On prescribed change of profile for solutions of parabolic equations *

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Abstract

Parabolic equations with homogeneous Dirichlet conditions on the boundary are studied in a setting where the solutions are required to have a prescribed change of the profile in fixed time, instead of a Cauchy condition. It is shown that this problem is well-posed in L_2-setting. Existence and regularity results are established, as well as an analog of the maximum principle.

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1 Introduction

Parabolic diffusion equations have fundamental significance for natural and social sciences, and various boundary value problems for them were widely studied including inverse and ill-posed problems; see examples in Miller (1973), Tikhonov and Arsenin (1977), Glasko (1984), Prilepko *et al* (1984), Beck (1985), Seidman (1996). According to Hadamard criterion, a boundary value problem is well-posed if there is existence and uniqueness of

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the solution, and if there is continuous dependence of the solution on the boundary data. Otherwise, a problem is ill-posed.

Apparently there are boundary value problems that do not fit the framework given by the classical theory of well-posedness (see examples in Dokuchaev (2007,2010)).

For parabolic equations, it is commonly recognized that the type of the boundary conditions usually defines if a problem is well-posed or ill-posed. A classical example is the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t \in [0, T].$$

The problem for this equation with the Cauchy condition at initial time $t = 0$ is well-posed in usual classes of solutions, including classical, Hölder and square integrable solutions. In contrast, the problem with the Cauchy condition at terminal time $t = T$ is ill-posed for this heat equation for all these classes. In particular, this means that a prescribed profile of temperature at time $t = T$ cannot be achieved via an appropriate selection of the initial temperature. In addition, L_2-norms of solutions cannot be estimated by L_2-norms of the boundary data (i.e, the dependence on boundary data is not continuous). This makes this problem ill-posed, despite the fact that solvability and uniqueness still can be achieved for some very smooth analytical boundary data or for special selection of the domains (see, e.g., Miranker (1961), Dokuchaev (2007, 2010)).

The paper investigates parabolic equation with homogeneous Dirichlet boundary condition on the boundary of a domain $D \subset \mathbf{R}^n$ and with mixed in time condition that connects the values of solutions at different times, similarly to the setting introduced in Dokuchaev (2008) for stochastic equations. The present paper considers a special mixed in time conditions requiring that the solutions have a prescribed change of profile in fixed time. Formally, this problem does not fit the framework given by the classical theory of well-posedness for parabolic equations based on the correct selection of Cauchy condition. However, it is shown below that this problem is well-posed in L_2-setting, and that some analog of Maximum Principle holds. In addition, it is shown that, for any nonnegative and non-trivial function $\gamma \in L_2(D)$, there exists a unique non-negative initial function $p(\cdot, 0)$ and a number $\alpha > 0$ such that $p(x, 0) \equiv p(x, T) + \alpha\gamma(x)$ and such that $\int_D p(x, 0)dx = 1$. This can be interpreted as an existence of a diffusion with prescribed change of the concentration profile. An interesting consequence is that, in the model of heat propagation, a prescribed change of temperature during time interval $[0, T]$ can be achieved via selection of some appropriate initial temperature, and this problem is well-posed. On the contrary, a prescribed profile of temperature at time $t = T$ cannot be achieved via selection of the initial temperature; this problem is ill-posed.

2 Definitions

Let $D \subset \mathbf{R}^n$ be an open bounded domain with C^2 - smooth boundary ∂D. The case when D is not connected or not simply connected is not excluded.

Let $T > 0$ be a fixed number. We consider the boundary value problems

$$\begin{aligned} \tfrac{\partial u}{\partial t} &= Au, \qquad \text{for} \quad (x,t) \in D \times (0,T) \\ u(x,t) &= 0, \qquad \text{for} \quad (x,t) \in \partial D \times (0,T) \end{aligned} \tag{2.1}$$

with some additional conditions imposed at times $t = 0$ or $t = T$. Here

$$Au \triangleq \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) + \sum_{i=1}^{n} f_i(x,t) \frac{\partial u}{\partial x_i}(x,t) - q(x,t)u(x,t).$$

The functions $f(x,t) : D \times (0,T) \to \mathbf{R}^n$ and $q(x,t) : D \times (0,T) \to [0,+\infty)$ are measurable and bounded, such that there exist bounded derivatives $\partial f(x,t)/\partial x_i$, $i = 1,...,n$. The function $a(x,t) : D \times (0,T) \to \mathbf{R}^{n\times n}$ is continuous, bounded, and such that there exist bounded derivatives $\partial a(x,t)/\partial x_i$, $i = 1,...,n$. In addition, we assume that the matrix $a(x,t)$ is symmetric and $a(x,t) \geq \delta I_n$ for all $(x,t) \in D \times (0,T)$, where $\delta > 0$ is a constant, and I_n is the unit matrix in $\mathbf{R}^{n\times n}$.

Problem (2.1) describes diffusion processes in domain D that are absorbed (killed) on the boundary and, with some rate, inside D. The matrix a represents the diffusion coefficients, the vector f describes the drift (advection), and q describes the rate of absorption inside D. The assumption that $q \geq 0$ ensures that there is absorption (loss of energy) inside the domain rather than generation of energy.

Spaces and classes of functions

For a Banach space X, we denote the norm by $\| \cdot \|_X$.

Let $H^0 \triangleq L_2(D)$ and $H^1 \triangleq \overset{0}{W}{}_2^1(D)$ be the standard Sobolev Hilbert spaces; H^1 is the closure in the $W_2^1(D)$-norm of the set of all smooth functions $u : D \to \mathbf{R}$ such that $u|_{\partial D} \equiv 0$.

Let H^{-1} be the dual space to H^1, with the norm $\| \cdot \|_{H^{-1}}$ such that if $u \in H^0$ then $\|u\|_{H^{-1}}$ is the supremum of $(u,v)_{H^0}$ over all $v \in H^1$ such that $\|v\|_{H^1} \leq 1$. H^{-1} is a Hilbert space.

We denote the Lebesgue measure and the σ-algebra of Lebesgue sets in $\mathbf{R}^n$ by $\bar{\ell}_n$ and $\bar{\mathcal{B}}_n$, respectively.

Introduce the spaces

$$C(s,T) \triangleq C\left([s,T];H^0\right), \quad \mathcal{W}^1(s,T) \triangleq L^2([s,T],\bar{\mathcal{B}}_1,\bar{\ell}_1;H^1),$$

and the space

$$\mathcal{V}(s,T) \triangleq \mathcal{W}^1(s,T)\cap C(s,T),$$

with the norm $\|u\|_{\mathcal{V}(s,T)} \triangleq \|u\|_{\mathcal{W}^1(s,T)} + \|u\|_{C(s,T)}$.

We denote the space $\mathcal{V}(0,T)$ as $\mathcal{V}$.

Definition 1 We say that equation (2.1) is satisfied for $u \in \mathcal{V}$ if, for any $t \in [0,T]$,

$$u(\cdot,t) = u(\cdot,0) + \int_0^t Au(\cdot,s)ds. \tag{2.2}$$

The equality here is assumed to be an equality in the space H^{-1}.

Note that the condition on ∂D is satisfied in the sense that $u(\cdot,t) \in H^1$ for a.e. t. Further, $Au(\cdot,s) \in H^{-1}$ for a.e. s. Hence the integral in (2.2) is defined as an element of H^{-1}. Therefore, Definition 1 requires that this integral is equal to an element of H^0 in the sense of equality in H^{-1}.

3 The result

Theorem 1 *For any $\gamma \in L_2(D)$, there exists a unique solution $u \in \mathcal{V}$ of (2.1) such that*

$$u(\cdot,0) = u(\cdot,T) + \gamma. \tag{3.1}$$

Moreover, there exists $c > 0$ such that

$$\|u\|_{\mathcal{V}} \le c\|\gamma\|_{L_2(D)} \tag{3.2}$$

for all $\gamma \in L_2(D)$.

Note that, for $u \in \mathcal{V}$, the value of $u(\cdot,t)$ is uniquely defined in $L_2(D)$ given t, by the definitions of the corresponding spaces. This makes condition (3.1) meaningful as an equality in $L_2(D)$. By Theorem 1, problem (2.1),(3.1) is well-posed in the sense of Hadamard.

Theorem 2 *For any non-negative and non-trivial $\gamma \in L_2(D)$, the solution $u \in \mathcal{V}$ in Theorem 1 is non-negative in $D \times (0,T)$, and there exists a number $\alpha = \alpha(\gamma) > 0$ and a unique nonnegative solution $p \in \mathcal{V}$ of (2.1) such that*

$$p(\cdot,0) = p(\cdot,T) + \alpha\gamma, \qquad \int_D p(x,0)dx = 1. \tag{3.3}$$

Moreover, there exists $c > 0$ such that

$$\|p\|_{\mathcal{V}} \leq c\alpha\|\gamma\|_{L_2(D)} \tag{3.4}$$

for all $\gamma \in L_2(D)$.

The statement in Theorem 2 regarding non-negativeness of the solution is an analog of the Maximum Principle known for classical Dirichlet problems for parabolic equations (see, e.g. Chapter III in Ladyzhenskaja *et al* (1968)).

Theorem 2 can be applied to the model of heat propagation in D, with the loss of energy on the boundary and inside D with the rate defined by q. The process $p(x,t)$ can be interpreted as the temperature at point $x \in D$ at time t. Therefore, Theorem 2 establishes existence of the initial temperature $p(x,0)$ that ensures the prescribed change of temperature during time interval $[0,T]$.

4 Proofs

For $s \in [0,T)$ and $\xi \in H^0$, consider the following auxiliary boundary value problem:

$$\begin{aligned} &\tfrac{\partial v}{\partial t} = Av, && \text{for} \quad (x,t) \in D \times (s,T) \\ &v(x,t) = 0, && \text{for} \quad (x,t) \in \partial D \times (s,T) \\ &v(x,s) = \xi(x) && \text{for} \quad x \in D. \end{aligned}$$

Introduce operators $\mathcal{L}_s : H^0 \to \mathcal{V}(s,T)$, such that $\mathcal{L}_s\xi = v$, where v is the solution in $\mathcal{V}(s,T)$ of this problem. These linear operators are continuous (see, e.g., Theorem III.4.1 in Ladyzhenskaja *et al* (1968)). Introduce an operator $\mathcal{Q} : H^0 \to H^0$, such that $\mathcal{Q}\xi = v(\cdot,T)$, where $v = \mathcal{L}_0\xi$. Clearly, this operator is linear and continuous.

Lemma 1 (i) *The operator $\mathcal{Q} : H^0 \to H^0$ is compact;*

(ii) *If the equation $\mathcal{Q}\xi = \xi$ has the only solution $\xi = 0$ in H^0, then the operator $(I - \mathcal{Q})^{-1} : H^0 \to H^0$ is continuous.*

Proof of Lemma 1. Let $\xi \in H^0$ and $v \triangleq \mathcal{L}_0\xi$, i.e. v is the solution of the problem (4.1). We have that $v = \mathcal{L}_s v(\cdot, s)$ in $D \times (s, T)$ for all $s \in [0, T]$. From the second fundamental inequality for parabolic equations, it follows that

$$\|v(\cdot, T)\|_{H^1} \le C_1 \|v(\cdot, s)\|_{H^1}, \tag{4.1}$$

where C_1 is a positive number that is independent from ξ and s (see, e.g., Theorem IV.9.1 in Ladyzhenskaja *et al* (1968)). Hence

$$\begin{aligned}\|v(\cdot, T)\|_{H^1} \le C_1 \inf_{t\in[0,T]} \|v(\cdot, t)\|_{H^1} \le \frac{C_1}{\sqrt{T}} \left(\int_0^T \|v(\cdot, t)\|^2_{H^1} dt \right)^{1/2} &\le \frac{C_2}{\sqrt{T}} \|v\|_{\mathcal{W}^1(0,T)} \\ &\le \frac{C_3}{\sqrt{T}} \|\xi\|_{H^0},\end{aligned}$$

for some $C_i > 0$ that are independent from ξ. Hence the operator $\mathcal{Q} : H^0 \to H^1$ is continuous. The embedding of H^1 to H^0 is a compact operator (see e.g. Yosida (1965), Ch. 10.3). Then statement (i) follows. Statement (ii) follows from Fredholm Theorem. This completes the proof of Lemma 1. □

Proof of Theorem 1. For $\varphi \in L_2(Q)$, consider the problem

$$\begin{aligned}\tfrac{\partial u}{\partial t} = Au + \varphi, \quad &\text{for} \quad (x, t) \in D \times (s, T) \\ u(x, t) = 0, \quad &\text{for} \quad (x, t) \in \partial D \times (s, T) \\ u(x, 0) = u(x, T) \quad &\text{for} \quad x \in D.\end{aligned} \tag{4.2}$$

By Theorem 2.2 from Dokuchaev (2004), there exists $c > 0$ such that, for any solution $u \in \mathcal{V}$,

$$\|u\|_{\mathcal{V}} \le c\|\varphi\|_{L_2(Q)} \quad \forall \varphi \in L_2(Q).$$

Therefore, if $\gamma = 0$ then the only solution of (3.1) in $\mathcal{V}$ is $u = 0$. By Lemma 1, it follows that the operator $(I - \mathcal{Q})^{-1} : H^0 \to H^0$ is continuous. It follows that, for any $\gamma \in H^0$, there exists $\zeta = (I - \mathcal{Q})^{-1}\gamma \in H^0$, and this ζ is unique. Let $u \triangleq \mathcal{L}_0\zeta$. By the definitions of $\mathcal{L}_0$ and $\mathcal{Q}$, it follows that $u(\cdot, T) = \mathcal{Q}u(\cdot, 0)$. We have that $u(\cdot, 0) - u(\cdot, T) = \gamma$, i.e.,

$$u(\cdot, 0) - \mathcal{Q}u(\cdot, 0) = \gamma.$$

Thus, $u \triangleq \mathcal{L}_0\zeta = \mathcal{L}_0(I - \mathcal{Q})^{-1}\gamma$ is the unique solution of (3.1) for any $\gamma \in H^0 = L_2(D)$. Estimate (3.2) follows from the continuity of operators $(I - \mathcal{Q})^{-1} : H^0 \to H^0$ and $\mathcal{L}_0 : H^0 \to \mathcal{V}$. The uniqueness follows from estimate (3.2). This completes the proof of Theorem 1. □

Proof of Theorem 2. The following definition will be useful.

Definition 2 *A function* $\gamma : D \to \mathbf{R}$ *is said to be piecewise continuous if there exists a integer* $N > 0$ *and a set of open domains* $\{D_i\}_{i=1}^N$ *such that the following holds:*

- $\cup_{i=1}^N D_i \subseteq D \subseteq \cup_{i=1}^N \bar{D}_i$, *and* $D_i \cap D_j = \emptyset$ *for* $i \neq j$. *Here* $\bar{D}_i = D_i \cup \partial D_i$.
- *For any* $i \in \{1, ..., N\}$, *the function* $\gamma|_{D_i}$ *is continuous and can be extended as a continuous function* $\bar{\gamma}_i : \bar{D}_i \cup \partial D_i \to \mathbf{R}$.
- *For any* $x \in \cup_{i=1}^N \partial D_i$, *there exists* $j \in \{1, ..., N\}$ *such that* $x \in \partial D_j$ *and* $\bar{\gamma}_j(x) = \gamma(x)$.

Clearly, the set of piecewise continuous functions is everywhere dense in $L_2(D)$, and the set of non-negative functions is closed in $L_2(D)$. Therefore, it suffices to consider piecewise continuous functions γ only.

Let $\gamma(x) \geq 0$ be a piecewise continuous function, and let $u \triangleq \mathcal{L}_0(I - \mathcal{Q})^{-1}\gamma$ be the solution of problem (3.1). Since the operator $\mathcal{L}_0 : H^0 \to \mathcal{V}$ is continuous, we have that $\|u\|_{\mathcal{V}} \leq c\|u(\cdot, 0)\|_{L_2(D)}$ for some $c > 0$. It follows that if $u(\cdot, 0) = 0$ then $u(\cdot, T) = 0$ and $\gamma = 0$. By the assumptions, $\gamma \neq 0$. Hence $u(\cdot, 0) \neq 0$ and $u \neq 0$.

Remind that $u = \mathcal{L}_0\zeta$, where $\zeta = u(\cdot, 0) \in H^0$. By Theorem III.8.1 from Ladyzhenskaja *et al* (1968), it follows that, for any $\varepsilon > 0$, we have that $\operatorname{ess\,sup}_{(x,t)\in Q'} |u(x, t)| \leq c_0$, where $Q' = \{(x, t) \in Q : \ t > \varepsilon\}$, and where $c_0 > 0$ depends only on ε, a, f, q, D, and $\|u(\cdot, 0)\|_{L_2(D)}$. We use here the part of the cited theorem that deals with solutions that are bounded on a part of the boundary; in our case, the solution vanishes on $\partial D \times (0, T]$. It follows that

$$\|u(\cdot, T)\|_{L_\infty(D)} \leq c_1, \tag{4.3}$$

where $c_1 > 0$ depends only on a, f, q, D, and $\|u(\cdot, 0)\|_{L_2(D)}$.

Consider a sequence of functions $u_i \in \mathcal{V}$ being solutions of (2.1) such that $u_i(\cdot, 0) \in C^2(\bar{D})$, where $\bar{D} = D \cup \partial D$, $u_i|_{\partial D} = 0$, and that $\|u(\cdot, 0) - u_i(\cdot, 0)\|_{L_2(D)} \to 0$ as $i \to +\infty$. By Theorem IV.9.1 from Ladyzhenskaja *et al* (1968), $u_i(\cdot, T) \in C(\bar{D})$. (More precisely, there exists a representative $\bar{u}_i(\cdot, T)$ of the corresponding element of $H^0 = L_2(D)$ which is a class of $\bar{\ell}_n$-equivalent functions). By (4.3) and by the linearity od the problem, we have that $\|u(\cdot, T) - u_i(\cdot, T)\|_{L_\infty(D)} \to 0$ as $i \to +\infty$. Since the set $C(\bar{D})$ is closed in $L_\infty(D)$, it follows that there exists a representative $\bar{u}$ of the corresponding element of $\mathcal{V}$ such that $u(\cdot, T)$ is continuous in $\bar{D}$. We have that $u(\cdot, 0) = u(\cdot, T) + \gamma$, hence there exists a piecewise continuous representative of $u(\cdot, 0) \in H^0$.

Let us show that $\operatorname{ess\,inf}_{x\in D} u(x, 0) \geq 0$. Suppose that

$$\operatorname*{ess\,inf}_{x\in D} u(x, 0) < 0. \tag{4.4}$$

If (4.4) holds, then there exists a piecewise continuous representative $\bar{u}(\cdot,0)$ of $u(\cdot,0)$, such that there exists $\widehat{x} \in D$ such that

$$\bar{u}(\widehat{x},0) < 0, \quad \bar{u}(\widehat{x},0) \le \bar{u}(x,0) \quad \text{for a.e. } x \in D.$$

Let $\widehat{v} \triangleq \bar{u}(\widehat{x},0)$ be considered as an element of $L_2(D)$. We have that

$$\bar{u}(\cdot,T) = \mathcal{Q}\bar{u}(\cdot,0) = \mathcal{Q}\widehat{v} + \mathcal{Q}(\bar{u}(\cdot,0) - \widehat{v}). \tag{4.5}$$

By the assumptions, $\bar{u}(\cdot,0) - \widehat{v} \ge 0$. Let us show that

$$(\mathcal{Q}[\bar{u}(\cdot,0) - \widehat{v}])(\widehat{x}) > 0. \tag{4.6}$$

For this, it suffices to show that $\bar{u}(\cdot,0) \ne \widehat{v}$, since a nonnegative solution of parabolic equation (2.1) is either identically zero or strictly positive everywhere in $D \times (0,T]$. Suppose that $\bar{u}(x,0) \equiv \widehat{v}$. By the Maximum Principle for parabolic equations (see, e.g. Theorem III.7.2 from Ladyzhenskaja *et al* (1968)), $\bar{u}(x,T) = (\mathcal{Q}\widehat{v})(x) \ge \widehat{v}$ for all x; we apply the version of Maximum Principle for non-positive solutions given that $\widehat{v} < 0$. It follows that if $\bar{u}(x,0) \equiv \widehat{v}$ then $\bar{u}(x,t)$ does not satisfy (3.1) with non-negative $\gamma \ne 0$. Thus, $\bar{u}(\cdot,0) \ne \widehat{v}$. Hence (4.6) holds.

Further, by the Maximum Principle for non-positive solutions again, it follows that

$$(\mathcal{Q}\widehat{v})(\widehat{x}) \ge \widehat{v} = \bar{u}(\widehat{x},0). \tag{4.7}$$

By (4.5)-(4.7), we have that $\bar{u}(\widehat{x},T) > \bar{u}(\widehat{x},0)$. It follows that if (4.4) holds then $\bar{u}$ does not satisfy (3.1) with $\gamma(x) \ge 0$. Thus, $u(x,0) \ge 0$ a.e.

Let

$$\alpha \triangleq \left(\int_D u(x,0)dx \right)^{-1}, \quad p \triangleq \alpha u.$$

We have that (3.3) holds. By the linearity of problem (3.1), it follows that and $p(\cdot,0) - p(\cdot,T) = \alpha\gamma$ and that (2.1) holds for p. Therefore, p is such as required. Estimate (3.4) follows immediately from estimate (3.2) and from the selection $p = \alpha u$.

Finally, let us show that p is unique. Let (p_i, α_i) be such that (2.1), (3.3) hold for $p_i \in \mathcal{V}$, $\alpha_i > 0$, $i = 1,2$. Let $u_i = p_i/\alpha_i$. Clearly, u_i is solution of (3.1). By the uniqueness established in Theorem 1, we have that $u_1 = u_2$. Hence $p_1 = p_2\alpha_2/\alpha_1$. If $\alpha_1 \ne \alpha_2$, then it is not possible to have that $\int_D p_1(x,0)dx = \int_D p_2(x,0)dx = 1$. Therefore, $\alpha_1 = \alpha_2$ and $p_1 = p_2$. This completes the proof of Theorem 2. $\square$

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