Mean kernels to improve gravimetric geoid determination based on modified Stokes's integration

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Abstract

Gravimetric geoid computation is often based on modified Stokes's integration, where Stokes's integral is evaluated with some stochastic or deterministic kernel modification. Accurate numerical evaluation of Stokes's integral requires the modified kernel to be integrated across the area of each discretised grid cell (mean kernel). Evaluating the modified kernel at the centre of the cell (point kernel) is an approximation which may result in larger numerical integration errors near the computation point, where the modified kernel exhibits a strongly nonlinear behaviour. The present study deals with the computation of whole-of-the-cell mean values of modified kernels, exemplified here with the Featherstone-Evans-Olliver (1998) kernel modification (Featherstone, W.E., Evans, J.D., Olliver, J.G., 1998. A Meissl-modified Vaníček and Kleusberg kernel to reduce the truncation error in gravimetric geoid computations. Journal of Geodesy 72(3), 154-160). We investigate two approaches (analytical and numerical integration) which are capable of providing accurate mean kernels. The analytical integration approach is based on kernel weighting factors which are used for the conversion of point to mean kernels. For the efficient numerical integration, Gauss-Legendre Quadrature is applied. The comparison of mean kernels from both approaches shows a satisfactory mutual agreement at the level of 10^{-4} and better, which is considered to be sufficient for practical geoid computation requirements. Closed-loop tests based on the EGM2008 geopotential model demonstrate that using mean instead of point kernels reduces numerical integration errors by ~65%. The use of mean kernels is recommended in remove-compute-restore geoid determination with the Featherstone-Evans-Olliver (1998) kernel or any other kernel modification under the condition that the kernel changes rapidly across the cells in the neighbourhood of the computation point.

1 Introduction

Many strategies used in gravity field modelling were developed at a time when the "*goal for geoid determination was at a precision of a least one order of magnitude less than it is today, i.e., ~10 cm or worse*" (Sjöberg, 2005). For today's geoid and quasigeoid modelling at the centimetre-level and better, it is therefore required to thoroughly assess and – if necessary – correct for approximations that are still inherent in the techniques used.

Regional geoid computations are often based on numerical integration of gravity anomalies using modified Stokes's integration where Stokes's integral is evaluated with some kernel modification (e.g., Featherstone, 2003; Ellmann, 2005). Inevitably, the practical evaluation of Stokes's integral is subject to approximations. This is because Stokes's integral is evaluated by numerical integration of gravity anomalies, given for small surface elements (aka blocks or cells), cf. Heiskanen and Moritz (1967) and Torge (2001). Not only are gravity anomalies required, but also values of the integral kernel which are 'most' representative for the cells (Heiskanen and Moritz, 1967; Vaníček and Krakiwsky, 1986).

Sometimes, the integral kernel is computed at the centre-of-the-cell, but this may be a coarse approximation in the neighbourhood of the computation point, where the kernel changes nonlinearly across the cell (Strang van Hees, 1990). Rigorously, whole-of-the-cell means of the kernel (herein abbreviated to mean kernels) are required. These can be obtained through integration of the kernel over the cell (Vaníček and Krakiwsky, 1986). Using mean instead of centre-of-cell kernels near the computation point may considerably reduce numerical integration errors (Hirt et al. 2011).

For Stokes's integral and Stokes's kernel, Strang van Hees (1990) and de Min (1994) have developed approaches to compute estimates of mean kernels across the cell. Recently, Hirt et al. (2011) have combined these approaches, yielding a generalized computation procedure for mean kernels used in Stokes's integral and other geodetic convolution integrals, such as the integrals of Vening-Meinesz and Poisson (Torge, 2001) or Hotine (Hotine, 1969). However, the computation procedures and benefits of mean kernels are generally not or little addressed by scholars for *modifications of Stokes's function*, such as the

deterministic modifications of Wong and Gore (1969), Heck and Grüninger (1987), Vaníček and Kleusberg (1987) and Featherstone et al. (1998) and the stochastic modifications of Wenzel (1982) and Sjöberg (1984, 1991). An exception is de Min (1996, p. 169) mentioning the use of cell-mean values for the Wong and Gore (1969) kernel modification.

The aim of the present study is to demonstrate that the use of mean kernel estimates is an important issue for the accurate evaluation of Stokes's integral using modified integration kernels. Section 2 briefly summarizes the basic theory of geoid computation using Stokes's integral, modified Stokes's integration and the Featherstone et al. (1998) kernel as an example for integral kernel modifications. Section 3 then describes and compares one analytical and one numerical approach capable of providing whole-of-the-cell mean values of modified kernels. Results of closed-loop tests are presented in Section 4, revealing the benefits of mean kernels in practice. The Featherstone et al. (1998) kernel has been chosen to serve as example, however, we consider our study representative for other kernel modifications used in practice (e.g., Featherstone, 2003; Ellmann, 2005). The present study is complementary to the paper by Hirt et al. (2011), that discusses the computation of mean kernels for geodetic convolution integrals in general and for the (unmodified) Stokes's integral in particular.

2. Basic theory

2.1 Stokes's integral

Stokes's integral (aka Stokes's formula) of 1849 allows computation of geoid heights N from gravity anomalies Δg which – at least theoretically – are required to be continously given for small cells of size $d\sigma$ covering the whole of the Earth's surface σ (Torge, 2001):

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} \Delta g \cdot S(\psi) d\sigma \qquad (1)$$

with R denoting the radius of Earth and γ the normal gravity. The term $S(\psi)$ is Stokes's function (aka Stokes's kernel) which is a function of the spherical distance ψ between the computation point P (where the geoid height N is sought) and the data points Q (where gravity anomalies Δg are given). In closed-form, Stokes's function reads (Heiskanen and Moritz, 1967):

$$S(\psi) = \frac{1}{\sin(\psi/2)} - 6\sin(\psi/2) + 1 - 5\cos\psi - 3\cos\psi \ln[\sin(\psi/2) + \sin^2(\psi/2)] \ . \quad (2)$$

To contrast with modified kernels in Sections 2.2 and 2.3, $S(\psi)$ is also called *unmodified* Stokes's kernel. The spherical distance ψ is computed as a function of the spherical coordinates (latitude φ_P and longitude λ_P) of the computation point P and the coordinates (φ_Q, λ_Q) of the data points Q (Strang van Hees 1990, p. 236):

$$\sin^2\left(\frac{\psi}{2}\right) = \sin^2\left(\frac{\varphi_P - \varphi_Q}{2}\right) + \sin^2\left(\frac{\lambda_P - \lambda_Q}{2}\right)\cos\varphi_P\cos\varphi_Q. \quad (3)$$

Practically, Stokes's integral formula is evaluated numerically by means of summation of products of Stokes's kernel $S(\psi)$ and gravity anomalies Δg over small, but finite cells, e.g., of 1 arc min x 1 arc min size. The required discretisation of Earth's surface is usually done by means of grid lines along meridians and parallels, giving a subdivision into approximately rectangular cells q (Heiskanen and Moritz, 1967, p. 118). Next, we use Q to denote the centre coordinates of the cells q.

Stokes's function $S(\psi)$ is singular for the gravity anomaly of the computation point P (i.e., $\psi=0$). The cell centred to the computation point P is also denoted *innermost zone* (e.g., Featherstone, 2002; Torge, 2001). The contribution of the gravity anomaly Δg_P of the innermost zone to the geoid height N can be taken into account with approximations given e.g., in Haagmans et al. (1993) or Torge (2001), p. 283.

Evaluating the $S(\psi)$ at the centre of the cells Q is sufficiently accurate beyond some distance, say 1° from the computation point P, while larger numerical integration errors are caused by centre-of-cell kernels in the neighbourhood of the computation point P. This is related to the first term of Stokes's function

$$S^{1st}(\psi) = \frac{1}{\sin(\psi/2)}, \quad (4)$$

which, as dominating component of $S(\psi)$ in the neighbourhood of the computation point P, is chiefly responsible for the non-linear across-the-cell-variation of the kernel (see, e.g., Hirt et al., 2011). To accurately evaluate Stokes's integral near the computation point P, the kernel

function needs to be integrated across the cell (Vaníček and Krakiwsky, 1986), which gives whole-of-the-cell mean kernels (Section 3).

2.2 Modified Stokes's integration

It is well known that geoid computation based on Stokes's integral [Eq. (1)] requires the gravity data Δg to be provided *continuously* over the whole of the Earth's surface. This is an impractical requirement, given that the global gravity data coverage (e.g., at 1 arc min resolution) is still incomplete and irregular, thus impeding the direct and accurate use of Stokes's formula (e.g., Vaníček and Sjöberg, 1991). A commonly applied strategy to circumvent this problem is the remove-compute-restore (RCR) technique together with modified Stokes's integration (after Vaníček and Sjöberg, 1991, Featherstone et al. 2001):

$$N = N^L + N^{res} = N^L + \frac{R}{4\pi\gamma} \iint_\sigma (\Delta g - \Delta g^L) \cdot S^*(\psi) d\sigma \ . \tag{5}$$

In the RCR-technique (e.g., Sjöberg, 2005; Featherstone et al., 2001), the observed gravity anomalies Δg are reduced ("remove") by the contribution Δg^L of a global geopotential model (GGM) expanded to spherical harmonic degree *L*, yielding residual anomalies $\Delta g^{res} = \Delta g - \Delta g^L$. These are transformed ("compute") to the residual geoid undulation N^{res} of the computation point *P* using modified Stokes's integration, i.e., evaluation of Stokes's integral in Eq. (5) with some modified kernel $S^*(\psi)$. Finally, the GGM geoid contribution N^L is added ("restore") to the residual geoid undulation N^{res}. We refer the reader to, e.g., Featherstone et al. (2001) and Torge (2001), p. 271f, for the spherical harmonic expansions to compute N^L and Δg^L from the GGM coefficients. As an important benefit of the RCR-technique, the computation area may be limited to radius ψ_0, typically a few degrees, around the computation point *P* (e.g., Vaníček and Kleusberg, 1987; Featherstone et al., 2001; Torge, 2001, p. 285).

The neglect of gravity data outside the computation area ($\psi > \psi_0$) results in a truncation error, which is kept manageably small by using a high-degree GGM reference field (e.g., *L*=360) in the RCR-approach. In case of using a low-degree GGM reference field (usually well below spherical harmonic degree 360), the truncation error is often estimated and corrected for based on a GGM. This technique, also known as University of New Brunswick approach (e.g., Vaníček and Kleusberg, 1987; Featherstone et al., 2003) is not further discussed here.

From a range of commonly used kernel modifications (see, e.g., Featherstone, 2003; Ellmann, 2005 for an overview), we select the Featherstone-Evans-Olliver (1998) deterministic modification (herein abbreviated to FEO) to serve as modified kernel $S^*(\psi)$ for this study. The Featherstone et al. (1998) modified kernel was used, e.g., in recent RCR-based geoid computations for Australia (Featherstone et al., 2011) and for New Zealand (Claessens et al., 2011).

2.3 Featherstone et al. (1998) modified kernel

The FEO modified kernel is based on the Vaníček and Kleusberg (1987) kernel modification and depends on the parameters (i) degree of modification M and (ii) the integration radius (aka integration cap radius or truncation radius) ψ_0. In a first step, the Wong and Gore (1969) modified kernel $S_{WG}(\psi)$ is computed (Featherstone, 2003)

$$S_{WG}(\psi) = S(\psi) - \sum_{n=2}^{M} \frac{2n+1}{n-1} P_n(\cos\psi) \tag{6}$$

where $P_n(\cos\psi)$ denotes the Legendre Polynomials (Torge 2001, p.68) and M the degree of modification. In a second step, the Vaníček and Kleusberg (1987) modified kernel $S_{VK}(\psi)$ is calculated

$$S_{VK}(\psi) = S_{WG}(\psi) - \sum_{n=2}^{M} \frac{2n+1}{2} t_n(\psi_0) P_n(\cos\psi) \tag{7}$$

where the modification coefficients $t_n(\psi_0)$ are obtained as solution of a set of $M-1$ linear equations (e.g., Featherstone et al., 2001; Featherstone, 2003)

$$\sum_{k=2}^{M} \frac{2k+1}{2} t_n(\psi_0) e_{nk}(\psi_0) = Q_n(\psi_0) - \sum_{k=2}^{M} \frac{2k+1}{k-1} e_{nk}(\psi_0). \tag{8}$$

The quantities $e_{nk}(\psi_0)$ and $Q_n(\psi_0)$ (see Featherstone et al. 2001)

$$e_{nk}(\psi_0) = \int_{\psi_0}^{\pi} P_k(\cos\psi) P_n(\cos\psi) \sin\psi \, d\psi \tag{9}$$

$$Q_n(\psi_0) = \int_{\psi_0}^{\pi} S(\psi) P_n(\cos\psi) \sin\psi \, d\psi \tag{10}$$

are computed using the algorithms by Paul (1973). In a third step, the FEO modified kernel $S_{FEO}(\psi)$ is obtained by forcing the Vaníček and Kleusberg kernel to zero at the integration radius ψ_0 (Featherstone et al. 1998, p. 157):

$$S_{FEO}(\psi) = S_{VK}(\psi) - S_{VK}(\psi_0) . \tag{11}$$

Eqs. (7) and (11) are valid for $0 < \psi \le \psi_0$. The modified kernel function $S_{FEO}(\psi)$ can be computed with the publicly available software modkern.f (Featherstone, 2003), allowing the reader to replicate the computation of the modified kernel.

The degree of modification M often ranges between 2 and 360 and the integration radius ψ_0 usually amounts to a few degrees (e.g., Vaníček and Featherstone, 1998; Featherstone et al., 2011; Claessens et al., 2011). Importantly, the spherical harmonic degree L of the GGM reference field and degree of modification M do not need to be equal (e.g., Featherstone et al., 2001). For instance, in practical RCR-based geoid computations using ultra-high-degree GGM reference fields (L=2160), the kernel parameters M=20..60 and ψ_0=2.5°..5° have been found to produce accurate geoid heights (e.g., Featherstone et al., 2011; Claessens et al., 2011). This serves as a justification for using the FEO kernel modification with M=40 and ψ_0=5° in the numerical tests of our study. To speed up the geoid computation using modified kernels, it is beneficial to establish a look-up-table (LUT) of pre-calculated kernel values ($0 < \psi \le \psi_0$) which are cubically interpolated.

The FEO kernel modification $S_{FEO}(\psi)$ (with L= 40 and ψ_0= 5°), the [unmodified] Stokes's kernel $S(\psi)$ and the first term of Stokes's function $S^{1st}(\psi)$ are shown together in Fig. 1 (left), illustrating the relationship among the functions. For small ψ-values, say less than ~0.5-1°, the FEO kernel modification is dominated by the first term of Stokes's function $S^{1st}(\psi)$, however not as strongly as the unmodified Stokes's kernel $S(\psi)$ (Fig. 1 right).

Please place Figure 1 near here

3. Computation of mean kernels

Whole-of-the-cell mean values of modified kernels can be obtained either through analytical or numerical integration. In the analytical integration approach, a planar approximation of the

first term of Stokes's function is integrated over the area of the cell, serving as an aid in computing so-called *kernel weighting factors* (e.g., Strang van Hees, 1990; Featherstone and Olliver, 1997; Hirt et al., 2011). These are conversion factors allowing transformation from point to mean kernel values (of Stokes's function or modified versions thereof). The numerical integration approach used here is based on *Gauss-Legendre Quadrature* (e.g., Abramowitz and Stegun, 1972; Hamming, 1986). Both approaches are detailed and compared next.

3.1 Analytical integration

The method presented here is a hybrid of the analytical integration approach developed for the unmodified Stokes's kernel by Strang van Hees (1990) and de Min (1994). Strang van Hees (1990) introduced kernel weighting factors *W* allowing transformation from point to mean kernels. He derived mean Stokes's kernels from a one-dimensional analytical integration assuming square-shaped cells, which limits the accuracy of his method in practice (Hirt et al., 2011). Without using kernel weighting factors, de Min (1994) applied two-dimensional analytical integration (2D-AI) of Stokes's function, taking into account the usually rectangular shape of cells. Combining the kernel weighting factor concept with the 2D-AI is a suitable approach to provide analytically mean values of modified kernels.

We start by introducing rectangular coordinates (x,y) to describe the distance between the computation point (φ_P, λ_P) and centre of the cell under evaluation (φ_Q, λ_Q) in planar approximation:

$$
\begin{aligned}
x &= (\lambda_Q - \lambda_P)\cos\varphi_Q, \\
y &= \varphi_Q - \varphi_P.
\end{aligned}
\tag{12}
$$

Planar approximation of the first term of Stokes's function $S^{1st}(\psi) \approx 2/\psi$ and planar approximation of the distance $\psi \approx \sqrt{x^2 + y^2}$ (see Torge, 2001, p. 287) gives Stokes's kernel in planar approximation (e.g., de Min 1994):

$$
K(x, y) = \frac{2}{\sqrt{x^2 + y^2}}.
\tag{13}
$$

For a given cell of size $\Delta\varphi$ x $\Delta\lambda$ (e.g., 1 arc min x 1 arc min), bounded by corner coordinates (x_1, y_1) and (x_2, y_2)

$$x_1 = (\lambda_Q - \lambda_P - \frac{\Delta\lambda}{2})\cos\varphi_Q, \; y_1 = \varphi_Q - \varphi_P - \frac{\Delta\varphi}{2},$$

$$x_2 = (\lambda_Q - \lambda_P + \frac{\Delta\lambda}{2})\cos\varphi_Q, \; y_2 = \varphi_Q - \varphi_P + \frac{\Delta\varphi}{2},$$

(14)

the whole-of-the-cell mean kernel $\overline{K}(x_1, y_1, x_2, y_2)$ in *planar approximation* is obtained from two-dimensional analytical integration (after Vaníček and Krakiwsky, 1986; de Min, 1994):

$$\overline{K}(x_1, y_1, x_2, y_2) = \frac{1}{a}\int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2} K(x, y)dxdy$$

(15)

where $a = (x_2 - x_1)(y_2 - y_1)$ denotes the area of the cell. Eq. (15) can be solved analytically (de Min, 1994):

$$\overline{K}(x_1, y_1, x_2, y_2) = \frac{1}{a}\left\|F(x, y)\right|_{y_1}^{y_2}\right|_{x_1}^{x_2} =$$

$$\frac{1}{a}[F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1) + F(x_2, y_2)]$$

(16)

using the antiderivative (de Min, 1994)

$$F(x, y) = 2[x\ln(y + \sqrt{x^2 + y^2}) + y\ln(x + \sqrt{x^2 + y^2})].$$

(17)

Comparing the mean value $\overline{K}(x_1, y_1, x_2, y_2)$ with the point (centre-of-cell) value $K(x, y)$ (both in planar approximation) gives the kernel weighting factor (Hirt et al., 2011)

$$W(x_1, y_1, x_2, y_2) = \overline{K}(x_1, y_1, x_2, y_2) / K(x, y)$$

(18)

which is adapted here to transform point values of FEO kernels S_{FEO} to mean values $\overline{S}_{FEO}^{2DAI}$ for a given cell (x_1, y_1, x_2, y_2):

$$\overline{S}_{FEO}^{2DAI} \approx WS_{FEO}.$$

(19)

Because the first term of Stokes's function $S^{1st}(\psi)$ 'governs' the modified kernel S_{FEO} near the computation point (say well within 1° radius, see also Fig. 1), the kernel weighting factors $\overline{S}_{FEO}^{2DAI} / S_{FEO}$ and $\overline{K} / K$ have very similar values. This makes the transformation in Eq. (19) possible. A numerical confirmation of Eq. (19) is given in Section 3.3. It should be noted

that the planar approximations $\overline{K}$ or K of Stokes's kernel are not used as a substitute of the modified kernel S_{FEO} . The direct use of the planar approximations (in place of FEO mean kernels $\overline{S}_{FEO}^{2DAI}$) would significantly worsen the accuracy of Stokes's integration.

3.2 Numerical integration

Gauss-Legendre Quadrature (GLQ) (e.g., Stark, 1970; Abramowitz and Stegun, 1972; Hamming, 1986; Alberts and Klees, 2004; Wild-Pfeiffer, 2008) is a very efficient integration technique for the numerical computation of mean kernel estimates. The basic idea is to evaluate the kernel function at carefully selected points (being a function of the Legendre Polynomials), located within the area of the cell q. The GLQ numerical mean kernel estimate is obtained as weighted average of the point kernel evaluations. For a given cell q, bounded by coordinates $\lambda_1, \lambda_2, \varphi_1, \varphi_2$, the GLQ numerical whole-of-the-cell mean value $\overline{S}_{FEO}^{GLQ}$ of the FEO kernel is obtained through n^2 evaluations of the modified kernel function $S_{FEO}(\psi)$ (see also Hirt et al., 2011):

$$\overline{S}_{FEO}^{GLQ} = \frac{1}{w_i w_j} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j S_{FEO}(\psi(\varphi_P, \lambda_P, \varphi_{Qi}, \lambda_{Qj})) \tag{20}$$

where n is the quadrature degree, w_i, w_j are the weights (not to confuse with kernel weighting factors W from Section 3.1) and $\varphi_{Qi}, \lambda_{Qj}$ are the coordinates of the evaluation points.. The evaluation points $\varphi_{Qi}, \lambda_{Qj}$ are located within the area of cell q (i.e., $\lambda_1 < \lambda_{Qj} < \lambda_2$ and $\varphi_1 < \varphi_{Qi} < \varphi_2$) and obtained from (Hirt et al., 2011)

$$\begin{aligned} \lambda_{Qi} &= \frac{\lambda_2 - \lambda_1}{2} x_i + \frac{\lambda_2 + \lambda_1}{2} \\ \varphi_{Qj} &= \frac{\varphi_2 - \varphi_1}{2} x_j + \frac{\varphi_2 + \varphi_1}{2} \end{aligned} \tag{21}$$

where $x_i(x_j)$ is the i-th (j-th) zero crossing of the n-th Legendre polynomial (e.g., Abramowitz and Stegun, 1972). The weights w_i, w_j are computed from (Abramowitz and Stegun, 1972, p. 887; Hirt et al., 2011):

$$w_i = \frac{\lambda_2 - \lambda_1}{(1 - x_i^2)[P_n'(x_i)]^2}$$

$$w_j = \frac{\varphi_2 - \varphi_1}{(1 - x_j^2)[P_n'(x_j)]^2}$$

(22)

with P_n' being the derivative of the n-th Legendre polynomial, evaluated at x_i and x_j, respectively. The computation of the coordinates of the evaluation points $\varphi_{Qi}, \lambda_{Qj}$ and the weights w_i, w_j can be accomplished with computer code from the Numerical Recipes (Press et al., 2002) or other mathematical software libraries. Naturally, the described approach can be used to compute mean kernel estimates of any other kernel modification.

For the practical application of GLQ, it is useful to introduce some convergence threshold. Given that the geoid signal in RCR-based geoid computation rarely exceeds an order of a few metres, a threshold of $\varepsilon = 10^{-4}$ would result in an acceptable uncertainty well below the millimetre level. To investigate the quadrature degree n required, Eq. (20) was evaluated as a function of n, ranging from 1 to 20, and compared against a 'reference value' $\overline{S}_{FEO}^{GLQ}(ref)$ from $n = 1000$. The convergence error

$$\varepsilon = \frac{\overline{S}_{FEO}^{GLQ}(n) - \overline{S}_{FEO}^{GLQ}(ref)}{\overline{S}_{FEO}^{GLQ}(ref)}$$

(23)

is shown in Fig. 2 for cells located 1, 5 and 30 arc min North of the innermost zone (cell size is 1 arc min x 1 arc min and latitude of the computation point P is $-35°$). For cells directly adjoining the innermost zone, a degree 3 quadrature (i.e., 9 evaluation points) is required to provide GLQ numerical mean kernels accurate to 10^{-4}. In 5 arc min distance from the computation point, 4 evaluation points per cell are sufficient to meet the 10^{-4} criterion, whereas the point kernel (equivalent to quadrature degree 1 with the cell centre Q as only evaluation point) is yet good enough in 30 arc min distance and beyond. For higher quadrature degrees, say $n = 5$ and higher, convergence errors are at the level of $\sim 10^{-8}$ (Fig. 2). This very small level of uncertainty would be more than sufficient for any practical application of mean kernels in modified Stokes's integration. The results of this convergence experiment demonstrate that GLQ is a very efficient numerical integration technique for the accurate computation of whole-of-the-cell mean values of modified kernels.

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3.3 Comparisons

To evaluate the 2D-AI analytical and GLQ numerical computation approach, we computed FEO kernel values for a grid of 1 arc min x 1 arc min cells over Australia. Fig. 3A shows the FEO point kernel matrix (with the computation point P as the centre). The percentage differences between the 2D-AI mean kernel $\overline{S}_{FEO}^{2DAI}$ and point kernel S_{FEO} are shown in Fig. 3B and those between the GLQ numerical mean $\overline{S}_{FEO}^{GLQ}$ and point kernel S_{FEO} in Fig. 3C. Both approaches exhibit very similar difference patterns with maximum differences of ~5% between FEO mean and point kernels for cells adjoining the computation point. This demonstrates across-the-cell variations of the FEO mean kernel for small ψ-values. Fig. 3D shows the percentage differences between 2D-AI and GLQ mean kernels. The differences are largest adjacent to the innermost zone (about 0.02%) and otherwise below or well below the level of 10^{-4}. This good mutual agreement is considered a sufficient validation of both mean kernel computation strategies. Further to this, the small discrepancies in Fig. 3D implicitly confirm the validity of Eq. (19). As a conclusion, the comparisons demonstrate the capability of both approaches to accurately compute FEO mean kernel estimates.

With both approaches, it will be usually sufficient to restrict the mean kernel computation to a zone of ~30 arc min radius around the innermost zone (cf. Fig. 2). The increase in computation time (with respect to point kernels) is below ~10 % for both methods, which should be well acceptable in practical applications.

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4. Numerical closed-loop tests

Closed-loop tests were used to compare and evaluate the performance of FEO analytical/numerical mean and point kernels in RCR-based geoid computation. The recent

high-resolution EGM2008 global gravity model (Pavlis et al., 2008) served as source to generate self-consistent pairs of gravity anomalies $\Delta g_{EGM2008}$ and geoid heights $N_{EGM2008}$ using the state-of-the-art harmonic_synth spherical harmonic synthesis software (Holmes and Pavlis 2008). Transformation of gravity anomalies $\Delta g_{EGM2008}$ to geoid heights $N(\Delta g_{EGM2008})$ using different FEO kernel variants and subsequent comparision against the 'error free' geoid heights $N_{EGM2008}$ allows assessment of the mean and point kernel performance.

Australia, Europe and the Himalaya mountains were chosen as test areas to cover different latitudes and signal strengths of gravity field features. The boundaries of the computation areas and signal strengths of the geoid heights are found in the captions of Tables 1–3. For any of the three test areas, the EGM2008 gravity anomalies $\Delta g_{EGM2008}$ and geoid heights were synthesized on 1 arc min grids in the spectral band 361..2190. This band was chosen because it 'replicates' the frequently used RCR-scheme with a degree L=360 GGM reference field removed. Given that a degree 2190 spherical harmonic expansion represents the geoid signal almost completely [the missing portion, known as omission error, is globally estimated to be ~4 cm, cf. Jekeli et al. (2009)], the chosen spectral band is considered a realistic simulation of the RCR-approach.

The FFT1Dmod2010 software of the Western Australian Centre of Geodesy was applied to evaluate Stokes's integral [Eq. (5)], using the FEO modified kernel [Eq. (9)] with parameters M= 40 and ψ_0 = 5°. This software is based on the one-dimensional Fast Fourier Transform technique (Haagmans et al. 1993) that allows very efficient numerical evaluation of Stokes's integral, see also Featherstone et al. (2001). To allow Stokes's integral to be correctly evaluated over the whole cap radius, also near the edges of our test area, the EGM2008 gravity grids were extended by an auxilliary 5° boundary strip. The descriptive statistics of the closed-loop differences $N_{EGM2008} - N(\Delta g_{EGM2008})$ show an almost identical performance of the FEO 2D-AI analytical and FEO GLQ numerical mean kernel (Tables 1-3). The practical equivalence of both variants in our tests is as expected, given the ~10^{-4} mutual agreement between both mean kernel computation approaches found in Sect. 3.3.

In any of the test areas, FEO mean kernels produce RMS errors which are factor ~2.5 to 3 lower than those of FEO point kernels (Tables 1-3). Likewise, it is seen that the maximum discrepancies drop from the cm to the mm-level (Australia) from ~2 cm to the sub-cm-level

(Europe) and from ~3-4 cm to the 1 cm-level (Himalaya Mountains). This suggests to use mean kernel estimates in Stokes' numerical integration with modified kernels.

A detail display of the FEO mean and point kernel performance is given from the closed-loop differences $N_{EGM\,2008} - N(\Delta g_{EGM\,2008})$ along parallel of $-35°$ (test area Australia) in Fig. 4, showing the larger numerical integration errors in case of point kernels. Finally, Fig. 5 shows the closed-loop differences $N_{EGM\,2008} - N(\Delta g_{EGM\,2008})$ for the Himalaya test area, illustrating the error patterns caused by using point kernels instead of mean kernels.

Please place Figure 4 near here

Please place Figure 5 near here

5. Conclusions and recommendations

The present study has investigated the computation of whole-of-the-cell means for the Featherstone-Evans-Olliver (1998) kernel, serving as an example of modified kernels used in gravimetric geoid determination based on Stokes's integration. We have described (i) the analytical approach to compute FEO mean kernels adapting the concept of kernel weighting factors and (ii) the numerical approach which is based on Gauss-Legendre-Quadrature. The comparison of both methods showed a satisfactory mutual consistency on the level of $~10^{-4}$ and better.

The performance of our mean kernel approaches was compared against centre-of-cell (point) kernels by means of closed-loop tests based on self-consistent sets of gravity anomalies and geoid heights from EGM2008 in the spectral band 361..2190. Any of the test demonstrated that whole-of-the-cell mean values of the FEO kernel modification reduce the numerical integration errors by ~65%. In many cases, maximum closed-loop errors were reduced from some cm to the sub-cm level, which is a relevant improvement for accurate geoid determination.

Our study demonstrates that mean kernels should be used instead of point kernels to accurately evaluate Stokes's integral with kernel modifications. We have exemplified this with Featherstone-Evans-Olliver (1998) kernel modification, one of many kernel modifications used in practical geoid computations. Our findings are considered

representative for any other kernel modification under the condition that the modified kernel rapidly varies across the cell in the neighbourhood of the innermost zone. Most of the modified kernels will meet this condition. As a general conclusion, using mean kernels in modified Stokes's integration considerably reduces numerical integration errors.

Acknowledgements

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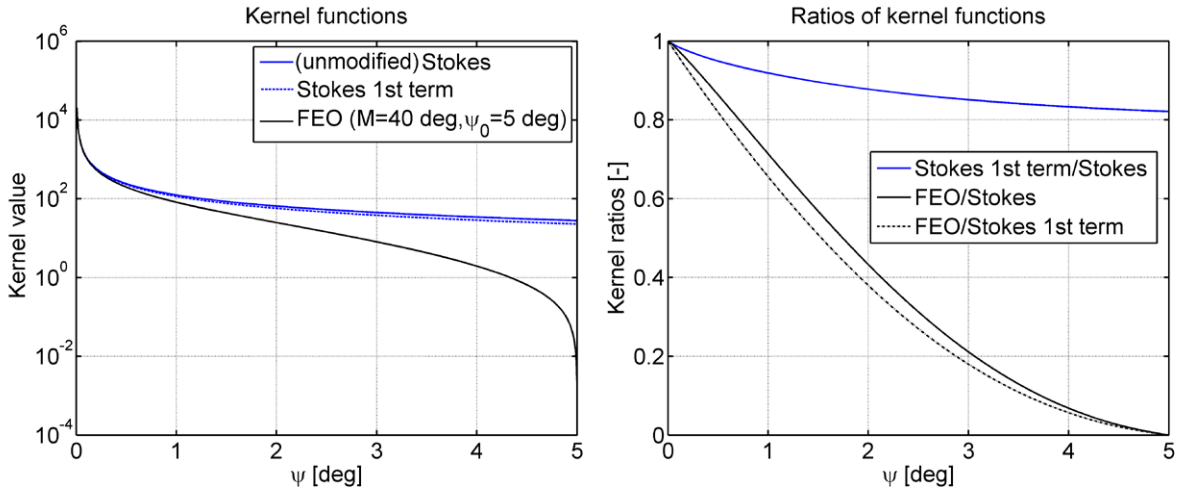
List of Figure captions:

Fig. 1. Left: (unmodified) Stokes's function $S(\psi)$, first term of Stokes's function $S^{1st}(\psi)$ and FEO modified kernel $S_{FEO}(\psi)$ with $M = 40$ and $\psi_0 = 5°$. Right: Ratios $S^{1st}(\psi)/S(\psi)$, $S_{FEO}(\psi)/S(\psi)$ and $S_{FEO}(\psi)/S^{1st}(\psi)$.
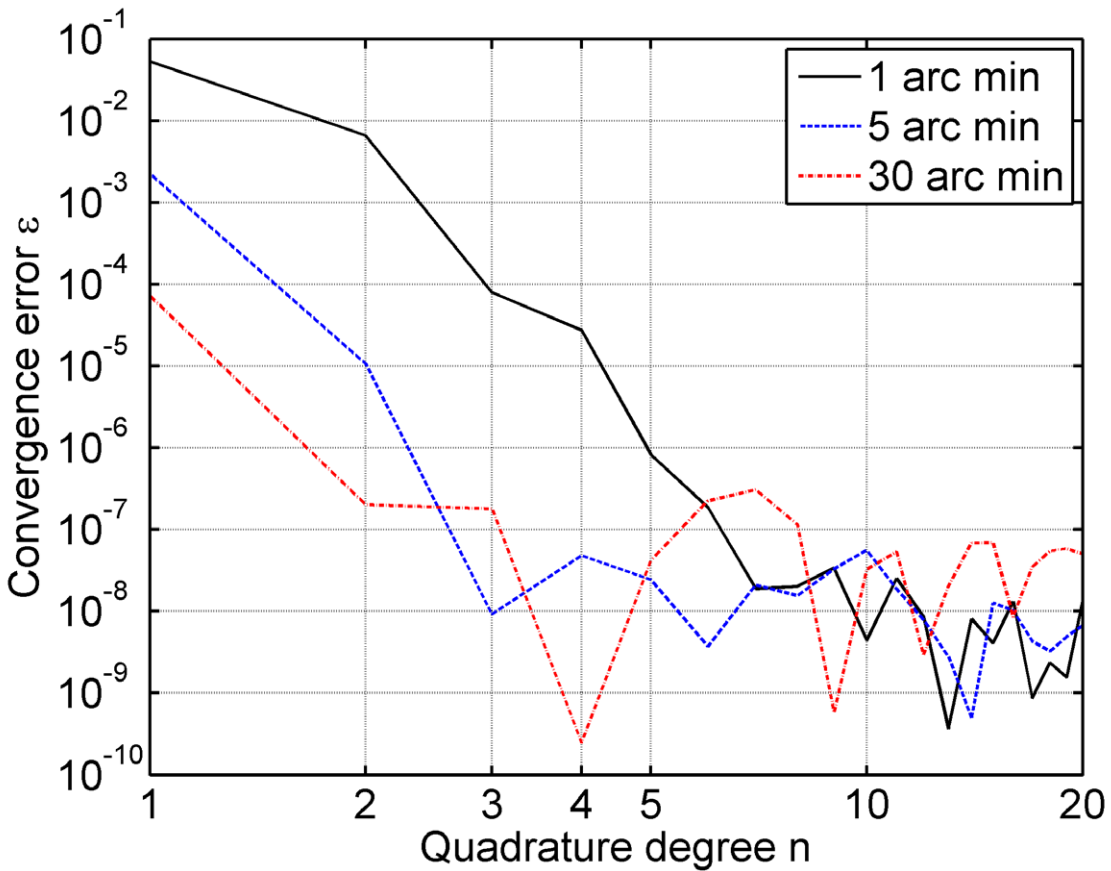
Fig. 2. Convergence error ε of GLQ numerical mean kernels as a function of the quadrature degree n and as a function of the distance ψ between the innermost zone and the cell under evaluation (1, 5 and 30 arcmin). Cell resolution is 1 arc min x 1 arc min and latitude of the computation point P is -35°. Kernel is the FEO modification (M= 40, ψ_0 = 5 deg).
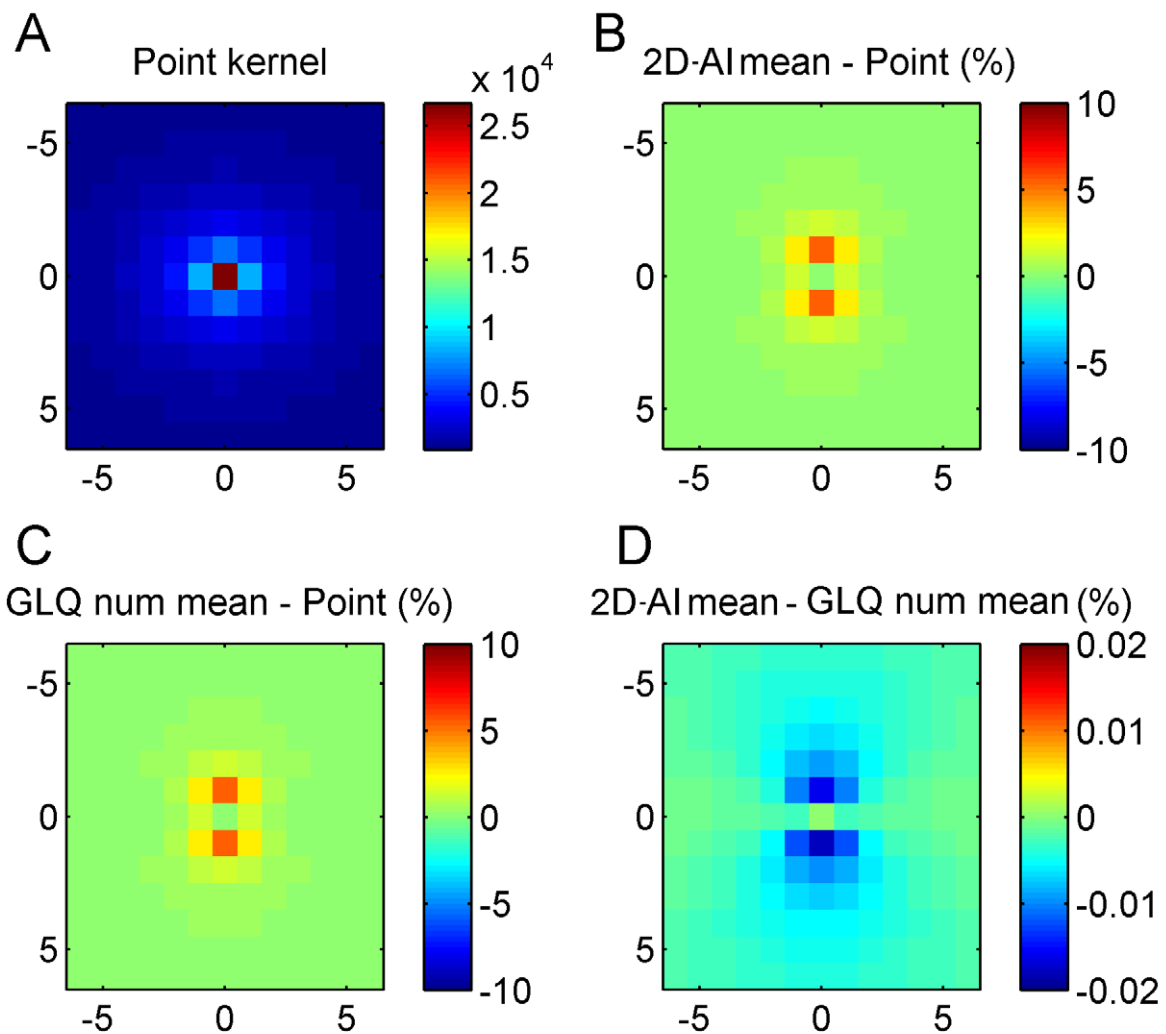

Fig. 3. A: Centre-of-cell (point) values of the FEO kernel. B: Percentage differences between FEO 2D-AI mean and point kernel. C: Percentage differences between GLQ numerical mean and point kernel. D: Percentage differences between 2D-AI and GLQ numerical mean kernel. The panels show 13 x 13 cell arrays with the centre being the computation point P. Cell resolution is 1 arc min x 1 arc min and latitude of the computation point P is -35°. FEO kernel parameters are M= 40 and ψ_0 = 5 deg.
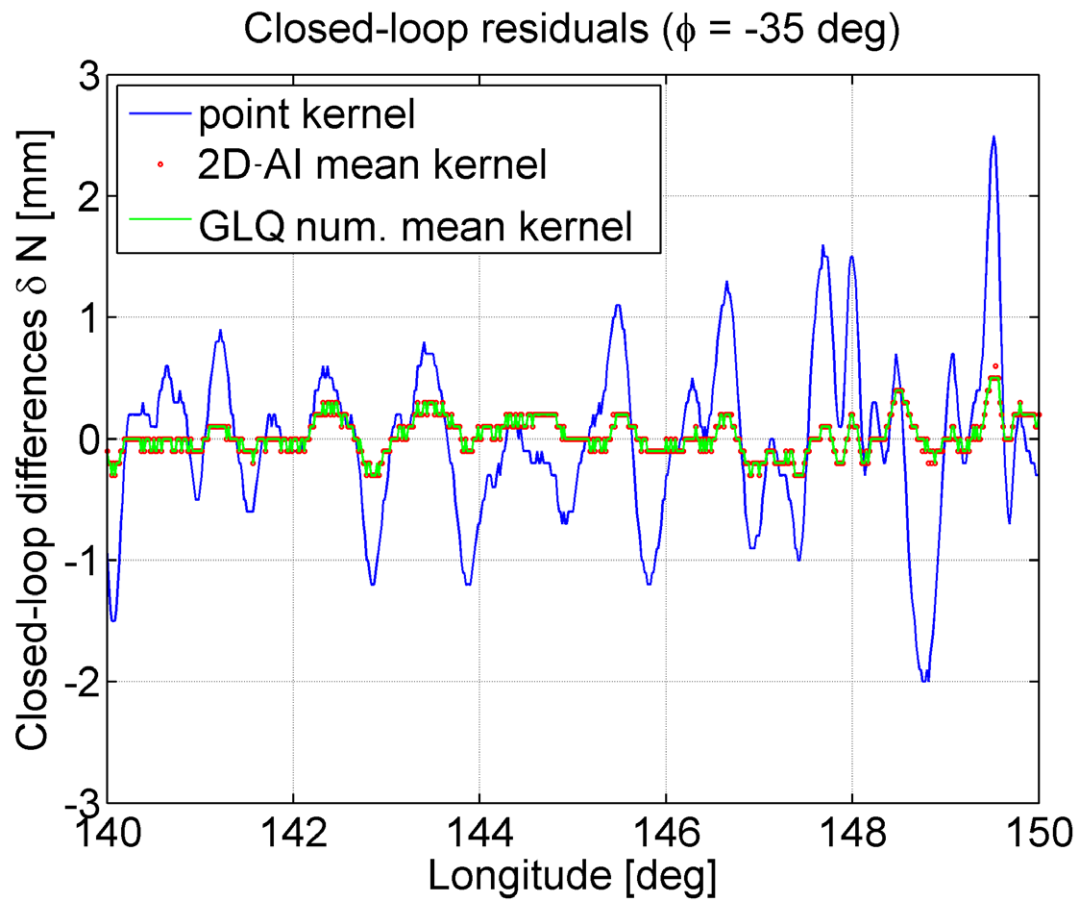
Fig. 4. Closed-loop differences $N_{EGM\,2008} - N(\Delta g_{EGM\,2008})$ over Australia along parallel -35° as a function of the FEO kernel variant (point, 2D-AI mean and GLQ numerical mean).
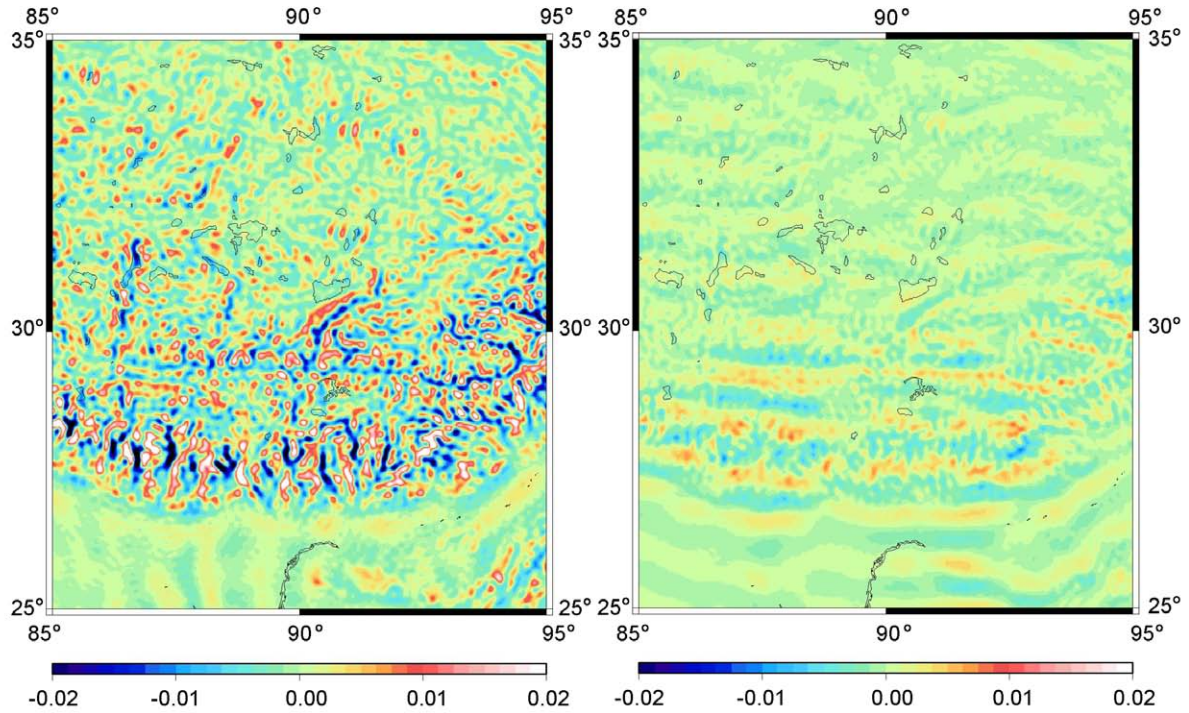
Fig. 5. Closed-loop differences $N_{EGM\,2008} - N(\Delta g_{EGM\,2008})$ for the Himalaya test area. Left: results of modified Stokes's integration using point kernels, Right: results of modified Stokes's integration using GLQ numerical mean kernels. Kernel used is the FEO modification (M= 40, ψ_0 = 5 deg). Units in metres.

Tables

Table 1. Descriptive statistics of closed-loop geoid differences $N_{EGM\,2008} - N(\Delta g_{EGM\,2008})$ for test area Australia (-40°≤φ≤-30° and 140°≤λ≤150°, 361201 pts, EGM2008 361..2190 truncation, $N_{EGM\,2008}$ min/max/RMS: -0.82/1.10/0.13 m). Units in mm.

Kernel type	Min	Max	Mean	RMS
Point (centre-of-cell)	-9.0	14.3	0.0	1.2
2D-AI analytical mean	-3.4	3.6	0.0	0.4
GLQ numerical mean	-3.1	3.0	0.0	0.4

Table 2. Descriptive statistics of closed-loop geoid differences $N_{EGM\,2008} - N(\Delta g_{EGM\,2008})$ for test area Europe (35°≤φ≤55° and 5°≤λ≤25°, 1442401 pts, EGM2008 361..2190 truncation, $N_{EGM\,2008}$ min/max/RMS: -1.81/1.80/0.22 m). Units in mm.

Kernel type	Min	Max	Mean	RMS
Point (centre-of-cell)	-20.3	23.0	0.0	2.1
2D-AI analytical mean	-8.0	8.3	0.0	0.8
GLQ numerical mean	7.1	7.4	0.0	0.8

Table 3. Descriptive statistics of closed-loop geoid differences $N_{EGM\,2008} - N(\Delta g_{EGM\,2008})$ for test area Himalaya ($25° \leq \varphi \leq 35°$ and $85° \leq \lambda \leq 95°$, 361201 pts, EGM2008 361..2190 truncation, $N_{EGM\,2008}$ min/max/RMS: -2.84/3.67/0.49 m). Units in mm.

Kernel type	Min	Max	Mean	RMS
Point (centre-of-cell)	-35.7	37.8	0.0	5.2
2D-AI analytical mean	-10.5	8.5	0.0	1.7
GLQ numerical mean	-11.3	9.0	0.0	1.8