On functional equations leading to exact solutions for standing internal waves

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Abstract

The Dirichlet problem for the wave equation is a classical example of a problem which is ill-posed. Nevertheless, it has been used to model internal waves oscillating harmonically in time, in various situations, standing internal waves amongst them. We consider internal waves in two-dimensional domains bounded above by the plane $z = 0$ and below by $z = -d(x)$ for depth functions d. This paper draws attention to the Abel and Schröder functional equations which arise in this problem and use them as a convenient way of organizing analytical solutions. Exact internal wave solutions are constructed for a selected number of simple depth functions d.

Keywords: Internal waves, analytical solutions, Schröder functional equation, Abel functional equation

1. Introduction

Internal gravity waves form the final chapter of a classic book on "Waves in Fluids" [14]. Equation (22) at [14] states that the the upward component of the mass flux, denoted there by q but here by w, satisfies

$$\Delta(\frac{\partial^2 w}{\partial t^2}) = -N(z)^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),$$

where Δ is the 3-dimensional Laplacian, and z is the vertical coordinate. Here $N(z)$ is the Brunt-Väisälä frequency. For 2-dimensional flows, i.e. no y dependence, there is a stream function, and several problems of physical interest involve solutions of the form $w(x, z, t) = \psi(x, z) \exp(i\omega t)$, and when,

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additionally, the Brunt-Väisälä frequency is constant, ψ satisfies the one-dimensional wave equation in the space variables. (See equation (2.1).)

The problem we treat in this paper - standing internal waves - is ill-posed, and, in particular, solutions when they exist are not unique. The same pde but with different boundary conditions describes two-dimensional internal waves generated by an oscillating cylinder in a uniformly stratified fluid and a few comments on such local wave generation are given in our §9. A photograph of the wave pattern for local wave generation is given in Figure 76 on page 314 of [14] and a diagram indicating the beams of internal waves is given in Figure 2 of [10]. The characteristic directions of the pde are very evident. For our standing wave problem, once again the characteristic directions are often evident in the flow fields: see, for example, our Figure 3 and other publications on the subject, including photographs of experiments.

For general plane domains standing waves are treated in [1]: see the sections in [1] starting with that on Sobolev's equation. In this paper we specialise to fluid domains confined by a flat surface $z = 0$ and a bottom boundary $z = -d(x)$ for a given non-negative depth function d. Exact solutions for certain depth functions d are known, e.g. Wunsch's solution for a subcritical wedge [23], Barcilon's solution in a semi-ellipse [2] and a self-similar solution in a specific trapezoid [16], among many others. It is known that analytical solutions to the wave equation (2.1) with Dirichlet boundary conditions can be constructed from functions which satisfy the functional equation

$$f\left(x + \frac{d(x)}{\nu}\right) = f\left(x - \frac{d(x)}{\nu}\right) + Q,$$

for $\nu > 0$ and Q given constants. Of course, when $Q > 0$ the preceding equation can be scaled and if a solves equation (1.1a) below, then Qa will solve the preceding equation. Results concerning the following linear functional equations are central to our study of standing internal waves:

$$a\left(x + \frac{d(x)}{\nu}\right) = a\left(x - \frac{d(x)}{\nu}\right) + 1, \tag{1.1a}$$

$$f\left(x + \frac{d(x)}{\nu}\right) = f\left(x - \frac{d(x)}{\nu}\right), \tag{1.1b}$$

These functional equations have been used for internal wave studies for several decades: see [15] and references therein. The physical interpretation of Q non-zero is a constant mass-flux through the domain and it is considered in [17, 3] in the context of tidal conversion. The zero-flux boundary condition $Q = 0$ as in equation (1.1b) is the physical condition appropriate to standing waves (and blinking modes) and is the main topic of this article. It has been noticed by [18] (their Theorem 2) and [21] that there are reformulations of equation (1.1b) such that one can associate solutions to equation (1.1b) with solutions to equation (1.1a). However, to date, very little use of advantages associated with these reformulations seems to have been made in the construction of analytical internal wave solutions.

For a large class of depth functions d one can invert the arguments in the functional equations (1.1) and formulate them as the functional equations (3.1) presented in §3, which corresponds to a special case of Schröder's functional equation for $Q = 0$ and Abel's functional equation $Q \neq 0$. The Schröder and Abel functional equations are well-studied functional equations [11, 12]. In this article known properties of these functional equations are put into context for the construction of internal waves. A selection of analytical internal wave solutions constructed from solutions to these functional equations is presented. Besides the application to internal waves, there are other wave phenomena described by the same boundary-value problem: we mention some of these at the end of §2.

The structure of this paper is as follows. In §2 we present the partial differential equation boundary-value problem that models the internal waves and in §3 we present the corresponding functional equations. We present in §4 Wunsch's solution for a subcritical wedge, and follow this in §5 with various solutions for standing waves with everywhere subcritical bottom profiles. Our treatment in §6 and in §7 indicates results for bottom profiles that have some supercritical parts. The latter of these two sections, §7, treats a particularly simple solution method appropriate when d is related in a certain way to involutions. We are confident that the methods allow both for further application and further development. The question of what other wave problems lead to similar functional equations, a topic which takes us away from internal waves, is addressed in §8. We return to internal waves in §9 and propose related problems where the functional equation methods might be used.

There is no claim that any new solutions in this paper – or indeed any other solutions from our functional equation approach – can only be obtained by the methods of this paper. Our paper is an exposition of the easier results associated with the functional equations (1.1) and (3.1), and we hope that others will develop the approach. We expect that future developments are most likely to be useful in establishing general qualitative aspects of the solutions. For the present, we wish to remind researchers in the area of the spectacular nonuniqueness of solutions, and the methods of generating more, as given in Theorem 2. This result and some others in this paper are given in [21], albeit without noting the relation to the standard functional equation literature. We expect future developments will treat 'attractor' solutions, as in [15, 16] and will establish results, particularising to domains with $z = 0$ as part of their boundary, using functional-equation and dynamical-systems approaches as in [1]. These matters concern bottom profiles which contain both subcritical and supercritical parts (as defined in §2.1) and situations where for some values of ν the only solution is the zero flow solution (f is constant); then, as exemplified in §6.1 one is required to determine for which values of ν there are nontrivial solutions, and find f then. We have chosen to organize our paper around a selection of exact solutions as, despite the large number of different methods available for solving functional equations, this seems a relatively easy way of introducing the functional equations to researchers familiar with internal waves, and internal waves to researchers familiar with functional equations.

2. Internal wave differential equation

2.1. The boundary-value problem

Let the bottom topography $d(x)$ be a positive function defined on the interval $I = [b_-, b_+] \subset \mathbf{R}$. If $b_\pm$ are finite, then $d(b_\pm) = 0$. Define the simply-connected open domain D in the plane by

$$D = \{(x, z) \in \mathbf{R}^2 \,|\, b_- < x < b_+, -d(x) < z < 0\},$$

with x and z representing the horizontal and vertical coordinates respectively. For a constant Brunt-Väisälä frequency, the streamfunction ψ of small-amplitude internal waves in D is governed by

$$\begin{aligned} &\frac{\partial^2\psi}{\partial x^2} - \nu^2 \frac{\partial^2\psi}{\partial z^2} = 0 \qquad \text{in } D, \\ &\psi(x, 0) = 0 \qquad\qquad \text{for } b_- < x < b_+ \\ &\psi(x, -d(x)) = Q \qquad \text{for } b_- < x < b_+ \end{aligned} \tag{2.1}$$

where $\nu > 0$ and Q are given constants. A derivation of (2.1) can be found in many books on fluid dynamics, e.g. Chapter VI §4 on Sobolev's equation in [1]. See also [15] equations (2.4) and (2.5)-(2.6), the latter specifically for the case $Q = 0$. For Q nonzero, see [17], in particular the paragraph containing his equation (2.1).

The quantity ν can be interpreted as the inclination of the characteristics (internal wave rays or beams) relative to the horizontal. A point x on the bottom of the domain D is called *subcritical* if the bottom topography function d satisfies $|d'(x)| < \nu$, where d' denotes the derivative of d, and *supercritical* if the reverse holds. If all points on the bottom are subcritical (supercritical), then the bottom profile d and the domain D are each refered to as being subcritical (supercritical).

Notice that it is always possible to stretch the z-coordinate such that ν takes the value 1 in the problem with the scaled bottom topography $d(x)/\nu$. In the following, unless ν is explicitly referenced, the parameter $\nu > 0$ is assumed to be 1.

We will consider $Q \neq 0$ when it is appropriate. This happens when all points on the bottom are subcritical (see §5), and in some other instances (see §6.2). For bounded domains D the physical interpretation has (harmonically oscillating) sources and sinks at $(b_\pm, 0)$.

Various comments are appropriate. The standing wave solutions, i.e. those with $Q = 0$, harmonic in time, can be used to solve initial-boundary-value problems for the Sobolev equation. Related problems occur in other applications, for example, in some theoretical physics applications (e.g. [9]), and other moving boundary problems for the wave equation (e.g. [8]).

2.2. Preparing for functional equations; the 'extension of f' to ψ

Assume a solution of the differential equation in (2.1) is represented by

$$\psi(x,z) = f\left(x - \frac{z}{\nu}\right) - f\left(x + \frac{z}{\nu}\right) \qquad \text{for } (x,z) \in D \tag{2.2}$$

for some differentiable real function f. The boundary condition $\psi(x, d(x)) = Q$ is satisfied if f satisfies the functional equation given in §1. Note that $\psi(x, 0) = 0$ is already satisfied by the definition (2.2).

With ψ defined from (2.2), ψ will inherit smoothness properties from f. Piecewise linear functions f will produce piecewise linear ψ.

We have used the term 'extends' merely to indicate the following. Given a function f defined on an interval (c_-, c_+) one can view equation (2.2) as extending the one-dimensional domain (c_-, c_+) to a domain in the plane. (Strictly speaking f itself might better be thought of as extending to the hyperbolic conjugate of ψ [15] as this is such that its restriction to $z = 0$ is, except for a factor of 2, the function f.) This extension defines the function

ψ in the triangle in $z \le 0$ with its other sides the characteristics through $(c_\pm, 0)$, namely the lines $z = c_\pm \mp \nu x$. When d is everywhere subcritical, we can take $c_\pm = b_\pm$ and, when both b_+ and b_- are bounded, the triangle so formed contains the whole of the domain D. The extension via (2.2) might well lead to a ψ defined over a larger set than the domain D. In the case $Q = 0$, the curve $z = -d(x)$ is then a nodal curve of ψ defined over the larger set.

Suppose now that $b_- = -b_+$. When f is an even function the corresponding ψ is odd in x. When f is an odd function the corresponding ψ is even in x.

3. Functional equations

The functional equations in this paper are all linear; the $Q = 0$ case being homogeneous. Some properties hold for any Q zero or nonzero. If one has a solution f then $f + c$ is also a solution for any constant c. Suppose f_0 and f_1 are solutions at the same Q. The minimum of f_0 and f_1 is also a solution. The convex combination $(1-t)f_0 + tf_1$ is also a solution. Consequences of these are used without further comment in this paper.

Equations (1.1) can sometimes be transformed to the much more widely studied pair of equations (3.1) and this section is a review of these. The results are applied in §4, §5 and §7. Some results for equations (3.1) extend, in obvious ways, to equations (1.1), and it is appropriate to use equations (1.1) in parts of §sec:partSuper.

3.1. The forward map T

Define the functions $\delta_\pm := x \pm d(x)/\nu$. If the δ_- in equations (1.1) is invertible, then one can (provided the domain of δ_+ includes the image of δ_-^{-1}) define the map $T_+ := \delta_+ \circ \delta_-^{-1}$ and rewrite the functional equations (1.1) as the functional equation

$$f(T_+(x)) = f(x) + Q.$$

In the same way, when appropriate conditions are satisfied, defining the map $T_- := \delta_- \circ \delta_+^{-1}$, one is led to the functional equation $f(x) = f(T_-(x)) + Q$. Let $d(b_\pm) = 0$ for the remainder of this section, so that $\delta_\pm(b_\pm) = b_\pm$. The domains of both δ_- and δ_+ are the same as the domain of d namely $[b_-, b_+]$, It remains to specify the domains of T_+, T_- and of f. It is simplest to consider a subcritical bottom d. Then (i) both δ_- and δ_+ are monotonic increasing so invertible, (ii) the maps $T_\pm$ are bijective on $[b_-, b_+]$ – in fact increasing on (b_-, b_+) with $T_\pm(b_\pm) = b_\pm$. To simplify notation, where this is appropriate, we omit the subscript $+$, and the equations we study are

$$a(T(x)) = a(x) + 1, \tag{3.1a}$$

$$f(T(x)) = f(x). \tag{3.1b}$$

For more on the case of subcritical bottoms, see the beginning of §5. Partly or entirely supercritical domains are more complicated: see §6.

There are geometric and physical relations between the functions d and T. A rightwards ray starting from $(x, 0)$ reflects from a subcritical bottom d and is next incident at the top at $(T(x), 0)$. (For partly supercritical bottoms, we view $(T(x), 0)$ as the point where the reflected ray – possibly prolonged through the bottom profile – meets $z = 0$, possibly with $T(x) > b_+$.) The reflection at the bottom takes place halfway between x and $T(x)$ along the x-coordinate and at the depth $-\nu\frac{T(x)-x}{2}$, so

$$d\left(\frac{x + T(x)}{2}\right) = \nu\frac{T(x) - x}{2}. \tag{3.2}$$

From this, with

$$X = \frac{x + T(x)}{2}, \qquad T(X - \frac{d(X)}{\nu}) = X + \frac{d(X)}{\nu}.$$

Provided the range of T is a subset of the domain of T, repeated composition – iterates of T– can be defined. When T is (strictly) increasing, with $T(b_+) = b_+$, repeated compositions of the map T applied to any $x \in (b_-, b_+)$ give a sequence $\{T^{[k]}(x)\}_{k\in\mathbf{N}}$ which converges to the fixed point $T(b_+) = b_+$ for $k \to \infty$. Similarly, when $T(b_-) = b_-$, one gets a sequence $\{T^{[-k]}(x)\}_{k\in\mathbf{N}}$ converging to $T(b_-) = b_-$ for repeated compositions of the inverse map $T^{[-1]}$ to any $x \in (b_-, b_+)$.

3.2. Schröder functional equation (3.1b)

Equation (3.1b) is a special case of the Schröder functional equation

$$f(T(x)) = s \cdot f(x)$$

for $s = 1$ [11, 12]. This subsection presents a few properties of solutions to (3.1b). A comprehensive list of known properties of Schröder functional equation - sometimes also referred to as Schröder-Konig's functional equation - can be found in Chapter VI of [11] and at various parts of [12].

One comment on the case $s > 0$ is appropriate (and will be used in § 5.2: see equation (5.6)). The following old result is standard: see, for example, [11] p163, [12] p128.

Theorem 1. *If f is a positive solution of the Schröder functional equation $f(T(y)) = s \cdot f(y)$ for $s > 0$, $s \neq 1$, then $a(x) = \log(f(x))/\log(s)$ is a solution of the Abel equation* FET(1)*.*

Some properties of solutions of (3.1b) are easy to see. If T is not the identity function $T(x) = x$ (or equivalently if d is not the zero function), no solution of (3.1b) (or of equation (1.1b) can be monotonic. Hence any solution must have a local maximum or minimum in (b_-, b_+). The solutions we present for f have various numbers of maxima and minima – sometimes finitely many, e.g. §6.1, sometimes countably infinitely many, e.g. the domains treated in §5.

Theorem 2. *If $f : I \to f(I) \subset \mathbf{R}$ is a solution to* FET(0) *and F is any real function whose domain contains the image $f(I)$ of f, then the composition $F \circ f$ is also a solution to* FET(0)*.*

Proof. If f is a solution of (3.1b), then $f(x) = f(T(x))$. F works on the image $f(I)$ of f, so it follows directly that $F(f(x)) = F(f(T(x))$. This shows that the composition $F \circ f$ also satisfies (3.1b) and completes the proof.

The nodal curves for ψ_f associated with f according to (2.2) remain nodal curves for $\psi_{F\circ f}$ associated with $F \circ f$. There may be more nodal curves for $\psi_{F\circ f}$ unless F is invertible.

So if a solution to Schröder's functional equation (3.1b) exists, then it is not unique - and one can be more constructive on this point: one is free to choose a function on some subset I_0 of the interval I on which (3.1b) must hold. This subset I_0 is refered to as a *fundamental interval* [15]. Once a choice for a solution f on some fundamental interval I_0 is made, then f is uniquely defined on all of I. Notice that a solution f to (3.1b) takes the same value for each element of the set $\{T^{[k]}(x)\}_{k\in\mathbf{Z}}$ for each $x \in (b_-, b_+)$. So if $f(x)$ is prescribed for one $x \in \{T^{[k]}(x)\}_{k\in\mathbf{Z}}$, then so it is for the entire set $\{T^{[k]}(x)\}_{k\in\mathbf{Z}}$. Together with the property $T(x) > x$ it shows that $I_0 = [x_0, T(x_0))$ is a fundamental interval for any $x_0 \in (b_-, b_+)$. Such a connected fundamental interval (with $x_0 = 0$) is considered at the beginning of §5. Be aware that it is not necessary for a fundamental interval I_0 to be a connected.

The solvability of Schröder functional equations (3.1b) depends crucially on the property $T^{[k]}(x) \neq x$ for all x in the open interval on which (3.1b) holds and for every positive $k \in \mathbf{N}$ [11, 12]. The following (easily proved) theorem deals with the consequences of fixed points of the map T on the solvability of (3.1b).

Theorem 3. *Let T be a strictly increasing continuous function on (b_-, b_+) for which $T(b_\pm) = b_\pm$. Suppose also that $T^{[k]}(x) \to b_\pm$ as $k \to \pm\infty$ for $b_- < x < b_+$. Then the only solutions of* (3.1b) *which are continuous on the closed interval $[b_-, b_+]$ are the constant solutions.*

3.3. Abel functional equation (3.1a)

Abel's functional equation (3.1a) is appropriate for problems with $Q \neq 0$. In some theoretical physics papers, e.g. [9], it is called Moore's equation. The physical interpretation of $Q \neq 0$ is a constant non-zero flux Q through the bottom $z = -d(x)$. Mathematically one can treat Q as a non-zero constant and associate it with the no-flux condition $Q = 0$ of Schröder's functional equations (3.1b), as motivated in the following observation.

Any solution f to the Schröder's functional equation (3.1b) has to be identical on the endpoints x_0 and $T(x_0)$ of a connected fundamental interval $I_0 = [x_0, T(x_0))$. This is the motivation to consider any solution f to (3.1b) to be a composition of a periodic function P with an argument function a. The function $f(x) = P(a(x))$ with P having period $Q > 0$ then satisfies the (3.1b) if and only if the argument function a satisfies one of the functional equations

$$a(T(x)) = a(x) + Q \cdot n \qquad \text{for n} \in \mathbf{Z}. \tag{3.3}$$

It is always possible to scale $a(x)$ such that $Q = 1$.

The fundamental interval introduced in the previous subsection applies in the same way to Abel's functional equation, e.g. if a solution exists, then it is uniquely determined if and only if it is prescribed on a fundamental interval. (See the beginning of §5 for an existence result.)

Theorem 4. *Let $a \in C^1$ be a strictly increasing solution of FET(1).*
(1) The general solution a_{gen} of FET(1) is given by

$$a_{\text{gen}}(x) = a(x) + P(a(x))$$

where P is a periodic function with period 1.
(2) If a^* is another strictly increasing C^1 solution of FET(1) then there exists some periodic function P with period 1 such that $P'(x) > -1$ for all x and

$$a^*(x) = a(x) + P(a(x)). \tag{3.4}$$

Conversely any a^* of the form (3.4) is an invertible solution of FET(1).

Part (1) is Theorem 1 of [20]. Part (2) is from [22] who attributes it to Abel (1881). Part (2), with its condition $P'(x) > -1$ is developed for C^k solutions in Theorem 2 of [20], with further development in his Theorem 3.

Theorem 5. *Let a and f be C^1 solutions to respectively* FET(1) *and* (3.1b) *on I. Assume further that a is injective and $T : I \to I$ bijective. Then there exists some periodic function P, with period 1, such that $f(x) = P(a(x))$.*

A direct consequence of Theorem 5 is that for subcritical bottom topographies all continuous solutions to (3.1b) are constructed by applying the set of all continuous periodic functions with period 1 to any continuous injective solution to (3.1a).

Theorem 6. *Given a strictly increasing continuous map T on (b_-, b_+) with $T(b_\pm) = b_\pm$, some fundamental interval $I_0 = [x_0, T(x_0))$ and a strictly increasing continuous function a_0 on I_0, then the unique continuous solution a to (3.1a) with $a = a_0$ on I_0 and $a_0(T(x_0)) - a_0(x_0) = 1$ satisfies*

$$a(x) = a_0(T^{[-k]}(x)) + k \tag{3.5}$$

for all $x \in I_k := [T^{[k-1]}(x_0), T^{[k]}(x_0))$ and $k \in \mathbf{Z}$.

This theorem is a special case of Theorem 4.1 in [13], which proves that $a(x) = a_0(T^{[-k]}(x)) + k$ for $x \in I_k$ if a is continuous solution satisfying (3.1a). In [13] the function a_0 satisfying $a_0(T(x_0)) - a_0(x_0) = 1$ is assumed to be linear, which is in fact not necessary for the proof.
The solution $a(x)$ to (3.1a) is clearly continous in all points x in the interior of some interval I_k. For the boundary points $x_k := T^{[k]}(x_0)$ study the limits $x \to x_k$ for $x > x_k$ and $x < x_k$: If $x > x_k$, $x \in I_k = [x_k, x_{k+1})$, then

$$\lim_{x \to x_k} a(x) = a_0(T^{[-k]}(x_k)) + kQ = a_0(x_0) + kQ.$$

For $x < x_k$, $x \in I_{k-1} = [x_{k-1}, x_k)$ it follows that

$$\lim_{x \to x_k} a(x) = a_0(T^{[-k+1)]}(x_k)) + (k-1)Q = a_0(T(x_0)) + (k-1)Q.$$

These two expressions are equal because $a(T(x_0)) = a(x_0) + Q$ by the definition of Q.

To prove uniqueness observe that for every $x \in (b_-, b_+)$ there exists a unique $k \in \mathbf{Z}$ such that $x \in I_k$ because $\bigcup_{k=-\infty}^{\infty} I_k = [b_-, b_+]$ and all I_k are disjunct. So for every $x \in (b_-, b_+)$ the function $a(x)$ is uniquely defined by the expression (3.5) since T, a_0 and Q are given.

3.4. Comments on equations (1.1)

Some of the results of §3.2 and of §3.3 have analogues for the equations (1.1b) and (1.1a) respectively. In particular, we remark that if P is a periodic function with period 1 and a solves (1.1a), then the composition $P \circ a$ solves the Schröder-like equation (1.1b).

4. Wunsch's solution: subcritical wedge

Let $b_- = -\infty$, $b_+ \in \mathbf{R}$ and $\nu = 1$. For a subcritical wedge $d(x) = \tau(b_+ - x)$ with $\tau \in (0, \nu)$ the map T is the linear function $T(x) = px + s$ where $p = \frac{1-\tau}{1+\tau}$ and $s = b_+ \frac{2\tau}{1+\tau}$. The Schröder functional equation (3.1b)

$$f(px + s) = f(x) \qquad \text{for } x < b_+ \tag{4.1}$$

can be formulated as the Abel's functional equation FET(1) under the assumption $f = P \circ a$ with P any period-1 function:

$$a(px + s) = a(x) + 1 \qquad \text{for } x < b_+. \tag{4.2}$$

A continuous, strictly increasing solution to (4.2) is $a(x) = \log(-x + b_+)/\log(p)$. So the Schröder functional equation (4.1) is solved by functions

$$f(x) = P\left(\frac{\log(-x + b_+)}{\log(p)}\right)$$

for any arbitrary continuous period-1 function P.

The solution given by [23] had P as a sine or cosine function. The nodal curves which intersect $z = 0$ in these solutions are hyperbolae. Of course there are many other periodic functions. For certain piecewise exponential P all the nodal curves are straight lines: for appropriate P some nodal lines are vertical straight lines. This makes a connection with this section and §5.1.

5. Symmetric domains with subcritical bottom profiles

Our treatment of the functional equations in §3 deliberately avoided general existence matters as these can be rather intricate, except in the context of subcritical bottoms. The existence result in the next paragraph is stated as it provides a lead-in to §5.1.

In the existence result below we have a genuine interval as a fundamental interval. (That this is not always the case is mentioned in §3.2.) For a symmetric domain, take as the domain of x the interval $[b_-, b_+] = [-b, b]$ for some $b > 0$. The following is stated in [20] (giving references for the proof, including [11]).

Theorem 7. *If T is a continuous strictly increasing real-valued function defined on a half- open interval $[0, b)$, $0 < b \leq \infty$,*
$T([0, b)) = [c, b)$ with $c > 0$, (so we can extend, by continuity, the domain of T so $T(b) = b$) and
$T(x) > x$ for $0 \leq x < b$
then there exists a solution for FET(1). *Furthermore under the above conditions, there is a unique solution a with prescribed values on the interval $[0, T(0))$. If, moreover, it is continuous on $[0, T(0))$ and (taking the limit from above)*

$$\lim_{x \to T(0)} a(x) = a(0) + 1$$

then a is continuous on $[0, b)$.

All the conditions on T above are satisfied by the forward maps T of symmetric domains with subcritical bottom profiles. (A hydrodynamic interpretation is that, for a given bottom profile d, there is a solution for all ν satisfying $\nu > \max(|d'(x)|)$.)

Any such solution a necessarily tends to minus infinity as x tends to b_-, and to plus infinity as x tends to b_+. (If a were to be continuous on the closed interval $[b_-, b_+]$ the solutions of the Schröder equation generated from it could also be continuous, contradicting Theorem 3.)

In the context of the symmetric domains and $Q \neq 0$ our main interest is in odd solutions a.

5.1. Subcritical isosceles triangle

In this section we construct all possible solutions to (3.1b) for the isosceles triangle with bottom topography function $d(x) = \tau(1 - |x|)$ with $\tau \in (0, 1)$ for $x \in (b_-, b_+) = (-1, 1)$ and $\nu = 1$. To the best of our knowledge this is the first exact description of all possible solutions for isosceles triangle. According to Theorem 5 one can construct all solutions f to (3.1b) via the relation $f = P \circ a$ with P all periodic functions with period Q (=length of connected fundamental interval I_0 when, as here, $\nu = 1$) and a a continuous, strictly increasing solution to Abel's functional equation (3.1a). The goal is therefore to construct one solution to (3.1a) for some $Q \neq 0$ using the expression (3.5). The map $T = \delta_+ \circ \delta_-^{-1}$ and its inverse $T^{[-1]}$ associated with $\delta_\pm = x \pm d(x)$ are given by

$$\begin{aligned}
T(x) &= p^{-1}x + s_- && \text{for } -1 \leq \mathrm{x} \leq -\tau \\
T(x) &= px + s_+ && \text{for } -\tau \leq \mathrm{x} \leq +1 \\
T^{[-1]}(x) &= px - s_+ && \text{for } -1 \leq \mathrm{x} \leq +\tau \\
T^{[-1]}(x) &= p^{-1}x - s_- && \text{for } +\tau \leq \mathrm{x} \leq +1
\end{aligned} \tag{5.1}$$

where $p = \frac{1-\tau}{1+\tau} < 1$, $s_+ = \frac{2\tau}{1+\tau}$ and $s_- = \frac{2\tau}{1-\tau}$. A fundamental interval is given by $I_0 = [-\tau, \tau)$, as can be verified by checking that $T(-\tau) = \tau$. Repeated compositions of function T or its inverse $T^{[-1]}$ map this fundamental interval I_0 onto the intervals $I_k := T^{[k]}(I_0)$, $k \in \mathbf{Z}$. So for $x \in I_k$ and $k \leq -1$ a solution $a(x)$ to the Abel equation $FET(Q)$ is given by $a(x) = a_0(T^{[k]}(x)) - kQ$ where a_0 is an arbitrary strictly increasing choice for a on I_0 which satisfies $a_0(\tau) - a_0(-\tau) = Q$. Similarly for $k \geq 1$ and $x \in I_k$ one gets $a(x) = a_0(T^{[-k]}(x)) + kQ$.

Compositions of the maps T, and $T^{[-1]}$, give respectively

$$\begin{aligned}
T^{[k]}(x) &= 1 + p^{-k}(x - 1) && \text{for } -\tau < x \\
T^{[-k]}(x) &= -1 + p^{-k}(x + 1) && \text{for } x < +\tau.
\end{aligned} \tag{5.2}$$

For the simple choice $a_0(x) = x$ on the fundamental interval I_0, which implies $Q = a_0(\tau) - a_0(-\tau) = 2\tau$, the continuous solution a is given by

$$\begin{aligned}
a(x) &= p^{-n}(x - 1) + 1 + 2\tau n && \text{for } \mathrm{x} \in \mathrm{I_n}, \mathrm{n} \in \mathbf{N} \\
a(x) &= p^{-n}(x + 1) - 1 - 2\tau n && \text{for } \mathrm{x} \in \mathrm{I_{-n}}, \mathrm{n} \in \mathbf{N}.
\end{aligned} \tag{5.3}$$

In Figure 1 a continuously differentiable streamfunction solution $\Psi(x, z) = f(x - z) - f(x + z)$ for the choice $P(x) = \cos(\frac{\pi}{\tau}x)$ is presented. The black line shows the bottom $d(x) = \tau(|x| - 1)$. There are many nodal curves. The plotted solution is also a solution for many bottom topographies, including

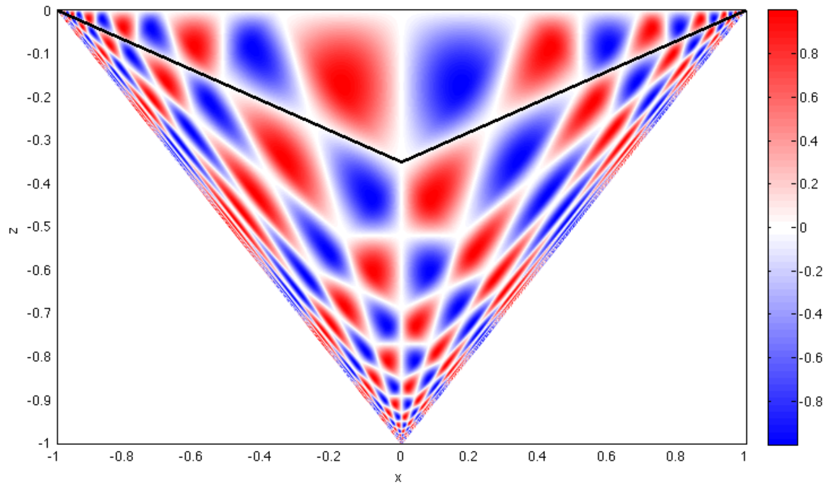

Figure 1: This figure shows the analytical streamfunction solution for $\tau = 0.35$ with $P(x) = \cos(\frac{\pi}{\tau}x)$. The bottom of the isosceles triangle is indicated by the black line. All streamfunction values $z < |x| - 1$ are set to zero.

partly and entirely supercritical bottom topographies. It is speculated that some of these nodal curves are independent of the choice of the periodic function P, e.g. streamfunction solutions to the bottom topographies along these isoclines can be constructed from $f = P \circ a$ for arbitrary period-2τ function P and a satisfying (5.3).

5.2. Subcritical symmetric hyperbolae

5.2.1. Symmetric hyperbolic lens

Again, set $\nu = 1$. For the subcritical bottom topography

$$d(x) = c - \sqrt{c^2 - 1 + x^2} \qquad \text{for } -1 < x < 1 \qquad \text{with } c > 1 \tag{5.4}$$

the corresponding map T is given by

$$T(x) = \frac{1 + cx}{c + x} = x + \frac{1 - x^2}{c + x} \qquad \text{for } -1 < x < 1. \tag{5.5}$$

The map T is fractional linear. Defining another fractional linear map r and motivated by the fact that compositions of fractional linear maps are fractional linear,

$$r(x) = \frac{1 + x}{1 - x} \quad \text{gives} \quad r(T(x)) = r\left(\frac{1}{c}\right) r(x).$$

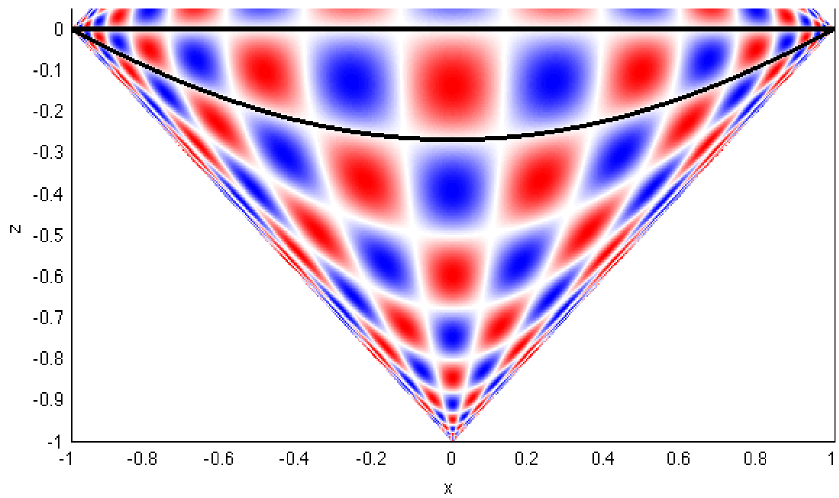

Figure 2: The streamfunction soltuion $\Psi(x,z) = f(x-z) + f(x+z)$ is plotted with f being the composition of $P(x) = \sin(\frac{2\pi}{\text{arctanh}(1/c)}x)$ for $c = 2$ and $a(x) =$arctanh(x) (which solves FET$(a(1/c))$). The color bar is as in Figure 1.

(The function r satisfies a Schröder functional equation with $s = r(1/c)$ positive.) Take logarithms of $r(x)$ and notice that $a(x) = \frac{1}{2}\log(r(x)) =$arctanh$(x)$ satisfies

$$a(T(x)) = a(x) + a(\frac{1}{c}). \tag{5.6}$$

This solution has been suggested by [21]. The solution $a(x) =$arctanh(x) is injective on the fundamental interval $I_0 = [0, \frac{1}{c})$ because $\frac{1}{c} < 1$. So according to Theorem 5 all solutions f to (3.1b) can be derived by applying arbitrary periodic function P with period $a(\frac{1}{c}) = \frac{1}{2}\log\left(\frac{1+c}{-1+c}\right)$ to $a(x)$: $f(x) = P(\text{arctanh}(x))$. The streamfunction solution for a sinusoidal choice for P is shown in Figure 2.

There are infinitely many nodal curves intersecting $z = 0$ at points in $-1 < x < 1$. Modes with different numbers of cells stacked vertically are easily constructed.

5.3. Some other subcritical bottom profiles

The entries in the table indicate some other subcritical bottom profiles for which we have solutions (with $\nu = 1$). The column headed a gives solutions of the Abel functional equation for the given T (from which one can generate all standing-wave solutions). A banal comment – useful when both a and its inverse a^{-1} have simple forms – is the simple formula for T given a solving (3.1a):

With $Q = 1$ in $T(x,Q) = a^{-1}\left(a(x) + Q\right)$, $T^{[k]}(x,Q) = a^{-1}\left(a(x) + kQ\right)$:

$[b_-, b_+]$	T	a	Comments
$[0, 1/2]$	$2x(1-x)$	$\frac{\log\left(\frac{\log(1-2x)}{\log(1-2c)}\right)}{\log(2)}$	Unsymmetrical parabolic segment
$(-\infty, \infty)$	See below	$\operatorname{arcsinh}(x)$	Symmetric hyperbolic hump
See below	$\frac{x}{1+x}$	$\frac{1}{x}$	Source where a hyperbolic slope intersects $z = 0$

• For the symmetric hyperbolic hump, for an appropriate value of τ with $0 < \tau < 1$,

$$T_\tau(x) = \frac{(1+\tau^2)x + 2t\sqrt{1+x^2}}{1-\tau^2}, \qquad d_\tau(x) = \tau\sqrt{\frac{1}{1-\tau^2} + x^2}.$$

• The entry in the table corresponding to $a(x) = 1/x$ can be viewed as a singular flow corresponding to a dipole located at the origin. (The domain of a is no longer an interval.) All streamlines are hyperbolas passing through the origin and located in the wedge shapes containing $z = 0$ and bounded by characteristics through the origin.

There are many other solutions in the literature e.g. in [6, 9]. A symmetrically placed fully submerged subcritical (isosceles) wedge will yield to the methods of §5.1.

6. Some domains where part or all of the bottom is supercritical

Here we are concerned with solutions of equation (1.1b)

$$f(x + \frac{d(x)}{\nu}) = f(x - \frac{d(x)}{\nu}))$$

where the function f may need to be defined on a larger interval than is the function d. $[b_-, b_+] \times \{-1, +1\}$: I.e. we are treating the case $Q = 0$. However in §6.2, we solve (1.1a) with $Q > 0$ as part of the method of solving (1.1b). In this section we use the FEd formulations and in §7 the FET version. When the domain of f is larger than that of d it restricts us to functions which extend to a ψ with a domain larger than D and vanishing on $z = 0$ over more than that part which is on the boundary of D: we might find just some of the solutions of the differential equation problem (2.1). By treating the problem in the form (1.1a) rather than (3.1a) we avoid some of the difficulties associated with the lack of invertibility of one or other of δ_+ or δ_-.

There are other methods of solving the problem, some of which are mentioned at the end of this section.

6.1. Barcilon's solutions for the semi-ellipse

Let the bottom topography be a semi-ellipse: $d(x) = \sqrt{1-x^2}$ for $x \in (-1,1)$. The functional equation (1.1b) then becomes

$$f(x - \frac{\sqrt{1-x^2}}{\nu}) - f(x + \frac{\sqrt{1-x^2}}{\nu}) = 0.$$

With this restriction the preceding functional equation can be re-written

$$f(\cos(\theta) - \sin(\theta)/\nu) - f(\cos(\theta) + \sin(\theta)/\nu) = 0 \tag{6.1}$$

A family of solutions, involving Chebyshev polynomials is given in [2]. These solutions have been rediscovered several times, e.g. [15].

6.1.1. Reduction to a constant coefficient functional equation

We now indicate one method to solve the functional equation (6.1), and find, amongst others, the Chebyshev function solutions. We begin with seeking solutions to

$$f_+ = f(\cos(\theta) - \sin(\theta)/\nu) = f(\cos(\theta) + \sin(\theta)/\nu) = f_-.$$

Next define $\cos(\theta_\nu) = \nu/\sqrt{1+\nu^2}$. Define also $\tilde{f}(\tilde{\theta}) = f(\sqrt{1+\nu^2}\cos(\tilde{\theta})/\nu)$. The functional equation in terms of $\tilde{f}$ is:

$$\tilde{f}(\theta + \theta_\nu) = \tilde{f}(\theta - \theta_\nu)$$

or, equivalently

$$\tilde{f}(\theta) = \tilde{f}(\theta + 2\theta_\nu).$$

This is solved, for $\tilde{f}$, by *any* periodic function P with period $2\theta_\nu$. However restrictions on ν may be required to ensure that the extension of f to ψ leads to a physically acceptable ψ. Barcilon's Chebyshev solutions, with integer m and k, result from

$$\tilde{f}(\tilde{\theta}) = \cos(\frac{m\pi\tilde{\theta}}{\theta_\nu}) \qquad \text{with} \qquad \theta_\nu = \frac{m\pi}{k}.$$

Returning to the general $2\theta_\nu$-periodic $\tilde{f}$, having found $\tilde{f}$ we can determine f as follows. Set

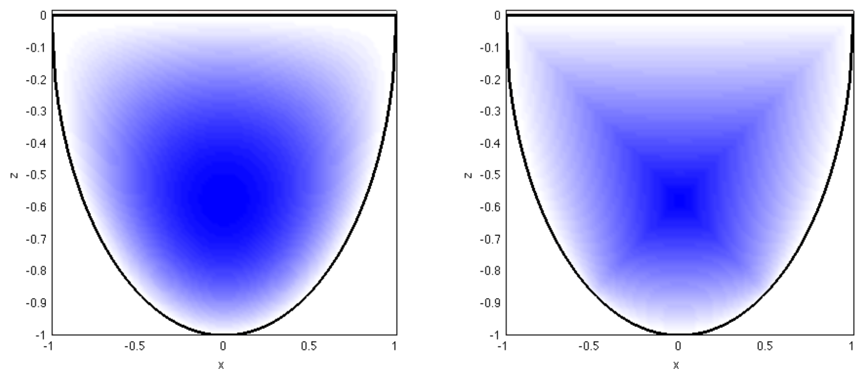

Figure 3: Different solutions with $k = 3$, $m = 1$. At left the periodic function is cos. At right, the periodic function replaces cos with a 2π periodic even triangle wave. In the same way as a triangle wave can be expressed as a Fourier cosine series, the solution at right can be represented as an infinite series superposition of polynomial solutions.

$$X = \frac{\sqrt{1+\nu^2}}{\nu}\cos(\tilde\theta) = \frac{\cos(\tilde\theta)}{\cos(\theta_\nu)}, \qquad \tilde\theta = \arccos(X\cos(\theta_\nu)),$$

$$f(X) = \tilde f(\arccos(X\cos(\theta_\nu)).$$

For Barcilon's solutions this is

$$\nu = \cot(\frac{m\pi}{k}) \qquad f(X) = \cos(k\arccos(\cos(\frac{m\pi}{k})X))$$

A couple of solutions for the lowest mode – no interior nodal curves – (and $\nu = 1/\sqrt{3}$) are shown in Figure 3.

For plots of some other modes, see [2, 15].

6.1.2. Taylor series methods for (1.1b) and (3.1b)

There are other methods that can be used to solve (1.1b) with $d(x) = \sqrt{1-x^2}$. One can form a Taylor series about $x = 0$ of each of $f(x \pm d(x)/\nu)$. If one is to seek a polynomial solution the Taylor series is a finite sum, and furthermore only even powers of $d(x)$ enter the equation to be solved. It is easy to recover Barcilon's Chebyshev polynomial solutions from this approach. One can also find other $d(x)$ which lead to polynomial f. The method can also be adapted to shapes other than the semiellipse, finding rational functions f, and to solving the Abel's functional equation (Q non-zero) not merely the $Q = 0$ Schröder functional equations.

6.1.3. A forward map T with range bigger than $[-1,1]$

T (determined using equation (3.2)) is

$$T(X) = \frac{2\sqrt{1-\nu^2\left(X^2-1\right)}+\left(\nu^2-1\right)X}{\nu^2+1}$$

Barcilon's Chebyshev solutions of f satisfying $f(X) = f(T(X))$ are readily verified. (An easy example is $f(X) = 2X^4 - 4X^2 + 1 = T_4(X\sin(\pi/4))$ corresponding to $\nu = 1$ and $T(X) = \sqrt{2-X^2}$. Here T_4 denotes the Chebyshev polynomial of degree 4.)

6.2. Dai's solutions for hyperbolae

The case of a hyperbolic bottom profile $d(x) = r/x$ for $x > 0$ is treated in [5]. One readily verifies that (1.1a)

$$a\left(x+\frac{r}{\nu x}\right) = a\left(x-\frac{r}{\nu x}\right)+1 \quad \text{is solved by} \quad a(x) = \frac{\nu x^2}{4r}.$$

The streamfunction associated with this a has fluid entering from $(\infty, 0)$ and exiting via $(0, -\infty)$.

In this case it happens that the problem can be recast using the forward map $T(x) = \sqrt{4r/\nu + x^2}$ for $x > 0$ into an Abel equation (3.1a). The solution appears elsewhere. For example, [9], near his equation (9), gives the solution with

$$d(x) = \frac{1}{d_0 + rx} \text{ and } \nu = 1, \qquad a(x) = -\frac{1}{2}(d_0 x + \frac{r}{2}x^2).$$

Solutions to the Schröder problem are found, in the usual method, by composing a period-1 function, P, with a. A typical example with P chosen to be a cosine is shown in figure 4. The plotted streamfunction has many interesting nodal curves in addition to the nodal curve along the bottom topography $d(x) = 1/|x|$ (black line). With the cosine P there are elliptic nodal lines around the origin.

7. Involutions, and a particularly simple family of solutions

Involutions are functions which when composed with themselves give the identity function:

$$\text{invol}(\text{invol}(x)) = x$$

for all x in the domain of the function.

It has already been noted, e.g. [15], that everywhere subcritical symmetric profiles lead to functional equations $f(x) = f(T(x))$ where $T(x) = -\text{invol}(x)$:

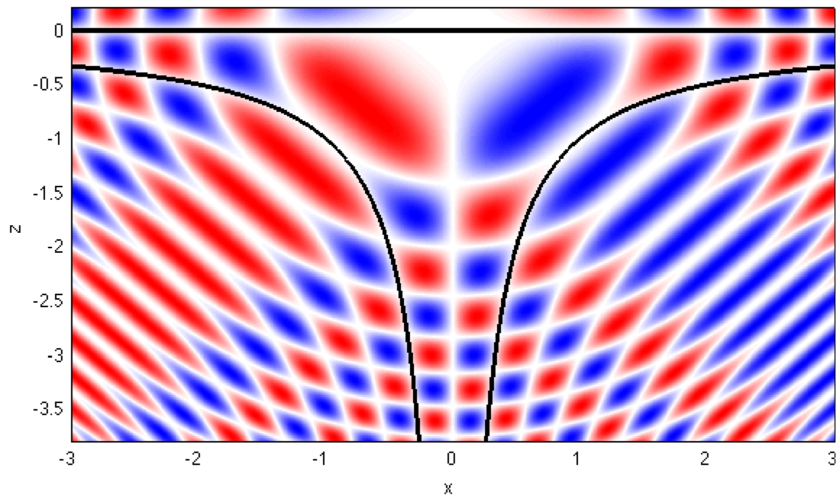

Figure 4: Dai's streamfunction solution for hyperbolic bottom profile $d(x) = 1/|x|$ corresponding to the solution $f(x) = \cos(\frac{\pi}{2}x^2)$ to (3.1b).

various examples are treated in §5. We do not know of any general method which is convenient to apply for all equations of this type. If one simply changes the minus to a plus, we will see that the equation is extremely easy to solve.

Theorem 8. *There are no solutions to the Abel functional equation, with* $Q \neq 0$

$$a(\text{invol}(x)) - a(x) = Q$$

Proof. Suppose there were to be a solution to the Abel functional equation above, then we also have

$$a(x) - a(\text{invol}(x)) = a(\text{invol}(\text{invol}(x))) - a(\text{invol}(x)) = Q$$

Adding the two preceding equations gives $0 = 2Q$ which contradicts the assumption $Q \neq 0$.

Because of the preceding result, the approach – using a solution of the Abel equation to generate solutions to the Schröder equation by compositions with periodic functions – fails here. However an alternative approach is available:

Theorem 9. *Let S be any symmetric function of two variables, meaning that $S(u,v) = S(v,u)$ for all u, v. Then the function $f(x) = S(x, \mathrm{invol}(x))$ solves the Schröder equation*

$$f(\mathrm{invol}(x)) = f(x) \quad \text{with invol} \quad \text{an involution.} \tag{7.1}$$

Proof.
$f(\mathrm{invol}(x)) = S(\mathrm{invol}(x), \mathrm{invol}(\mathrm{invol}(x))) = S(\mathrm{invol}(x), x) = S(x, \mathrm{invol}(x)) = f(x)$.

For $\mathrm{invol}(x)$ to correspond to a forward map T we need to make sure that its domain is such that $\mathrm{invol}(x) > x$.

The entries in the table below indicate some flows associated with the involutions given. We take $\nu = 1$. The entry d is the solution of $\mathrm{invol}(x-d) = x + d$. There are many possibilities for S; our descriptions of the flow are for $S(u,v) = u + v$. (Any streamfunction ψ defined by the usual extension of f is zero on $z = -d(x)$.)

$\mathrm{invol}(x)$	d	Comments
$\frac{1}{x}$	$\sqrt{x^2-1}$ for $x < -1$	corner flow with a hyperbolic boundary
$\frac{x_0-x}{1+bx}$	$\sqrt{(x+\frac{1}{b})^2 - \frac{1+x_0 b}{b^2}}$	further flows with hyperbolic d
$\sqrt{2b^2-x^2}$	$\sqrt{b^2-x^2}$	d: portion of ellipse
$\mathrm{PL}(x_0, m, x)$ with $m > 1$	$\frac{(m+1)(x-x_0)}{m-1}$	piecewise linear ψ giving a corner flow in a supercritical wedge

Some comments on the table above follow:

• Concerning the third entry in the table, we remark that Barcilon's solution in a circular quadrant with $\nu = 1$ can be constructed using the discontinuous involution $\mathrm{sign}(x)\sqrt{1-x^2}$ and $f(x) = x^4 + \mathrm{invol}(x)^4$.

• In the fourth entry in the table, the piecewise linear involution PL is defined, with $m > 1$, by

$$\mathrm{PL}(x_0, m, x) = \frac{1}{2}\left(m - \frac{1}{m}\right)|x_0 - x| + \frac{1}{2}\left(m + \frac{1}{m}\right)(x_0 - x) + x_0$$

There are several ways to generate the piecewise linear ψ corner flow. One might take the symmetric function S as $S(u,v) = u + v$ or, alternatively, as $S(u,v) = \min(u,v)$. Let Γ be the characteristic through $(x_0, 0)$ extending downwards and to the right. The flow has its streamlines parallel to $z = 0$ in the triangle below the top boundary and above Γ and parallel to the bottom profile $z = -d(x)$ in the triangle above it and below Γ. Taking $f = x + \mathrm{PL}(x_0, m, x)$ generates a similar corner flow.

The corner flows, with no interior nodal lines, can be composed with other functions, e.g. periodic functions, and then the ψ has nodal curves – the flow exhibiting cells as in many of our earlier examples.

Functions whose k-th iterate, $k \geq 2$ is the identity are called *involutions of order* k. The account above treats the case $k = 2$, and it generalises. For any $k \geq 2$ there are no solutions to the involution Abel equations with $Q \neq 0$. Also, let S be a function of k arguments which is invariant as one cycles through them,

$$S(u_1, u_2, u_3, \ldots, u_k) = S(u_2, u_3 \ldots, u_k, u_1),$$

and define

$$f(x) = S(x, \mathrm{invol}_k(x), \mathrm{invol}_k^{[2]}(x), \ldots \mathrm{invol}_k^{[k-1]}(x)).$$

Then, for any $k \geq 2$, f solves (3.1b) when $T = \mathrm{invol}_k$ is an involution of order k. (Examples of S include symmetric functions such as the sum of k variables, etc..)

8. Other hyperbolic equations

At the end of §2.1 we noted that the pde problem of this paper arose in contexts other than that of standing internal waves under rather special conditions. Broadly similar pdes arise when the buoyancy frequency is a function of z, i.e. ν^2 depends on z, or where the waves arise superposed on some steady base flow. The question arises as to what extent the functional equation approach of this paper might be applied to other hyperbolic pdes. To the best of the author's knowledge, the pdes in this subsection are not related to internal waves, but the subsection is here to indicate that other hyperbolic pdes are amenable to similar approaches, and may have some application to other wave phenomena. The method is applicable when the general solution of the pde is of the form

$$\psi(x, z) = \Psi(x, z) \left(f_-(X(x) - Z(z)) - f_+(X(x) + Z(z)) \right),$$

often with some condition like $Z(0) = 0$.

Rather than beginning with the immediately preceding solution and finding pdes that it satisfies, we note here various equations whose solutions are particular cases of the form above. A special case of the telegrapher's equation

$$\frac{\partial^2 u}{\partial x^2} - \nu^2 \frac{\partial^2 u}{\partial z^2} - b\nu \frac{\partial u}{\partial z} - \frac{b^2 u}{4} = 0$$

has as its general solution

$$u(x,z) = \exp\left(\frac{-bz}{2\nu}\right)\left(f_-(x-\frac{z}{\nu}) + f_+(x+\frac{z}{\nu})\right).$$

Variable coefficient pdes can also be treated. A very simple example is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\nu}{Z'(z)}\frac{\partial}{\partial z}\left(\frac{\nu}{Z'(z)}\frac{\partial u}{\partial z}\right) + \frac{a\nu}{Z'(z)}\frac{\partial u}{\partial z} + \frac{a^2 u}{4} = 0,$$

and its solution is

$$u(x,z) = \exp\left(\frac{-aZ(z)}{2\nu}\right)\left(f_-(x-\frac{Z(z)}{\nu}) + f_+(x+\frac{Z(z)}{\nu})\right).$$

Another widely studied wave equation concerns 'spherically' symmetric waves in polar coordinates

$$\frac{\partial^2 u}{\partial x^2} - \frac{\mu^2}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial u}{\partial r}\right) + \frac{\mu^2 a_0 u}{4r^2} = 0.$$

(When $a_0 = 0$ this is equivalent to the Euler-Poisson-Darboux equation. Copson p98.) With a_0 as given, it's general solution is

$$a_0 = (N-1)(N-3), \qquad u(x,r) = \frac{f_-(x-\frac{r}{\mu}) - f_+(x+\frac{r}{\mu})}{r^{(N-1)/2}}.$$

The case $N = 1$ is the pde of this paper. The Dirichlet problem with $u = 0$ on $x = 0$ and on $r = \pm\sqrt{1-x^2}$ leads to the functional equation solved in §6.1. The polynomial f of §6.1 lead to solutions at other values of N.

There are many other pdes for which the general solution can be found, including examples with first derivatives with respect to x. A simple example of this, generalizing the special case of the telegrapher's equation noted at the beginning of this subsection, is the equation – with α and β functions of x and z –

$$\frac{\partial^2 u}{\partial x^2} - \nu^2\frac{\partial^2 u}{\partial z^2} + \frac{\partial(\alpha u)}{\partial x} - \nu\frac{\partial(\beta u)}{\partial z} - \left(\frac{\partial \alpha}{\partial x} - \nu\frac{\partial \beta}{\partial z} - \frac{\alpha^2 - \beta^2}{2}\right)\frac{u}{2} = 0 \text{ with } \beta_x = \nu\alpha_z.$$

The last condition ensures that there is a function ϕ for which $\beta = -2\nu\phi_z/\phi$ and $\alpha = -2\phi_x/\phi$, Then the general solution is

$$u(x,z) = \phi(x,z)\ \left(f_-(x-\frac{z}{\nu}) + f_+(x+\frac{z}{\nu})\right).$$

For appropriate boundary value problems for any of the pdes of this subsection, functional equation methods may prove to be useful.

9. Discussion

9.1. Conclusion

Solutions to the functional equations (1.1) and (3.1) can be used to construct exact two-dimensional standing internal wave solutions. Several approaches for subcritical and (partly) supercritical domains are presented making use of the functional equations. There are others, e.g. the iterative methods due to Levy and others (see [11, 12]). We believe that our exposition of the methods is satisfactory in the case of everywhere subcritical bottom profiles, our §4 and §5: these are solutions where the 'rays focus to the end-points'. For partly supercritical bottom profiles – where the determination of the values of ν for which there are solutions is also part of the problem – our examples suggest that the functional equation approach may have value. Our work on this in §6 and §7 is as much intended to publicise the problem as to present solutions.

The functional equations (1.1b) and (3.1b) have been used in the past to construct exact internal wave solutions, and [21] has also pointed out that one can associate solutions to (3.1b) with solutions to (3.1a). What is new with respect to earlier work on internal waves is to link (3.1a) to Abel's functional equation and to make use for known properties and solutions of Abel's functional equation. Theorem 5 guarantees that for subcritical bottom topographies all solutions to (3.1b) are derived by applying the set of periodic function with period 1 to any injective continuous solution of (3.1a). We are convinced that there is more to be elaborated, especially with the results on Abel's functional equation in [11, 12].

9.2. Anticipating applications to other internal-wave problems

We expect that functional equation techniques may prove useful for some other internal wave problems in which $z = 0$ is a streamline.

1) One such situation concerns the generation of internal waves by horizontal oscillations of a symmetric cylinder. The usual formulation has the stream function ψ_{gen} nonzero on the cylinder: $\psi_{\text{gen}} = -Uz$ on the cylinder $z = \pm d(x)$: see equation (2.7) of [10]. The pde remains the wave equation as in our equations (2.1), but the boundary conditions, except for $\psi_{\text{gen}}(x, 0) = 0$ are different. The representation of solutions as in equation (2.2) with the boundary condition on the cylinder yields the functional equation

$$f_{\text{gen}}\left(x - \frac{d(x)}{\nu}\right) - f_{\text{gen}}\left(x + \frac{d(x)}{\nu}\right) = -Ud(x).$$

One solution of this is of the form $f_{\text{gen}}(x) = c_{\text{gen}}x$ with the constant $c_{\text{gen}} = \nu U/2$. If f solves the homogeneous equation (1.1b) it follows that the general solution is $f_{\text{gen}}(x) = c_{\text{gen}}x + f(x)$. The problem now requires complex-valued solutions of the functional equation with appropriate behaviour at infinity, a radiation boundary condition there. Several special cases have been investigated, and some solved by other techniques.

- Elliptical cylinders with axes aligned with the coordinate axes are a particular case of the more general treatment in [10]. Here consider only the case when $V = 0$ in equation (3.42). The $\sigma_\pm$ in [10] is a multiple of our $x \pm z/\nu$: see his equation (3.3). Barcilon's (real) polynomial solutions correspond to blinking modes. For the wave-generation problem of [10] the complex-valued f requires careful treatment of branch cuts in order that the radiation conditions at infinity are satisfied.

- An experimental treatment of a square cylinder is given in [7].

2) Tidal conversion is treated in [17, 3]. Another situation where complex f, and radiation conditions, are involved is the propagation, transmission and reflection of monochromatic internal waves in a channel with a rigid upper lid $\{(x, 0)| -\infty < x < \infty\}$ and an everywhere subcritical bottom, see [19, 4].

Vitae: Felix Beckebanze (FB) recently completed Masters degrees in Mathematics and Physics at Utrecht University. The internal waves group at Utrecht includes FB's supervisor, Prof. L.R.M. Maas, who introduced FB to GK.

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