (4) holds, by (8) and (9), we have

$$\frac{|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)}{|a_{ts\cdots s}|}\frac{|z| - P_s^t(\mathcal{A})}{R_s(\mathcal{A}) - P_s^t(\mathcal{A})} \leq \frac{|z| - P_t^s(\mathcal{A})}{R_t(\mathcal{A}) - P_t^s(\mathcal{A})}\frac{|z| - P_s^t(\mathcal{A})}{R_s(\mathcal{A}) - P_s^t(\mathcal{A})} \leq 1,$$

equivalently,

$$\big[|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\big]\big(|z| - P_s^t(\mathcal{A})\big) \leq |a_{ts\cdots s}|\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big),$$

which implies that $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. On the other hand, when (5) holds and $|z| < P_s^t(\mathcal{A})$, we have $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ if

$$P_t^s(\mathcal{A}) \le |z| < R_t(\mathcal{A}) - |a_{ts\cdots s}|,$$

and $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ from

$$\left[|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\right]\left(|z| - P_s^t(\mathcal{A})\right) \le 0 \le |a_{ts\cdots s}|\left(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\right)$$

if $R_t(\mathcal{A}) - |a_{ts\cdots s}| \le |z| \le R_t(\mathcal{A})$.

If $|a_{ts\cdots s}| = 0$, by $|z| \le R_t(\mathcal{A})$, we have

$$|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|) \le 0 = |a_{ts\cdots s}|. \tag{10}$$

When (4) holds, by (10), we can obtain

$$\left[|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\right]\left(|z| - P_s^t(\mathcal{A})\right) \le 0 = |a_{ts\cdots s}|\left(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\right),$$

which implies that $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. On the other hand, when (5) holds and $|z| < P_s^t(\mathcal{A})$, we easily get $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ if

$$P_t^s(\mathcal{A}) \le |z| < R_t(\mathcal{A}) - |a_{ts\cdots s}|,$$

and $z \in \mathcal{M}_{t,s}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ from

$$\left[|z| - (R_t(\mathcal{A}) - |a_{ts\cdots s}|)\right]\left(|z| - P_s^t(\mathcal{A})\right) \le 0 = |a_{ts\cdots s}|\left(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\right)$$

if

$$R_t(\mathcal{A}) - |a_{ts\cdots s}| \le |z| \le R_t(\mathcal{A}).$$

The conclusion follows from Case I and Case II. ∎

Remark 1. Theorem 2.2 shows that the set $\Omega(\mathcal{A})$ in Theorem 2.1 is tighter than $\mathcal{K}(\mathcal{A})$ in Theorem 1.1 and $\mathcal{M}(\mathcal{A})$ in Theorem 1.2, that is, $\Omega(\mathcal{A})$ can capture all Z-eigenvalues of $\mathcal{A}$ more precisely than $\mathcal{K}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$.

3. A new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors

As an application of the results in Section 2, a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is given.

Theorem 3.1. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then*

$$\varrho(\mathcal{A}) \le \Omega_{max} = \max\left\{\hat{\Omega}_{max}, \tilde{\Omega}_{max}\right\},$$

where

$$\hat{\Omega}_{max} = \max_{i,j\in N, j\neq i} \min\{P_i^j(\mathcal{A}), P_j^i(\mathcal{A})\},$$
$$\tilde{\Omega}_{max} = \max_{i,j\in N, j\neq i} \min\left\{R_i(\mathcal{A}), \Delta_{i,j}(\mathcal{A})\right\},$$

and

$$\Delta_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{P_i^j(\mathcal{A}) + P_j^i(\mathcal{A}) + \sqrt{\left(P_i^j(\mathcal{A}) - P_j^i(\mathcal{A})\right)^2 + 4\left(R_i(\mathcal{A}) - P_i^j(\mathcal{A})\right)\left(R_j(\mathcal{A}) - P_j^i(\mathcal{A})\right)}\right\}.$$

Proof. From Lemma 4.4 in [4], we know that $\varrho(\mathcal{A})$ is the largest Z-eigenvalue of $\mathcal{A}$. By Theorem 2.1, we have

$$\varrho(\mathcal{A}) \in \bigcup_{i,j\in N, j\neq i} \left(\hat{\Omega}_{i,j}(\mathcal{A}) \bigcup \left(\tilde{\Omega}_{i,j}(\mathcal{A}) \bigcap \mathcal{K}_i(\mathcal{A})\right)\right),$$

that is, there are $t, s \in N, t \neq s$ such that $\varrho(\mathcal{A}) \in \hat{\Omega}_{t,s}(\mathcal{A})$ or $\varrho(\mathcal{A}) \in \left(\tilde{\Omega}_{t,s}(\mathcal{A}) \bigcap \mathcal{K}_t(\mathcal{A})\right)$.

If $\varrho(\mathcal{A}) \in \hat{\Omega}_{t,s}(\mathcal{A})$, i.e., $\varrho(\mathcal{A}) < P_t^s(\mathcal{A})$ and $\varrho(\mathcal{A}) < P_s^t(\mathcal{A})$, we have $\varrho(\mathcal{A}) < \min\{P_t^s(\mathcal{A}), P_s^t(\mathcal{A})\}$. Furthermore,

$$\varrho(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} \min\{P_i^j(\mathcal{A}), P_j^i(\mathcal{A})\}. \tag{11}$$

If $\varrho(\mathcal{A}) \in \left(\tilde{\Psi}_{t,s}(\mathcal{A}) \bigcap \mathcal{K}_t(\mathcal{A})\right)$, i.e., $\varrho(\mathcal{A}) \leq R_t(\mathcal{A})$ and

$$\big(\varrho(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(\varrho(\mathcal{A}) - P_s^t(\mathcal{A})\big) \leq \big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big), \tag{12}$$

then solving $\varrho(\mathcal{A})$ in (12) gives

$$\varrho(\mathcal{A}) \leq \frac{1}{2}\left\{P_t^s(\mathcal{A}) + P_s^t(\mathcal{A}) + \sqrt{\big(P_t^s(\mathcal{A}) - P_s^t(\mathcal{A})\big)^2 + 4\big(R_t(\mathcal{A}) - P_t^s(\mathcal{A})\big)\big(R_s(\mathcal{A}) - P_s^t(\mathcal{A})\big)}\right\} = \Delta_{t,s}(\mathcal{A}),$$

and furthrermore

$$\varrho(\mathcal{A}) \leq \min\left\{R_t(\mathcal{A}), \Delta_{t,s}(\mathcal{A})\right\} \leq \max_{i,j\in N, j\neq i} \min\left\{R_i(\mathcal{A}), \Delta_{i,j}(\mathcal{A})\right\}. \tag{13}$$

The conclusion follows from (11) and (13). ■

By Theorem 2.2, Theorem 4.6 and Corollary 4.2 in [4], the following comparison theorem can be derived easily.

Theorem 3.2. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then the upper bound in Theorem 3.1 is sharper than those in Theorem 4.6 of [4] and Corollary 4.5 of [5], that is,*

$$\begin{aligned}
\varrho(\mathcal{A}) &\leq \Omega_{max} \\
&\leq \max_{i,j\in N, i\neq j} \left\{\frac{1}{2}\Big(R_i(\mathcal{A}) - a_{ij\cdots j} + P_j^i(\mathcal{A}) + \Lambda^{\frac{1}{2}}(\mathcal{A})\Big), R_i(\mathcal{A}) - a_{ij\cdots j}, P_j^i(\mathcal{A})\right\} \\
&\leq \max_{i\in N} R_i(\mathcal{A}),
\end{aligned}$$

where

$$\Lambda_{i,j}(\mathcal{A}) = (R_i(\mathcal{A}) - a_{ij\cdots j} - P_j^i(\mathcal{A}))^2 + 4a_{ij\cdots j}(R_j(\mathcal{A}) - P_j^i(\mathcal{A})).$$

Finally, we show that the upper bound in Theorem 3.1 is sharper than those in [4–10] in some cases by the following two examples.

Example 3.1. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor defined by

$$a_{1111} = \frac{1}{2},\ a_{2222} = 3,\ a_{ijkl} = \frac{1}{3} \text{ elsewhere}.$$

By Corollary 4.5 of [5], we have

$$\varrho(\mathcal{A}) \leq 5.3333.$$

By Theorem 2.7 of [10], we have

$$\varrho(\mathcal{A}) \leq 5.2846.$$

By Theorem 3.3 of [6], we have

$$\varrho(\mathcal{A}) \leq 5.1935.$$

By Theorem 4.5, Theorem 4.6 and Theorem 4.7 of [4], we all have

$$\varrho(\mathcal{A}) \leq 5.1822.$$

By Theorem 3.5 of [7] and Theorem 6 of [8], we both have

$$\varrho(\mathcal{A}) \leq 5.1667.$$

By Theorem 2.9 of [9], we have

$$\varrho(\mathcal{A}) \leq 4.5147.$$

By Theorem 3.1, we obtain

$$\varrho(\mathcal{A}) \leq 4.3971.$$

Example 3.2. Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ with entries defined as follows:

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 0 & 3 & 3 \\ 2.5 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}, \ \mathcal{A}(:,:,2) = \begin{pmatrix} 2 & 0.5 & 1 \\ 0 & 2 & 0 \\ 1 & 0.5 & 0 \end{pmatrix}, \ \mathcal{A}(:,:,3) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

It is not difficult to verify that $\mathcal{A}$ is a weakly symmetric nonnegative tensor. By Corollary 4.5 of [5] and Theorem 3.3 of [6], we both have

$$\varrho(\mathcal{A}) \leq 14.5000.$$

By Theorem 3.5 of [7], we have

$$\varrho(\mathcal{A}) \leq 14.2650.$$

By Theorem 4.6 of [4], we have

$$\varrho(\mathcal{A}) \leq 14.2446.$$

By Theorem 4.5 of [4], we have

$$\varrho(\mathcal{A}) \leq 14.1027.$$

By Theorem 6 of [8], we have

$$\varrho(\mathcal{A}) \leq 14.0737.$$

By Theorem 4.7 of [4], we have

$$\varrho(\mathcal{A}) \leq 13.2460.$$

By Theorem 2.9 of [9], we have

$$\varrho(\mathcal{A}) \leq 13.2087.$$

By Theorem 3.1, we obtain

$$\varrho(\mathcal{A}) \leq 11.7268.$$

Remark 2. It is easy to see that in some cases the upper bound in Theorem 3.1 is sharper than those in [4-10] from Example 3.1 and Example 3.2.

4. Conclusions

In this paper, we establish a new Z-eigenvalue localization set $\Omega(\mathcal{A})$ and prove that this set is tighter than those in [4]. As an application, we obtain a new upper bound Ω_{max} for the Z-spectral radius of weakly symmetric nonnegative tensors, and show that this bound is sharper than those in [4–10] in some cases by two numerical examples.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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