ICOCO2010

X.M. Yang and G.Y. Chen (eds)

1 AN EFFICIENT GOLDEN SECTION METHOD FOR THE FLEET COMPOSITION PROBLEM

Ryan Loxton[a]

[a]Department of Mathematics and Statistics
Curtin University of Technology, Perth, Australia

Abstract: Most large organizations operate a vehicle fleet for transporting materials and personnel. When establishing such a fleet, an organization must decide, based on forecasts of its future requirements, how many vehicles to purchase and how many to hire. Hiring vehicles is much more expensive than operating owned vehicles. On the other hand, purchasing vehicles incurs a large opportunity cost. In this paper, we consider an optimization problem in which the number of purchased vehicles is chosen to minimize the total cost of owning and hiring vehicles. This problem is called the fleet composition problem (FCP). Our main contribution is to prove that the FCP's cost function is convex. We then exploit this result to show that the FCP can be solved efficiently using the well-known golden section method.

Key words: Fleet composition, logistics, convex optimization, golden section method.

1 INTRODUCTION

Consider an organization using a certain type of vehicle. This organization has forecasted its vehicle requirements for the upcoming planning horizon (divided into periods). It now needs to decide how many vehicles to purchase.

Define:

$$\begin{aligned}
n &= \text{number of periods in the planning horizon.} \\
p_t &= \text{number of vehicles required during period } t \in \{1, \ldots, n\}. \\
p &= \text{number of vehicles to be purchased (decision variable).} \\
p_{\max} &= \text{maximum number of vehicles that can be purchased.} \\
c_f &= \text{fixed cost per period of an owned vehicle.} \\
c_v &= \text{variable cost per period of an owned vehicle.} \\
c_h &= \text{cost per period of hiring one vehicle.}
\end{aligned}$$

The fixed cost of an owned vehicle includes the initial cost of purchasing the vehicle (less the salvage value) plus other costs such as insurance premiums and registration fees. The variable cost is generally due to maintenance (servicing, replacing tires, etc.) and is only incurred when the vehicle is used.

The cost of owning and operating a vehicle for one period must be less than the cost of hiring a vehicle for one period—otherwise, there would be no reason to purchase vehicles. Hence,

$$c_f + c_v < c_h.$$

Since c_f is the fixed cost per period of an owned vehicle,

$$\text{Total fixed cost} = nc_f p.$$

If $p_t > p$, then during time period t the organization needs to use all p of its own vehicles plus $p_t - p$ hired vehicles. On the other hand, when $p_t \leq p$, the organization only uses p_t of its own vehicles (no vehicles are hired). Thus,

$$\text{Number of owned vehicles used during period } t \;=\; \min(p_t, p)$$

and

$$\text{Number of vehicles hired during period } t \;=\; \max(p_t - p, 0).$$

Consequently,

$$\text{Total variable cost} \;=\; c_v \sum_{t=1}^{n} \min(p_t, p)$$

and

$$\text{Total hiring cost} \;=\; c_h \sum_{t=1}^{n} \max(p_t - p, 0).$$

The key question that now arises is: what value of p minimizes the overall cost? This question leads to the following optimization problem.

Fleet Composition Problem (FCP)
Find an integer $p \in [0, p_{\max}]$ *that minimizes the cost function*

$$C(p) = \underbrace{nc_f p}_{\text{Fixed cost}} + \underbrace{c_v \sum_{t=1}^{n} \min(p_t, p)}_{\text{Variable cost}} + \underbrace{c_h \sum_{t=1}^{n} \max(p_t - p, 0)}_{\text{Hiring cost}}.$$

A rudimentary approach to solving the FCP is just to evaluate C at every integer in $[0, p_{\max}]$ and then select the integer with the smallest cost. This is obviously extremely inefficient when n and $p_{\max}$ are large. The purpose of this paper is to develop a more sophisticated method for solving the FCP.

Ghiani et al. (2004) formulated the FCP and suggested solving it by differentiating C and setting C' equal to zero. This yields the following optimality condition:

$$m = \frac{nc_f}{c_h - c_v}, \tag{1.1}$$

where m is the number of time periods in which $p_t > p$. Unfortunately, equation (1.1) only holds when C is differentiable (see the next section). Furthermore, equation (1.1) only makes sense when its right-hand side is an integer. This is often not the case. For example, if $n = 52$, $c_f = 50$, $c_v = 40$, and $c_h = 100$, then (1.1) gives $m = 43.3333$, which is impossible.

2 MAIN RESULTS

We first derive some important results. In this section, we assume that the FCP's cost function C is defined for all real numbers.

For each $p \in \mathbb{R}$, define the following sets:

$$\mathcal{R}(p) \triangleq \{\, t : \ p_t > p \,\},$$
$$\mathcal{S}(p) \triangleq \{\, t : \ p_t < p \,\},$$
$$\mathcal{T}(p) \triangleq \{\, t : \ p_t = p \,\}.$$

Clearly, $\mathcal{R}(p)$, $\mathcal{S}(p)$, and $\mathcal{T}(p)$ partition $\{1, \ldots, n\}$.

Now, let $f : \mathbb{R} \to \mathbb{R}$ be any function. Recall that the left derivative of f is defined by

$$D_- f(x) \triangleq \lim_{\epsilon \to 0-} \frac{f(x+\epsilon) - f(x)}{\epsilon},$$

provided this limit exists. Similarly, the right derivative of f is defined by

$$D_+ f(x) \triangleq \lim_{\epsilon \to 0+} \frac{f(x+\epsilon) - f(x)}{\epsilon}.$$

In our first result, we show that C has a left derivative at every point.

Theorem 2.1 *For each* $p \in \mathbb{R}$,

$$D_- C(p) = nc_f + (c_v - c_h)|\mathcal{R}(p)| + (c_v - c_h)|\mathcal{T}(p)|.$$

Proof. Let $p \in \mathbb{R}$ be arbitrary but fixed. Furthermore, define

$$\epsilon' \triangleq \begin{cases} \max\{\, p_t - p : \ t \in \mathcal{S}(p) \,\}, & \text{if } \mathcal{S}(p) \neq \emptyset, \\ -\infty, & \text{if } \mathcal{S}(p) = \emptyset. \end{cases}$$

Since $p_t < p$ for each $t \in \mathcal{S}(p)$, we have $\epsilon' < 0$.

Now, let $\epsilon \in (\epsilon', 0)$. Then

$$C(p+\epsilon) = nc_f p + nc_f \epsilon + c_v \sum_{t=1}^{n} \min(p_t, p+\epsilon) + c_h \sum_{t=1}^{n} \max(p_t - p - \epsilon, 0). \tag{2.1}$$

If $t \in \mathcal{R}(p) \cup \mathcal{T}(p)$, then $p_t \geq p > p + \epsilon$. Therefore,

$$\min(p_t, p+\epsilon) = p + \epsilon, \qquad t \in \mathcal{R}(p) \cup \mathcal{T}(p), \tag{2.2}$$

and

$$\max(p_t - p - \epsilon, 0) = p_t - p - \epsilon, \qquad t \in \mathcal{R}(p) \cup \mathcal{T}(p). \tag{2.3}$$

On the other hand, if $t \in \mathcal{S}(p)$ then

$$p + \epsilon > p + \epsilon' \geq p + p_t - p = p_t.$$

Hence,

$$\min(p_t, p+\epsilon) = p_t, \qquad t \in \mathcal{S}(p), \tag{2.4}$$

and

$$\max(p_t - p - \epsilon, 0) = 0, \qquad t \in \mathcal{S}(p). \tag{2.5}$$

By equations (2.2) and (2.4),

$$\begin{aligned} \sum_{t=1}^{n} \min(p_t, p+\epsilon) &= \sum_{t \in \mathcal{R}(p) \cup \mathcal{T}(p)} \min(p_t, p+\epsilon) + \sum_{t \in \mathcal{S}(p)} \min(p_t, p+\epsilon) \\ &= \sum_{t \in \mathcal{R}(p) \cup \mathcal{T}(p)} (p+\epsilon) + \sum_{t \in \mathcal{S}(p)} p_t \\ &= \epsilon|\mathcal{R}(p)| + \epsilon|\mathcal{T}(p)| + \sum_{t \in \mathcal{R}(p) \cup \mathcal{T}(p)} p + \sum_{t \in \mathcal{S}(p)} p_t \\ &= \epsilon|\mathcal{R}(p)| + \epsilon|\mathcal{T}(p)| + \sum_{t=1}^{n} \min(p_t, p). \end{aligned} \tag{2.6}$$

Now, by equations (2.3) and (2.5),

$$\begin{aligned} \sum_{t=1}^{n} \max(p_t - p - \epsilon, 0) &= \sum_{t \in \mathcal{R}(p) \cup \mathcal{T}(p)} (p_t - p - \epsilon) \\ &= -\epsilon|\mathcal{R}(p)| - \epsilon|\mathcal{T}(p)| + \sum_{t \in \mathcal{R}(p) \cup \mathcal{T}(p)} (p_t - p) \\ &= -\epsilon|\mathcal{R}(p)| - \epsilon|\mathcal{T}(p)| + \sum_{t=1}^{n} \max(p_t - p, 0). \end{aligned} \tag{2.7}$$

Substituting equations (2.6) and (2.7) into equation (2.1) yields

$$C(p+\epsilon) = nc_f p + nc_f\epsilon + \epsilon(c_v - c_h)|\mathcal{R}(p)| + \epsilon(c_v - c_h)|\mathcal{T}(p)| \\ + c_v \sum_{t=1}^{n} \min(p_t, p) + c_h \sum_{t=1}^{n} \max(p_t - p, 0).$$

Thus,

$$C(p+\epsilon) = C(p) + nc_f\epsilon + \epsilon(c_v - c_h)|\mathcal{R}(p)| + \epsilon(c_v - c_h)|\mathcal{T}(p)|,$$

which immediately implies

$$C(p+\epsilon) - C(p) = nc_f\epsilon + \epsilon(c_v - c_h)|\mathcal{R}(p)| + \epsilon(c_v - c_h)|\mathcal{T}(p)|.$$

Dividing both sides by ϵ (recall that $\epsilon < 0$) gives

$$\frac{C(p+\epsilon) - C(p)}{\epsilon} = nc_f + (c_v - c_h)|\mathcal{R}(p)| + (c_v - c_h)|\mathcal{T}(p)|.$$

This equation holds for all $\epsilon \in (\epsilon', 0)$. Taking the limit as $\epsilon \to 0-$ gives

$$D_-C(p) = \lim_{\epsilon \to 0-} \frac{C(p+\epsilon) - C(p)}{\epsilon} = nc_f + (c_v - c_h)|\mathcal{R}(p)| + (c_v - c_h)|\mathcal{T}(p)|,$$

which completes the proof. □

We now determine the right derivative of C.

Theorem 2.2 *For each $p \in \mathbb{R}$,*

$$D_+C(p) = nc_f + (c_v - c_h)|\mathcal{R}(p)|.$$

Proof. Let $p \in \mathbb{R}$ be arbitrary but fixed. Furthermore, define

$$\epsilon' \triangleq \begin{cases} \min\{\, p_t - p : \ t \in \mathcal{R}(p)\,\}, & \text{if } \mathcal{R}(p) \neq \emptyset, \\ +\infty, & \text{if } \mathcal{R}(p) = \emptyset. \end{cases}$$

Since $p_t > p$ for each $t \in \mathcal{R}(p)$, we have $\epsilon' > 0$.

Let $\epsilon \in (0, \epsilon')$. Then

$$C(p+\epsilon) = nc_f p + nc_f\epsilon + c_v \sum_{t=1}^{n} \min(p_t, p+\epsilon) + c_h \sum_{t=1}^{n} \max(p_t - p - \epsilon, 0). \quad (2.8)$$

If $t \in \mathcal{R}(p)$, then

$$p + \epsilon < p + \epsilon' \leq p + p_t - p = p_t.$$

Therefore,

$$\min(p_t, p+\epsilon) = p + \epsilon, \qquad t \in \mathcal{R}(p), \quad (2.9)$$

and

$$\max(p_t - p - \epsilon, 0) = p_t - p - \epsilon, \qquad t \in \mathcal{R}(p). \tag{2.10}$$

On the other hand, if $t \in \mathcal{S}(p) \cup \mathcal{T}(p)$ then $p_t \le p < p + \epsilon$. Thus,

$$\min(p_t, p + \epsilon) = p_t, \qquad t \in \mathcal{S}(p) \cup \mathcal{T}(p), \tag{2.11}$$

and

$$\max(p_t - p - \epsilon, 0) = 0, \qquad t \in \mathcal{S}(p) \cup \mathcal{T}(p). \tag{2.12}$$

By equations (2.9) and (2.11),

$$\begin{aligned}\sum_{t=1}^{n} \min(p_t, p + \epsilon) &= \sum_{t \in \mathcal{R}(p)} (p + \epsilon) + \sum_{t \in \mathcal{S}(p) \cup \mathcal{T}(p)} p_t \\ &= \epsilon |\mathcal{R}(p)| + \sum_{t \in \mathcal{R}(p)} p + \sum_{t \in \mathcal{S}(p) \cup \mathcal{T}(p)} p_t \\ &= \epsilon |\mathcal{R}(p)| + \sum_{t=1}^{n} \min(p_t, p). \end{aligned} \tag{2.13}$$

By equations (2.10) and (2.12), we have

$$\begin{aligned}\sum_{t=1}^{n} \max(p_t - p - \epsilon, 0) &= \sum_{t \in \mathcal{R}(p)} (p_t - p - \epsilon) \\ &= -\epsilon |\mathcal{R}(p)| + \sum_{t \in \mathcal{R}(p)} (p_t - p) \\ &= -\epsilon |\mathcal{R}(p)| + \sum_{t=1}^{n} \max(p_t - p, 0). \end{aligned} \tag{2.14}$$

Substituting equations (2.13) and (2.14) into equation (2.8) gives

$$\begin{aligned} C(p + \epsilon) = n c_f p + n c_f \epsilon + \epsilon (c_v - c_h) |\mathcal{R}(p)| \\ + c_v \sum_{t=1}^{n} \min(p_t, p) + c_h \sum_{t=1}^{n} \max(p_t - p, 0). \end{aligned}$$

Therefore,

$$C(p + \epsilon) - C(p) = n c_f \epsilon + \epsilon (c_v - c_h) |\mathcal{R}(p)|.$$

Dividing both sides by ϵ gives

$$\frac{C(p + \epsilon) - C(p)}{\epsilon} = n c_f + (c_v - c_h) |\mathcal{R}(p)|.$$

Hence,

$$D_+ C(p) = \lim_{\epsilon \to 0+} \frac{C(p + \epsilon) - C(p)}{\epsilon} = n c_f + (c_v - c_h) |\mathcal{R}(p)|.$$

This completes the proof. □

Ghiani et al. (2004) claim that the derivative of C is

$$C'(p) = nc_f + (c_v - c_h)|\mathcal{R}(p)|. \tag{2.15}$$

However, by Theorems 2.1 and 2.2,

$$D_+C(p) - D_-C(p) = -(c_v - c_h)|\mathcal{T}(p)|. \tag{2.16}$$

Thus, since $c_v - c_h < 0$ (recall that $c_f + c_v < c_h$), the left and right derivatives of C differ when $\mathcal{T}(p) \neq \emptyset$. This means that C is not differentiable at $p = p_t$, $t = 1, \ldots, n$. Equation (2.15) is obviously invalid at these points.

The following result is proved in Chapter 5 of Royden (1988).

Lemma 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose right derivative D_+f exists at every point. If D_+f is non-decreasing, then f is convex.*

We can use Lemma 2.1 to show that C is a convex function. First, let $p_1 < p_2$. Then obviously

$$\mathcal{R}(p_2) \subset \mathcal{R}(p_1).$$

Thus,

$$|\mathcal{R}(p_2)| \leq |\mathcal{R}(p_1)|.$$

Since $c_v - c_h < 0$,

$$(c_v - c_h)|\mathcal{R}(p_2)| \geq (c_v - c_h)|\mathcal{R}(p_1)|$$

and consequently

$$D_+C(p_1) = nc_f + (c_v - c_h)|\mathcal{R}(p_1)| \leq nc_f + (c_v - c_h)|\mathcal{R}(p_2)| = D_+C(p_2).$$

This shows that D_+C is non-decreasing. Note also that C is a continuous function. Thus, it follows immediately from Lemma 2.1 that C is convex. In the next section, we will exploit this result to devise an efficient computational method for solving the FCP.

3 APPLYING THE GOLDEN SECTION METHOD

To solve the FCP directly, we need to evaluate C at every integer in $[0, p_{\max}]$. This is obviously extremely inefficient. In this section, we will develop a superior method for solving the FCP.

We first need to introduce the FCP's *continuous relaxation*, which is obtained by dropping the integer constraints on p.

Continuous Relaxation of the FCP

Find a real number $p \in [0, p_{\max}]$ that minimizes the cost function

$$C(p) = nc_f p + c_v \sum_{t=1}^{n} \min(p_t, p) + c_h \sum_{t=1}^{n} \max(p_t - p, 0).$$

We now show that, since C is convex, a solution of the original FCP can be easily obtained from a solution of its continuous relaxation.

Theorem 3.1 *Let p^* be an optimal solution of the FCP's continuous relaxation. Then either $\lfloor p^* \rfloor$ or $\lceil p^* \rceil$ is an optimal solution of the original FCP.*

Proof. Obviously, both $\lfloor p^* \rfloor$ and $\lceil p^* \rceil$ are feasible for the FCP. Suppose that neither $\lfloor p^* \rfloor$ nor $\lceil p^* \rceil$ is optimal. Then there exists an integer $p' \in [0, p_{\max}]$ such that

$$C(p') < \min\left\{C(\lfloor p^* \rfloor), C(\lceil p^* \rceil)\right\}. \tag{3.1}$$

Now, suppose that $p' = p^*$. Then p^* is an integer and hence

$$p' = p^* = \lfloor p^* \rfloor = \lceil p^* \rceil.$$

But this clearly contradicts (3.1), so $p' = p^*$ is impossible. Hence, $p' \neq p^*$.

Consider the line between p' and p^*:

$$\lambda p^* + (1-\lambda)p', \qquad \lambda \in [0,1].$$

This line must contain either $\lfloor p^* \rfloor$ or $\lceil p^* \rceil$; we assume without loss of generality that it contains $\lfloor p^* \rfloor$. Then there exists a $\lambda' \in (0,1]$ such that

$$\lfloor p^* \rfloor = \lambda' p^* + (1-\lambda')p'.$$

Thus, since C is convex,

$$C(\lfloor p^* \rfloor) = C(\lambda' p^* + (1-\lambda')p') \le \lambda' C(p^*) + (1-\lambda')C(p').$$

It is clear that $C(p^*) \le C(p')$. Therefore,

$$C(\lfloor p^* \rfloor) \le \lambda' C(p^*) + (1-\lambda')C(p') \le C(p').$$

But this contradicts (3.1). Thus, either $\lfloor p^* \rfloor$ or $\lceil p^* \rceil$ is optimal. □

Since the FCP's continuous relaxation is just a one-dimensional convex optimization problem, it can be solved efficiently using the well-known golden section method (see Bazaraa et al. (2006) and Luenberger & Ye (2008)). We can then obtain a solution of the original FCP using Theorem 3.1. Solving the FCP in this way is much more efficient than evaluating C at every integer in $[0, p_{\max}]$.

The golden section method works by computing C at various *test points* and using this information to continually reduce the *interval of uncertainty*. Initially, the only information we have is that the optimal solution lies somewhere in $[0, p_{\max}]$. Thus, the initial interval of uncertainty is

$$\mathcal{I}_0 = [\alpha_0, \beta_0] = [0, p_{\max}].$$

We define initial test points p_1^1 and p_2^1 as follows:

$$p_1^1 = p_{\max} - r p_{\max}$$

and

$$p_2^1 = rp_{\max},$$

where

$$r = \frac{\sqrt{5}-1}{2} = 0.618.$$

Now, given the $(k-1)$th interval of uncertainty $\mathcal{I}_{k-1} = [\alpha_{k-1}, \beta_{k-1}]$ and the test points p_1^k and p_2^k, we determine the new interval of uncertainty $\mathcal{I}_k$ according to the following rules. If $C(p_1^k) < C(p_2^k)$, then because C is convex, the optimal solution must lie in $[\alpha_{k-1}, p_2^k]$. Hence, the new interval of uncertainty is

$$\mathcal{I}_k = [\alpha_{k-1}, p_2^k].$$

On the other hand, if $C(p_1^k) \geq C(p_2^k)$, then the optimal solution must lie in the interval $[p_1^k, \beta_{k-1}]$. Hence, the new interval of uncertainty is

$$\mathcal{I}_k = [p_1^k, \beta_{k-1}].$$

The two test points p_1^{k+1} and p_2^{k+1} for this new interval of uncertainty are:

$$p_1^{k+1} = \begin{cases} p_2^k - r|\mathcal{I}_k|, & \text{if } \mathcal{I}_k = [\alpha_{k-1}, p_2^k], \\ p_2^k, & \text{if } \mathcal{I}_k = [p_1^k, \beta_{k-1}], \end{cases}$$

and

$$p_2^{k+1} = \begin{cases} p_1^k, & \text{if } \mathcal{I}_k = [\alpha_{k-1}, p_2^k], \\ p_1^k + r|\mathcal{I}_k|, & \text{if } \mathcal{I}_k = [p_1^k, \beta_{k-1}]. \end{cases}$$

Note that one of the new test points coincides with a test point from the previous interval of uncertainty. Choosing the test points in this way ensures that

$$|\mathcal{I}_k| = r|\mathcal{I}_{k-1}| < |\mathcal{I}_{k-1}|.$$

Consequently, the length of the kth interval of uncertainty is

$$|\mathcal{I}_k| = \beta_k - \alpha_k = r^k p_{\max}.$$

How many iterations of the golden section method are needed to solve the FCP's continuous relaxation? The next result answers this question.

Theorem 3.2 *Let N be an integer such that*

$$N > -\frac{\ln p_{\max}}{\ln r}. \tag{3.2}$$

Furthermore, let $\mathcal{I}_N = [\alpha_N, \beta_N]$ denote the Nth interval of uncertainty when the golden section method is applied to the FCP's continuous relaxation. Then either $\lfloor \alpha_N \rfloor$, $\lfloor \alpha_N \rfloor + 1$, or $\lfloor \alpha_N \rfloor + 2$ is a solution of the original FCP.

Proof. From (3.2), we obtain

$$r^N < \frac{1}{p_{\max}}.$$

Thus,

$$\beta_N - \alpha_N = r^N p_{\max} < 1. \tag{3.3}$$

Now, let p^* denote the solution of the FCP's continuous relaxation. Then

$$\lfloor \alpha_N \rfloor \leq \alpha_N \leq p^* \leq \beta_N \leq \lceil \beta_N \rceil.$$

Thus,

$$\lfloor \alpha_N \rfloor \leq \lfloor p^* \rfloor \leq \lceil p^* \rceil \leq \lceil \beta_N \rceil. \tag{3.4}$$

It follows from inequality (3.3) that $\beta_N < \lceil \alpha_N \rceil + 1$. Thus,

$$\lceil \beta_N \rceil \leq \lceil \alpha_N \rceil + 1 \leq \lfloor \alpha_N \rfloor + 1 + 1 = \lfloor \alpha_N \rfloor + 2. \tag{3.5}$$

Combining (3.4) and (3.5) gives

$$\lfloor \alpha_N \rfloor \leq \lfloor p^* \rfloor \leq \lceil p^* \rceil \leq \lfloor \alpha_N \rfloor + 2. \tag{3.6}$$

But by Theorem 3.1, either $\lfloor p^* \rfloor$ or $\lceil p^* \rceil$ is a solution of the original FCP. Thus, inequality (3.6) implies that either $\lfloor \alpha_N \rfloor$, $\lfloor \alpha_N \rfloor + 1$, or $\lfloor \alpha_N \rfloor + 2$ is a solution of the FCP. □

In Theorem 3.2, we can choose

$$N = \left\lceil -\frac{\ln p_{\max}}{\ln r} \right\rceil.$$

Then solving the FCP using the golden section method entails at most

$$\left\lceil -\frac{\ln p_{\max}}{\ln r} \right\rceil + 4$$

cost function evaluations ($N + 1$ evaluations for the golden section method, 3 more evaluations to decide which of $\lfloor \alpha_N \rfloor$, $\lfloor \alpha_N \rfloor + 1$, and $\lfloor \alpha_N \rfloor + 2$ is optimal). This is far less than the $p_{\max} + 1$ cost function evaluations needed to solve the FCP directly. For example, consider a FCP with $p_{\max} = 10,000$. Then

$$\left\lceil -\frac{\ln p_{\max}}{\ln r} \right\rceil + 4 = 24 \ll 10,001 = p_{\max} + 1.$$

Thus, the golden section method is very efficient for large-scale FCP's.

References

M. S. Bazaraa, H. D. Sherali, and C. M. Shetty (2006), *Nonlinear Programming: Theory and Algorithms*, third edition, John Wiley, New Jersey, USA.

G. Ghiani, G. Laporte, and R. Musmanno (2004), *Introduction to Logistics Systems Planning and Control*, John Wiley, Chichester, England.

D. G. Luenberger and Y. Ye (2008), *Linear and Nonlinear Programming*, third edition, Springer, New York, USA.

H. L. Royden (1988), *Real Analysis*, third edition, Prentice Hall, New Jersey, USA.