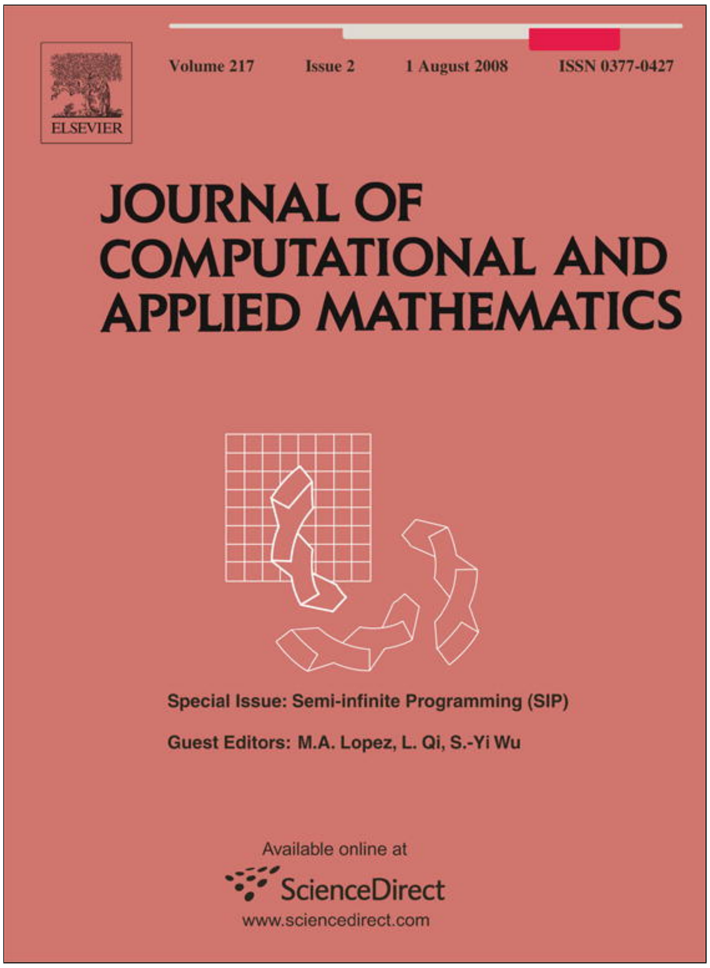
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Numerical method for a class of optimal control problems subject to nonsmooth functional constraints

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Abstract

In this paper, we consider a class of optimal control problems which is governed by nonsmooth functional inequality constraints involving convolution. First, we transform it into an equivalent optimal control problem with smooth functional inequality constraints at the expense of doubling the dimension of the control variables. Then, using the Chebyshev polynomial approximation of the control variables, we obtain an semi-infinite quadratic programming problem. At last, we use the dual parametrization technique to solve the problem.

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1. Introduction

Optimal control problems arise in a wide variety of disciplines. Apart from traditional areas such as aerospace engineering, robotics and chemical engineering, optimal control theory has also been used with great success in areas as diverse as economics to biomedicine. In practice, many optimal control problems are subject to constraints on the state and/or control variables. It is well known that constrained optimal control problems are very difficult to solve. In particular, their analytical solutions are in many cases out of the question. Thus, numerical methods are needed for solving many of these real world problems. There are now many numerical methods available in the literature for various optimal control problems. See, for example [1,13,2,5,7,4,15,11]. In particular, an efficient optimal control software package, known as MISER3.2 [6], was developed based on the control parametrization technique [13] and control parametrization enhancing transform [14].

For constrained optimal control problems, there are various types of constraints such as bound constraints on the control variables and continuous state inequality constraints (see [13]). In [13], a constraint transcription method is introduced to deal with the optimal control problems subject to continuous state inequality constraints. In [17], an optimal design of an envelop-constrained filter with input uncertainty is considered, where the constraints are expressed

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by nonsmooth functional inequality constraints involving convolution. This class of optimal envelope-constrained filter design problems has been extensively studied by many researchers in the past two decades. See, for example [17,3,16]. In this paper, we consider a class of optimal control problems subject to the above mentioned nonsmooth functional inequality constraints. These constraints, which are expressed in terms of convolution, are to be satisfied at each time point. This optimal control problem, which can be considered as a generalized optimal envelope-constrained filter design problem, cannot be solved as such using any standard optimal control techniques. Here, we shall make use of the idea of Teo et al. [12] to transform these nonsmooth functional inequality constraints into equivalent smooth functional inequality constraints at the expense of doubling the dimension of the control variables, which leads to an equivalent optimal control problem subject to smooth functional inequality constraints. Consequently, by using the Chebyshev polynomial approximation of the control variables, we obtain an approximate optimal control problem, which is further reduced into an equivalent semi-infinite quadratic programming problem. Then, the dual parametrization technique obtained in [8] is used to develop an efficient method for solving the equivalent semi-infinite programming problem. The convergence analysis supporting this computational method will also be presented.

The rest of the paper is organized as follows. In Section 2, the problem to be studied is formulated, which is an optimal control problem subject to nonsmooth functional inequality constraints involving convolution. In Section 3, the idea appeared in [12] is used to transform this nonsmooth functional inequality constrained optimal control problem into an equivalent optimal control problem subject to smooth functional constraints. Then, by using the Chebyshev polynomial approximation of the control variables, we convert the optimal control problem into an equivalent semi-infinite programming problem. Subsequently, the dual parametrization technique reported in [8] (or its improved version reported in [9]) is used to develop an efficient computational method for solving the semi-infinite programming problem. In Section 4, a numerical example is solved using the proposed approach. Section 5 concludes the paper.

2. Problem formulation

Consider a process described by the following system of linear ordinary differential equations defined on a fixed time interval $(t_0, t_f]$:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{2.1a}$$

where

$$x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n, \quad u = [u_1, u_2, \ldots, u_m]^\top \in \mathbb{R}^m$$

are, respectively, the state and control vectors; the superscript $^\top$ denotes the transpose.

The initial condition for the differential (2.1a) is

$$x(t_0) = x_0, \tag{2.1b}$$

where $x_0 = [x_1^0, x_2^0, \ldots, x_n^0]^\top \in \mathbb{R}^n$ is a given vector.

Define

$$U = \{v = [v_1, v_2, \ldots, v_m]^\top \in \mathbb{R}^m : a_i \leqslant v_i \leqslant b_i, i = 1, \ldots, m\},$$

where a_i and b_i, $i = 1, \ldots, m$, are real numbers.

A continuous function $u = [u_1, u_2, \ldots, u_m]^\top$ from $[0, T]$ into $\mathbb{R}^m$ is said to be an admissible control if $u(t) \in U$ for $t \in (0, T]$. Let $\mathscr{U}$ be the class of all such admissible controls. For each $u \in \mathscr{U}$, let $x(\cdot|u)$, $t \in [0, T]$, be the solution of differential equation (2.1a) with initial condition (2.1b) corresponding to the control $u \in \mathscr{U}$.

We consider the following nonsmoothing functional inequality constraint on the control variables:

$$\left| \int_{t_0}^{t_f} \varphi^\top(t-\tau)u(\tau)\,\mathrm{d}\tau - \mathrm{d}(t) \right| + \int_{t_0}^{t_f} \sum_{i=1}^{m} \theta_i(t-\tau)|u_i(\tau)|\,\mathrm{d}\tau - \varepsilon(t) \leqslant 0, \tag{2.2}$$

where $\varphi(t)$, $\varepsilon(t)$, and $\theta_i(t)$, $i = 1, \ldots, m$, are given real-valued functions.

If $u \in \mathscr{U}$ is such that the constraint (2.2) is satisfied, then it is called a feasible control. Let $\mathscr{F}$ be the class of all such feasible controls.

We may now state our optimal control problem as follows:

Problem (P). Given system (2.1), find a control $u \in \mathscr{F}$ such that the cost functional

$$J(u) = x^\top(t_f)R_0x(t_f) + \int_{t_0}^{t_f}\int_{t_0}^{t_f} x^\top(\tau)R_1(t-\tau)x(t)\,\mathrm{d}\tau\,\mathrm{d}t + \int_{t_0}^{t_f}\int_{t_0}^{t_f} u^\top(\tau)R_2(t-\tau)u(t)\,\mathrm{d}\tau\,\mathrm{d}t \tag{2.3}$$

is minimized over $\mathscr{F}$, where R_0, R_1 and R_2 are given real-valued functions.

The following conditions are imposed in the rest of the paper.

(A1) There exists a solution for Problem (P).
(A2) The matrix valued functions $A(t)$ and $B(t)$ are continuously differentiable with respect to t.
(A3) The functions $\varphi(t)$, $\varepsilon(t)$, $d(t)$ and $\theta_i(t)$, $i = 1, \ldots, m$, are all continuously differentiable with respect to t.
(A4) The integral kernel operators

$$\mathscr{K}_1x(t) = \int_{t_0}^{t_f} R_1(t-\tau)x(\tau)\,\mathrm{d}\tau, \quad \mathscr{K}_2u(t) = \int_{t_0}^{t_f} R_2(t-\tau)u(\tau)\,\mathrm{d}\tau$$

are all positive definite and the matrix R_0 is also positive definite, i.e.,

$$\int_{t_0}^{t_f}\int_{t_0}^{t_f} x(t)R_1(t-\tau)x(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \geqslant 0, \quad \int_{t_0}^{t_f}\int_{t_0}^{t_f} u(t)R_2(t-\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \geqslant 0.$$

(A5) The slater condition is satisfied, i.e., there exists a control $u(t)$, such that

$$\left|\int_{t_0}^{t_f} \varphi^\top(t-\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau - d(t)\right| + \int_{t_0}^{t_f} \sum_{i=1}^{m} \theta_i(t-\tau)|u_i(\tau)|\,\mathrm{d}\tau - \varepsilon(t) < 0$$

for all $t \in [t_0, t_f]$, where

$$u(t) \neq 0 \quad \text{for all } t \in [t_0, t_f],$$

and

$$a_i < u_i(t) < b_i \quad \text{for } t \in [t_0, t_f], \quad i = 1, \ldots, m.$$

Remark 2.1. Note that Problem (P) with $A(t) = 0$, $n = 1$, $b(t) = 1$, $R_0 = 0$ and $R_1 = 0$ reduces to the optimal design of envelope-constrained filters with input uncertainty considered in [17]. This class of envelope-constrained filter design problems with input uncertainty has been extensively studied in the past two decades, see, for example [9–11] .

The functional inequality constraints (2.2), which are nondifferentiable with respect to u, are in the form of convolution of the control variable u. The cost functional is in the form of double integral. Thus, Problem (P) cannot be solved directly by standard optimal control methods such as those developed in [13].

3. Problem transformation

In this section, we shall show that the nondifferentiable inequality constraint on the control variable can be converted into equivalent differentiable inequality constraints. For this, we need the following lemma, which is quoted from [12].

Lemma 3.1. *For each $x \in \mathbb{R}^n$, let $p : \mathbb{R}^n \to \mathbb{R}^n$, and q: $\mathbb{R}^n \to \mathbb{R}^n$ be defined by*

$$p(x) = [\max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_n, 0\}]^\top,$$

$$q(x) = [\max\{-x_1, 0\}, \max\{-x_2, 0\}, \ldots, \max\{-x_n, 0\}]^\top,$$

Then

(1) $p(x) \geqslant 0$, $q(x) \geqslant 0$,
(2) $p(x)^\top q(x) = 0$,
(3) $p(x) - q(x) = x$, $p(x) + q(x) = |x| = [|x_1|, |x_2|, \ldots, |x_n|]^\top$,
(4) *for each* $x, y \in \mathbb{R}^n$ *with* $x, y \geqslant 0$, *there exists a unique nonnegative function* $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, *such that* $|x - y| = x + y - r(x, y)$, *where* $x = [x_1, x_2, \ldots, x_n]^\top \geqslant 0$ *means* $x_i \geqslant 0$, $i = 1, \ldots, n$.

We may now define formally the following optimal control problem.

Problem (EP)**.** Given the system

$$\dot{x}(t) = A(t)x(t) + B(t)(u_1(t) - u_2(t)), \quad t \in (t_0, t_f], \tag{3.1a}$$

with the initial condition

$$x(t_0) = x_0 \tag{3.1b}$$

find a continuous control $\tilde{u}(t) = [(u^1(t))^\top, (u^2(t))^\top]^\top \in C(t_0, t_f; \mathbb{R}^{2m})$, where $C(t_0, t_f; \mathbb{R}^{2m})$ is the set of continuous functions from $[t_0, t_f]$ to $\mathbb{R}^{2m}$, such that the cost functional

$$\begin{aligned}\tilde{J}(\tilde{u}) = {} & x^\top(t_f)R_0x(t_f) + \int_{t_0}^{t_f}\int_{t_0}^{t_f} x^\top(\tau)R_1(t-\tau)x(t)\,\mathrm{d}\tau\,\mathrm{d}t \\ & + \int_{t_0}^{t_f}\int_{t_0}^{t_f} (u^1(\tau) - u^2(\tau))^\top R_2(t-\tau)(u^1(t) - u^2(t))\,\mathrm{d}\tau\,\mathrm{d}t\end{aligned} \tag{3.2}$$

is minimized subject to the functional inequality constraints:

$$\int_{t_0}^{t_f} \varphi^\top(t-\tau)(u^1(\tau) - u^2(\tau))\,\mathrm{d}\tau - d(t) + \int_{t_0}^{t_f}\sum_{i=1}^m \theta_i(t-\tau)(u_i^1(\tau) + u_i^2(\tau))\,\mathrm{d}\tau - \varepsilon(t) \leqslant 0, \tag{3.3a}$$

$$-\int_{t_0}^{t_f} \varphi^\top(t-\tau)(u^1(\tau) - u^2(\tau))\,\mathrm{d}\tau + d(t) + \int_{t_0}^{t_f}\sum_{i=1}^m \theta_i(t-\tau)(u_i^1(\tau) + u_i^2(\tau))\,\mathrm{d}\tau - \varepsilon(t) \leqslant 0, \tag{3.3b}$$

$$a \leqslant [I_m, -I_m]\tilde{u}(t) \leqslant b, \tag{3.3c}$$

$$\beta \geqslant \tilde{u}(t) = [(u^1(t))^\top, (u^2(t))^\top]^\top \geqslant 0 \quad \text{for all } t \in [t_0, t_f] \tag{3.3d}$$

where I_m denotes the $m \times m$ identity matrix, $u_i = [u_1^i, u_2^i, \ldots, u_m^i]^\top$, $i = 1, 2$, $\beta = [\beta_1, \ldots, \beta_{2m}]^\top$, $\beta_i = |a_i| + |b_i|$, $i = 1, \ldots, m$, and $\beta_{i+m} = |a_i| + |b_i|$, $i = 1, \ldots, m$.

Now we have the following theorem:

Theorem 3.1. *Problem* (*P*) *is equivalent to Problem* (*EP*) *in the sense that if* $\tilde{u}^* = [u^{1,*}, u^{2,*}]^\top$ *is the optimal solution of Problem* (*EP*), *then* $u = u^{1,*} - u^{2,*}$ *is an optimal solution of Problem* (*P*). *On the other hand, if* u^* *is an optimal solution of Problem* (*P*) , *then* $[p(u^*), q(u^*)]$ *is an optimal solution of Problem* (*EP*) *and*

$$J(u^*) = \tilde{J}(\tilde{u}^*).$$

Proof. Suppose that $\tilde{u}^* = ((u^{1,*})^\top, (u^{2,*})^\top)^\top$ is an optimal solution of Problem (*EP*), and u^* is an optimal solution of Problem (P). Let

$$u = u^{1,*} - u^{2,*}, \quad \tilde{u} = [p^\top(u^*), q^\top(u^*)].$$

Clearly, $u \in \mathscr{U}$, and $\tilde{u} \in C(t_0, t_f; \mathbb{R}^{2m})$ is such that (3.3c) and (3.3d) are satisfied. Note that

$$\begin{aligned}
&\left|\int_{t_0}^{t_f} \varphi^\top(t-\tau)u(\tau)\,\mathrm{d}\tau - d(t)\right| + \int_{t_0}^{t_f} \sum_{i=1}^{m} \theta_i(t-\tau)|u_i(\tau)|\,\mathrm{d}\tau - \varepsilon(t) \\
&\quad = \left|\int_{t_0}^{t_f} \varphi^\top(t-\tau)(u^{1,*}(\tau) - u^{2,*}(\tau))\,\mathrm{d}\tau - d(t)\right| \\
&\qquad + \int_{t_0}^{t_f} \sum_{i=1}^{m} \theta_i(t-\tau)|u_i^{1,*}(\tau) - u_i^{2,*}(\tau)|\,\mathrm{d}\tau - \varepsilon(t) \\
&\quad = \left|\int_{t_0}^{t_f} \varphi^\top(t-\tau)(u^{1,*}(\tau) - u^{2,*}(\tau))\,\mathrm{d}\tau - d(t)\right| \\
&\qquad + \int_{t_0}^{t_f} \sum_{i=1}^{m} \theta_i(t-\tau)(u_i^{1,*}(\tau) + u_i^{2,*}(\tau) - r(u_i^{1,*}(\tau), u_i^{2,*}(\tau)))\,\mathrm{d}\tau - \varepsilon(t) \\
&\quad \leqslant \left|\int_{t_0}^{t_f} \varphi^\top(t-\tau)(u^{1,*}(\tau) - u^{2,*}(\tau))\,\mathrm{d}\tau - d(t)\right| \\
&\qquad + \int_{t_0}^{t_f} \sum_{i=1}^{m} \theta_i(t-\tau)(u_i^{1,*}(\tau) + u_i^{2,*}(\tau))\,\mathrm{d}\tau - \varepsilon(t) \\
&\quad \leqslant 0.
\end{aligned}$$

This implies that $u = u^{1,*} - u^{2,*}$ satisfies the inequality constraint (2.2) of Problem (P), and hence $u \in \mathscr{F}$. At the same time, the solution $\tilde{x}(\cdot \mid u^{1,*}, u^{2,*})$ of the dynamical system (3.1a) and (3.1b) is the same as the solution $x(\cdot \mid u)$ of the dynamical system (2.1a) and (2.1b). Thus,

$$\begin{aligned}
J(u) &= x^\top(t_f \mid u)R_0 x(t_f|u) + \int_{t_0}^{t_f}\int_{t_0}^{t_f} x^\top(\tau \mid u)R_1(t-\tau)x(t \mid u)\,\mathrm{d}\tau\,\mathrm{d}t \\
&\quad + \int_{t_0}^{t_f}\int_{t_0}^{t_f} u^\top(\tau)R_2(t-\tau)u(t)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \tilde{x}^\top(t_f \mid u^{1,*}, u^{2,*})R_0\tilde{x}(t_f \mid u^{1,*}, u^{2,*}) \\
&\quad + \int_{t_0}^{t_f}\int_{t_0}^{t_f} \tilde{x}^\top(\tau \mid u^{1,*}, u^{2,*})R_1(t-\tau)\tilde{x}(t \mid u^{1,*}, u^{2,*})\,\mathrm{d}\tau\,\mathrm{d}t \\
&\quad + \int_{t_0}^{t_f}\int_{t_0}^{t_f} (u^{1,*}(\tau) - u^{2,*}(\tau))^\top R_2(t-\tau)(u^{1,*}(t) - u^{2,*}(t))\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \tilde{J}(u^{1,*}, u^{2,*})
\end{aligned}$$

and hence

$$J(u^*) \leqslant J(u) = \tilde{J}(u^{1,*}, u^{2,*}). \tag{3.4}$$

By the same token, we can show that

$$\tilde{J}(u^{1,*}, u^{2,*}) \leqslant \tilde{J}(\tilde{u}) = J(u^*). \tag{3.5}$$

Therefore, the conclusion of the theorem follows from (3.4) and (3.5). □

Denote

$$\tilde{B}(t)=B(t)[I_m,-I_m],\quad \tilde{R}_2(t-\tau)=\begin{bmatrix} I_m \\ -I_m \end{bmatrix},\quad R_2(t-\tau)[I_m,-I_m],$$

$$\tilde{\varphi}^\top(t-\tau)=\varphi^\top(t-\tau)[I_m,-I_m],\quad \theta^\top(t-\tau)[I_m,I_m].$$

Then, Problem (EP) can be written more concisely as

Given the dynamical system

$$\dot{x}(t)=A(t)x(t)+\tilde{B}(t)\tilde{u}(t),\quad t\in(t_0,t_f], \tag{3.6a}$$

with the initial condition

$$x(t_0)=x_0 \tag{3.6b}$$

find a continuous control $\tilde{u}(t)=[(u^1(t))^\top,(u^2(t))^\top]^\top$ such that the cost functional

$$\tilde{J}(\tilde{u})=x^\top(t_f)R_0x(t_f)+\int_{t_0}^{t_f}\int_{t_0}^{t_f}x^\top(\tau)R_1(t-\tau)x(t)\,\mathrm{d}\tau\,\mathrm{d}t+\int_{t_0}^{t_f}\int_{t_0}^{t_f}\tilde{u}^\top(\tau)\tilde{R}_2(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \tag{3.7}$$

is minimized over $C(t_0,t_f;\mathbb{R}^{2m})$ and subject to the functional inequality constraints:

$$\int_{t_0}^{t_f}(\tilde{\varphi}(t-\tau))^\top\tilde{u}(\tau)\,\mathrm{d}\tau-d(t)+\int_{t_0}^{t_f}(\theta(t-\tau))^\top\tilde{u}(\tau)\,\mathrm{d}\tau-\varepsilon(t)\leqslant 0, \tag{3.8a}$$

$$-\int_{t_0}^{t_f}(\tilde{\varphi}(t-\tau))^\top\tilde{u}(\tau)\,\mathrm{d}\tau+d(t)+\int_{t_0}^{t_f}(\theta(t-\tau))^\top\tilde{u}(\tau)d\tau-\varepsilon(t)\leqslant 0 \tag{3.8b}$$

$$a\leqslant[I_m,-I_m]\tilde{u}(t)\leqslant b, \tag{3.8c}$$

$$\beta\geqslant\tilde{u}(t)=[(u^1(t))^\top,(u^2(t))^\top]^\top\geqslant 0\quad\text{for all } t\in[t_0,t_f]. \tag{3.8d}$$

This problem is again denoted by Problem (*EP*). We have the following theorem.

Theorem 3.2. *The cost functional $\tilde{J}(\tilde{u})$ is a strictly convex quadratic functional of u on $C(t_0,t_f;\mathbb{R}^{2m})$, and the Slater condition is still satisfied for Problem* (*EP*), *i.e., there exits a $\tilde{u}\in C(t_0,t_f;\mathbb{R}^{2m})$, such that the following conditions are satisfied*:

$$\int_{t_0}^{t_f}\tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau-d(t)+\int_{t_0}^{t_f}\theta^\top(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau-\varepsilon(t)<0,$$

$$-\int_{t_0}^{t_f}\tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau+d(t)+\int_{t_0}^{t_f}\theta^\top(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau-\varepsilon(t)<0,$$

$$a<[I_m,-I_m]\tilde{u}(t)<b,$$

$$\beta>\tilde{u}(t)=[u_1^\top(t),u_2^\top(t)]^\top>0\quad\textit{for all } t\in[t_0,t_f].$$

Proof. By Assumption (A5), the integral kernel operator $\mathscr{K}_2u(t)$ is positive definite. Thus, the integral kernel operator

$$\tilde{\mathscr{K}}_2u(t)=\int_{t_0}^{t_f}\tilde{R}_2(t-\tau)\tilde{u}(\tau)$$

is also positive definite. Note the form of the cost functional $\tilde{J}(\tilde{u})$, it is easy to verify the validity of the conclusion of the theorem. The Slater condition is clearly satisfied. □

4. The method of Chebyshev approximation

In this section, we shall first approximate the optimal control problem (EP) which is a semi-infinite quadratic programming problem by Chebyshev series approximation, then we use the dual parametrization method developed in [7] to deal with the functional inequality constraints.

Let $\Phi(t,\tau)$ denote the fundamental matrix of the dynamical system (3.6). For a given control $\tilde{u} \in C(t_0, t_f; \mathbb{R}^{2m})$, the corresponding trajectory of dynamical system (3.6) is given by

$$x(t \mid \tilde{u}) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)\tilde{B}(\tau)\tilde{u}(\tau)\,\mathrm{d}\tau. \tag{4.1}$$

To convert the optimal control problem (EP) into a quadratic programming problem, we parameterize each element $\tilde{u}_i(t)$, $i=1,\ldots,2m$, of the control variable $\tilde{u}(t)$ by a finite length Chebyshev series of the first class, $\{T_n(t)=\cos(n\cos^{-1}t)\}$, with unknown parameters. But before approximating the control variables, it is necessary to transform the time horizon $t \in [t_0, t_f]$ of Problem (EP) into the new interval $s \in [-1, 1]$, because the Chebyshev series are defined on the interval $[-1, 1]$. This can be achieved by using the scaling

$$t = \frac{t_f}{2}(s+1).$$

For the simplicity of the notation, we assume that $t_0 = -1$ and $t_f = 1$.

Approximate each element $\tilde{u}_i(t)$, $i = 1,\ldots,2m$, of the control variable $\tilde{u}(t)$ is approximated by a finite Chebyshev series with unknown parameters gives

$$\tilde{u}_i(t \mid \alpha^{N,i}) = \sum_{j=0}^{N} \alpha_j^{N,i} T_j(t), \quad i=1,\ldots,2m, \tag{4.2}$$

where $T_j(t)$ is the jth order Chebyshev polynomial of the first type and $\alpha_j^{N,i}$, $i=1,\ldots,2m$; $j=1,\ldots,N$, $\alpha^{N,i} = [\alpha_1^{N,i}, \alpha_2^{N,i}, \ldots \alpha_N^{N,i}]$ are the unknown parameters. Define

$$\alpha^N = [\alpha_1^{N,1}, \alpha_2^{N,1}, \ldots, \alpha_N^{N,1}, \alpha_1^{N,2}, \alpha_2^{N,2}, \ldots, \alpha_N^{N,2}, \ldots, \alpha_1^{N,2m}, \alpha_2^{N,2m}, \ldots, \alpha_N^{N,2m}]^\top$$

and

$$\tilde{u}(t \mid \alpha^N) = [\tilde{u}_1(t \mid \alpha^{N,1}), \tilde{u}_2(t \mid \alpha^{N,2}), \ldots, \tilde{u}_{2m}(t \mid \alpha^{N,2m})]^\top,$$

where $\tilde{u}_i(t \mid \alpha^{N,i})$, $i=1,\ldots,2m$, are given by (4.2). Clearly, $\tilde{u}(\cdot \mid \alpha^N)$ depends on the choice of α^N. Thus, the solution of (4.1) corresponding to $\tilde{u}(\cdot \mid \alpha^N)$ can be written as $x(\cdot \mid \alpha^N)$. Now, by approximation the control in the form of (4.2), Problem (EP) reduces to the following sequence of semi-infinite quadratic programming problem.

Problem (EP(N)).

$$\begin{aligned} \min \quad & J_N(\alpha^N) \\ \text{s.t} \quad & h_N(\alpha^N, t) \leqslant 0,\ t \in [-1, 1], \end{aligned} \tag{4.3}$$

where $\alpha^N \in \mathbb{R}^{2mN}$,

$$\begin{aligned} J_N(\alpha^N) = {} & x^\top(1 \mid \tilde{u}(t \mid \alpha^N))R_0 x(1 \mid \tilde{u}(t \mid \alpha^N)) \\ & + \int_{-1}^{1}\int_{-1}^{1} x^\top(\tau \mid \tilde{u}(t \mid \alpha^N))R_1(t-\tau)x(t \mid \tilde{u}(t \mid \alpha^N))\,\mathrm{d}\tau\,\mathrm{d}t \\ & + \int_{-1}^{1}\int_{-1}^{1} \tilde{u}^\top(\tau \mid \alpha^N)\tilde{R}_2(t-\tau)\tilde{u}(\tau \mid \alpha^N)\,\mathrm{d}\tau\,\mathrm{d}t, \end{aligned}$$

$$h_N(\alpha^N,t)=\begin{bmatrix}\int_{-1}^{1}\tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau\mid\alpha^N)\,\mathrm{d}\tau-d(t)+\int_{-1}^{1}\theta^\top(t-\tau)\tilde{u}(\tau\mid\alpha^N)\,\mathrm{d}\tau-\varepsilon(t)\\ -\int_{-1}^{1}\tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau\mid\alpha^N)\,\mathrm{d}\tau+d(t)+\int_{-1}^{1}\theta^\top(t-\tau)\tilde{u}(\tau\mid\alpha^N)\,\mathrm{d}\tau-\varepsilon(t)\\ [I_m,-I_m]\tilde{u}(t\mid\alpha^N)-b\\ [-I_m,I_m]\tilde{u}(t\mid\alpha^N)+a\\ \tilde{u}(t\mid\alpha^N)-\beta\\ -\tilde{u}(t\mid\alpha^N)\end{bmatrix}.$$

Theorem 4.1. *Consider Problem (EP(N)). Then,*

$$J_N(\alpha^N)=\tfrac{1}{2}(\alpha^N)^\top Q_N\alpha^N+(p_N)^\top\alpha^N+e_N, \tag{4.4}$$

and

$$h_N(\alpha^N,t)=A_N(t)\alpha^N-q(t), \tag{4.5}$$

where

$$\begin{aligned}Q_N={}&2\left[\int_{-1}^{1}F(1,\tau)(I_{2m}\otimes T^N(\tau))\,\mathrm{d}\tau\right]^\top R_0\int_{-1}^{1}F(1,\tau)(I_{2m}\otimes T^N(\tau))\,\mathrm{d}\tau\\ &+2\int_{-1}^{1}\int_{-1}^{1}\left[\int_{-1}^{t}F(t,s)(I_{2m}\otimes T^N(s))\,\mathrm{d}s\right]^\top R_1(t-\tau)\\ &\times\int_{-1}^{\tau}F(\tau,s)(I_{2m}\otimes T^N(s))\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}t\\ &+2\int_{-1}^{1}\int_{-1}^{1}(I_{2m}\otimes T^N(t))^\top\tilde{R}_2(t-\tau)(I_{2m}\otimes T^N(\tau))\,\mathrm{d}\tau\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}(p_N)^\top={}&2[\Phi(1,-1)x_0]^\top\int_{-1}^{1}F(1,\tau)(I_{2m}\otimes T^N(\tau))\,\mathrm{d}\tau\\ &+2\int_{-1}^{1}\int_{-1}^{1}[\Phi(t,-1)x_0]^\top R_1(t-\tau)\int_{-1}^{\tau}F(\tau,s)(I_{2m}\otimes T^N(s))\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}e_N={}&[\Phi(1,-1)x_0]^\top R_0\Phi(1,-1)x_0\\ &+\int_{-1}^{1}\int_{-1}^{1}[\Phi(t,-1)x_0]^\top R_1(t-\tau)\Phi(\tau,-1)x_0\,\mathrm{d}\tau\,\mathrm{d}t,\end{aligned}$$

$$A_N(t)=\begin{bmatrix}\int_{-1}^{1}[\tilde{\varphi}^\top(t-\tau)+\theta^\top(t-\tau)](I_{2m}\otimes T^N(\tau))\,\mathrm{d}\tau\\ \int_{-1}^{1}[-\tilde{\varphi}^\top(t-\tau)+\theta^\top(t-\tau)](I_{2m}\otimes T^N(\tau))\,\mathrm{d}\tau\\ [I_m,-I_m](I_{2m}\otimes T^N(t))\\ [-I_m,I_m](I_{2m}\otimes T^N(t))\\ I_{2m}\otimes T^N(t)\\ -I_{2m}\otimes T^N(t)\end{bmatrix},$$

$$q(t)=[d(t)+\varepsilon(t),-d(t)+\varepsilon(t),b,-a,\beta,0]^\top,$$

$$T^N(t)=[T_1(t),T_2(t),\ldots,T_N(t)],$$

$$F(t,\tau)=\Phi(t,\tau)\tilde{B}(\tau),$$

and $\otimes$ denotes the Kronecker product.

Proof. Note that (4.2) can be written in terms of the Kronecker product as

$$\tilde{u}(t \mid \alpha^N) = (I_{2m} \otimes T^N(t))\alpha^N,$$

where I_{2m} denotes the $2m \times 2m$ identity matrix. By direct computation and using (4.1), it follows that $J_N(\alpha^N)$ and $h_N(\alpha^N, t)$ are given by (4.4) and (4.5), respectively. $\square$

Remark 4.1. Note that the cost function (4.4) is a strictly positive quadratic function, while the constraints given by (4.5) are linear. Thus, Problem (EP(N)) is a strictly convex quadratic semi-infinite programming problem.

The next theorem shows that the Slater condition is satisfied if (A5) holds.

Theorem 4.2. *Suppose that (A5) is satisfied. Then, there exists an integer N_0 such that for $N > N_0$, the Slater condition is also satisfied for Problem (EP(N)), i.e., there exists an $\alpha^N \in \mathbb{R}^{2mN}$, such that*

$$h_N(\alpha^N, t) = A_N(t)\alpha^N - q(t) < 0 \quad \textit{for all } t \in [-1, 1].$$

Proof. Since (A5) is satisfied, it follows from Theorem 4.2 that there exists a control $\tilde{u} \in C(-1, 1; \mathbb{R}^{2m})$, such that

$$\int_{-1}^{1} \tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - d(t) + \int_{-1}^{1} \theta^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - \varepsilon(t) < 0,$$
$$-\int_{-1}^{1} \tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau + d(t) + \int_{-1}^{1} \theta^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - \varepsilon(t) < 0,$$

and

$$a < [I_m, -I_m]\tilde{u}(t|\alpha^N) < b,$$
$$\beta > \tilde{u}(t)|\alpha^N = [u_1^\top(t|\alpha^N), u_2^\top(t|\alpha^N)]^\top > 0 \quad \text{for all } t \in [t_0, t_f].$$

Since $\tilde{u} \in C(-1, 1; \mathbb{R}^{2m})$ and $\tilde{\varphi}(t)$ and $\theta(t)$ are continuous on the interval $[-1, 1]$, there exists a δ, $\frac{1}{2} > \delta > 0$, such that

$$\int_{-1}^{1} \tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - d(t) + \int_{-1}^{1} \theta^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - \varepsilon(t) \leqslant -\delta,$$
$$-\int_{-1}^{1} \tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau + d(t) + \int_{-1}^{1} \theta^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - \varepsilon(t) \leqslant -\delta,$$
$$\tilde{u}(t|\alpha^N) \geqslant \delta \quad \text{for all } t \in [-1, 1],$$

and

$$\max_{-1 \leqslant t \leqslant 1} \int_{-1}^{1} |\tilde{\varphi}_i(t-\tau) + \theta_i(t-\tau)|\,\mathrm{d}\tau \leqslant \frac{1}{4m\delta}, \quad i = 1, \ldots, 2m,$$
$$\max_{-1 \leqslant t \leqslant 1} \int_{-1}^{1} |-\tilde{\varphi}_i(t-\tau) + \theta_i(t-\tau)|\,\mathrm{d}\tau \leqslant \frac{1}{4m\delta}, \quad i = 1, \ldots, 2m.$$

By Jackson Theorem ([18, p. 19, Chapter 2]), there exists an integer N_0, such that for $N \geqslant N_0$,

$$\max_{-1 \leqslant t \leqslant 1} |\tilde{u}_i(t) - \tilde{u}_i(t|\alpha^{N,i})| \leqslant \delta^2 \leqslant \frac{\delta}{2}, \quad i = 1, \ldots, 2m,$$

and the following inequalities are satisfied:

$$a < [I_m, -I_m]\tilde{u}(t|\alpha^N) < b,$$

$$\beta > \tilde{u}(t|\alpha^N) = [(u^1(t|\alpha^N))^\top, (u^2(t|\alpha^N))^\top]^\top > 0 \quad \text{for all } t \in [t_0, t_f].$$

Therefore, we have

$$\begin{aligned}
&\int_{-1}^{1} \tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - d(t) + \int_{-1}^{1} \theta^\top(t-\tau)\tilde{u}(\tau|\alpha^N)\,\mathrm{d}\tau - \varepsilon(t)\\
&\quad = \sum_{i=1}^{2m}\int_{-1}^{1} (\tilde{\varphi}_i(t-\tau) + \theta_i(t-\tau))(\tilde{u}_i(\tau|\alpha^{N,i}) - (\tilde{u}_i(t|\alpha^{N,i}) - \tilde{u}_i(t|\alpha^{N,i})))\,\mathrm{d}\tau - d(t) - \varepsilon(t)\\
&\quad \leqslant \int_{-1}^{1} (\tilde{\varphi}^\top(t-\tau) + \theta^\top(t-\tau))\tilde{u}(\tau|\alpha^N) - d(t) - \varepsilon(t)\\
&\qquad + \sum_{i=1}^{2m}\int_{-1}^{1} |\tilde{\varphi}_i(t-\tau) + \theta_i(t-\tau)||\tilde{u}_i(t|\alpha^{N,i}) - \tilde{u}_i(t|\alpha^{N,i})|\\
&\quad \leqslant \int_{-1}^{1} (\tilde{\varphi}^\top(t-\tau) + \theta^\top(t-\tau))\tilde{u}(\tau) - d(t) - \varepsilon(t) + \sum_{i=1}^{2m}\frac{1}{4m\delta}\cdot\delta^2\\
&\quad \leqslant -\delta + \frac{\delta}{2} = -\frac{\delta}{2}.
\end{aligned}$$

By the same token, we obtain

$$-\int_{-1}^{1} \tilde{\varphi}^\top(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau + d(t) + \int_{-1}^{1} \theta^\top(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau - \varepsilon(t) \leqslant -\frac{\delta}{2}.$$

This completes the proof. □

Since e_N is a constant in Problem (EP(N)), it can be dropped. Hence, we only need to consider the following equivalent problem:

Problem (MEP(N)).

$$\min \quad J_N(\alpha^N) = \tfrac{1}{2}(\alpha^N)^\top Q_N \alpha^N + p_N^\top \alpha^N \tag{4.6}$$

$$\text{s.t.} \quad h_N(\alpha^N, t) = A_N(t)\alpha^N - q(t) \leqslant 0. \tag{4.7}$$

The Dorn's dual of Problem (MEP(N)) is

Problem (DMEP(N)).

$$\min_{\alpha^N, v} \quad \frac{1}{2}(\alpha^N)^\top Q_N \alpha^N + \int_{-1}^{1} q^\top(t)\,\mathrm{d}v_N(t) \tag{4.8}$$

$$\text{s.t.} \quad Q_N\alpha^N + p_N + \int_{-1}^{1} A_N^\top(t)\,\mathrm{d}v_N(t) = 0, \tag{4.9}$$

$$v_N \in M^+([-1,1]), \quad \alpha^N \in \mathbb{R}^{2mN},$$

where $M^+([-1,1])$ is the set of all nonnegative bounded regular Borel measures on $[-1,1]$.

To proceed further, we need some crucial results obtained in [9]. They are presented in the following theorem.

Theorem 4.3. *Let the Slater constraint qualification be satisfied. The minimum of Problem (MEP(N)) is achieved at* α^N *if and only if* α^N *is feasible and there exists a* $v_N^* \in M^+([-1,1])$, *such that*

$$Q_N\alpha^N + p_N + \int_{-1}^{1} A_N^\top(t)\,\mathrm{d}v_N^*(t) = 0,$$

$$\int_{-1}^{1} (A_N^\top(t)\alpha^N - q(t))\,\mathrm{d}v_N^*(t) = 0,$$

$$v_N^* \geqslant 0.$$

Proof. See the proof given for Lemma 2.1 of [9]. □

Theorem 4.4. *Let Assumption* (*A5*) *be satisfied, and assume that the minimum of Problem* (*MEP*(*N*)) *is achieved at* α^N. *Then the solution of the dual problem* (*DMEP*(*N*)) *contains a solution* (α^N, v_N^*) *of which the measure* v_N^* *has a finite support of no more than 2mN points.*

Proof. See the proof given for Theorem 2.1 of [9] . □

On the basis of the above theorems, solving the semi-infinite programming problem (MEP(N)) is equivalent to solving the dual problem (DMEP(N)), which is a finite dimensional optimization problem. If we restrict the discrete measure with finite support of $k \leqslant 2mN$ points, then Problem (DMEP(N)) is reduced to the following Problem $((\mathrm{DMEP(N)})_k$.

Problem $((\mathrm{DMEP(N)})_k$.

$$\min_{\alpha^N, v, \tau} \quad J_N^k(\alpha^N, v_N, \tau_k) = \frac{1}{2}(\alpha^N)^\top Q_N \alpha^N + \sum_{i=1}^{k} q(t_i)^\top v_i$$

$$\text{s.t.} \quad Q_N\alpha^N + p_N + \sum_{i=1}^{k} A_N^\top(t_i) v_i = 0,$$

$$v_i \geqslant 0, \quad -1 \leqslant t \leqslant 1, \quad i = 1, 2, \ldots, k,$$

where $v_i = v(t_i)$, $\tau_k = (t_1, t_2, \ldots, t_k)$ and $v_N = (v_1, v_2, \ldots, v_k)$.

Once a solution $(\alpha^N, v_N^*, \tau_k^*)$ of Problem $(\mathrm{DMEP(N)})_k$ which satisfies the condition in Step 7 of Algorithm, is achieved, then $\tilde{u}(t|\alpha^N) = (I_{2m} \otimes T^N(t))\alpha^N$ is an optimal solution of Problem (EP(N)).

Next theorem presents the relationship between the solution of Problem (EP) and that of Problem (EP(N)).

Theorem 4.5. *Let* $\tilde{u}^*(t)$ *be the optimal solution of Problem* (*EP*), *and* α^N, *the optimal solution of Problem* (*EP*(*N*)). *Then,* $J(\alpha^N) \to \tilde{J}(\tilde{u}^*(t))$ *as* $N \to \infty$. *For sufficiently large N,* $\tilde{u}(t|\alpha^N) = (I_{2m} \otimes T^N(t))\alpha^N$ *can be viewed as an optimal solution of Problem* (*EP*).

Proof. Let $\alpha^{*,N} = [\alpha^{1,*}, \alpha^{2,*}, \ldots, \alpha^{N,*}]$, $\tilde{u}^{*,N}(t) = \sum_{n=1}^{N} \alpha^{n,*} T_n(t)$. It is clear that

$$\tilde{J}(\tilde{u}^*(t)) \leqslant \tilde{J}(\tilde{u}^{*,N}(t)) = J(\alpha^N) \leqslant J(\alpha^{*,N}).$$

On the other hand, we know that $\tilde{J}(\tilde{u}^{*,N}(t)) \to \tilde{J}(\tilde{u}^*(t))$. Thus, the proof is complete. □

We now propose an algorithm to solve Problem (P) as follows:

Algorithm
1. Transform Problem (P) with nonsmooth inequality constraints into equivalent Problem (EP).
2. Rescale time $t \in [t_0, t_f]$ into the new time scale $s \in [-1, 1]$.
3. Set $N = 1$.
4. Approximate the control variable $\tilde{u}(t)$ in the form of (4.2). Then, express Problem (EP) in the form of (4.5) and (4.6).
5. Set k = 1.
6. Compute the optimal solution $(\alpha^N, v_N^{k*}, \tau_k^*)$ and $J_N^k(\alpha^N, v_N^{k*}, \tau_k^*)$ of Problem $((\mathrm{DMEP(N)})_k$.
7. If $k \geqslant 2$ and $|J_N^k - J_N^{k-1}|$ is sufficiently small, set $\alpha^N = \alpha^N$ and goto Step 8, otherwise, set $k = k + 1$, goto Step 6.
8. Compute $J_N(\alpha^N)$.
9. If $N \geqslant 2$ and $|J_N(\alpha^N) - J_{N-1}(\alpha^N)|$ is sufficiently small, goto Step 10. Otherwise, set $N = N + 1$, goto Step 4.
10. Let $\tilde{u} = (I_{2m} \otimes T^N(t))\alpha^N$. Then,$u = [I_m, -I_m]\tilde{u}$ is an approximate optimal solution of Problem (P).

5. Numerical example

To illustrate the efficiency of the proposed method, we will present a numerical example in this section.

Example 5.1. *Given the dynamical system*

$$\dot{x}_1(t) = x_2(t), \tag{5.1a}$$

$$\dot{x}_2(t) = -x_1(t) + u(t), \tag{5.1b}$$

with initial condition

$$x_1(-1) = 0, \quad x_2(-1) = -1. \tag{5.1c}$$

The cost functional is

$$J(u) = x_1^2(1) + x_2^2(1) + \int_{-1}^{1}\int_{-1}^{1} u(t)e^{-(t-\tau)^2}u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t + \int_{-1}^{1}\int_{-1}^{1}(x_1(t) + x_2(t))\cos\left(\frac{t-\tau}{2}\right)(x_1(\tau) + x_2(\tau))\,\mathrm{d}\tau\,\mathrm{d}t. \tag{5.2}$$

Our problem is to find a control $u \in C[t_0, t_f]$, such that $J(u)$ is minimized subject to the dynamical system (5.1a)–(5.1c), *and the following functional nonsmooth inequality constraint*:

$$\int_{-1}^{1}(t-\tau)^2|u(\tau)|\,\mathrm{d}\tau \leqslant 1. \tag{5.3}$$

We transform it into the following equivalent optimal control problem with smooth inequality constraints.

Problem (EEP). The dynamical system is

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix},$$

with initial condition

$$x(0) = [0, -1]^\top.$$

The cost functional is

$$\tilde{J}(\tilde{u}) = x^\top(1)R_0x(1) + \int_{-1}^{1}\int_{-1}^{1} x(t)R_1(t-\tau)x(\tau)\,\mathrm{d}\tau\,\mathrm{d}t + \int_{-1}^{1}\int_{-1}^{1}\tilde{u}(t)R_2(t-\tau)\tilde{u}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$

and the constraints are

$$\int_{-1}^{1}(t-\tau)^2[1,1]\tilde{u}(\tau)\,\mathrm{d}\tau \leqslant 1,$$

$$\tilde{u}(t) \geqslant 0, \quad t \in [-1,1],$$

where

$$x(t)=[x_1(t),x_2(t)]^\top, \quad \tilde{u}(t)=[u_1(t),u_2(t)]^\top,$$

$$R_0=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad R_1(t-\tau)=\begin{bmatrix}\cos\left(\dfrac{t-\tau}{2}\right) & \cos\left(\dfrac{t-\tau}{2}\right)\\ \cos\left(\dfrac{t-\tau}{2}\right) & \cos\left(\dfrac{t-\tau}{2}\right)\end{bmatrix},$$

$$R_2(t-\tau)=\begin{bmatrix}\mathrm{e}^{-(t-\tau)^2} & -\mathrm{e}^{-(t-\tau)^2}\\ -\mathrm{e}^{-(t-\tau)^2} & \mathrm{e}^{-(t-\tau)^2}\end{bmatrix}.$$

We approximate the control variable $\tilde{u}(t)$ in Problem (EEP) by an Nth order Chebyshev series, and the trajectory $x(t)$ is obtained from (4.1). Problem (EEP) is transformed into the standard form expressed by (4.4) and (4.5) as stated in Theorem 4.1. Then, by using the dual parametrization method developed in [7], Problem (EEP) can be solved. We solve the problem for $N=6, 8, 10$. The results are given as follows:

Case $N=6$: The value of the cost functional is 1.83117801, $\alpha_6^*=[0.0369, -0.0390, 0.0484, -0.0065, -0.00258, -0.0283, -0.0061, -00.0514, -0.0484, -0.0065, -0.0258, 0.0285]$;

Case $N=8$: The value of the cost functional is 1.83115079, α_8^*=$[0.0337, -0.0382, 0.0528, 0.0051, -0.0264, -0.0296, 0.0931, 0.0500, 0.0159, -0.0412, -0.0528, -0.0051, -0.0264, 0.0296, -0.0931, 0.0500]$.

Case $N=10$: The value of the cost functional is 1.83115078, $\alpha_{10}^*=[0.0342, -0.0379, 0.0528, 0.0051, -0.0262, -0.0298, 0.0932, 0.0478, 0.0204, -0.0511, -0.0582, -0.0047, -0.0265, 0.0287, -0.0932, 0.0501, 0.0004, -0.0012, 0.0001, 0.0003]$.

The corresponding optimal controls obtained are depicted in Figs. 1–3.

From the numerical experience, the method is developed in the paper is effective.

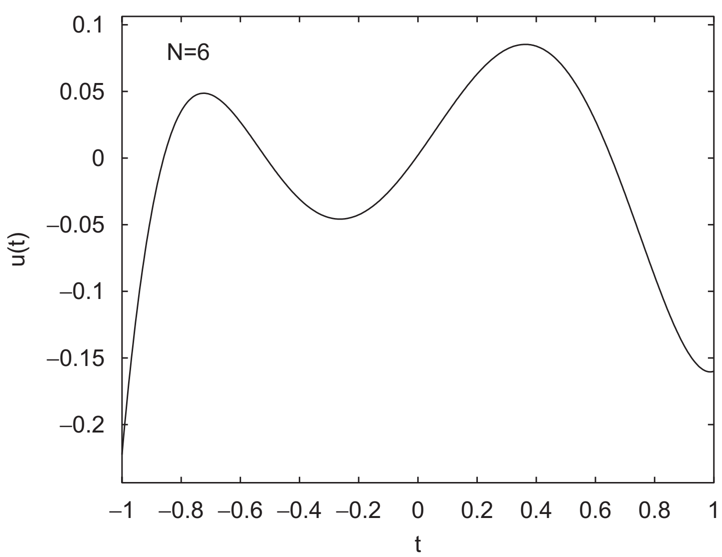

Fig. 1. The optimal control when $N=6$.

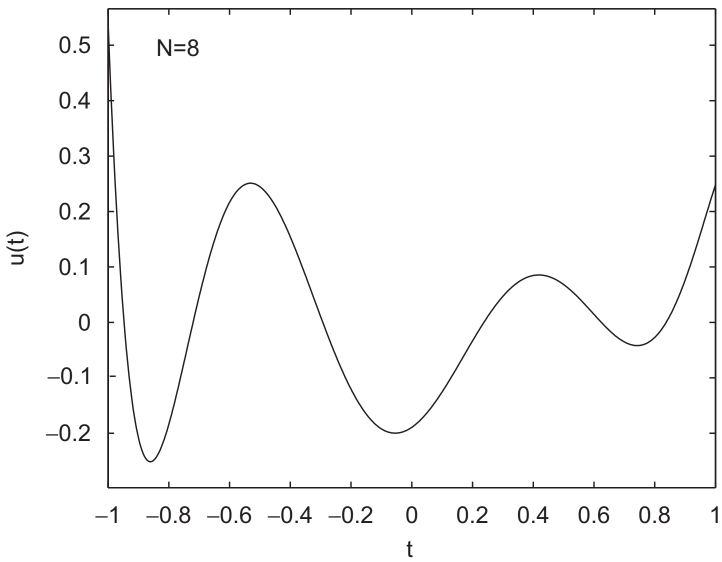

Fig. 2. The optimal control when $N = 8$.

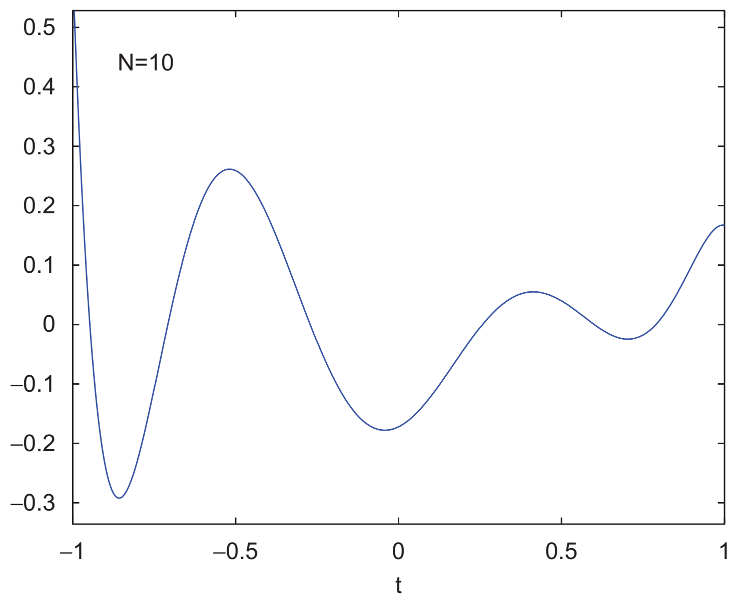

Fig. 3. The optimal control when $N = 10$.

6. Conclusion

In this paper, we considered an optimal control problem subjected to nonsmooth continuous inequality constraints. We first showed that it is equivalent to an optimal control problem subject to smooth continuous inequality constraints. Then, we developed an efficient algorithm to solve the equivalent problem using the parametrization of control variable by finite Chebyshev series expansion. A numerical example was then solved using the method proposed. We see that the method is highly efficient.

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