The Discrete-Time Generalized Algebraic Riccati Equation: Order Reduction and Solutions' Structure*

Lorenzo Ntogramatzidis[⋆] Augusto Ferrante[†]

[⋆]Department of Mathematics and Statistics,
Curtin University, Perth (WA), Australia
L.Ntogramatzidis@curtin.edu.au

[†]Dipartimento di Ingegneria dell'Informazione,
Università di Padova, via Gradenigo, 6/B – 35131 Padova, Italy
augusto@dei.unipd.it

Abstract

In this paper we discuss how to decompose the constrained generalized discrete-time algebraic Riccati equation arising in optimal control and optimal filtering problems into two parts corresponding to an additive decomposition $X = X_0 + \Delta$ of each solution X: The first part is trivial, in the sense that it is an explicit expression of the addend X_0 which is common to all solutions, so that it does not depend on the particular X. The second part can be — depending on the structure of the considered generalized Riccati equation — either a reduced-order discrete-time *regular* algebraic Riccati equation whose associated closed-loop matrix is non-singular, or a symmetric Stein equation. The proposed reduction is explicit, so that it can be easily implemented in a software package that uses only standard linear algebra procedures.

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1 Introduction

Ever since Kalman described the crucial role of Riccati equations in the solution of the linear quadratic (LQ) optimal control and filtering problems [14, 15], the range of problems where Riccati equations have been discovered to play a crucial role has been increasing impressively. In the last fifty years Riccati equations have been found to arise also in linear dynamic games with quadratic cost criteria, spectral factorization problems, singular perturbation theory, stochastic realization theory and identification, boundary value problems for ordinary differential equations, invariant embedding and scattering theory. For this reason, Riccati equations are universally regarded as one of the fundamental cornerstones of modern systems and control theory, see e.g. the monographs [20, 17, 13, 1] which have been entirely devoted to providing a general and systematic framework for the study of Riccati equations.

This paper is concerned with the following relations

$$X \;=\; A^\top X A - (A^\top X B + S)(R + B^\top X B)^\dagger (S^\top + B^\top X A) + Q, \tag{1}$$

$$\ker(R + B^\top X B) \subseteq \ker(A^\top X B + S) \tag{2}$$

where the symbol † denotes the Moore-Penrose pseudo-inverse operation.[1] Equation (1) subject to the constraint (2) arises for example in discrete-time LQ problems – see [18] and [5] for the finite and infinite-horizon cases, respectively. Here, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times m}$ are such that the *Popov matrix* Π satisfies

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \Pi^\top \geq 0. \tag{3}$$

The set of matrices $\Sigma = (A, B; \Pi)$ is often referred to as *Popov triple*, and (1) is known as the *generalized discrete-time algebraic Riccati equation* GDARE(Σ). This equation, together with the additional constraint (2), is usually referred to as *constrained generalized discrete-time algebraic Riccati equation*, and it is herein denoted by CGDARE(Σ). This equation generalizes the standard *discrete-time algebraic Riccati equation* DARE(Σ)

$$X \;=\; A^\top X A - (A^\top X B + S)(R + B^\top X B)^{-1} (S^\top + B^\top X A) + Q, \tag{4}$$

as the natural equation arising in LQ optimal control and filtering problems. In fact, it is only when the underlying linear system — obtained by a full-rank factorization $\Pi = \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$ and considering a system described by the quadruple (A, B, C, D) — is left invertible that the

[1] We recall that given an *arbitrary* matrix $M \in \mathbb{R}^{h\times k}$, there exists a unique matrix $M^\dagger \in \mathbb{R}^{k\times h}$ that satisfies the following four properties: **(1)** $M M^\dagger M = M$; **(2)** $M^\dagger M M^\dagger = M^\dagger$; **(3)** $M^\dagger M$ is symmetric; **(4)** $M M^\dagger$ is symmetric. By definition, the matrix $M^\dagger$ is the *Moore-Penrose pseudo-inverse* of the matrix M.

standard DARE(Σ) admits solutions. The dynamic optimization problem, however, may still admit solutions in the more general setting where the underlying linear system is not left-invertible so that the corresponding spectral density

$$\Phi(z) \stackrel{\text{def}}{=} [G(z^{-1})]^\top \Pi\, G(z), \qquad \text{with} \quad G(z) \stackrel{\text{def}}{=} \left[\begin{array}{c} (zI-A)^{-1}B \\ I_m \end{array}\right], \tag{5}$$

is singular. In these cases, however, the standard DARE(Σ) does not admit solutions and the correct equation that must be used to address the original optimization problem is the CGDARE(Σ), see e.g. [5]. As discussed in [1, Chapt. 6], these general situations are particularly relevant in the context of stochastic control problems, see also [2, 9] and the references cited therein. On the other hand, whenever the standard DARE(Σ) admits solutions, the set of its solutions coincides with the set of solutions of CGDARE(Σ), so that the latter is a genuine generalization of the former (here and in the rest of the paper, we are only considering *symmetric* solutions X both for the DARE(Σ) and the CGDARE(Σ)).

In the literature, several efforts have been devoted by many authors to the task of reducing the order and difficulty of the standard DARE(Σ) by means of different techniques, [16, 10, 11, 12, 3, 8]. This interest is motivated by the fact that the standard DARE(Σ) is richer than the structure of its continuous-time counterpart, the continuous-time algebraic Riccati equation. In particular, in [3] a method was presented which, differently from earlier contributions presented on this topic, aimed at iteratively decomposing DARE(Σ) into a trivial part and a reduced DARE whose associated closed-loop matrix is non-singular. The subsequent contribution [8] achieves a similar goal by avoiding the need for an iterative procedure.

The development of reduction procedures for generalized Riccati equations has received much less attention in the literature. This is in part likely to be due to the technical difficulties associated with generalized Riccati equations in the discrete time. In [3], a hint is given on how the iterative reduction detailed therein could be extended to the case of an equation in the form (1), provided that the attention is restricted to the set of positive semidefinite solutions, for which condition (2) is automatically satisfied. On the other hand, CGDARE(Σ) may well admit solutions that are not positive semidefinite, see e.g. [5, 6]. In [12], a Riccati equation in the form of a CGDARE(Σ) is considered, and a reduction technique is proposed to the end of computing the stabilizing solution of CGDARE(Σ). The main goal of this paper is to combine the generality of the framework considered in [12] with the ambition of achieving a reduction for the entire set of solutions of CGDARE(Σ). This task is accomplished by developing an iterative procedure that is similar in spirit to that of [3], but which presents a richer and more articulated structure. Indeed, not only do several technical difficulties and structural differences arise in extending the results of [3] to the case of CGDARE(Σ) when the set of solutions is not restricted to semidefinite ones, but also, differently from the iterations needed in [3], which are essentially performed via changes of coordinates in the state space, in the general case of a CGDARE(Σ), it is necessary to also resort to changes of coordinates in the input space. The problem of obtaining a systematic procedure to decompose generalized Riccati equations into a trivial part and a reduced, "well-behaved", part described by a *regular* DARE (or at times, differently from the standard case, by a symmetric Stein equation), becomes much more interesting and challenging in the case of generalized Riccati equations. Our reduction method is based on the computation of null spaces of given matrices so that it can be easily implemented in a software procedure that uses only standard linear algebra procedures which are robust and available in any numerical software package. Therefore a relevant outcome of the presented procedure is what we believe to be the first systematic numerical procedure to compute the solutions of CGDARE.

2 Problem formulation and preliminaries

First, in order to simplify the notation, for any $X = X^\top \in \mathbb{R}^{n\times n}$ we define the matrices

$$\begin{array}{lllllllll} R_X & \stackrel{\text{def}}{=} & R+B^\top X\,B & \quad G_X & \stackrel{\text{def}}{=} & I_m - R_X^\dagger R_X & & & \\ S_X & \stackrel{\text{def}}{=} & A^\top X\,B+S & \quad K_X & \stackrel{\text{def}}{=} & R_X^\dagger S_X^\top & \quad A_X & \stackrel{\text{def}}{=} & A-B\,K_X \end{array}$$

so that (2) in CGDARE(Σ) can be written concisely as $\ker R_X \subseteq \ker S_X$. The term $R_X^\dagger R_X$ is the orthogonal projector that projects onto $\operatorname{im} R_X^\dagger = \operatorname{im} R_X$ so that G_X is the orthogonal projector that projects onto $\ker R_X$. Hence, $\ker R_X = \operatorname{im} G_X$.

As already mentioned, in this paper we present a procedure that reduces CGDARE(Σ) to another discrete-time algebraic Riccati equation with the same structure but smaller order and in which both $A_0 \stackrel{\text{def}}{=} A - B R^\dagger S^\top$ and R are non-singular. On the other hand, this means that the Riccati equation thus obtained is indeed a standard DARE, i.e., it has the structure shown in (4), as the following result shows.

Proposition 1 *Suppose that the matrix R is non-singular, and let $X = X^\top$ be any symmetric solution of CGDARE(Σ). Then $R_X = R + B^\top X B$ is non-singular.*

Proof: As shown in [5, Lemma 4.1], for any symmetric solution $X = X^\top$ of CGDARE(Σ) the inclusion $\ker R_X \subseteq \ker R$ holds. As a consequence, if R is non-singular, its null-space $\ker R$ is zero, and therefore so is the null-space of R_X. This is equivalent to the fact that R_X is non-singular. $\blacksquare$

The reduction technique presented in this paper can also be viewed from the perspective of the so-called extended symplectic pencil $N_\Sigma - zM_\Sigma$, where

$$M_\Sigma \stackrel{\text{def}}{=} \begin{bmatrix} I_n & 0 & 0 \\ 0 & -A^\top & 0 \\ 0 & -B^\top & 0 \end{bmatrix} \qquad \text{and} \qquad N_\Sigma \stackrel{\text{def}}{=} \begin{bmatrix} A & 0 & B \\ Q & -I_n & S \\ S^\top & 0 & R \end{bmatrix}.$$

The case in which the matrix pencil $N_\Sigma - zM_\Sigma$ is regular (i.e., if there exists $z \in \mathbb{C}$ such that $\det(N_\Sigma - zM_\Sigma) \neq 0$) corresponds to the case in which CGDARE(Σ) is indeed a DARE(Σ), whereas the one in which $N_\Sigma - zM_\Sigma$ is singular (i.e., the determinant of $N_\Sigma - zM_\Sigma$ is the zero polynomial) corresponds to a case in which DARE(Σ) does not admit solutions. It is shown in [3] for DARE(Σ) and in [7] for CGDARE(Σ) that if A_X is singular, the Jordan structure of A_X associated with the eigenvalue $\lambda = 0$ is completely determined by $N_\Sigma - zM_\Sigma$, and is independent of the particular solution X of DARE(Σ) or CGDARE(Σ). It is shown in [3] that in the case where the matrix pencil $N_\Sigma - zM_\Sigma$ is regular — or, equivalently, the CGDARE(Σ) and the standard DARE(Σ) have the same solutions— the following statements are equivalent:

(1) N_Σ is singular;

(2) $N_\Sigma - zM_\Sigma$ has a generalized eigenvalue at zero;

(3) there exists a solution X of CGDARE(Σ) such that the corresponding closed-loop matrix A_X is singular;

(3′) for any solution X of CGDARE(Σ), the corresponding closed-loop matrix A_X is singular;

(4) at least one of the two matrices R and $A_0 = A - BR^\dagger S^\top$ is singular.

The case where the matrix pencil $N_\Sigma - zM_\Sigma$ is possibly singular was investigated in [7], where it was proved that in this more general case these four facts are not equivalent. In particular, **(1)** is not equivalent to **(2)**. Moreover, in the case where $N_\Sigma - zM_\Sigma$ is singular, **(1)** and **(3)** are not equivalent, nor are **(3)** and **(4)**. However, it was shown in [7, Lemma 3.1] that it is still true that **(1)** is equivalent to **(4)**. Furthermore, it is shown in [7, Proposition 3.4] that $r \stackrel{\text{def}}{=} \text{rank} R_X$ is constant for any solution X of CGDARE(Σ), and that A_X is singular if and only if at least one of the following two conditions holds: (i) $\text{rank} R < r = \text{rank} R_X$ and (ii) $A_0 = A - BR^\dagger S^\top$ is singular. It is clear that this condition reduces to **(4)** in the case where R_X is invertible, i.e., in the case where $N_\Sigma - zM_\Sigma$ is regular. Notice also that since both conditions are independent of the particular solution X of the CGDARE(Σ), the singularity of the closed-loop matrix A_X is invariant with respect to the particular solution X.

To summarize, in the case where the matrix pencil $N_\Sigma - zM_\Sigma$ is singular, the following statements are equivalent:

(1′) N_Σ is singular;

(2′) at least one of the two matrices R and $A_0 = A - BR^\dagger S^\top$ is singular;

and the following statements are equivalent:

(1″) there exists a solution X of CGDARE(Σ) such that the corresponding closed-loop matrix A_X is singular;

(2″) for any solution X of CGDARE(Σ), the corresponding closed-loop matrix A_X is singular;

(3″) at least one of the two conditions

(a) $\mathrm{rank} R < r = \mathrm{rank} R_X$; or

(b) $A_0 = A - BR^\dagger S^\top$ is singular;

is satisfied.

We recall again that in [5, Lemma 4.1] it was shown that for any solution X of CGDARE(Σ), we have $\ker R_X \subseteq \ker R$. This means that if R is non-singular, such is also R_X, and therefore the condition $\mathrm{rank} R < \mathrm{rank} R_X$ is not satisfied. Thus, in this case, the closed-loop matrix A_X is non-singular for some solution X of the CGDARE(Σ) if and only if it is non-singular for each solution X of the CGDARE(Σ) and this is in turn equivalent to A_0 being non-singular.

3 Mathematical preliminaries

We begin this section by recalling a standard linear algebra result that is used in the derivations throughout the paper.

Lemma 1 *Consider* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} = P^\top \geq 0$. *Then,*

(i) $\ker P_{12} \supseteq \ker P_{22}$*;*

(ii) $P_{12} P_{22}^\dagger P_{22} = P_{12}$*;*

(iii) $P_{12}(I - P_{22}^\dagger P_{22}) = 0$*;*

(iv) $P_{11} - P_{12} P_{22}^\dagger P_{12}^\top \geq 0$.

Proof: Let us start by proving **(i)**. Since $P = P^\top \geq 0$, two matrices C and D exist such that $P = [\, C \;\; D \,]^\top [\, C \;\; D \,]$ so that $P_{12} = C^\top D$ and $P_{22} = D^\top D$. Let $x \in \ker P_{22}$. Then, $0 = x^\top D^\top D x = \|Dx\|^2$ gives $Dx = 0$, which in turn implies that $x \in \ker P_{12}$. Now consider **(ii)**. The inclusion $\ker P_{12} \supseteq \ker P_{22}$ can be rewritten as $\mathrm{im}\, P_{12}^\top \subseteq \mathrm{im}\, P_{22}$. Thus, a matrix K of suitable size exists such that $P_{12} = K P_{22}$. On post-multiplying both sides of this identity by $P_{22}^{\dagger} P_{22}$ we obtain $P_{12} P_{22}^{\dagger} P_{22} = K P_{22} P_{22}^{\dagger} P_{22} = K P_{22} = P_{12}$. Now we prove **(iii)**. Since as already proved $\ker P_{22} \subseteq \ker P_{12}$, a matrix K exists such that $P_{12} = K P_{22}$. Therefore, $P_{12}\,(I - P_{22}^{\dagger} P_{22}) = K P_{22}\,(I - P_{22}^{\dagger} P_{22}) = K\,(P_{22} - P_{22} P_{22}^{\dagger} P_{22}) = K\,(P_{22} - P_{22}) = 0$. Notice that **(iv)** follows directly from $P_{11} - P_{12} P_{22}^{\dagger} P_{12}^{\top} = \begin{bmatrix} I & -P_{12} P_{22}^{\dagger} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\top} & P_{22} \end{bmatrix} \begin{bmatrix} I \\ -P_{22}^{\dagger} P_{12}^{\top} \end{bmatrix} \geq 0$. ■

We now generalize a well-known result of the classic Riccati theory — which essentially shows how to eliminate the cross-penalty matrix S — to the case of a constrained generalized Riccati equation.

Lemma 2 *Let* $A_0 \stackrel{def}{=} A - B R^{\dagger} S^{\top}$ *and* $Q_0 \stackrel{def}{=} Q - S R^{\dagger} S^{\top}$. *Moreover, let* $\Pi_0 \stackrel{def}{=} \begin{bmatrix} Q_0 & 0 \\ 0 & R \end{bmatrix}$ *and* $\Sigma_0 \stackrel{def}{=} (A_0, B, \Pi_0)$. *Then, the following statements hold true:*

(i) *CGDARE(Σ) has the same set of solutions as CGDARE(Σ_0)*

$$X \;=\; A_0^{\top} X A_0 - A_0^{\top} X B (R + B^{\top} X B)^{\dagger} B^{\top} X A_0 + Q_0, \tag{6}$$

$$\ker(R + B^{\top} X B) \subseteq \ker(A_0^{\top} X B); \tag{7}$$

(ii) *for any symmetric solution X of CGDARE(Σ), we have*

$$A_X \;=\; A_{0X} \stackrel{def}{=} A_0 - B\,(R + B^{\top} X B)^{\dagger} B^{\top} X A_0;$$

(iii) $Q_0 \geq 0$.

Proof: We start proving **(i)**. Inserting the expressions for A_0 and Q_0 into (6) yields

$$\begin{aligned}
X \;=\;& A^{\top} X A - A^{\top} X B R^{\dagger} S^{\top} - S R^{\dagger} B^{\top} X A + S R^{\dagger} B^{\top} X B R^{\dagger} S^{\top} \\
& - A^{\top} X B R_X^{\dagger} B^{\top} X A + A^{\top} X B R_X^{\dagger} B^{\top} X B R^{\dagger} S^{\top} + S R^{\dagger} B^{\top} X B R_X^{\dagger} B^{\top} X A \\
& - S R^{\dagger} B^{\top} X B R_X^{\dagger} B^{\top} X B R^{\dagger} S^{\top} + Q - S R^{\dagger} S^{\top} \\
=\;& A^{\top} X A - A^{\top} X B R^{\dagger} S^{\top} - S R^{\dagger} B^{\top} X A + S R^{\dagger} B^{\top} X B R^{\dagger} S^{\top} \\
& - A^{\top} X B R_X^{\dagger} B^{\top} X A + A^{\top} X B R_X^{\dagger} (B^{\top} X B + R - R) R^{\dagger} S^{\top} \\
& + S R^{\dagger} (B^{\top} X B + R - R) R_X^{\dagger} B^{\top} X A \\
& - S R^{\dagger} (B^{\top} X B + R - R) R_X^{\dagger} (B^{\top} X B + R - R) R^{\dagger} S^{\top} + Q - S R^{\dagger} S^{\top} \\
=\;& A^{\top} X A - A^{\top} X B R^{\dagger} S^{\top} - S R^{\dagger} B^{\top} X A + S R^{\dagger} B^{\top} X B R^{\dagger} S^{\top} \\
& - A^{\top} X B R_X^{\dagger} B^{\top} X A + A^{\top} X B R_X^{\dagger} R_X R^{\dagger} S^{\top} - A^{\top} X B R_X^{\dagger} S^{\top} \\
& + S R^{\dagger} R_X R_X^{\dagger} B^{\top} X A - S R_X^{\dagger} B^{\top} X A - S R^{\dagger} R_X R^{\dagger} S^{\top} \\
& + S R^{\dagger} R_X R_X^{\dagger} S^{\top} + S R_X^{\dagger} R_X R^{\dagger} S^{\top} - S R_X^{\dagger} S^{\top} + Q - S R^{\dagger} S^{\top}.
\end{aligned} \tag{8}$$

From $\ker R_X \subseteq \ker S_X$, it follows that there exists K such that $S_X = K R_X$, which gives

$$S_X R_X^\dagger R_X = K R_X R_X^\dagger R_X = K R_X = S_X. \tag{9}$$

Using this identity and its transpose, we can develop the terms in the right hand-side of the last equality sign of (8) as

$$A^\top X B R_X^\dagger R_X R^\dagger S^\top + S R_X^\dagger R_X R^\dagger S^\top = S_X R_X^\dagger R_X R^\dagger S^\top = S_X R^\dagger S^\top,$$

$$S R^\dagger R_X R_X^\dagger B^\top X A + S R^\dagger R_X R_X^\dagger S^\top = S R^\dagger R_X R_X^\dagger S_X^\top = S R^\dagger S_X^\top.$$

and

$$S R^\dagger B^\top X B R^\dagger S^\top - S R^\dagger R_X R^\dagger S^\top = -S R^\dagger R R^\dagger S^\top = -S R^\dagger S^\top.$$

Using these new simplified expressions back into (8) gives

$$\begin{aligned} X &= -A^\top X B R^\dagger - S R^\dagger S^\top - S R^\dagger B^\top X A - S R^\dagger S^\top + S_X R^\dagger S^\top - S R^\dagger S_X^\top \\ &= A^\top X A - A^\top X B R_X^\dagger B^\top X A - S R_X^\dagger B^\top X A - A^\top X B R_X^\dagger S^\top - S R_X^\dagger S^\top + Q \\ &\quad -(A^\top X B + S) R^\dagger S^\top - S R^\dagger (B^\top X A + S^\top) + S_X R^\dagger S^\top - S R^\dagger S_X^\top \\ &= A^\top X A - (A^\top X B + S)(R + B^\top X B)^\dagger (B^\top X A + S^\top) + Q, \end{aligned}$$

which is indeed (1). We conclude the proof of **(i)** showing that (2) is equivalent to (7). We write (7) as

$$\begin{aligned} \ker R_X &\subseteq \ker(A_0^\top X B) \\ &= \ker(A^\top X B - S R^\dagger B^\top X B) \\ &= \ker[A^\top X B - S R^\dagger (R + B^\top X B - R)] \\ &= \ker(A^\top X B + S - S R^\dagger R_X), \end{aligned}$$

since $S R^\dagger R = S$ in view of the second point in Lemma 1. Suppose (2) holds. Let $\omega \in \ker R_X$. Then $S_X \omega = (S + A^\top X B)\,\omega = 0$. Thus, we have also $(A^\top X B + S - S R^\dagger R_X)\,\omega = 0$ since $\omega \in \ker R_X$. Conversely, suppose that (7) holds true, and take $\omega \in \ker R_X$. Then, $(A^\top X B + S - S R^\dagger R_X)\,\omega = 0$ implies $(S + A^\top X B)\,\omega = 0$.

Let us now consider **(ii)**. We first show that $(R_X^\dagger R_X - I_m) R^\dagger = 0$. To prove this fact — which is trivial in the case of the standard DARE(Σ) — we use the inclusion $\ker R_X \subseteq \ker R$, which holds true for any symmetric solution X of CGDARE(Σ), see [5, Lemma 4.1]. In a suitable basis of the input space, R_X can be written as $R_X = \begin{bmatrix} R_{X,1} & 0 \\ 0 & 0 \end{bmatrix}$, where $R_{X,1}$ is invertible; let μ be the order of $R_{X,1}$. In this basis, R is written as $R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$, where R_1 may or may not be singular, and we obtain

$$(R_X^\dagger R_X - I_m)\, R^\dagger \;=\; \left(\begin{bmatrix} R_{X,1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{X,1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} I_\mu & 0 \\ 0 & I_{m-\mu} \end{bmatrix} \right) \begin{bmatrix} R_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} = 0. \qquad (10)$$

Thus,

$$\begin{aligned} A_{0X} = A_0 - B\,(R + B^\top X B)^\dagger B^\top X A_0 \;&=\; (A - BR^\dagger S^\top) - B\,(R + B^\top X B)^\dagger B^\top X\,(A - BR^\dagger S^\top) \\ &= A - BR^\dagger S^\top - BR_X^\dagger B^\top X A + BR_X^\dagger (R + B^\top X B - R)\, R^\dagger S^\top \\ &= A_X + B\,(R_X^\dagger R_X - I_m)\, R^\dagger S^\top = A_X. \end{aligned}$$

To prove **(iii)** it suffices to observe that Q_0 is the generalized Schur complement of R in Π. Since Π is assumed to be positive semidefinite, then such is also Q_0. ■

Another useful result is the following generalization of a classic property of DARE(Σ).

Lemma 3 *Let $T \in \mathbb{R}^{n\times n}$ be invertible. Let*

$$A_T \overset{def}{=} T^{-1} A_0\, T, \quad B_T \overset{def}{=} T^{-1} B, \quad Q_T \overset{def}{=} T^{-1} Q_0\, T. \qquad (11)$$

Let also $\Pi_T \overset{def}{=} \begin{bmatrix} Q_T & 0 \\ 0 & R \end{bmatrix}$ and $\Sigma_T \overset{def}{=} (A_T, B_T, \Pi_T)$. Then, X is a solution of CGDARE(Σ) – and therefore also of CGDARE(Σ_0) – if and only if $X_T = T^{-1} X\, T$ is a solution of CGDARE(Σ_T)

$$X_T \;=\; A_T^\top X_T\, A_T - A_T^\top X_T\, B_T\, (R + B_T^\top X_T B_T)^\dagger B_T^\top X_T\, A_T + Q_T \qquad (12)$$

$$\ker(R + B_T^\top X_T\, B_T) \subseteq \ker(A_T^\top X_T\, B_T) \qquad (13)$$

Proof: The equations obtained by multiplying (6) to the left by T^{-1} and to the right by T coincides with (12) with $X_T \overset{\text{def}}{=} T^{-1} X\, T$. Moreover, since T is invertible, $\ker(R + B^\top X\, B) \subseteq \ker(A_0^\top X\, B)$ is equivalent to $\ker(R + B^\top X\, B) \subseteq \ker(T^{-1} A_0^\top X\, B)$, which is equivalent to (13). ■

4 Main results

4.1 Reduction corresponding to a singular A_0

In this section, we present the first fundamental result of this paper, that can be exploited as a basis for an iterative procedure – to be used whenever A_0 is singular – to the end of decomposing the set of solutions of CGDARE(Σ) into a trivial part and a part given by the set of solutions of a reduced order CGDARE.

Theorem 1 *Let $\nu \stackrel{def}{=} \dim(\ker A_0)$. Let $U = [\ U_1 \quad U_2\ ]$ be an orthonormal change of coordinates in $\mathbb{R}^n$, where $\mathrm{im}\, U_2 = \ker A_0$. Let $A_U \stackrel{def}{=} U^\top A_0 U = [\ \tilde{A} \quad 0_{n\times\nu}\ ]$ where $\tilde{A} = \begin{bmatrix} A_1 \\ A_{21} \end{bmatrix}$ with $A_1 \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ and $A_{21} \in \mathbb{R}^{\nu\times(n-\nu)}$. Let also $B_U = U^\top B$ and $Q_U = U^\top Q_0 U$ be partitioned conformably, i.e., $B_U = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $Q_U = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix}$, with $B_1 \in \mathbb{R}^{(n-\nu)\times m}$, $B_2 \in \mathbb{R}^{\nu\times m}$, $Q_{11} \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ and $Q_{22} \in \mathbb{R}^{\nu\times\nu}$. Finally, let $Q_1 \stackrel{def}{=} \tilde{A}^\top Q_U \tilde{A}$, $S_1 \stackrel{def}{=} \tilde{A}^\top Q_U B_U$ and $R_1 \stackrel{def}{=} R + B_U^\top Q_U B_U$.*

1. *Let X be a solution of CGDARE(Σ), and partition $X_U \stackrel{def}{=} U^\top X U$ as $X_U = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^\top & X_{22} \end{bmatrix}$, with $X_{11} \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ and $X_{22} \in \mathbb{R}^{\nu\times\nu}$. Then,*

   (i) *there hold*

$$X_{12} = Q_{12} \quad \text{and} \quad X_{22} = Q_{22}$$

   (ii) *The new Popov matrix $\Pi_1 \stackrel{def}{=} \begin{bmatrix} Q_1 & S_1 \\ S_1^\top & R_1 \end{bmatrix}$ is positive semidefinite.*

   (iii) *Let $\Sigma_1 \stackrel{def}{=} (A_1, B_1, \Pi_1)$. Then, $\Delta_1 \stackrel{def}{=} X_{11} - Q_{11}$ satisfies CGDARE(Σ_1)*

$$\Delta_1 = A_1^\top \Delta_1 A_1 - (A_1^\top \Delta_1 B_1 + S_1)(R_1 + B_1^\top \Delta_1 B_1)^\dagger (B_1^\top \Delta_1 A_1 + S_1^\top) + Q_1 \tag{14}$$

$$\ker(R_1 + B_1^\top \Delta_1 B_1) \subseteq \ker(S_1 + A_1^\top \Delta_1 B_1). \tag{15}$$

2. *Conversely, if Δ_1 is a solution of (14-15), then*

$$X = U \begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} U^\top \tag{16}$$

   is a solution of CGDARE(Σ).

Proof: We begin proving the first point. In view of Lemma 3, X is a solution of CGDARE(Σ) if and only if $X_U = U^\top X U$ is a solution of CGDARE(Σ_U)

$$X_U = A_U^\top X_U A_U - A_U^\top X_U B_U (R + B_U^\top X_U B_U)^\dagger B_U^\top X_U A_U + Q_U \tag{17}$$

$$\ker(R + B_U^\top X_U B_U) \subseteq \ker(A_U^\top X_U B_U), \tag{18}$$

where $\Pi_U = \begin{bmatrix} Q_U & 0 \\ 0 & R \end{bmatrix}$ and $\Sigma_U = (A_U, B_U, \Pi_U)$. Multiplying (17) to the left by $[\ 0 \quad I_\nu\ ]$ yields

$$[\ 0 \quad I_\nu\ ] \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^\top & X_{22} \end{bmatrix} = [\ 0 \quad I_\nu\ ] \begin{bmatrix} A_1^\top & A_{21}^\top \\ 0 & 0 \end{bmatrix} X_U A_U$$
$$- [\ 0 \quad I_\nu\ ] \begin{bmatrix} A_1^\top & A_{21}^\top \\ 0 & 0 \end{bmatrix} X_U B_U (R + B_U^\top X_U B_U)^\dagger B_U^\top X_U A_U + [\ 0 \quad I_\nu\ ] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix},$$

which gives $[\ X_{12}^\top \quad X_{22}\ ] = [\ Q_{12}^\top \quad Q_{22}\ ]$. This proves the first statement. To prove **(ii)** we observe that

$$\Pi_1 = \begin{bmatrix} Q_1 & S_1 \\ S_1^\top & R_1 \end{bmatrix} = \begin{bmatrix} \tilde{A}^\top \\ B^\top \end{bmatrix} Q_0 [\ \tilde{A} \quad B\ ] + \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \geq 0, \tag{19}$$

since, as shown in Lemma 2, $Q_0 \geq 0$. We now prove **(iii)**. Substitution of $X_U = Q_U + \left[\begin{smallmatrix} \Delta_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ obtained in the proof of **(i)** into (17) gives

$$\begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_1^\top \Delta_1 A_1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} S_1 + A_1^\top \Delta_1 B_1 \\ 0 \end{bmatrix} (R_1 + B_1^\top \Delta_1 B_1)^\dagger [\ S_1^\top + B_1^\top \Delta_1 A_1 \quad 0\ ],$$

which is equivalent to (14). We now prove that Δ_1 satisfies $\ker(R_1 + B_1^\top \Delta_1 B_1) \subseteq \ker(S_1 + A_1^\top \Delta_1 B_1)$. Substitution of $X_U = Q_U + \left[\begin{smallmatrix} \Delta_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ into (18) gives

$$\ker(R_1 + B_1^\top \Delta_1 B_1) \subseteq \ker\left(\begin{bmatrix} \tilde{A}^\top \\ 0 \end{bmatrix} Q_U B_U + \begin{bmatrix} A_1^\top \Delta_1 B_1 \\ 0 \end{bmatrix} \right) = \ker \begin{bmatrix} S_1 + A_1^\top \Delta_1 B_1 \\ 0 \end{bmatrix},$$

which is equivalent to (15). We now prove the converse. Let X be as in (16). Substituting $X_U = U^\top X U = \left[\begin{smallmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{smallmatrix}\right]$ into CGDARE(Σ_U) gives

$$\begin{aligned}
\begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} &= \begin{bmatrix} A_1^\top & A_{21}^\top \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_{21} & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} A_1^\top & A_{21}^\top \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \left(R + [\ B_1^\top \quad B_2^\top\ ] \begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)^\dagger \\
&\quad \times \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_{21} & 0 \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix}
\end{aligned}$$

Developing the products and recalling that we have defined $Q_1 = \tilde{A}^\top Q_U \tilde{A}$, $S_1 = \tilde{A}^\top Q_U B_U$ and $R_1 = R + B_U^\top Q_U B_U$ gives

$$\begin{aligned}
\begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} &= \begin{bmatrix} A_1^\top \Delta_1 A_1 + Q_1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_1^\top \Delta_1 B_1 + S_1 \\ 0 \end{bmatrix} (R_1 + B_1^\top \Delta_1 B_1)^\dagger [\ B_1^\top \Delta_1 A_1 + S_1^\top \quad 0\ ] \\
&\quad + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix},
\end{aligned}$$

which is satisfied since Δ_1 is a solution of (14-15). ■

The following property, which considers the structure of the closed-loop matrix in the basis described by U, is stated separately from properties **(i-iii)** in Theorem 1 to emphasize the differences between this first reduction and the second reduction that will be presented in the next section. In fact, while in the standard case of DARE(Σ) this property of the closed-loop matrix applies to both the first and the second reduction procedure, in the general case of CGDARE(Σ) the structure of the closed-loop matrix described in the following property is maintained only for the first reduction procedure.

Proposition 2 *Given a solution X of CGDARE(Σ) and the associated solution Δ_1 of (14-15), let A_X and A_{Δ_1} be the associated closed-loop matrices. Then,*

$$U^\top A_X U = \begin{bmatrix} A_{\Delta_1} & 0 \\ \star & 0_{\nu\times\nu} \end{bmatrix}.$$

Proof: We first observe that the last ν columns of $U^\top A_X U$ are also zero, i.e.,

$$\begin{aligned} U^\top A_X U &= U^\top (A_0 - B R_X^\dagger B^\top X A_0) U \\ &= A_U - B_U (R + B_U^\top X_U B_U)^\dagger B_U^\top X_U A_U = [\ \star \quad 0\ ], \end{aligned}$$

in view of the fact that the last ν columns of A_U are zero. Moreover,

$$\begin{aligned} U^\top A_X U &= \begin{bmatrix} A_1 & 0 \\ A_{21} & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \left[R + [\ B_1^\top \quad B_2^\top\ ] \left(Q_U + \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right]^\dagger B_U X_U A_U \\ &= \begin{bmatrix} A_1 & 0 \\ A_{21} & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_U^\top X_U A_U \end{aligned}$$

and

$$\begin{aligned} A_{\Delta_1} &= A_1 - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger (B_1^\top \Delta_1 A_1 + S_1^\top) - B_1 R_1^\dagger S_1^\top + B_1 R_1^\dagger S_1^\top \\ &= A_1 - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 A_1 - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger R_1 R_1^\dagger S_1^\top \\ &\quad - B_1 R_1^\dagger S_1^\top + B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger (R_1 + B_1^\top \Delta_1 B_1) R_1^\dagger S_1^\top, \end{aligned}$$

where the last equality follows from the identity $(R_1 + B_1^\top \Delta_1 B_1)^\dagger (R_1 + B_1^\top \Delta_1 B_1) R_1^\dagger = R_1^\dagger$, which can be proved exactly in the same way as (10). [2] Thus,

$$\begin{aligned} A_{\Delta_1} &= A_1 - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 A_1 - B R_1^\dagger S_1^\top - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger (R_1 - R_1 - B_1^\top \Delta_1 B_1) R_1^\dagger S_1^\top \\ &= A_1 - B R_1^\dagger S_1^\top - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 A_1 + B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 B_1 R_1^\dagger S_1^\top \\ &= A_1 - B R_1^\dagger S_1^\top - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 (A_1 - B_1 R_1^\dagger S_1^\top). \end{aligned}$$

Then, denoting by Γ the upper-left block submatrix of order $n - \nu$ within $U^\top A_X U$, we find

$$\begin{aligned} \Gamma - A_{\Delta_1} &= B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger (B_1^\top \Delta_1 A_1 - B_U^\top X_U \tilde{A}) \\ &\quad + B_1 R_1^\dagger S_1^\top - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 B_1 R_1^\dagger S_1^\top. \end{aligned} \tag{20}$$

[2] Indeed, in CGDARE(Σ_1) the matrices R_1 and $R_1 + B_1^\top \Delta_1 B_1$ play the same role of R and $R + B^\top X B$ in CGDARE(Σ), so that $\ker(R_1 + B_1^\top \Delta_1 B_1) \subseteq \ker R_1$.

A simple calculation shows also that

$$\begin{aligned}
B_1^\top \Delta_1 A_1 - B_U^\top X_U \tilde{A} &= B_1^\top \Delta_1 A_1 - \begin{bmatrix} B_1^\top & B_2^\top \end{bmatrix} \begin{bmatrix} (Q_{11}+\Delta_1) & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_{21} \end{bmatrix} \\
&= -\begin{bmatrix} B_1^\top & B_2^\top \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_{21} \end{bmatrix} - B_U^\top Q_U \tilde{A} = -S_1^\top.
\end{aligned}$$

We can use this identity in (20) and we obtain

$$\begin{aligned}
\Gamma - A_{\Delta_1} &= -B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger S_1^\top + B_1 R_1^\dagger S_1^\top - B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 B_1 R_1^\dagger S_1^\top \\
&= -B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger S_1^\top + B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger (R_1 + B_1^\top \Delta_1 B_1) R_1^\dagger S_1^\top \\
&\quad -B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 B_1 R_1^\dagger S_1^\top \\
&= B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger \left[(R_1 + B_1^\top \Delta_1 B_1) R_1^\dagger S_1^\top - S_1^\top - B_1^\top \Delta_1 B_1 R_1^\dagger S_1^\top \right] \\
&= B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger (R_1 R_1^\dagger S_1^\top - S_1^\top) = 0.
\end{aligned}$$

■

In view of **(i)** of Theorem 1, all solutions of CGDARE(Σ) coincide along the subspace $\mathscr{U} \stackrel{\text{def}}{=} \ker\left(\begin{bmatrix} I_{n-\nu} & 0 \\ 0 & 0 \end{bmatrix} U^\top \right)$. This means that given any two solutions X and Y of CGDARE(Σ), we have $X|_{\mathscr{U}} = Y|_{\mathscr{U}} = Q_0|_{\mathscr{U}}$.

The following result gives a property of the set of solutions of CGDARE(Σ), and a procedure to solve CGDARE(Σ) in terms of the reduced order DARE(Σ).

Corollary 1 *The set $\mathscr{X}$ of solutions of CGDARE(Σ) is parameterized as the set of matrices that can be expressed as*

$$X = U \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} U^\top + Q_0$$

where $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ is defined as in Theorem 1 and Δ_1 is solution of (14-15).

After the reduction described in Theorem 1, it may still happen that $A_1 - B_1 R_1^\dagger S_1$ is singular. However, since we have proved that CGDARE(Σ_1) has exactly the same structure of CGDARE(Σ), because $\Pi_1 = \Pi_1^\top \geq 0$, if $A_1 - B_1 R_1^\dagger S_1$ is singular we can iterate the procedure by rewriting (14-15) as

$$\Delta_1 = A_{0,1}^\top \Delta_1 A_{0,1} - A_{0,1}^\top \Delta_1 B_1 (R_1 + B_1^\top \Delta_1 B_1)^\dagger B_1^\top \Delta_1 A_{0,1} + Q_{0,1} \tag{21}$$

$$\ker(R_1 + B_1^\top \Delta_1 B_1) \subseteq \ker(A_{0,1}^\top \Delta B_1), \tag{22}$$

where $A_{0,1} \stackrel{\text{def}}{=} A_1 - B_1 R_1^\dagger S_1^\top$ and $Q_{0,1} \stackrel{\text{def}}{=} Q_1 - S_1 R_1^\dagger S_1^\top$, and choosing a basis where $A_{0,1} = \begin{bmatrix} \tilde{A}_1 & 0 \end{bmatrix}$ and $\tilde{A}_1$ is of full column-rank. By following iteratively the procedure that led from CGDARE(Σ) to CGDARE(Σ_1), we eventually obtain a CGDARE(Σ_k) of the form

$$\Delta_k = A_{0,k}^\top \Delta_k A_{0,k} - A_{0,k}^\top \Delta_k B_k \left(R_k + B_k^\top \Delta_k B_k\right)^\dagger B_k^\top \Delta_k A_{0,k} + Q_{0,k} \tag{23}$$

$$\ker(R_k + B_k^\top \Delta_k B_k) \subseteq \ker(A_{0,k}^\top \Delta_k B_k), \tag{24}$$

where now $A_{0,k}$ is non-singular. Notice also that this reduction procedure can be carried out only using the problem data A, B, Q, R, S, so that it holds for any solution X of CGDARE(Σ). In other words, this procedure (and the one that will follow in the next section) can be performed without the need to compute a particular solution of the Riccati equation.

Once we have obtained the reduced-order CGDARE, if the corresponding matrix R is singular, we can proceed with the second reduction procedure outlined in the next section.

4.2 Reduction corresponding to a singular R

Consider CGDARE(Σ), either in the form given by (1-2) or (6-7). Suppose R is singular. We assume that we have already performed the reduction described in the previous section. Hence, we may assume that A_0 is now non-singular. To deal with this situation, we address separately two different cases: the first leads either to a reduced-order DARE or to a symmetric Stein equation depending on the rank of R, and the second leads to a reduced-order CGDARE. We first consider the case in which $A_0^{-1} B \ker R = \{0\}$, i.e., $B \ker R = \{0\}$. This case can in turn be divided into two sub-cases. The first is the one in which R is not the zero matrix. In this case, denoting by r the rank of R, we can consider a change of coordinates in the input space that brings R in the form

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where R_1 is non-singular, and r is its order. With respect to this basis, since $\ker R = \operatorname{im} \begin{bmatrix} 0 \\ I_{m-r} \end{bmatrix}$, matrix B can be written as $B = [\; B_1 \quad 0_{n\times(m-r)} \;]$, and (6-7) written in this basis

$$X = A_0^\top X A_0 - A_0^\top [\; X B_1 \quad 0 \;] \left(\begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1^\top X B_1 & 0 \\ 0 & 0 \end{bmatrix} \right)^\dagger [\; B_1^\top X \quad 0 \;] A_0 + Q_0$$

$$\ker \left(\begin{bmatrix} R_1 + B_1^\top X B_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \subseteq \ker(A_0^\top X [\; B_1 \quad 0 \;]),$$

reduces to

$$X = A_0^\top X A_0 - A_0^\top X B_1 (R_1 + B_1^\top X B_1)^\dagger B_1^\top X A_0 + Q_0$$

$$\operatorname{im} \begin{bmatrix} 0 \\ I_{m-r} \end{bmatrix} \subseteq \ker[\; \star \quad 0_{n\times(m-r)} \;]$$

where now R_1 is invertible as required, so that $R_1 + B_1^\top X B_1$ is positive definite. Hence, the latter is in fact a DARE

$$X = A_0^\top X A_0 - A_0^\top X B_1 (R_1 + B_1^\top X B_1)^{-1} B_1^\top X A_0 + Q_0.$$

If $r = 0$, i.e., if R is the zero matrix, then $B \ker R = \{0\}$ implies that B is also the zero matrix. In this case, CGDARE(Σ) reduces to a symmetric Stein equation[3]

$$X = A_0^\top X A_0 + Q_0.$$

We now consider the case in which $A_0^{-1} B \ker R \neq \{0\}$.

Theorem 2 *Let* $\eta \stackrel{def}{=} \dim(A_0^{-1} B \ker R)$. *Let* $V = [\ V_1 \quad V_2\ ]$ *be an orthonormal change of coordinates in* $\mathbb{R}^n$ *where* $\operatorname{im} V_2 = A_0^{-1} B \ker R$. *Let* $Q_V \stackrel{def}{=} V^\top A_0 V$ *and* $A_V \stackrel{def}{=} V^\top A_0 V = \begin{bmatrix} A_1 & \star \\ \star & \star \end{bmatrix}$, $B_V \stackrel{def}{=} V^\top B = \begin{bmatrix} B_1 \\ \star \end{bmatrix}$, $R_1 \stackrel{def}{=} R + B^\top Q_0 B$, *with* $A_1 \stackrel{def}{=} V_1^\top A_0 V_1 \in \mathbb{R}^{(n-\eta)\times(n-\eta)}$ *and* $B_1 \stackrel{def}{=} V_1^\top B \in \mathbb{R}^{(n-\eta)\times m}$. *Let* $Q_V \stackrel{def}{=} V^\top Q_0 V = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix}$, $A_V^\top Q_V A_V = \begin{bmatrix} Q_1 & \star \\ \star & \star \end{bmatrix}$, $A_V^\top Q_V B_V = \begin{bmatrix} S_1 \\ \star \end{bmatrix}$, *where* $Q_{11}, Q_1 \in \mathbb{R}^{(n-\eta)\times(n-\eta)}$ *and* $S_1 \in \mathbb{R}^{(n-\eta)\times m}$. *Then,*

1. *Let* X *be a solution of CGDARE(Σ), and partition* $X_V \stackrel{def}{=} V^\top X V$ *as* $X_V = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^\top & X_{22} \end{bmatrix}$. *Then,*

   (i) *there hold*

   $$X_{12} = Q_{12} \quad \textit{and} \quad X_{22} = Q_{22}$$

   (ii) *The Popov matrix* $\Pi_1 \stackrel{def}{=} \begin{bmatrix} Q_1 & S_1 \\ S_1^\top & R_1 \end{bmatrix}$ *is positive semidefinite.*

   (iii) *Let* $\Sigma_1 \stackrel{def}{=} (A_1, B_1, \Pi_1)$. *Then,* $\Delta_1 \stackrel{def}{=} X_{11} - Q_{11}$ *satisfies CGDARE(Σ_1)*

   $$\Delta_1 = A_1^\top \Delta_1 A_1 - (A_1^\top \Delta_1 B_1 + S_1)(R_1 + B_1^\top \Delta_1 B_1)^\dagger (B_1^\top \Delta_1 A_1 + S_1^\top) + Q_1 \tag{25}$$
   $$\ker(R_1 + B_1^\top \Delta_1 B_1) \subseteq \ker(S_1 + A_1^\top \Delta B_1). \tag{26}$$

2. *Conversely, if* Δ_1 *is a solution of (25-26), then*

   $$X = V \begin{bmatrix} \Delta_1 + Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} V^\top$$

   is a solution of CGDARE(Σ).

[3] For a discussion on the properties of symmetric Stein equations we refer to [17, Section 5.3] and [13, Section 1.5].

Proof: We prove the first point. As already observed in the beginning of Section 4.1, X is a solution of (1-2) – and therefore also of (6-7) – if and only if $X_V = V^\top X V$ is a solution of CGDARE(Σ_V)

$$X_V = A_V^\top X_V A_V - A_V^\top X_V B_V (R + B_V^\top X_V B_V)^\dagger B_V^\top X_V A_V + Q_V \tag{27}$$

$$\ker(R + B_V^\top X_V B_V) \subseteq \ker(A_V^\top X_V B_V), \tag{28}$$

where $\Pi_V = \left[\begin{smallmatrix} Q_V & 0 \\ 0 & R \end{smallmatrix}\right]$ and $\Sigma_V = (A_V, B_V, \Pi_V)$. We can re-write (27) as

$$X_V = A_V^\top X_V V^\top [I_n - B(R + B^\top X B)^\dagger B^\top X] A V + Q_V.$$

Post-multiplying the latter by $\left[\begin{smallmatrix} 0 \\ I_\eta \end{smallmatrix}\right]$ and considering a basis matrix K_R for $\ker R$, so that we can write $V_2 = A^{-1} B K_R$, gives

$$\begin{aligned}
\begin{bmatrix} X_{12} \\ X_{22} \end{bmatrix} &= A_V^\top X_V V^\top [I_n - B(R + B^\top X B)^\dagger B^\top X] A V_2 + \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} \\
&= V^\top A_0^\top X B [I_m - R_X^\dagger (B^\top X B + R - R)] K_R + \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} \\
&= V^\top A_0^\top X B (I_m - R_X^\dagger R_X - R_X^\dagger R) K_R + \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} \\
&= V^\top A_0^\top X B (I_m - R_X^\dagger R_X) K_R + \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} = V^\top A_0^\top X B G_X K_R + \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix}.
\end{aligned}$$

Recalling that $\operatorname{im} G_X = \ker R_X$, and that by virtue of (7) there holds $\ker R_X \subseteq \ker(A_0^\top X B)$, we get $V^\top A_0^\top X B G_X K_R = 0$, from which **(i)** immediately follows. To prove **(ii)** we observe that

$$\Pi_1 = \begin{bmatrix} I_{n-\eta} & 0 & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} A_V^\top \\ B_V^\top \end{bmatrix} Q_V [\, A_V \quad B_V \,] \begin{bmatrix} I_{n-\eta} & 0 \\ 0 & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \geq 0. \tag{29}$$

In order to prove **(iii)**, we first observe that in view of the previous considerations we have $X_V = Q_V + \left[\begin{smallmatrix} \Delta_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Substitution of this expression into (27-28) yields

$$\begin{aligned}
\begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_1^\top \Delta_1 A_1 & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} S_1 + A_1^\top \Delta_1 B_1 \\ 0 \end{bmatrix} (R_1 + B_1^\top \Delta_1 B_1)^\dagger [\, S_1^\top + B_1^\top \Delta_1 A_1 \quad 0 \,],
\end{aligned}$$

whose block in position (1,1) is exactly (25). We now prove that Δ_1 satisfies (26). Substitution of $X_V = Q_V + \left[\begin{smallmatrix} \Delta_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ into (28) gives

$$\ker(R_1 + B_1^\top \Delta_1 B_1) \subseteq \ker \begin{bmatrix} S_1 + A_1^\top \Delta_1 B_1 \\ \star \end{bmatrix},$$

from which (26) immediately follows.

The second point can be proved by reversing these arguments along the same lines of the second part of the proof of Theorem 1. ■

In view of **(i)** of Theorem 2, all solutions of CGDARE(Σ) coincide along $\mathscr{V} \stackrel{\text{def}}{=} \ker\left(\left[\begin{smallmatrix} I_{n-\eta} & 0 \\ 0 & 0 \end{smallmatrix}\right] V^\top\right)$. This means that given any two solutions X and Y of CGDARE(Σ), we have $X|_{\mathscr{V}} = Y|_{\mathscr{V}} = Q_0|_{\mathscr{V}}$.

Corollary 2 *The set $\mathscr{X}$ of solutions of CGDARE(Σ) is parameterized as the set of matrices*

$$X = V \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} V^\top + Q_0$$

where $V = [\; V_1 \quad V_2 \;]$ is defined as in Theorem 2 and Δ_1 is solution of (25-26).

Remark 1 In [3] it is shown that if X is a solution of DARE(Σ) and we consider the associated solution Δ_1 of the reduced DARE(Σ_1), and if we denote by A_X and A_{Δ_1} the associated closed-loop matrices, there holds

$$V^\top A_X V = \begin{bmatrix} A_{\Delta_1} & 0 \\ \star & 0_{\eta\times\eta} \end{bmatrix}. \tag{30}$$

This is a simple consequence of the fact that in the case of a solution X of DARE(Σ), the matrix R_X is invertible. We now show via a simple example that this fact does not hold in general in the case of CGDARE(Σ). Consider a Popov triple Σ described by the matrices

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 24 \end{bmatrix}, \quad R = 0, \quad S = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In this case $A_0 = A$ is invertible, and $A_0^{-1} B \ker R = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$. Let $V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $V = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then, we compute

$$A_V = V^\top A_0 V = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -5 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad B_V = V^\top B = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad Q_V = V^\top Q_0 V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_V^\top Q_V A_V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 600 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_V^\top Q_V B_V = 0,$$

so that the matrices of the reduced CGDARE(Σ_1) are

$$A_1 = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 600 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_1 = 0.$$

A simple direct calculation shows that the only solution of this reduced CGDARE is $X_1 = \begin{bmatrix} 0 & 0 \\ 0 & -25 \end{bmatrix}$. Thus, the only solution of the original CGDARE(Σ) is $X = V\left(Q_V + \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\right) V^\top =$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The corresponding closed-loop matrix coincides with A, i.e., $A_X = A$. Now,

$$V^\top A_X V = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -5 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

This shows that neither of the two zero submatrices in the second block-column of (30) is zero in the general case of CGDARE(Σ). While the submatrix in the upper left block of A_X still coincides with A_{Δ_1}, in the case of CGDARE(Σ) it is also no longer true that the spectrum of A_{Δ_1} is contained in that of A_X. Indeed, in this case $\sigma(A_{\Delta_1}) = \{-5, 3\}$ whereas $\sigma(A_X) = \{-5, 1 \pm \sqrt{5}\}$. This difference between DARE and CGDARE is related to the fact that in this generalized case the reduction can correspond simply to the singularity of R which does not imply the singularity of A_X as discussed in Section 2.

Remark 2 As for the reduction described in Theorem 1, it may occur that, as a result of the reduction illustrated in Theorem 2, $A_1 - B_1 R_1^\dagger S_1^\top$ and/or R_1 be still singular. However, we have showed that Π_1 is symmetric and positive semidefinite. This means that if $A_1 - B_1 R_1^\dagger S_1^\top$ is singular, we can repeat the reduction procedure described in Theorem 1, while if $A_1 - B_1 R_1^\dagger S_1^\top$ is non-singular but R_1 is singular, we can repeat the reduction procedure described in Theorem 2. Since the order of the Riccati equation lowers at each reduction step, after at most n steps, either we have computed the unique solution of the original CGDARE(Σ), or we have obtained a symmetric Stein equation (which is linear), or we obtained a "well-behaved" DARE of maximally reduced order where the corresponding R and $A - BR^\dagger S^\top$ matrices are non-singular.

5 Numerical examples

Example 5.1 Using the reduction techniques developed in the previous sections, we want to study the set of solutions of the CGDARE(Σ) where Σ is given by the matrices

$$A = \begin{bmatrix} 0 & -4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

First notice that since S is the zero matrix, A_0 and Q_0 coincide with A and Q, respectively. Thus, in this case both A_0 and R are singular. We begin with a reduction that corresponds to the singularity of A_0. Since $\ker A_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, we can consider a basis matrix $U = [\ U_1 \mid U_2\ ]$ given by $U = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, so that

$$A_U = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 4 & 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \\ 4 & 0 \end{bmatrix}, \quad B_U = \begin{bmatrix} -3 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$A_1 = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 16 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

In view of Corollary 1, X is a solution of CGDARE(Σ) if and only if it can be written as

$$X = Q_0 + U \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} U^\top,$$

where Δ_1 is an arbitrary solution of (14-15). To maintain the notations as consistent as possible to those employed in Section 4.2, we define $\overline{A} \stackrel{\text{def}}{=} A_1$, $\overline{B} \stackrel{\text{def}}{=} B_1$, $\overline{Q} \stackrel{\text{def}}{=} Q_1$, $\overline{S} \stackrel{\text{def}}{=} S_1$, $\overline{R} \stackrel{\text{def}}{=} R_1$ and $\overline{X} \stackrel{\text{def}}{=} \Delta_1$. With this notation, (14-15) can be re-written as

$$\overline{X} = \overline{A}_0^\top \overline{X}\,\overline{A}_0 - \overline{A}_0^\top \overline{X}\,\overline{B}(\overline{R} + \overline{B}^\top \overline{X}\,\overline{B})^\dagger \overline{B}^\top \overline{X}\,\overline{A}_0 + \overline{Q}_0 \tag{31}$$

$$\ker(\overline{R} + \overline{B}^\top \overline{X}\,\overline{B}) \subseteq \ker(\overline{A}_0^\top \overline{X}\,\overline{B}), \tag{32}$$

where $\overline{A}_0 = \overline{A} - \overline{B}\,\overline{R}^\dagger \overline{S}^\top = \overline{A}$ and $\overline{Q}_0 = \overline{Q} - \overline{S}\,\overline{R}^\dagger \overline{S}^\top = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Matrix $\overline{A}_0$ is invertible, whereas $\overline{R}$ is singular. Thus, we can apply the reduction procedure in Section 4.2 (we will employ the same notation used in Section 4.2, with the only exception that all the letters will have a bar, to distinguish this second reduction from the first one). A simple calculation shows that $\operatorname{im}(\overline{A}_0^{-1}\overline{B}\ker\overline{R}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Thus, we can consider a basis matrix $V = [\; V_1 \;|\; V_2 \;]$ given by $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence, we define $\overline{X}_V \stackrel{\text{def}}{=} V^\top \overline{X} V$ along with

$$\overline{A}_V = V^\top \overline{A}_0 V = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad \overline{B}_V = V^\top \overline{B} = \begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}, \quad \overline{Q}_V = V^\top \overline{Q}_0 V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so that $\overline{A}_1 = -1$, $\overline{B}_1 = [\,0 \;\; 0\,]$, $\overline{S}_1 = [\,0 \;\; 0\,]$, $\overline{Q}_1 = 0$, $\overline{R}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. In view of Corollary 2, $\overline{X}$ is a solution of (31-32) if and only if

$$\overline{X} = \overline{Q}_0 + V \begin{bmatrix} \overline{\Delta}_1 & 0 \\ 0 & 0 \end{bmatrix} V^\top$$

with $\overline{\Delta}_1$ being an arbitrary solution of

$$\overline{\Delta}_1 = \overline{A}_1^\top \overline{\Delta}_1 \overline{A}_1 - \overline{A}_1^\top \overline{\Delta}_1 \overline{B}_1 (\overline{R}_1 + \overline{B}_1^\top \overline{\Delta}_1 \overline{B}_1)^\dagger \overline{B}_1^\top \overline{\Delta}_1 \overline{A}_1 + \overline{Q}_1 \tag{33}$$

$$\ker(\overline{R}_1 + \overline{B}_1^\top \overline{\Delta}_1 \overline{B}_1) \subseteq \ker(\overline{A}_1^\top \overline{\Delta}_1 \overline{B}_1). \tag{34}$$

We still have $\overline{R}_1$ singular, and $\overline{A}_1 - \overline{B}_1 \overline{R}_1^\dagger \overline{S}_1^\top = \overline{A}_1$ is invertible. On the other hand, $\overline{B}_1 \ker \overline{R}_1 = \{0\}$, so that the reduction associated to the singularity of $\overline{R}_1$ cannot be carried out. Using a change of coordinates in the input space given by $\Omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we obtain

$$\hat{R}_1 = \Omega^{-1} \overline{R}_1 \Omega = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B}_1 = \overline{B}_1 \Omega = [\,0 \;\; 0\,],$$

so that $\hat{R}_{1,0} = 1$ and $\hat{B}_{1,0} = 0$. Thus, (33-34) can be written in this basis as

$$\overline{\Delta}_1 = \overline{A}_1^\top \overline{\Delta}_1 \overline{A}_1 - \overline{A}_1^\top \overline{\Delta}_1 \hat{B}_{1,0} (\hat{R}_{1,0} + \hat{B}_{1,0}^\top \overline{\Delta}_1 \hat{B}_{1,0})^\dagger \hat{B}_{1,0}^\top \overline{\Delta}_1 \overline{A}_1 + \overline{Q}_1 \tag{35}$$

$$\ker(\hat{R}_{1,0} + \hat{B}_{1,0}^\top \overline{\Delta}_1 \hat{B}_{1,0}) \subseteq \ker \overline{A}_1^\top \overline{\Delta}_1 \hat{B}_{1,0}. \tag{36}$$

which reduce to the trivial equation $\overline{\Delta}_1 = \overline{\Delta}_1$ subject to the trivial constraint $\ker \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \subseteq \ker \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Any $\xi \stackrel{\text{def}}{=} \overline{\Delta}_1 \in \mathbb{R}$ satisfies this reduced Riccati equation. Thus, the solutions of (31-32) are given by $\overline{X} = V \begin{bmatrix} \xi & 0 \\ 0 & 0 \end{bmatrix} V^\top = \begin{bmatrix} 0 & 0 \\ 0 & \xi \end{bmatrix}$, $\xi \in \mathbb{R}$, so that – recalling that $Q_0 = Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ – the set of solutions of the original CGDARE(Σ) is parametrized by

$$X = Q_0 + U \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & \xi & 0 \\ \hline 0 & 0 & 0 \end{array}\right] U^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi \end{bmatrix}, \; \xi \in \mathbb{R}.$$

Example 5.2 Using the reduction techniques developed here, we want to study the set of solutions of the CGDARE(Σ) where Σ is given by the matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -5 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since S is the zero matrix, $A_0 = A$ and $Q_0 = Q$. Both A_0 and R are singular. We begin with a reduction that corresponds to the singularity of A_0. Since $\ker A_0 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, we can consider a basis matrix $U = [\; U_1 \;|\; U_2 \;]$ given by $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, so that

$$A_U = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ -3 & 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 4 & 0 \\ 0 & -3 \\ -3 & 0 \end{bmatrix}, \quad B_U = \begin{bmatrix} 3 & -5 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad Q_U = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$A_1 = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 36 & -60 \\ 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 48 & 0 \\ 0 & 144 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 27 & -45 \\ -45 & 75 \end{bmatrix}.$$

In view of Corollary 1, X is a solution of CGDARE(Σ) if and only if it can be written as

$$X = Q_0 + U \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} U^\top,$$

where Δ_1 is an arbitrary solution of (14-15). As in Example 5.1, to maintain the notations as consistent as possible to those employed in Section 4.2, we define $\overline{A} \stackrel{\text{def}}{=} A_1$, $\overline{B} \stackrel{\text{def}}{=} B_1$, $\overline{Q} \stackrel{\text{def}}{=} Q_1$, $\overline{S} \stackrel{\text{def}}{=} S_1$, $\overline{R} \stackrel{\text{def}}{=} R_1$ and $\overline{X} \stackrel{\text{def}}{=} \Delta_1$. With this notation, (14-15) can be re-written as in (31-32) where

$$\overline{A}_0 = \overline{A} - \overline{B}\,\overline{R}^\dagger \overline{S}^\top = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \quad \text{and} \quad \overline{Q}_0 = \overline{Q} - \overline{S}\,\overline{R}^\dagger \overline{S}^\top = \begin{bmatrix} 0 & 0 \\ 0 & 144 \end{bmatrix}.$$

Both $\overline{A}_0$ and $\overline{R}$ are singular. We can reapply the reduction procedure in Section 4.1 (we will employ the same notation used in Section 4.1, with the only exception that all the letters will have a tilde, to distinguish this reduction from the first one). Now $\ker \overline{A}_0 = \operatorname{span}\left\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right\}$. Thus, we can consider a basis matrix $\overline{U} = [\ \overline{U}_1 \mid \overline{U}_2\ ]$ given by $\overline{U} = \left[\begin{smallmatrix}0 & 1\\1 & 0\end{smallmatrix}\right]$. Hence, we define $\overline{X}_{\overline{U}} \stackrel{\text{def}}{=} \overline{U}^\top \overline{X}\,\overline{U}$ along with $\overline{A}_{\overline{U}} = \overline{U}^\top \overline{A}_0 \overline{U} = \left[\begin{smallmatrix}-3 & 0\\0 & 0\end{smallmatrix}\right]$, $\overline{B}_{\overline{U}} = \overline{U}^\top \overline{B} = \left[\begin{smallmatrix}0 & 0\\3 & -5\end{smallmatrix}\right]$, $\overline{Q}_{\overline{U}} = \overline{U}^\top \overline{Q}_0 \overline{U} = \left[\begin{smallmatrix}144 & 0\\0 & 0\end{smallmatrix}\right]$. We have thus obtained the matrices of the reduced-order Riccati equation

$$\overline{A}_1 = -3, \quad \overline{B}_1 = \begin{bmatrix}0 & 0\end{bmatrix}, \quad \overline{S}_1 = \begin{bmatrix}0 & 0\end{bmatrix}, \quad \overline{Q}_1 = 1296, \quad \overline{R}_1 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

In view of Corollary 2, $\overline{X}$ is a solution of (31-32) if and only if

$$\overline{X} = \overline{Q}_0 + \overline{U} \begin{bmatrix}\overline{\Delta}_1 & 0\\0 & 0\end{bmatrix} \overline{U}^\top$$

with $\overline{\Delta}_1$ being an arbitrary solution of (33-34). We still have $\overline{R}_1$ singular, and $\overline{A}_1 - \overline{B}_1 \overline{R}_1^\dagger \overline{S}_1^\top = \overline{A}_1$ is invertible. On the other hand, $\overline{B}_1 \ker \overline{R}_1 = \{0\}$, so that the reduction associated to the singularity of $\overline{R}_1$ cannot be carried out. Since $\overline{R}_1$ is the zero matrix, and so is $\overline{B}_1$, (35-36) can be written as the symmetric Stein equation

$$\overline{\Delta}_1 = \overline{A}_1^\top \overline{\Delta}_1 \overline{A}_1 + \overline{Q}_1$$

subject to the trivial constraint $\ker(0) \subseteq \ker(0)$. This equation therefore reduces to

$$\overline{\Delta}_1 = 9\,\overline{\Delta}_1 + 1296$$

which admits the solution $\overline{\Delta}_1 = -162$. Thus, the matrix $\overline{X} = \overline{U} \left[\begin{smallmatrix}-162 & 0\\0 & 0\end{smallmatrix}\right] \overline{U}^\top + \overline{Q}_0 = \left[\begin{smallmatrix}0 & 0\\0 & -18\end{smallmatrix}\right]$ satisfies (31-32), and, recalling that $Q_0 = Q = \left[\begin{smallmatrix}3 & 0 & 0\\0 & 0 & 0\\0 & 0 & 16\end{smallmatrix}\right]$ and $U = \left[\begin{smallmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{smallmatrix}\right]$, we find

$$X = Q_0 + U \left[\begin{array}{cc|c}0 & 0 & 0\\0 & -18 & 0\\\hline 0 & 0 & 0\end{array}\right] U^\top = \begin{bmatrix}3 & 0 & 0\\0 & 0 & 0\\0 & 0 & -2\end{bmatrix},$$

which is the only solution of the original CGDARE(Σ).

Concluding remarks

We have shown how a general CGDARE(Σ) may be reduced to a well-behaved DARE(Σ) of smaller order featuring a non-singular closed-loop matrix. This reduction may be performed through repeated steps each of which may be easily implemented via robust linear algebraic routines thus providing an effective tool to deal with generalized Riccati equations in practical situations.

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