GEOPHYSICS, VOL. 69, NO. 2 (MARCH-APRIL 2004); P. 393–397, 1 FIG.
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Short Note

A simple derivation of the effective stress coefficient for seismic velocities in porous rocks

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INTRODUCTION

The effect of confining stress and pore pressure on seismic velocities is important for such geophysical applications as overpressure prediction from seismic data (Eaton, 1975; Dutta, 2002; Huffman, 2002; Sayers et al., 2002) and, more recently, for hydrocarbon production monitoring using time-lapse seismic measurements (Tura and Lumley, 1999; Landrø, 2001). The dependence of seismic velocity on pressure has been confirmed for a variety of rocks by laboratory measurements of elastic wave velocities in samples with varying pressure in pore fluids (see, e.g., Wyllie et al., 1958; Todd and Simmons, 1972; Eberhart-Phillips et al., 1989; Prasad and Manghnani, 1997). In general, for a rock subjected to a given confining stress σ_c, higher pore pressures P correspond to lower compressional and shear velocities. Confining stress has a similar effect (but with opposite sign) on seismic velocities. Since both confining stress and pore pressure vary in the subsurface, knowledge of acoustic velocities in rocks as functions of both confining stress and pore pressure

$$v_{p,s} = v_{p,s}(\sigma_c, P) \tag{1}$$

is required. Central to the analysis of this relationship is the notion of effective stress, which postulates that since confining stress and pore pressure have similar but opposite effects, their cumulative effect on velocities can be expressed as a function of some linear combination of σ_c and P,

$$v_{p,s} = v_{p,s}(\sigma_c, P) = v_{p,s}(\sigma^e), \tag{2}$$

where the tensor

$$\sigma^e_{ij} = \sigma^c_{ij} - nP\delta_{ij} \tag{3}$$

is called effective stress, and n is called an effective stress coefficient. The concept of effective stress greatly simplifies the analysis of stress and pressure dependency of rock properties by reducing the number of independent variables from two to one. Thus, a detailed understanding of the effective stress and effective stress coefficient is in order.

While the concept of effective stress is central to the studies of stress- and pressure-dependent behavior of rocks, there exists a considerable confusion among different authors on this matter. Following Terzaghi (1943), it has been shown that the effective stress coefficient n should be equal to one (see, e.g., Gardner et al., 1965; Zimmerman, 1991). In turn, Biot's theory of poroelasticity (Biot, 1941; Geertsma, 1957) shows that effective stress coefficient for bulk volumetric strain is given by

$$n = 1 - K_0/K_s, \tag{4}$$

where K_0 denotes the bulk modulus of the solid matrix, and K_s is the bulk modulus of the solid grain material (Nur and Byerlee, 1971; Robin, 1973). Other authors even suggest that n is not always a constant and may itself depend on the confining stress and pressure (Gangi and Carlson, 1996; Prasad and Manghnani, 1997).

An in-depth analysis of this disparity and of the concept of effective stress in general has been performed by Robin (1973), Carroll and Katsube (1983), Zimmerman (1991), and Berryman (1992, 1993). These studies show that there is no universal effective stress coefficient for all rock properties, and different values of n apply for different physical quantities. In particular, Zimmerman (1991) has shown that for a rock composed of a single mineral constituent, the effective stress coefficient is equal to one for porosity and dry elastic moduli, and equal to the Biot coefficient $1 - K_0/K_s$ for bulk volumetric strain. Berryman (1992, 1993) derived effective stress coefficients for various properties of rocks composed of a number of mineral constituents, and summarized these results in a concise table that shows which effective stress coefficients apply to which physical quantity.

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As follows from these results, in a range of pressures encountered in the sedimentary crust, the effective stress coefficient for seismic velocity in clean sandstones must be equal to one with a high degree of accuracy. However, laboratory measurements often show effective stress coefficients systematically lower than one (Todd and Simmons, 1972; Christensen and Wang, 1985; Prasad and Manghnani, 1997; Siggins and Dewhurst, 2003). To explain these observations, some authors proposed models of effective stress which are inconsistent with theoretical results described above (Gangi and Carlson, 1996; Prasad and Manghnani, 1997).

These publications suggest that the controversy about effective stress coefficients still persists and requires further clarification. The purpose of this paper is to clarify the value of effective stress coefficients for elastic moduli and seismic velocity in a clean sandstone composed of a single linearly elastic solid material. Specifically, I derive the effective stress coefficient directly from the equations of linear elasticity and show that it indeed must be equal to one. This approach is not completely new; it may be regarded as an expanded version of the argument given by Gardner et al. (1965) and Zimmerman (1991). A somewhat different and more general approach based on the equations of Brown and Korringa (1975) was developed by Berryman (1992).

DEFINITIONS OF EFFECTIVE STRESS

Suppose that some property of the rock, say F, depends only on the current stress state irrespective of the stress history and stress path. This assumption obviously ignores hysteresis effects, and is often referred to as the *reversibility* assumption (Gardner et al., 1965; Zimmerman, 1991). In the context of geologic history, reversibility is most likely to be violated during the compaction, but to hold during the unloading stage of the stress history (Goulty, 1998). If this assumption is valid, than property F can be written as a function of the confining stress and pore pressure:

$$F = F(\sigma^c, P). \tag{5}$$

Generally, effective stress for property F is defined as a linear combination of confining stress and pore pressure $\sigma^e_{ij} = \sigma^c_{ij} - nP\delta_{ij}$ such that $F(\sigma^c, P)$ can be expressed as a function of σ^e only:

$$F = F(\sigma^c, P) = F(\sigma^e). \tag{6}$$

That is, if two stress–pressure conditions (σ^{c0}, P^0) and (σ^{c1}, P^1) are such that effective stress $\sigma^e_{ij} = \sigma^c_{ij} - nP\delta_{ij}$ is the same,

$$\sigma^{c0}_{ij} - nP^0\delta_{ij} = \sigma^{c1}_{ij} - nP^1\delta_{ij}, \tag{7}$$

then the property F is also the same;

$$F(\sigma^{c0}, P^0) = F(\sigma^{c1}, P^1). \tag{8}$$

In addition, effective pressure is also considered to be a linear combination of confining stress and pressure that enters a constitutive relation in the theory of poroelasticity.

As mentioned earlier, effective stress coefficients may be different for different rock properties, since different rock properties may depend on confining stress and pressure in different ways (Robin, 1973; Zimmerman, 1991; Berryman, 1992, 1993). To analyse the concept of effective stress for acoustic velocities, we recall that compressional and shear velocities in a saturated rock whose skeleton is made up of a single elastic material may be computed using the Gassmann (1951) equation from the bulk and shear moduli of the dry matrix, bulk moduli of the solid grain material and fluid, porosity, and solid and fluid densities. Thus, in order to understand the stress and pressure dependencies of these velocities, it is necessary to analyse the stress and pressure dependency of the bulk and shear moduli of the dry rock matrix. This is done in the next section.

EFFECTIVE STRESS FOR DRY-ROCK MODULI

Consider a porous rock sample shown in Figure 1. The rock consists of a solid part and a fluid-filled pore space, which are separated by an interface Γ. The solid part has bulk modulus K_s, shear modulus μ_s, and density ρ_s. Here and below we assume that, within the range of stress fields considered, the solid grain material is linearly elastic, that is, K_s and μ_s are constants. Note that for a rock defined in such way the reversibility assumption (5) is automatically satisfied, as linear elastic deformations are always reversible.

The outer surface of the sample is denoted by Γ_e. The sample is subjected to a confining stress σ^c on Γ_e and a uniform pore fluid pressure P on Γ. These boundary conditions create a stress field σ^0 which satisfies the elasticity equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0 \tag{9}$$

and boundary conditions $\sigma^0 = \sigma^c$ on Γ_e and $\sigma^0_n = P$ on Γ, where subscript n refers to the normal stress traction (here and below summation over repeated indices is assumed). Due to the stress σ^0, the solid material is in a deformed state (from an initial unstressed state). The corresponding strain e^0_{ij} satisfies

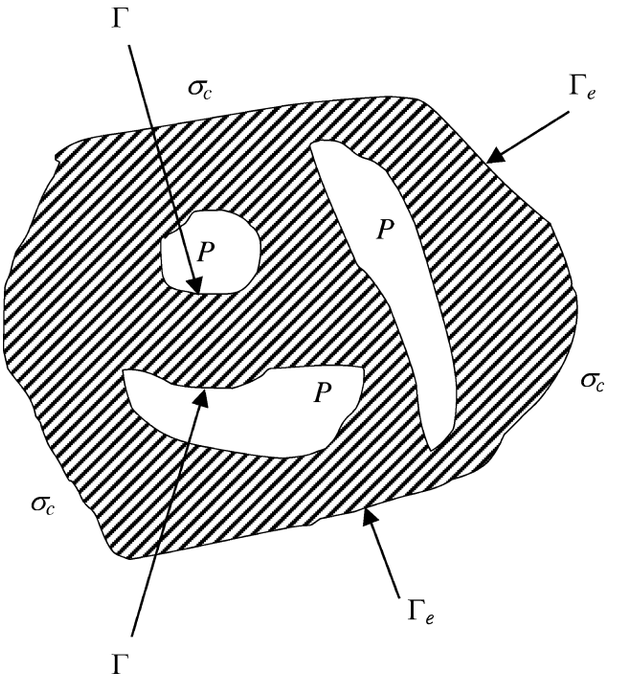

Figure 1. Idealised rock subjected to confining stress on the outer surface Γ_e and pore pressure on the inner surface Γ.

the elastic stress-strain relation (Hooke's law), (Landau and Lifshitz, 1959):

$$\sigma_{ij}^0 = K_s e_{mm}^0 \delta_{ij} + 2\mu_s \left(e_{ij}^0 - \frac{1}{3} e_{mm}^0 \delta_{ij} \right). \quad (10)$$

Note that this form of elastic stress-strain relations assumes that compressive stresses (and hence pressures) are negative.

Now suppose that we have applied a new pore pressure $P^1 = P + P'$ (that is, now there is an extra pressure P') and a new confining stress σ^{c1} such that

$$\sigma_{ij}^{c1} - P^1 \delta_{ij} = \sigma_{ij}^c - P\delta_{ij}. \quad (11)$$

Equation (11) can be rewritten in the form

$$\sigma_{ij}^{c1} - \sigma_{ij}^c = P^1 \delta_{ij} - P\delta_{ij} = P'\delta_{ij}. \quad (12)$$

This means that the new boundary conditions for the solid are $\sigma = \sigma^c + P'\delta_{ij}$ on Γ_e and $\sigma_n = P + P'$ on the internal boundary Γ.

A solution to the equilibrium equations (9) with these new boundary conditions can be written as

$$\sigma_{ij}^1 = \sigma_{ij}^0 + P'\delta_{ij}. \quad (13)$$

The corresponding strain e_{ij}^1 must satisfy the elastic stress-strain relation

$$\sigma_{ij}^1 = K_s e_{mm}^1 \delta_{ij} + 2\mu_s \left(e_{ij}^1 - \frac{1}{3} e_{mm}^1 \delta_{ij} \right). \quad (14)$$

Subtracting equation (10) from equation (14) yields

$$P'\delta_{ij} = K_s e'_{mm} \delta_{ij} + 2\mu_s \left(e'_{ij} - \frac{1}{3} e'_{mm} \delta_{ij} \right), \quad (15)$$

where $e' = e^1 - e^0$ denotes strain relative to the state defined by e^0. Taking trace of both sides of equation (15) yields

$$P' = K_s e'_{mm}. \quad (16)$$

Substituting P' from equation (16) back into equation (15), we obtain

$$2\mu_s \left(e'_{ij} - \frac{1}{3} e'_{mm} \delta_{ij} \right) = 0. \quad (17)$$

Therefore,

$$e'_{11} = e'_{22} = e'_{33} = \frac{1}{3}(e'_{11} + e'_{22} + e'_{33}) = \frac{P'}{3K_s} \quad (18)$$

and

$$e'_{12} = e'_{23} = e'_{13} = 0. \quad (19)$$

Equations (18) and (19) give the solution to our problem. They indicate that a change in confining stress and pore pressure such that the differential stress $\sigma_{ij}^d = \sigma_{ij} - P\delta_{ij}$ remains unchanged results in the extra uniform dilatation by $P'/3K_s$.

We now recall that for the rock whose solid material is linearly elastic, the dry matrix (or drained) bulk K_0 and shear μ_0 moduli are scale-invariant properties, that is, they depend on the geometry of the pore space but do not depend on its absolute spatial dimensions (Gardner et al., 1965; Christensen, 1979; Berryman, 1992). Therefore, these drained bulk and shear moduli are not affected by the uniform dilatation. Thus, for a given differential pressure $\sigma_{ij}^d = \sigma_{ij} - P\delta_{ij}$, bulk and shear drained moduli are independent of the confining stress or pore pressure. In other words,

$$K_0 = K_0(\sigma_{ij} - P\delta_{ij}) \quad (20)$$

and

$$\mu_0 = \mu_0(\sigma_{ij} - P\delta_{ij}). \quad (21)$$

We conclude that for a rock whose skeleton is made up of a single elastic grain material, the effective stress coefficient for bulk and shear drained moduli is exactly unity.

EFFECTIVE STRESS FOR OTHER EFFECTIVE PROPERTIES

In the previous section, we showed that effective stress coefficient for bulk and shear drained moduli is exactly unity, and hence effective stress is equal to Terzaghi's (1943) differential stress $\sigma_{ij} - P\delta_{ij}$. The same is true about porosity: porosity ϕ is also obviously a scale-invariant quantity. However, the same cannot be said about other effective rock properties. In particular, permeability is affected by dilatation because it is a scale-dependent quantity (proportional to the square of the linear dimensions of the pore space). Overall matrix density $\rho_0 = (1-\phi)\rho_s$ is also affected by uniform dilatation because the overall volume of the sample changes. As elastic wave velocities depend on density as well as elastic moduli, we observe that, strictly speaking, effective stress cannot be defined for velocities. However, dependency of the velocities on confining stress and pore pressure can still be easily constructed using effective stress coefficients for elastic moduli, porosity, total volume, and solid density.

Furthermore, we note that stresses and pressures up to several kilometers in depth do not exceed 100 MPa, while bulk and shear moduli of the grain material such as quartz or calcite are in the range of 20×10^3 to 80×10^3 MPa. Therefore, the dilatation caused by changes in confining stress and pressure corresponding to a constant differential stress does not exceed 0.003. Thus, variations of total volume and solid density due to stress variations should not exceed 0.3%. This variation is usually much smaller than variations in effective elastic moduli in the same stress range (usually caused by opening and closing of microcracks), and thus its effect on seismic velocities can be ignored.

Thus, we conclude that for moderate stresses, the differential stress represents the effective stress for acoustic velocities. When applied to the properties affected by the pore fluid (such as acoustic velocities), this result requires the fluid to also have compressibility independent of pressure. This assumption is valid to a high degree for water and dead oil, but is clearly invalid for any gas or live oil (Batzle and Wang, 1992).

As mentioned in the introduction, another definition of effective stress refers to the one that enters *macroscopic* constitutive relations for poroelastic media. Macroscopic infinitesimal strain can be defined as relative change of the overall volume

of the sample V_b:

$$d\epsilon_b = \frac{dV_b}{V_b}. \tag{22}$$

Biot (1941) showed that macroscopic stress-strain relation for small strains may be written as

$$d\epsilon_b = -\frac{d\left(\sigma_{ij}^c - n_b P\delta_{ij}\right)}{K_0}, \tag{23}$$

where $n_b = 1 - K_0/K_s$ is sometimes called Biot's effective stress coefficient or the Biot-Willis coefficient (Biot and Willis, 1957). Equation (23) means that for a small macroscopic strain, an appropriate definition of effective stress is

$$\sigma_{ij}^{eb} = \sigma_{ij}^c - n_b P\delta_{ij}, \tag{24}$$

with $n_b = 1 - K_0/K_s$ being the effective stress coefficient. Note that if σ^{eb} is constant, the overall volume of the sample and, hence, matrix density, are also constant as well. Thus, σ^{eb} represents exact effective stress for matrix density.

Generally speaking, effective stress given by equation (24) is different from the differential stress $\sigma_{ij}^d = \sigma_{ij} - P\delta_{ij}$. However, if the rock matrix is highly compressible ($K_0 \ll K_s$), then n_b is close to unity, and effective stress σ^{eb} becomes equal to the differential stress. The condition $K_0 \ll K_s$ is a good approximation for soils and therefore is widely used in soil mechanics. It is this equivalence of the effective stress to the differential stress for soils that is perhaps the original cause of the coexistence of several quite different definitions of effective stress.

DISCUSSION AND CONCLUSIONS

We have shown that, for an idealized model of the rock, within a margin of error of less than 1% effective stress coefficient n may be taken to be unity for the variety of rock properties. This theoretical result is supported by laboratory observations (Wyllie et al., 1958; Zimmerman, 1991), particularly on clean sandstones.

However, many authors have reported measurements of rock properties (e. g., elastic wave velocities) as a function of confining stress and pore pressure, in which much lower values of n were observed (Prasad and Manghnani, 1997; Siggins et al., 2001; Siggins and Dewhurst, 2003). As also noted by Zimmerman (1991), this deviation from the behavior predicted by the theory may be caused by the violation of the assumptions made in the theoretical treatment. Two main assumptions are the homogeneity of the grain matrix material and its linear elasticity within the range of stresses and pressures considered. The first assumption never holds exactly for real rocks. It may be a reasonable approximation for rocks which consist either predominantly of one mineral component or of a few minerals with not very different elastic properties. However, this approximation will not hold for very heterogeneous rocks, such as shaley sandstones or shales. Detailed analysis of the effect of matrix heterogeneity on effective stress coefficients for different physical properties based on the equations of Brown and Korringa (1975) has been performed by Berryman (1992, 1993). Assumptions of linear elasticity of the solid grain material should be acceptable for quartz and calcite, but may not be valid for clay minerals, some cements, or bound water. Further experimental and theoretical studies are required to establish effective stress coefficients for real rocks.

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