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A NEIGHBORING EXTREMAL SOLUTION FOR AN OPTIMAL SWITCHED IMPULSIVE CONTROL PROBLEM

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Abstract. This paper presents a neighboring extremal solution for a class of optimal switched impulsive control problems with perturbations in the initial state, terminal condition and system's parameters. The sequence of mode's switching is pre-specified, and the decision variables, i.e. the switching times and parameters of the system involved, have inequality constraints. It is assumed that the active status of these constraints is unchanged with the perturbations. We derive this solution by expanding the necessary conditions for optimality to first-order and then solving the resulting multiple-point boundary-value problem by the backward sweep technique. Numerical simulations are presented to illustrate this solution method.

1. **Introduction.** Real-world optimal control problems are often nonlinear and far too complex to be solved analytically. Thus, numerical methods are indispensable for solving such problems. However, solving optimal control problems numerically is often time consuming. Furthermore, existing numerical solution methods (see, for example, [1, 17]) only compute open-loop optimal controls, which are sensitive to disturbances and modelling uncertainties. The neighboring extremal (NE) method was proposed in the 1960s [2] to construct a NE control in a feedback form for an unconstrained optimal control problem involving nonlinear dynamics. In this method, it is assumed that a nominal optimal solution has been computed offline and the aim is to construct an approximate optimal control online when the initial state and terminal condition are slightly perturbed. This NE method was extended in [15, 16, 8, 13, 14] to optimal control problems involving nonlinear continuous dynamics subject to continuous inequality constraints. Recently, this method was extended in [6, 5] to singular control problems, and in [3, 4] to constrained discrete-time optimal control problems.

In this paper, we consider an optimal control problem for a class of switched impulsive systems with perturbations of initial state, terminal condition and system's parameters. These systems are operated by switching between different subsystems

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or modes, and may exhibit instantaneous state jumps during the mode switching. Switched impulsive systems arise in areas such as circuits [7, 12], etc. Impulsive systems [10, 18, 19, 11, 9] are a special class of them, which have one mode. The optimal control problem for switched impulsive systems is to choose the sequence of the modes, the times to switch between the modes, and the parameters controlling the state jumps to minimize a given cost function subject to constraints. This paper assumes that the sequence of the modes is pre-specified. With respect to a nominal solution for the optimal switched impulsive control problem, an approximation to the NE solution is derived in this paper when the initial state, terminal condition and system's parameters are slightly perturbed. It is assumed that the active status of the inequality constraints on the switching times and parameters is unchanged with the perturbations. We derive this NE solution by expanding the necessary conditions for optimality (NCO) to first-order and then solving the resulting multiple-point boundary-value problem (MPBVP). The backward sweep technique [2] is used to solve the MPBVP. Then, the design procedure of the controller consists of two stages. In the offline stage, we can integrate two groups of switched impulsive matrix differential equations along the trajectory of the nominal solution, and compute the Jacobian matrix of the switching times and parameters with respect to the initial state, terminal condition and system's parameters on the condition that a certain symmetric matrix is invertible. Then, with this Jacobian a first-order correction to the nominal solution can be computed online when the perturbations of the initial state, terminal condition and system's parameters are obtained. In this way, a closed-loop control law is constructed efficiently, which is adaptive to the small changes of the initial state, terminal condition and system's parameters, if the operation is repeated like in batch processes. Since the switching times and parameters are both constrained in the problem and a MPBVP is involved, the solving procedure is more complex than that in [2] for a two-point boundary-value problem (TPBVP). Furthermore, this paper also considers the perturbation of the system's parameters, which is not easy to deal with by the shooting method [15, 8, 13, 14].

The rest of this paper is organized as follows. The nominal optimal control problem for switched impulsive systems is presented in Section 2. Our main result on the NE solution of the optimal switched impulsive control problem with respect to small perturbations of initial state, terminal condition and system's parameters is presented in Section 3. After that, we present three groups of numerical simulations in Section 4 to verify our NE solution. Finally, Section 5 concludes the paper. Throughout this paper, '$*$' in a symmetric matrix denotes its symmetric term. For a continuously differentiable function $f(x,y):\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}\to\mathbb{R}$, $f_x\triangleq\partial f/\partial x$ is a row vector, and $f_{xy}\triangleq(\partial^2 f)/(\partial x\partial y)=\partial(\partial f/\partial y)^{\mathrm{T}}/\partial x$.

2. **Optimal control problem for switched impulsive systems.**

Consider the following switched impulsive system with $N+1$ subsystems:

$$\dot{x}(t)=f^i(x(t),\gamma,t),\quad t\in(t_{i-1},t_i),\quad i=1,\dots,N+1, \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the system's state at time t, $\gamma\in\mathbb{R}^q$ is the uncertain parametric vector of the system that can be measured or estimated, $f^i:\mathbb{R}^{n+q}\times(t_{i-1},t_i)\to\mathbb{R}^n$, $i=1,\dots,N+1$, are given functions, $t_0\triangleq 0$, $t_{N+1}\triangleq t_f>0$, and $t_i>0$, $i=1,\dots,N$, represent the subsystems' switching times. The subsystems are switched in a pre-specified sequence with index i from 1 to $N+1$, and these switchings are accompanied by instantaneous state jumps which are determined by

$$x(t_i^+) = \begin{cases} x^0, & i = 0, & \text{(2a)} \\ g^i\left(x(t_i^-), s_i, t_i\right), & i = 1, \ldots, N, & \text{(2b)} \end{cases}$$

where $x^0 \in \mathbb{R}^n$ is a given initial state, and $g^i : \mathbb{R}^{n+m} \times \mathbb{R}^+ \to \mathbb{R}^n$, $i = 1, \ldots, N$, are given functions governing the state jumps at $t = t_i$ with $s_i \in \mathbb{R}^m$, $i = 1, \ldots, N$, being the parameters controlling the jumps. In equation (2), $x(t_i^+)$ and $x(t_i^-)$, $i = 1, \ldots, N$, denote, respectively, the limits of $x(t)$ from the right and left at $t = t_i$.

The terminal state of system (1)-(2) is constrained by

$$\psi\left(x(t_f), t_f\right) = 0, \tag{3}$$

where t_f is free and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^p$ is given. Let

$$\bar{\psi}^i\left(x(t_i^+), x(t_i^-), s_i, t_i\right) \triangleq x(t_i^+) - g^i\left(x(t_i^-), s_i, t_i\right) = 0, \quad i = 1, \ldots, N. \tag{4}$$

In system (1)-(2), the terminal time $t_f = t_{N+1}$, the switching times t_i, $i = 1, \ldots, N$, and the parameters s_i, $i = 1, \ldots, N$, are decision variables. They are to be chosen such that the following constraints are satisfied:

$$\begin{cases} a^j \le s_i^j \le b^j, & j = 1, \ldots, m, \quad i = 1, \ldots, N, & \text{(5a)} \\ t_i - t_{i-1} \ge c, & i = 1, \ldots, N+1, \quad t_0 = 0, \quad t_{N+1} = t_f, & \text{(5b)} \end{cases}$$

where s_i^j is the jth element of the vector s_i, $a^j < b^j$ are given lower and upper bounds of the control parameters, and $c > 0$ is the given minimum duration of a subsystem. Constraints (5a) and (5b) are linear, which belong to the following more general nonlinear constraints

$$\begin{cases} \hat{\psi}^i\left(x(t_i^+), x(t_i^-), s_i, t_i\right) \le 0, & i = 1, \ldots, N, & \text{(6a)} \\ \eta\left(t_1, \ldots, t_{N+1}\right) \le 0 & & \text{(6b)} \end{cases}$$

respectively, where the inequalities are componentwise, and functions $\hat{\psi}^i : \mathbb{R}^{2n+m} \times \mathbb{R}^+ \to \mathbb{R}^{\hat{p}}$ and $\eta : \mathbb{R}^+ \times \cdots \times \mathbb{R}^+ \to \mathbb{R}^w$ are given.

Let

$$\sigma \triangleq \left[s_1^{\mathrm{T}}, \ldots, s_N^{\mathrm{T}}\right]^{\mathrm{T}} \quad \text{and} \quad \tau \triangleq \left[t_1, \ldots, t_{N+1}\right]^{\mathrm{T}}.$$

If the nominal value of the parameter γ is given, a nominal optimal control problem for the switched impulsive system (1)-(2) can be formally stated as follows.

Problem (N)**.** *For the given system (1)-(2), find a control pair $(\sigma, \tau) \in \mathbb{R}^{Nm} \times \mathbb{R}^{N+1}$ such that the cost function*

$$J(\sigma, \tau) \triangleq \phi\left(x(t_f), t_f\right) + \sum_{i=1}^{N} \bar{\phi}^i\left(x(t_i^+), x(t_i^-), t_i\right) \tag{7}$$

is minimized subject to the terminal constraint (3) and the inequality constraints in (6). In (7), $\phi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, $\bar{\phi}^i : \mathbb{R}^{2n} \times \mathbb{R}^+ \to \mathbb{R}$, $i = 1, \ldots, N$, are given functions.

Remark 1. we can easily incorporate an integral term into (7) by introducing a dummy state variable. For example, consider the integral term

$$\sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} L^i\left(x(t), t\right) \mathrm{d}t.$$

It is clear that this term can be replaced by $y(t_f)$, where $y(t)$ satisfies the dynamics

$$\dot{y}(t) = L^i\left(x(t), t\right), \quad t \in (t_{i-1}, t_i), \quad i = 1, \ldots, N+1,$$

and

$$y(t_i^+) = \begin{cases} 0, & i = 0, \\ y(t_i^-), & i = 1, \ldots, N. \end{cases}$$

For Problem (N), we need the following assumption:

Assumption 1. f^i, $i = 1, \ldots, N+1$, ϕ, ψ, and g^i, $\bar{\phi}^i$, $\hat{\psi}^i$, $i = 1, \ldots, N$ are twice continuously differentiable with respect to each of their arguments, and η is continuously differentiable with respect to its arguments. Furthermore, g^i, $i = 1, \ldots, N$, are bounded.

Define

$$\begin{aligned} H^i\left(x(t), \lambda(t), \gamma, t\right) &\triangleq \lambda^{\mathrm{T}}(t) f^i\left(x(t), \gamma, t\right), \quad i = 1, \ldots, N+1, \\ \Phi\left(x(t_f), \nu, t_f\right) &\triangleq \phi\left(x(t_f), t_f\right) + \nu^{\mathrm{T}} \psi\left(x(t_f), t_f\right), \\ \bar{\Phi}^i(x(t_i^+), x(t_i^-), s_i, \bar{\nu}_i, \hat{\nu}_i, t_i) &\triangleq \bar{\phi}^i\left(x(t_i^+), x(t_i^-), t_i\right) + \bar{\nu}_i^{\mathrm{T}} \bar{\psi}^i\left(x(t_i^+), x(t_i^-), s_i, t_i\right) \\ &\quad + \hat{\nu}_i^{\mathrm{T}} \hat{\psi}^i\left(x(t_i^+), x(t_i^-), s_i, t_i\right), \quad i = 1, \ldots, N, \end{aligned}$$

where $\lambda(t) \in \mathbb{R}^n$ is the costate, and $\nu \in \mathbb{R}^p$, $\bar{\nu}_i \in \mathbb{R}^n$, $\hat{\nu}_i \in \mathbb{R}^{\hat{p}}$, $i = 1, \ldots, N$, are Lagrange multipliers. Let $\pi \in \mathbb{R}^w$ be Lagrange multiplier corresponding to constraint (6b).

Let (σ^*, τ^*), where $\sigma^* = [s_1^{*\mathrm{T}}, \ldots, s_N^{*\mathrm{T}}]^{\mathrm{T}}$ and $\tau^* = [t_1^*, \ldots, t_{N+1}^*]^{\mathrm{T}}$, be an extremal solution to Problem (N) corresponding to the initial state x^0, the terminal condition (3), and the nominal value of parameter γ. Furthermore, let $x^*(t)$ and $\lambda^*(t)$ be the corresponding state and costate, and let ν^*, $\bar{\nu}_i^*$, $\hat{\nu}_i^*$, $i = 1, \ldots, N$, and π^* be, respectively, the Lagrange multipliers adjoining the constraints (3), (4), (6a) and (6b). Denote the nominal value of γ by γ^*. We call this solution the nominal solution to Problem (N), and this solution should satisfy the following NCO [2]:

$$\begin{cases} \dot{x}^*(t) = f^i\left(x^*(t), \gamma^*, t\right), \quad t \in (t_{i-1}^*, t_i^*), \quad x^*(t_0) = x^0, & \text{(9a)} \\ \dot{\lambda}^{*\mathrm{T}}(t) = -H_x^i\left(x^*(t), \lambda^*(t), \gamma^*, t\right), \quad t \in (t_{i-1}^*, t_i^*), & \text{(9b)} \end{cases}$$

for $i = 1, \ldots, N+1$; and

$$\begin{cases} \lambda^*(t_f^*) = \Phi_x^{\mathrm{T}}(x^*(t_f^*), \nu^*, t_f^*), & \text{(10a)} \\ 0 = \psi\left(x^*(t_f^*), t_f^*\right), & \text{(10b)} \\ 0 = \Phi_t(x^*(t_f^*), \nu^*, t_f^*) + H^{N+1}(x^*(t_f^*), \lambda^*(t_f^*), \gamma^*, t_f^*) + \pi^{*\mathrm{T}} \eta_{t_f}; & \text{(10c)} \end{cases}$$

$$\begin{cases} \lambda^{*\mathrm{T}}(t_i^{*+}) = -\bar{\Phi}_{x_i^+}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, \bar{\nu}_i^*, \hat{\nu}_i^*, t_i^*\right), & \text{(11a)} \\ \lambda^{*\mathrm{T}}(t_i^{*-}) = \bar{\Phi}_{x_i^-}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, \bar{\nu}_i^*, \hat{\nu}_i^*, t_i^*\right), & \text{(11b)} \\ 0 = \bar{\Phi}_{s_i}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, \bar{\nu}_i^*, \hat{\nu}_i^*, t_i^*\right), & \text{(11c)} \\ 0 = \bar{\psi}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, t_i^*\right), & \text{(11d)} \\ 0 \geq \hat{\psi}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, t_i^*\right), & \text{(11e)} \\ 0 = \hat{\nu}_i^{*\mathrm{T}} \hat{\psi}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, t_i^*\right), \quad \hat{\nu}_i^* \geq 0, & \text{(11f)} \\ 0 = H^i\left(x^*(t_i^{*-}), \lambda^*(t_i^{*-}), \gamma^*, t_i^*\right) - H^{i+1}\left(x^*(t_i^{*+}), \lambda^*(t_i^{*+}), \gamma^*, t_i^*\right) & \\ \quad + \bar{\Phi}_{t_i}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, \bar{\nu}_i^*, \hat{\nu}_i^*, t_i^*\right) + \pi^{*\mathrm{T}} \eta_{t_i} & \text{(11g)} \end{cases}$$

for $i = 1, \ldots, N$, where the subscripts x_i^+ and x_i^- denote, respectively, partial derivatives with respect to $x(t_i^+)$ and $x(t_i^-)$; and

$$\begin{cases} 0 \geq \eta, & (12a) \\ 0 = \pi^{*\mathrm{T}}\eta, \quad \pi^* \geq 0. & (12b) \end{cases}$$

Define $\Omega \triangleq \Phi_t + H^{N+1}$.

3. NE solution for the optimal switched impulsive control problem.

Suppose that the initial state, terminal condition and system's parameters γ are perturbed by small amounts. Let $\delta x^0 \in \mathbb{R}^n$, $\mathrm{d}\psi \in \mathbb{R}^p$ and $\mathrm{d}\gamma \in \mathbb{R}^q$ denote, respectively, arbitrary deviations from the initial state, terminal condition and parameter γ of Problem (N). Specifically, the initial state, terminal condition and γ become, respectively,

$$x(0) = x^0 + \delta x^0, \quad \mathrm{d}\psi = \psi\left(x(t_f), t_f\right) \quad \text{and} \quad \gamma = \gamma^* + \mathrm{d}\gamma. \tag{13}$$

Let (σ, τ), where $\sigma = [s_1^{\mathrm{T}}, \ldots, s_N^{\mathrm{T}}]^{\mathrm{T}}$ and $\tau = [t_1, \ldots, t_{N+1}]^{\mathrm{T}}$, $x(t)$, $\lambda(t)$, ν, $\bar{\nu}$, $\hat{\nu}$, π be an extremal solution to Problem (N) with the initial state, terminal condition and system's parameters γ perturbed according to (13). This solution has the following first-order approximation:

$$z(t) \approx z^*(t) + \delta z(t), \tag{14a}$$

$$z(t_f) \approx z^*(t_f^*) + \mathrm{d}z(t_f^*), \quad \mathrm{d}z(t_f^*) = \delta z(t_f^*) + \dot{z}(t_f^*)\mathrm{d}t_f, \tag{14b}$$

$$z(t_i^{\pm}) \approx z^*(t_i^{*\pm}) + \mathrm{d}z(t_i^{*\pm}), \quad \mathrm{d}z(t_i^{*\pm}) = \delta z(t_i^{*\pm}) + \dot{z}(t_i^{*\pm})\mathrm{d}t_i, \quad i = 1, \ldots, N, \tag{14c}$$

for $z = x, \lambda$, and

$$\begin{cases} t_f \approx t_f^* + \mathrm{d}t_f, \ \nu \approx \nu^* + \mathrm{d}\nu, \ \pi \approx \pi^* + \mathrm{d}\pi, \\ s_i \approx s_i^* + \mathrm{d}s_i, \ \bar{\nu}_i \approx \bar{\nu}_i^* + \mathrm{d}\bar{\nu}_i, \ \hat{\nu}_i \approx \hat{\nu}_i^* + \mathrm{d}\hat{\nu}_i, \ t_i \approx t_i^* + \mathrm{d}t_i, \quad i = 1, \ldots, N, \end{cases} \tag{15}$$

where $\delta z(t)$ denotes the first-order variation of $z(t)$ at time t, and all the variables with a prefix 'd' denote the differentials. The NE problem for the perturbed system may now be stated as follows.

Problem (NE). *Given the nominal extremal control pair (σ^*, τ^*) of Problem (N) and the nominal value γ^* of the uncertain parameters γ, find the NE solution (σ, τ) in the first-order approximation form (15) to Problem (N) when the initial state, terminal condition and parameters γ are perturbed according to (13).*

3.1. Linearization of the NCO.

For Problem (NE), we make the following assumption.

Assumption 2. The perturbations δx^0, $\mathrm{d}\psi$ and $\mathrm{d}\gamma$ are small enough such that the active status of the inequality constraints (6) are unchanged after the perturbations.

Let $\tilde{\psi}^i\left(x^*(t_i^{*+}), x^*(t_i^{*-}), s_i^*, t_i^*\right) = 0$ and $\tilde{\eta}\left(\tau^*\right) = 0$ denote, respectively, the active parts of the constraints (6a) and (6b) for the nominal solution. Let $\tilde{\nu}_i^*$ and $\tilde{\pi}^*$ be the corresponding Lagrange multipliers. As in [2], forcing the first-order variation of the NCO (9)-(12) to zero, we can derive the following four groups of equations.

1. For $t \in (t_{i-1}^*, t_i^*)$, $i = 1, \ldots, N+1$, we have

$$\delta\dot{x}(t) = f_x^i \delta x(t) + f_\gamma^i \mathrm{d}\gamma, \tag{16a}$$

$$\delta\dot{\lambda}(t) = -H_{xx}^i \delta x(t) - f_x^{i\mathrm{T}} \delta\lambda(t) - H_{\gamma x}^i \mathrm{d}\gamma, \tag{16b}$$

where all the partial derivatives are evaluated along the nominal trajectory;

2. at $t = t_f^*$, we have

$$\mathrm{d}\lambda(t_f^*) = \Phi_{xx}\mathrm{d}x(t_f^*) + \psi_x^{\mathrm{T}}\mathrm{d}\nu + \Phi_{tx}\mathrm{d}t_f, \tag{17a}$$

$$\mathrm{d}\psi = \psi_x\mathrm{d}x(t_f^*) + \psi_t\mathrm{d}t_f, \tag{17b}$$

$$0 = \Omega_x\mathrm{d}x(t_f^*) + \dot{\psi}^{\mathrm{T}}\mathrm{d}\nu + \Omega_t\mathrm{d}t_f + \tilde{\eta}_{t_f}^{\mathrm{T}}\mathrm{d}\tilde{\pi} + \lambda^{*\mathrm{T}}f_\gamma^{N+1}\mathrm{d}\gamma, \tag{17c}$$

where all the derivatives and partial derivatives are evaluated at $t = t_f^*$ along the nominal trajectory;

3. at $t = t_i^*$, $i = 1, \dots, N$, we have

$$\begin{aligned}\mathrm{d}\lambda(t_i^{*+}) = &-\bar{\Phi}^i_{x_i^+x_i^+}\mathrm{d}x(t_i^{*+}) - \bar{\Phi}^i_{x_i^-x_i^+}\mathrm{d}x(t_i^{*-})\\ &-\bar{\Phi}^i_{s_ix_i^+}\mathrm{d}s_i - \bar{\psi}^{i\mathrm{T}}_{x_i^+}\mathrm{d}\bar{\nu}_i - \tilde{\psi}^{i\mathrm{T}}_{x_i^+}\mathrm{d}\tilde{\nu}_i - \bar{\Phi}^i_{t_ix_i^+}\mathrm{d}t_i,\end{aligned} \tag{18a}$$

$$\begin{aligned}\mathrm{d}\lambda(t_i^{*-}) = &\bar{\Phi}^i_{x_i^+x_i^-}\mathrm{d}x(t_i^{*+}) + \bar{\Phi}^i_{x_i^-x_i^-}\mathrm{d}x(t_i^{*-})\\ &+\bar{\Phi}^i_{s_ix_i^-}\mathrm{d}s_i + \bar{\psi}^{i\mathrm{T}}_{x_i^-}\mathrm{d}\bar{\nu}_i + \tilde{\psi}^{i\mathrm{T}}_{x_i^-}\mathrm{d}\tilde{\nu}_i + \bar{\Phi}^i_{t_ix_i^-}\mathrm{d}t_i,\end{aligned} \tag{18b}$$

$$\begin{aligned}0 = &\bar{\Phi}^i_{x_i^+s_i}\mathrm{d}x(t_i^{*+}) + \bar{\Phi}^i_{x_i^-s_i}\mathrm{d}x(t_i^{*-})\\ &+\bar{\Phi}^i_{s_is_i}\mathrm{d}s_i + \bar{\psi}^{i\mathrm{T}}_{s_i}\mathrm{d}\bar{\nu}_i + \tilde{\psi}^{i\mathrm{T}}_{s_i}\mathrm{d}\tilde{\nu}_i + \bar{\Phi}^i_{t_is_i}\mathrm{d}t_i,\end{aligned} \tag{18c}$$

$$0 = \bar{\psi}^i_{x_i^+}\mathrm{d}x(t_i^{*+}) + \bar{\psi}^i_{x_i^-}\mathrm{d}x(t_i^{*-}) + \bar{\psi}^i_{s_i}\mathrm{d}s_i + \bar{\psi}^i_{t_i}\mathrm{d}t_i, \tag{18d}$$

$$0 = \tilde{\psi}^i_{x_i^+}\mathrm{d}x(t_i^{*+}) + \tilde{\psi}^i_{x_i^-}\mathrm{d}x(t_i^{*-}) + \tilde{\psi}^i_{s_i}\mathrm{d}s_i + \tilde{\psi}^i_{t_i}\mathrm{d}t_i, \tag{18e}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{s_i}\right)\mathrm{d}s_i + \dot{\bar{\psi}}^{i\mathrm{T}}\mathrm{d}\bar{\nu}_i + \dot{\tilde{\psi}}^{i\mathrm{T}}\mathrm{d}\tilde{\nu}_i + \kappa_i\mathrm{d}t_i + \tilde{\eta}^{\mathrm{T}}_{t_i}\mathrm{d}\tilde{\pi} + \epsilon_i\mathrm{d}\gamma, \tag{18f}$$

where

$$\begin{aligned}\kappa_i \triangleq &\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{t_i}\right) + H^i_{t_i}\left(x^*(t_i^{*-}), \lambda^*(t_i^{*-}), \gamma^*, t_i^*\right)\\ &- H^{i+1}_{t_i}\left(x^*(t_i^{*+}), \lambda^*(t_i^{*+}), \gamma^*, t_i^*\right),\\ \epsilon_i \triangleq &\lambda^{*\mathrm{T}}(t_i^{*-})f^i_\gamma\left(x^*(t_i^{*-}), \gamma^*, t_i^*\right) - \lambda^{*\mathrm{T}}(t_i^{*+})f^{i+1}_\gamma\left(x^*(t_i^{*+}), \gamma^*, t_i^*\right),\end{aligned}$$

and all the partial derivatives are evaluated at $t = t_i^*$ along the nominal extremal trajectory; and

4.

$$\sum_{i=1}^{N}\tilde{\eta}_{t_i}\mathrm{d}t_i + \tilde{\eta}_{t_{N+1}}\mathrm{d}t_f = 0. \tag{19}$$

Equation (18f) is obtained by considering

$$\begin{aligned}H^i_x\left(x(t_i^{*-}), \lambda(t_i^{*-}), \gamma^*, t_i^*\right)\mathrm{d}x(t_i^{*-}) &+ H^i_\lambda\left(x(t_i^{*-}), \lambda(t_i^{*-}), \gamma^*, t_i^*\right)\mathrm{d}\lambda(t_i^{*-})\\ &= -\dot{\lambda}^{*\mathrm{T}}(t_i^{*-})\mathrm{d}x(t_i^{*-}) + \dot{x}^{*\mathrm{T}}(t_i^{*-})\mathrm{d}\lambda(t_i^{*-})\end{aligned}$$

and

$$\begin{aligned}H^{i+1}_x\left(x(t_i^{*+}), \lambda(t_i^{*+}), \gamma^*, t_i^*\right)\mathrm{d}x(t_i^{*+}) &+ H^{i+1}_\lambda\left(x(t_i^{*+}), \lambda(t_i^{*+}), \gamma^*, t_i^*\right)\mathrm{d}\lambda(t_i^{*+})\\ &= -\dot{\lambda}^{*\mathrm{T}}(t_i^{*+})\mathrm{d}x(t_i^{*+}) + \dot{x}^{*\mathrm{T}}(t_i^{*+})\mathrm{d}\lambda(t_i^{*+}),\end{aligned}$$

and equations (18a), (18b), (11a) and (11b).

Rearranging (16a) and (16b), we have, for $t \in (t_{i-1}^*, t_i^*)$, $i = 1, \ldots, N+1$,

$$\begin{bmatrix} \delta\dot{x}(t) \\ \delta\dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} f_x^i & 0 \\ -H_{xx}^i & -f_x^{i\mathrm{T}} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} + \begin{bmatrix} f_\gamma^i \\ -H_{\gamma x}^i \end{bmatrix} \mathrm{d}\gamma$$
$$= \begin{bmatrix} A^i & 0 \\ C^i & -A^{i\mathrm{T}} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} + \begin{bmatrix} E^i \\ F^i \end{bmatrix} \mathrm{d}\gamma \tag{20}$$

where A^i, C^i, E^i and F^i are evaluated along the nominal extremal solution.

3.2. **NE solution.** Equations (16)-(19) form a linear MPBVP, which can be solved by the backward sweep technique [2]. First, define along the nominal solution

$$\Theta_i \triangleq \begin{cases} \begin{bmatrix} \mathrm{d}\nu^{\mathrm{T}} & \mathrm{d}t_f \end{bmatrix}^{\mathrm{T}}, & \text{if } i = N+1, \\ \begin{bmatrix} \mathrm{d}\nu^{\mathrm{T}} & \mathrm{d}t_f & \mathrm{d}s_N^{\mathrm{T}} & \mathrm{d}\tilde{\nu}_N^{\mathrm{T}} & \mathrm{d}t_N & \cdots & \mathrm{d}s_i^{\mathrm{T}} & \mathrm{d}\tilde{\nu}_i^{\mathrm{T}} & \mathrm{d}t_i \end{bmatrix}^{\mathrm{T}}, & \text{if } i = N, \ldots, 1, \end{cases} \tag{21}$$

$$\Psi_i \triangleq \begin{cases} \begin{bmatrix} \mathrm{d}\psi^{\mathrm{T}} & -\tilde{\eta}_{t_f}^{\mathrm{T}}\mathrm{d}\tilde{\pi} \end{bmatrix}^{\mathrm{T}}, & \text{if } i = N+1, \\ \begin{bmatrix} \mathrm{d}\psi^{\mathrm{T}} & -\tilde{\eta}_{t_f}^{\mathrm{T}}\mathrm{d}\tilde{\pi} & 0 & 0 & -\tilde{\eta}_{t_N}^{\mathrm{T}}\mathrm{d}\tilde{\pi} & \cdots & 0 & 0 & -\tilde{\eta}_{t_i}^{\mathrm{T}}\mathrm{d}\tilde{\pi} \end{bmatrix}^{\mathrm{T}}, & \text{if } i = N, \ldots, 1. \end{cases} \tag{22}$$

Next, consider two groups of auxiliary systems governed by switched impulsive differential equations:

$$\begin{cases} \dot{S}_{1,1}(t) = -S_{1,1}(t)A^i - A^{i\mathrm{T}}S_{1,1}(t) + C^i, & t \in (t_{i-1}^*, t_i^*), \quad i = N+1, \ldots, 1, \\ S_{1,1}(t_i^{*-}) = \Phi_{xx}, & i = N+1, \\ S_{1,1}(t_i^{*-}) = \bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,1}(t_i^{*+})\bar{\psi}_{x_i^-}^i + \bar{\Phi}_{x_i^- x_i^-}^i, & i = N, \ldots, 1, \end{cases} \tag{23a}$$

$$\begin{cases} \dot{S}_{1,2}(t) = -A^{i\mathrm{T}}S_{1,2}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N+1, \ldots, 1, \\ S_{1,2}(t_i^{*-}) = \psi_x^{\mathrm{T}}, & i = N+1, \\ S_{1,2}(t_i^{*-}) = -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,2}(t_i^{*+}), & i = N, \ldots, 1, \end{cases} \tag{23b}$$

$$\begin{cases} \dot{S}_{1,3}(t) = -A^{i\mathrm{T}}S_{1,3}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N+1, \ldots, 1, \\ S_{1,3}(t_i^{*-}) = \Omega_x^{\mathrm{T}}, & i = N+1, \\ S_{1,3}(t_i^{*-}) = -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,3}(t_i^{*+}), & i = N, \ldots, 1, \end{cases} \tag{23c}$$

$$\begin{cases} \dot{S}_{1,3j+1}(t) = -A^{i\mathrm{T}}S_{1,3j+1}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N-j+1, \ldots, 1, \\ S_{1,3j+1}(t_i^{*-}) = \bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,1}(t_i^{*+})\bar{\psi}_{s_i}^i + \bar{\Phi}_{s_i x_i^-}^i, & i = N-j+1, \\ S_{1,3j+1}(t_i^{*-}) = -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,3j+1}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{23d}$$

$$\begin{cases} \dot{S}_{1,3j+2}(t) = -A^{i\mathrm{T}}S_{1,3j+2}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N-j+1, \ldots, 1, \\ S_{1,3j+2}(t_i^{*-}) = \tilde{\psi}_{x_i^-}^{i\mathrm{T}} - \left(\tilde{\psi}_{x_i^+}^i \bar{\psi}_{x_i^-}^i\right)^{\mathrm{T}}, & i = N-j+1, \\ S_{1,3j+2}(t_i^{*-}) = -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,3j+2}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{23e}$$

$$\begin{cases} \dot{S}_{1,3j+3}(t) = -A^{i\mathrm{T}}S_{1,3j+3}(t), & t \in (t_{i-1}^*, t_i^*), \quad i = N-j+1, \ldots, 1, \\ S_{1,3j+3}(t_i^{*-}) = \bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,1}(t_i^{*+})\dot{\bar{\psi}}^i, & i = N-j+1, \\ S_{1,3j+3}(t_i^{*-}) = -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,3j+3}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{23f}$$

and

$$\begin{cases} \dot{R}_1(t) = -S_{1,1}(t)E^i - A^{i\mathrm{T}}R_1(t) + F^i, & t \in (t_{i-1}^*, t_i^*), \quad i = N+1, \ldots, 1, \\ R_1(t_i^{*-}) = 0, & i = N+1, \\ R_1(t_i^{*-}) = -\bar{\psi}_{x_i^-}^{i\mathrm{T}} R_1(t_i^{*+}), & i = N, \ldots, 1, \end{cases} \tag{24a}$$

$$\begin{cases} \dot{R}_2(t) = -S_{1,2}^{\mathrm{T}}(t)E^i, & t \in (t_{i-1}^*, t_i^*),\ i = N+1, \ldots, 1, \\ R_2(t_i^{*-}) = 0, & i = N+1, \\ R_2(t_i^{*-}) = R_2(t_i^{*+}), & i = N, \ldots, 1, \end{cases} \tag{24b}$$

$$\begin{cases} \dot{R}_3(t) = -S_{1,3}^{\mathrm{T}}(t)E^i, & t \in (t_{i-1}^*, t_i^*),\ i = N+1, \ldots, 1, \\ R_3(t_i^{*-}) = \lambda^{*\mathrm{T}}(t_i^*)f_\gamma^i, & i = N+1, \\ R_3(t_i^{*-}) = R_3(t_i^{*+}), & i = N, \ldots, 1, \end{cases} \tag{24c}$$

$$\begin{cases} \dot{R}_{3j+1}(t) = -S_{1,3j+1}^{\mathrm{T}}(t)E^i, & t \in (t_{i-1}^*, t_i^*),\ i = N-j+1, \ldots, 1, \\ R_{3j+1}(t_i^{*-}) = -\bar{\psi}_{s_i}^{i\mathrm{T}} R_1(t_i^{*+}), & i = N-j+1, \\ R_{3j+1}(t_i^{*-}) = R_{3j+1}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{24d}$$

$$\begin{cases} \dot{R}_{3j+2}(t) = -S_{1,3j+2}^{\mathrm{T}}(t)E^i, & t \in (t_{i-1}^*, t_i^*),\ i = N-j+1, \ldots, 1, \\ R_{3j+2}(t_i^{*-}) = 0, & i = N-j+1, \\ R_{3j+2}(t_i^{*-}) = R_{3j+2}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{24e}$$

$$\begin{cases} \dot{R}_{3j+3}(t) = -S_{1,3j+3}^{\mathrm{T}}(t)E^i, & t \in (t_{i-1}^*, t_i^*),\ i = N-j+1, \ldots, 1, \\ R_{3j+3}(t_i^{*-}) = -\dot{\bar{\psi}}^{i\mathrm{T}} R_1(t_i^{*+}) + \epsilon_i, & i = N-j+1, \\ R_{3j+3}(t_i^{*-}) = R_{3j+3}(t_i^{*+}), & i = N-j, \ldots, 1, \end{cases} \tag{24f}$$

where $j = 1, \ldots, N$, and all the coefficient matrices are evaluated along the nominal solution. Let Υ be a symmetric matrix, which has the definition

$$\Upsilon \triangleq \begin{bmatrix} \Upsilon_{1,1} & \Upsilon_{1,2} \\ \Upsilon_{1,2}^{\mathrm{T}} & 0 \end{bmatrix}, \tag{25}$$

where

$$\Upsilon_{1,1} \triangleq \begin{bmatrix} S_{2,2}(0) & \ldots & S_{2,3N+3}(0) \\ * & \ddots & \vdots \\ * & * & S_{3N+3,3N+3}(0) \end{bmatrix},$$

$$\Upsilon_{1,2}^{\mathrm{T}} \triangleq \begin{bmatrix} 0 & \tilde{\eta}_{t_f} & 0 & 0 & \tilde{\eta}_{t_N} & \ldots & 0 & 0 & \tilde{\eta}_{t_1} \end{bmatrix},$$

and $S_{\alpha,\beta}(0)$, $\alpha = 2, \ldots, 3N+3$, $\beta = \alpha, \ldots, 3N+3$, are defined along the nominal solution by

$$\begin{cases} S_{2,2}(0) = 0, \quad S_{2,3}(0) = \dot{\psi}, \quad S_{3,3}(0) = \dot{\Omega}, \\ S_{2,3j+1}(0) = -S_{1,2}^{\mathrm{T}}(t_{N-j+1}^{*+})\bar{\psi}_{s_{N-j+1}}^{N-j+1}, \quad S_{2,3j+2}(0) = 0, \\ S_{2,3j+3}(0) = -S_{1,2}^{\mathrm{T}}(t_{N-j+1}^{*+})\dot{\bar{\psi}}^{N-j+1}, \\ S_{3,3j+1}(0) = -S_{1,3}^{\mathrm{T}}(t_{N-j+1}^{*+})\bar{\psi}_{s_{N-j+1}}^{N-j+1}, \quad S_{3,3j+2}(0) = 0, \\ S_{3,3j+3}(0) = -S_{1,3}^{\mathrm{T}}(t_{N-j+1}^{*+})\dot{\bar{\psi}}^{N-j+1}, \\ S_{3j+1,3j+1}(0) = \left(\bar{\psi}_{s_{N-j+1}}^{N-j+1}\right)^{\mathrm{T}} S_{1,1}(t_{N-j+1}^{*+})\bar{\psi}_{s_{N-j+1}}^{N-j+1} + \bar{\Phi}_{s_{N-j+1}s_{N-j+1}}^{N-j+1}, \\ S_{3j+1,3j+2}(0) = \left(\tilde{\psi}_{s_{N-j+1}}^{N-j+1}\right)^{\mathrm{T}} - \left(\tilde{\psi}_{x_{N-j+1}^+}^{N-j+1}\bar{\psi}_{s_{N-j+1}}^{N-j+1}\right)^{\mathrm{T}}, \\ S_{3j+1,3j+3}(0) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}_{s_{N-j+1}}^{N-j+1}\right)^{\mathrm{T}} + \left(\bar{\psi}_{s_{N-j+1}}^{N-j+1}\right)^{\mathrm{T}} S_{1,1}(t_{N-j+1}^{*+})\dot{\bar{\psi}}^{N-j+1}, \\ S_{3j+2,3j+2}(0) = 0, \quad S_{3j+2,3j+3}(0) = \dot{\tilde{\psi}}^{N-j+1} - \tilde{\psi}_{x_{N-j+1}^+}^{N-j+1}\dot{\bar{\psi}}^{N-j+1}, \\ S_{3j+3,3j+3}(0) = \kappa_{N-j+1} + \left(\dot{\bar{\psi}}^{N-j+1}\right)^{\mathrm{T}} S_{1,1}(t_{N-j+1}^{*+})\dot{\bar{\psi}}^{N-j+1}, \\ S_{3k+r,3l+1}(0) = -S_{1,3k+r}^{\mathrm{T}}(t_{N-l+1}^{*+})\bar{\psi}_{s_{N-l+1}}^{N-l+1}, \quad S_{3k+r,3l+2}(0) = 0, \\ S_{3k+r,3l+3}(0) = -S_{1,3k+r}^{\mathrm{T}}(t_{N-l+1}^{*+})\dot{\bar{\psi}}^{N-l+1} \end{cases} \tag{26}$$

for $j = 1, \ldots, N$, $k = 1, \ldots, N-1$, $l = k+1, \ldots, N$, and $r = 1, 2, 3$.

With the definitions (21)-(25), we now have the main theorem for Problem (NE).

Theorem 3.1. *Suppose that Assumptions 1 and 2 are satisfied, the perturbations δx^0, $\mathrm{d}\psi$ and $\mathrm{d}\gamma$ are known, and the symmetric matrix Υ defined in (25) is invertible. Then, the NE solution of Problem (NE) is given by*

$$\sigma \approx \left[s_1^{*\mathrm{T}} + \mathrm{d}s_1^{\mathrm{T}}, \dots, s_N^{*\mathrm{T}} + \mathrm{d}s_N^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \tau \approx \left[t_1^* + \mathrm{d}t_1, \dots, t_{N+1}^* + \mathrm{d}t_{N+1}\right]^{\mathrm{T}}, \tag{27}$$

where $\mathrm{d}s_i$, $i = 1, \dots, N$, and $\mathrm{d}t_i$, $i = 1, \dots, N+1$, are obtained from Θ_1 of (21) derived by

$$\begin{bmatrix} \Theta_1 \\ \mathrm{d}\tilde{\pi} \end{bmatrix} = -\Upsilon^{-1} \begin{bmatrix} -I_p & S_{1,2}^{\mathrm{T}}(0) & R_2(0) \\ 0 & S_{1,3}^{\mathrm{T}}(0) & R_3(0) \\ \vdots & \vdots & \vdots \\ 0 & S_{1,3N+3}^{\mathrm{T}}(0) & R_{3N+3}(0) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}\psi \\ \delta x^0 \\ \mathrm{d}\gamma \end{bmatrix}. \tag{28}$$

Proof. The backward sweep technique in [2] was developed to solve linear TPBVPs. Its idea is to construct an auxiliary differential system according to the terminal condition, and then integrate the auxiliary system backward to determine the differential of the control at the initial point. Now, we extend this technique to linear MPBVPs.

1) Boundary condition at $t = t_f^*$. Using equations in (14b), we can rearrange [2] equations (17a)-(17c) into

$$\begin{bmatrix} \delta\lambda(t_f^*) \\ \mathrm{d}\psi \\ -\tilde{\eta}_{t_f}^{\mathrm{T}}\mathrm{d}\tilde{\pi} \end{bmatrix} = \begin{bmatrix} \Phi_{xx} & \psi_x^{\mathrm{T}} & \Omega_x^{\mathrm{T}} \\ * & 0 & \dot{\psi} \\ * & * & \dot{\Omega} \end{bmatrix} \begin{bmatrix} \delta x(t_f^*) \\ \mathrm{d}\nu \\ \mathrm{d}t_f \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda^{*\mathrm{T}} f_\gamma^{N+1} \end{bmatrix} \mathrm{d}\gamma. \tag{29}$$

Let

$$\begin{bmatrix} \delta\lambda(t) \\ \mathrm{d}\psi \\ -\tilde{\eta}_{t_f}^{\mathrm{T}}\mathrm{d}\tilde{\pi} \end{bmatrix} = \begin{bmatrix} S_{1,1}(t) & S_{1,2}(t) & S_{1,3}(t) \\ * & S_{2,2}(t) & S_{2,3}(t) \\ * & * & S_{3,3}(t) \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \mathrm{d}\nu \\ \mathrm{d}t_f \end{bmatrix} + \begin{bmatrix} R_1(t) \\ R_2(t) \\ R_3(t) \end{bmatrix} \mathrm{d}\gamma \tag{30}$$

for $t \in (t_N^*, t_f^*)$. Then, equation (20) can be written as

$$\begin{aligned} \begin{bmatrix} \delta\dot{x}(t) \\ \delta\dot{\lambda}(t) \end{bmatrix} = & \begin{bmatrix} A^{N+1} & 0 & 0 \\ C^{N+1} - (A^{N+1})^{\mathrm{T}} S_{1,1} & -(A^{N+1})^{\mathrm{T}} S_{1,2} & -(A^{N+1})^{\mathrm{T}} S_{1,3} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \mathrm{d}\nu \\ \mathrm{d}t_f \end{bmatrix} \\ & + \begin{bmatrix} E^{N+1} \\ -(A^{N+1})^{\mathrm{T}} R_1 + F^{N+1} \end{bmatrix} \mathrm{d}\gamma. \end{aligned} \tag{31}$$

Now, differentiate equation (30) with respect to t, treating $\mathrm{d}\psi$, $-\tilde{\eta}_{t_f}^{\mathrm{T}}\mathrm{d}\tilde{\pi}$, $\mathrm{d}\nu$, $\mathrm{d}t_f$ and $\mathrm{d}\gamma$ as constants. Doing this yields

$$\begin{bmatrix} \delta\dot{\lambda} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{S}_{1,1} & \dot{S}_{1,2} & \dot{S}_{1,3} \\ * & \dot{S}_{2,2} & \dot{S}_{2,3} \\ * & * & \dot{S}_{3,3} \end{bmatrix} \begin{bmatrix} \delta x \\ \mathrm{d}\nu \\ \mathrm{d}t_f \end{bmatrix} + \begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ * & S_{2,2} & S_{2,3} \\ * & * & S_{3,3} \end{bmatrix} \begin{bmatrix} \delta\dot{x} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{R}_1 \\ \dot{R}_2 \\ \dot{R}_3 \end{bmatrix} \mathrm{d}\gamma. \tag{32}$$

Then, by substituting $\delta\dot{x}$ and $\delta\dot{\lambda}$ obtained from equation (31) into (32), it follows that

$$\begin{bmatrix} \dot{S}_{1,1}+S_{1,1}A^{N+1}+(A^{N+1})^{\mathrm{T}}S_{1,1}-C^{N+1} & \dot{S}_{1,2}+(A^{N+1})^{\mathrm{T}}S_{1,2} & \dot{S}_{1,3}+(A^{N+1})^{\mathrm{T}}S_{1,3} \\ * & \dot{S}_{2,2} & \dot{S}_{2,3} \\ * & * & \dot{S}_{3,3} \end{bmatrix}\begin{bmatrix} \delta x(t) \\ \mathrm{d}\nu \\ \mathrm{d}t_f \end{bmatrix} + \begin{bmatrix} \dot{R}_1+S_{1,1}E^{N+1}+(A^{N+1})^{\mathrm{T}}R_1-F^{N+1} \\ \dot{R}_2+S_{1,2}^{\mathrm{T}}E^{N+1} \\ \dot{R}_3+S_{1,3}^{\mathrm{T}}E^{N+1} \end{bmatrix}\mathrm{d}\gamma=0. \tag{33}$$

To keep this identity valid for arbitrary $\delta x(t)$, $\mathrm{d}\nu$, $\mathrm{d}t_f$ and $\mathrm{d}\gamma$, it is necessary that

$$\begin{cases} \dot{S}_{1,1}=-S_{1,1}A^{N+1}-(A^{N+1})^{\mathrm{T}}S_{1,1}+C^{N+1}, \\ \dot{S}_{1,2}=-(A^{N+1})^{\mathrm{T}}S_{1,2}, \quad \dot{S}_{1,3}=-(A^{N+1})^{\mathrm{T}}S_{1,3}, \\ \dot{S}_{2,2}=0, \quad \dot{S}_{2,3}=0, \quad \dot{S}_{3,3}=0, \\ \dot{R}_1=-S_{1,1}E^{N+1}-(A^{N+1})^{\mathrm{T}}R_1+F^{N+1}, \\ \dot{R}_2=-S_{1,2}^{\mathrm{T}}E^{N+1}, \quad \dot{R}_3=-S_{1,3}^{\mathrm{T}}E^{N+1}, \end{cases} \quad t\in(t_N^*,t_f^*). \tag{34}$$

and

$$\begin{cases} S_{1,1}(t_f^*)=\Phi_{xx}, & S_{1,2}(t_f^*)=\psi_x^{\mathrm{T}}, & S_{1,3}(t_f^*)=\Omega_x^{\mathrm{T}}, \\ S_{2,2}(t_f^*)=0, & S_{2,3}(t_f^*)=\dot{\psi}, & S_{3,3}(t_f^*)=\dot{\Omega}, \\ R_1(t_f^*)=0, & R_2(t_f^*)=0, & R_3(t_f^*)=\lambda^{*\mathrm{T}}(t_f^*)f_\gamma^{N+1}. \end{cases} \tag{35}$$

2) Boundary conditions at $t=t_i^*$, $i=N,\ldots,1$. From equations (14c) and

$$\bar{\psi}^i_{x_i^+}=I_n,\ \bar{\Phi}^i_{x_i^+x_i^+}=0,\ \bar{\Phi}^i_{x_i^+x_i^-}=\bar{\Phi}^{i\mathrm{T}}_{x_i^-x_i^+}=0,\ \bar{\Phi}^i_{x_i^+s_i}=\bar{\Phi}^{i\mathrm{T}}_{s_ix_i^+}=0, \quad i=N,\ldots,1,$$

which can be derived from (4), we can rearrange equations (18a)-(18f) into

$$\begin{bmatrix} 0 \\ \delta\lambda(t_i^{*-}) \\ 0 \\ 0 \\ 0 \\ -\tilde{\eta}_{t_i}^{\mathrm{T}}\mathrm{d}\tilde{\pi} \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 & 0 & I_n & \tilde{\psi}^{i\mathrm{T}}_{x_i^+} & 0 \\ 0 & 0 & \bar{\Phi}^i_{x_i^-x_i^-} & \bar{\Phi}^i_{s_ix_i^-} & \bar{\psi}^{i\mathrm{T}}_{x_i^-} & \tilde{\psi}^{i\mathrm{T}}_{x_i^-} & 0 \\ 0 & 0 & \bar{\Phi}^i_{x_i^-s_i} & \bar{\Phi}^i_{s_is_i} & \bar{\psi}^{i\mathrm{T}}_{s_i} & \tilde{\psi}^{i\mathrm{T}}_{s_i} & \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^{i\mathrm{T}}_{s_i}\right) \\ 0 & I_n & \bar{\psi}^i_{x_i^-} & \bar{\psi}^i_{s_i} & 0 & 0 & \dot{\bar{\psi}}^i \\ 0 & \tilde{\psi}^i_{x_i^+} & \tilde{\psi}^i_{x_i^-} & \tilde{\psi}^i_{s_i} & 0 & 0 & \dot{\tilde{\psi}}^i \\ 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^i_{s_i}\right) & \dot{\bar{\psi}}^{i\mathrm{T}} & \dot{\tilde{\psi}}^{i\mathrm{T}} & \kappa_i \end{bmatrix} \times \begin{bmatrix} \delta\lambda(t_i^{*+}) \\ \delta x(t_i^{*+}) \\ \delta x(t_i^{*-}) \\ \mathrm{d}s_i \\ \mathrm{d}\bar{\nu}_i \\ \mathrm{d}\tilde{\nu}_i \\ \mathrm{d}t_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \epsilon_i \end{bmatrix}\mathrm{d}\gamma. \tag{36}$$

Then, from the fourth block row of (36), we have

$$\delta x(t_i^{*+})=-\bar{\psi}^i_{x_i^-}\delta x(t_i^{*-})-\bar{\psi}^i_{s_i}\mathrm{d}s_i-\dot{\bar{\psi}}^i\mathrm{d}t_i. \tag{37}$$

Note that the next boundary point is at $t = t_{i+1}^*$ as time increases. Let

$$\begin{aligned}
\delta\lambda(t_i^{*+}) \triangleq{}& S_{1,1}\left(t_i^{*+}\right)\delta x(t_i^{*+}) + S_{1,2}\left(t_i^{*+}\right)\mathrm{d}\nu + S_{1,3}\left(t_i^{*+}\right)\mathrm{d}t_f \\
&+ \sum_{j=1}^{N-i}\left[S_{1,3j+1}\left(t_i^{*+}\right)\mathrm{d}s_{N-j+1} + S_{1,3j+2}\left(t_i^{*+}\right)\mathrm{d}\tilde{\nu}_{N-j+1}\right. \\
&\left.+ S_{1,3j+3}\left(t_i^{*+}\right)\mathrm{d}t_{N-j+1}\right] + R_1(t_i^{*+})\mathrm{d}\gamma, \\
={}& -S_{1,1}\left(t_i^{*+}\right)\bar{\psi}^i_{x_i^-}\delta x(t_i^{*-}) + S_{1,2}\left(t_i^{*+}\right)\mathrm{d}\nu + S_{1,3}\left(t_i^{*+}\right)\mathrm{d}t_f \\
&+ \sum_{j=1}^{N-i}\left[S_{1,3j+1}\left(t_i^{*+}\right)\mathrm{d}s_{N-j+1} + S_{1,3j+2}\left(t_i^{*+}\right)\mathrm{d}\tilde{\nu}_{N-j+1}\right. \\
&\left.+ S_{1,3j+3}\left(t_i^{*+}\right)\mathrm{d}t_{N-j+1}\right] \\
&- S_{1,1}\left(t_i^{*+}\right)\bar{\psi}^i_{s_i}\mathrm{d}s_i - S_{1,1}\left(t_i^{*+}\right)\dot{\bar{\psi}}^i\mathrm{d}t_i + R_1(t_i^{*+})\mathrm{d}\gamma. \qquad (38)
\end{aligned}$$

Using equations (37) and (38), we can solve for $\mathrm{d}\bar{\nu}_i$ from the first block row of (36). Specifically,

$$\begin{aligned}
\mathrm{d}\bar{\nu}_i ={}& S_{1,1}\left(t_i^{*+}\right)\bar{\psi}^i_{x_i^-}\delta x(t_i^{*-}) - S_{1,2}\left(t_i^{*+}\right)\mathrm{d}\nu - S_{1,3}\left(t_i^{*+}\right)\mathrm{d}t_f \\
&- \sum_{j=1}^{N-i}\left[S_{1,3j+1}\left(t_i^{*+}\right)\mathrm{d}s_{N-j+1} + S_{1,3j+2}\left(t_i^{*+}\right)\mathrm{d}\tilde{\nu}_{N-j+1}\right. \\
&\left.+S_{1,3j+3}\left(t_i^{*+}\right)\mathrm{d}t_{N-j+1}\right] + S_{1,1}\left(t_i^{*+}\right)\bar{\psi}^i_{s_i}\mathrm{d}s_i - \tilde{\psi}^{i\mathrm{T}}_{x_i^+}\mathrm{d}\tilde{\nu}_i \\
&+ S_{1,1}\left(t_i^{*+}\right)\dot{\bar{\psi}}^i\mathrm{d}t_i - R_1(t_i^{*+})\mathrm{d}\gamma. \qquad (39)
\end{aligned}$$

In equations (38) and (39), summations that have upper limits less than lower limits are defined to be zero. Now, incorporate the following equation into (36),

$$\begin{aligned}
\Psi_{i+1} ={}& \begin{bmatrix} S_{2,1}(t_i^{*+}) & \dots & S_{2,3(N-i+1)}(t_i^{*+}) \\ \vdots & \ddots & \vdots \\ S_{3(N-i+1),1}(t_i^{*+}) & \dots & S_{3(N-i+1),3(N-i+1)}(t_i^{*+}) \end{bmatrix} \begin{bmatrix} \delta x(t_i^{*+}) \\ \Theta_{i+1} \end{bmatrix} \\
&+ \begin{bmatrix} R_2(t_i^{*+}) \\ \vdots \\ R_{3(N-i+1)}(t_i^{*+}) \end{bmatrix}\mathrm{d}\gamma, \qquad (40)
\end{aligned}$$

which reduces to the second and third block rows in (30) with $t = t_N^{*+}$ when $i = N$. After eliminating $\delta x(t_i^{*+})$, $\delta\lambda(t_i^{*+})$ and $\mathrm{d}\bar{\nu}_i$ by, respectively, equations (37), (38) and (39), the expanded equation (36) becomes

$$\begin{bmatrix} \delta\lambda(t_i^{*-}) \\ \Psi_i \end{bmatrix} = \Gamma^i \begin{bmatrix} \delta x(t_i^{*-}) \\ \Theta_i \end{bmatrix} + \Lambda^i\mathrm{d}\gamma, \qquad (41)$$

where Γ^i is a block symmetric matrix with $3(N-i+2)\times 3(N-i+2)$ blocks and Λ^i is a block matrix with $3(N-i+2)\times 1$ blocks. The blocks of both Γ^i and Λ^i are defined in Appendix A.

Similar to (30), we assume that

$$\begin{bmatrix} \delta\lambda(t) \\ \Psi_i \end{bmatrix} = S^i(t) \begin{bmatrix} \delta x(t) \\ \Theta_i \end{bmatrix} + R^i(t)\mathrm{d}\gamma, \quad t \in (t_{i-1}^*, t_i^*), \qquad (42)$$

where $S^i(t)$ is a $3(N-i+2)\times 3(N-i+2)$ block symmetric matrix with its (α,β) term, $\alpha,\beta \in \{1,\ldots,3(N-i+2)\}$, denoted by $S_{\alpha,\beta}$, and $R^i(t)$ is a $3(N-i+2)\times 1$ block matrix with its $\alpha \in \{1,\ldots,3(N-i+2)\}$ block denoted by R_α. Using a similar argument as that given to obtain (34) and (35), we can show that the following differential equations are valid:

$$\begin{cases} \dot{S}_{1,1} = -S_{1,1}A^i - A^{i\mathrm{T}}S_{1,1} + C^i, & \dot{S}_{1,j} = -A^{i\mathrm{T}}S_{1,j}, \quad \dot{S}_{j,k} = 0, \\ \dot{R}_1 = -S_{1,1}E^i - A^{i\mathrm{T}}R_1 + F^i, & \dot{R}_j = -S_{1,j}^{\mathrm{T}}E^i, \end{cases} \quad t \in (t_{i-1}^*, t_i^*), \tag{43}$$

where $i = N,\ldots,1$, $j = 2,\ldots,3(N-i+2)$, and $k = j,\ldots,3(N-i+2)$, with boundary conditions

$$\begin{cases} S_{\alpha,\beta}(t_i^{*-}) = \Gamma_{\alpha,\beta}^i, \\ R_\alpha(t_i^{*-}) = \Lambda_\alpha^i, \end{cases} \quad \alpha = 1,\ldots,3(N-i+2), \quad \beta = \alpha,\ldots,3(N-i+2). \tag{44}$$

Now, equations (34) and (43) with boundary conditions (35) and (44), $i = N,\ldots,1$, form the two groups of switched impulsive systems (23) and (24), which can be integrated backward with time from $t = t_f^*$ to $t = t_0^*$ and subsystem index i from $N+1$ to 1 to obtain $S_{1,\alpha}(t_0^*)$ and $R_\alpha(t_0^*)$, $\alpha = 1,\ldots,3N+3$. The other terms $S_{\alpha,\beta}(t_0^*)$, where $\alpha = 2,\ldots,3N+3$, and $\beta = \alpha,\ldots,3N+3$, can be obtained by equations in (26) at $t = t_i^*$, $i = N+1,\ldots,1$. Then, it follows from equations (42) (with $i = 1$ and $t_0^* = 0$) and (19) that the differentials of the parameters, i.e., Θ_1 and $\mathrm{d}\tilde{\pi}$, can be solved from (28) for given $\mathrm{d}\psi$, δx^0 and $\mathrm{d}\gamma$ if the symmetric matrix Υ is invertible. Then, (σ,τ) can be derived by (27). □

Remark 2. The results in [13, 14] demonstrate that the invertibility of Υ is equivalent to certain controllability of the linear system (16a) with perturbed boundary conditions (17b), (18d), (18e) and (19). In practice, the NE solution of Theorem 3.1 can be computed in a numerically stable way without explicitly inverting Υ by using the triangular or orthogonal decomposition of Υ.

The procedure to compute the NE solution to Problem (NE) can be summarized as follows.

Algorithm.

In the offline stage:

Step 1. With respect to the nominal values x^0 and γ^*, and the nominal terminal condition (3), solve Problem (N) by certain computational methods like that of [9] to obtain the nominal control pair (σ^*,τ^*). The active status of the inequality constraints (6) is known.

Step 2. The extremal state $x^*(t)$ for $t \in [0,t_f^*]$ can be obtained by integrating (9a) forward with the control pair (σ^*,τ^*) and boundary conditions (11d).

Step 3. The differential equation (9b) and equations (10a)-(10c), (11a)-(11d), (11g) and the active parts of (11e) and (12a) form a MPBVP from which the costate $\lambda^*(t)$ for $t \in [0,t_f^*]$ and the parameters ν^*, $\tilde{\pi}^*$, and $\bar{\nu}_i^*$ and $\tilde{\nu}_i^*$, $i = 1,\ldots,N$, can be determined. The components of π^* and $\hat{\nu}_i^*$, $i = 1,\ldots,N$, for the inactive inequality constraints are set to zero.

Step 4. Solve switched impulsive systems (23) and (26), and switched impulsive systems (24) backward with time from $t = t_f^*$ to $t = t_0$ and subsystem index from $i = N+1$ to $i = 1$ to obtain $S_{\alpha,\beta}(0)$ and $R_\alpha(0)$, where

$\alpha, \beta = 1, \ldots, 3N+3$. Then, the Jacobian of the control (σ, τ) and multipliers ν, $\tilde{\pi}$ and $\tilde{\nu}_i$, $i = 1, \ldots, N$, with respect to the initial state, terminal condition and system's uncertain parameters, which is defined in (28), is obtained.

In the online stage:

Step 5. When the perturbations δx^0, $\mathrm{d}\psi$ and $\mathrm{d}\gamma$ are available, the corrections of the control $\mathrm{d}s_i$, $i = 1, \ldots, N$, and $\mathrm{d}t_i$, $i = 1, \ldots, N+1$ can be computed from (28). The NE solution for Problem (NE) can be derived by (27).

4. **Numerical simulations.** To verify our NE solution method, we present two example problems in this section. One is the optimal shrimp harvesting problem discussed in [9], and the other is the optimal impulsive control problem discussed in [10, 9].

Example 1. Optimal shrimp harvesting. Let $x_1(t)$ be the number of shrimp at time t, and let $x_2(t)$ be the average weight of shrimp (in grams) at time t, where t is measured in weeks. The shrimp population growth can be modeled by the dynamics,

$$\begin{cases} \dot{x}_1(t) = -0.03x_1(t), & x_1(0) = 40000, \\ \dot{x}_2(t) = 3.5 - \gamma x_1(t)x_2(t), & x_2(0) = 1, \end{cases} \tag{45}$$

where $\gamma = 0.00001$. Suppose that shrimp are harvested at times $t = t_i$, $i = 1, \ldots, N$, and s_i is the fraction of the total shrimp stock harvested at $t = t_i$. Then, we have the following jump conditions at each time $t = t_i$:

$$\begin{cases} x_1(t_i^+) = (1 - s_i)x_1(t_i^-), \\ x_2(t_i^+) = x_2(t_i^-), \end{cases} \quad i = 1, \ldots, N. \tag{46}$$

The revenue obtained by harvesting a fraction s_i of the shrimp stock is given by

$$px_2(t_i^-)s_i x_1(t_i^-) - h,$$

where $p \triangleq \$0.008$ is the price per gram of shrimp and $h = 50$ is the fixed cost of harvesting. At the specified final time $t = t_f = 13.2$, all the remaining shrimp will be harvested. The first N harvesting times $\tau \triangleq [t_1, \ldots, t_N]$ and fractions $\sigma \triangleq [s_1, \ldots, s_N]$ are subject to the following constraints:

$$\begin{cases} 0.01 \le s_i \le 1, & i = 1, \ldots, N, \\ t_i - t_{i-1} \ge 0.01, & i = 1, \ldots, N+1, \ t_0 = 0, \ t_{N+1} = t_f. \end{cases} \tag{47}$$

The problem is to choose τ and σ to maximize the total revenue

$$R(\sigma, \tau) = \sum_{i=1}^{N} \left(px_2(t_i^-)s_i x_1(t_i^-) - h \right) + px_2(t_f)x_1(t_f) - h$$

subject to the constraints (47).

Table 1 presents the nominal solutions for $N = 1, 2, 3$, which are computed by the computational method in [9].

TABLE 1. Nominal solutions of Example 1

N	Solution	Revenue
1	$s_1^* = 0.584$, $t_1^* = 5.330$	3128
2	$s_1^* = 0.388$, $t_1^* = 4.270$	3189
	$s_2^* = 0.454$, $t_2^* = 7.810$	
3	$s_1^* = 0.289$, $t_1^* = 3.854$	3172
	$s_2^* = 0.323$, $t_2^* = 6.120$	
	$s_3^* = 0.374$, $t_3^* = 9.110$	

Example 2. Optimal control for a nonlinear impulsive system. In this example, the following impulsive system is considered:

$$\dot{x}_1 = \begin{cases} 0.01x_1^2 + 2.02x_1x_2 - 0.99x_2^2 - 2x_1 + 4x_2 + 1, & \text{if } 0 < t < 1.8, \\ 1.01x_1^2 + 0.02x_1x_2 + 0.01x_2^2 - 2x_1 + 4x_2 + 1, & \text{if } 1.8 < t < 2, \end{cases} \tag{48a}$$

$$\dot{x}_2 = \begin{cases} 0.01x_1x_2 + 1.01x_2^2 + 1.01x_1x_3 - 0.99x_2x_3 - 3x_1 - x_2 + 2x_3 + 1, \\ \qquad \text{if } 0 < t < 1.8, \\ 1.01x_1x_2 + 0.01x_2^2 + 0.01x_1x_3 + 0.01x_2x_3 - 3x_1 - x_2 + 2x_3 + 1, \\ \qquad \text{if } 1.8 < t < 2, \end{cases} \tag{48b}$$

$$\dot{x}_3 = \begin{cases} 0.01x_2^2 + 2.02x_2x_3 - 0.99x_3^2 - 6x_2 + 1, & \text{if } 0 < t < 1.8, \\ 1.01x_2^2 + 0.02x_2x_3 + 0.01x_3^2 - 6x_2 + 1, & \text{if } 1.8 < t < 2 \end{cases} \tag{48c}$$

with

$$x_1(0) = 0.1, \quad x_2(0) = 0, \quad x_3(0) = 25, \tag{49a}$$

$$x_1(t_i^+) = \begin{cases} \frac{4x_1(t_i^-) + x_1(t_i^-)x_3(t_i^-) - x_2^2(t_i^-)}{4x_1(t_i^-) - 4x_2(t_i^-) + x_3(t_i^-) + 4}, & \text{if } i = 1, \dots, N-1, \\ x_1(t_i^-), & \text{if } i = N, \end{cases} \tag{49b}$$

$$x_2(t_i^+) = \begin{cases} \frac{4x_2(t_i^-) + 2x_1(t_i^-)x_3(t_i^-) - 2x_2^2(t_i^-)}{4x_1(t_i^-) - 4x_2(t_i^-) + x_3(t_i^-) + 4}, & \text{if } i = 1, \dots, N-1, \\ x_2(t_i^-), & \text{if } i = N, \end{cases} \tag{49c}$$

$$x_3(t_i^+) = \begin{cases} \frac{4x_3(t_i^-) + x_1(t_i^-)x_3(t_i^-) - x_2^2(t_i^-)}{4x_1(t_i^-) - 4x_2(t_i^-) + x_3(t_i^-) + 4}, & \text{if } i = 1, \dots, N-1, \\ x_3(t_i^-), & \text{if } i = N, \end{cases} \tag{49d}$$

where the N switching times, $t_1, \dots, t_N$, satisfy

$$0 = t_0 < t_1 < \cdots < t_N = 1.8.$$

Furthermore, it is assumed that

$$t_i - t_{i-1} \geq 0.1, \qquad i = 1, \dots, N. \tag{50}$$

The problem is to choose the switching times, $\tau \triangleq \{t_1, \dots, t_{N-1}\}$, to minimize the cost function

$$J(\tau) = x_1^2(2) + 2x_2^2(2) + x_3^2(2)$$

subject to the dynamics (48), the boundary conditions (49), and the constraints (50).

This optimal control problem has the following nominal solution for $N = 3$ [9]:

$$t_1^* = 1.0972, \quad t_2^* = 1.7000, \quad J^* = 0.6844,$$

which satisfies

$$t_3^* - t_2^* = 0.1 \tag{51}$$

Hence, there is an active constraint along this nominal solution.

Now, we consider three simulation cases. Cases 1 and 2 are for problems without parametric perturbation, while Case 3 considers the situation where parametric perturbation is presented.

4.1. **Case 1.** In this case, we consider the optimal shrimp harvesting problem in Example 1. The initial state is perturbed for $N = 1, 2, 3$, and our method in Theorem 3.1 is used to compute the corresponding NE solutions. Table 2 presents results for four different perturbations in the initial conditions. For each perturbation, we show the nominal solution for the unperturbed system, the NE solution for the perturbed system, and the optimal open-loop solution for the perturbed system. We also give the revenue of each solution for the perturbed system.

TABLE 2. Simulation results of Case 1

No.	Initial perturbation	Nominal solution	NE solution	Optimal solution
1	$\delta x^0 = [-4000, 0.1]^{\mathrm{T}}$	$s_1^* = 0.584$	$s_1 = 0.566$	$s_1^{\mathrm{op}} = 0.565$
		$t_1^* = 5.330$	$t_1 = 5.545$	$t_1^{\mathrm{op}} = 5.558$
	Revenue	3001	3003	3003
2	$\delta x^0 = -[8000, 0.2]^{\mathrm{T}}$	$s_1^* = 0.388$	$s_1 = 0.359$	$s_1^{\mathrm{op}} = 0.355$
		$s_2^* = 0.454$	$s_2 = 0.420$	$s_2^{\mathrm{op}} = 0.414$
		$t_1^* = 4.270$	$t_1 = 4.814$	$t_1^{\mathrm{op}} = 4.902$
		$t_2^* = 7.810$	$t_2 = 8.245$	$t_2^{\mathrm{op}} = 8.298$
	Revenue	2858	2871	2871
3	$\delta x^0 = [10000, -0.25]^{\mathrm{T}}$	$s_1^* = 0.289$	$s_1 = 0.316$	$s_1^{\mathrm{op}} = 0.313$
		$s_2^* = 0.323$	$s_2 = 0.354$	$s_2^{\mathrm{op}} = 0.349$
		$s_3^* = 0.374$	$s_3 = 0.411$	$s_3^{\mathrm{op}} = 0.406$
		$t_1^* = 3.854$	$t_1 = 3.324$	$t_1^{\mathrm{op}} = 3.424$
		$t_2^* = 6.120$	$t_2 = 5.616$	$t_2^{\mathrm{op}} = 5.699$
		$t_3^* = 9.110$	$t_3 = 8.743$	$t_3^{\mathrm{op}} = 8.795$
	Revenue	3487	3499	3499
4	$\delta x^0 = [12000, 0.3]^{\mathrm{T}}$	$s_1^* = 0.289$	$s_1 = 0.323$	$s_1^{\mathrm{op}} = 0.319$
		$s_2^* = 0.323$	$s_2 = 0.362$	$s_2^{\mathrm{op}} = 0.356$
		$s_3^* = 0.374$	$s_3 = 0.422$	$s_3^{\mathrm{op}} = 0.415$
		$t_1^* = 3.854$	$t_1 = 3.034$	$t_1^{\mathrm{op}} = 3.164$
		$t_2^* = 6.120$	$t_2 = 5.366$	$t_2^{\mathrm{op}} = 5.471$
		$t_3^* = 9.110$	$t_3 = 8.576$	$t_3^{\mathrm{op}} = 8.639$
	Revenue	3556	3579	3580

As seen from Table 2, the difference between the revenue of the nominal solution and that of the optimal one enlarges with the increasing initial perturbation. On the contrary, the revenue of the NE solution is equal to the optimal one in the first three situations where the perturbation is less or equal to 25% of the corresponding nominal value. Clearly, our NE solutions approximate the optimal ones well in the presence of the initial state perturbation.

However, all of the constraints (47) in this example are inactive along the nominal solutions. We next consider Example 2 to verify our method in the case where some constraints are active.

4.2. **Case 2.** Along the nominal solution of Example 2 with $N = 3$, we compute the NE solutions using the method in Theorem 3.1 again. Simulation results are listed in Table 3.

TABLE 3. Simulation results of Case 2

No.	Initial perturbation	Nominal solution	NE solution	Optimal solution
1	$\delta x^0 = [-0.2, 1, 10]^{\mathrm{T}}$	$t_1^* = 1.0972$	$t_1 = 1.0482$	$t_1^{\mathrm{op}} = 1.0410$
		$t_2^* = 1.7000$	$t_2 = 1.7000$	$t_2^{\mathrm{op}} = 1.7000$
	Cost	0.6961	0.6939	0.6939
2	$\delta x^0 = [0.2, -1, -10]^{\mathrm{T}}$	$t_1^* = 1.0972$	$t_1 = 1.1462$	$t_1^{\mathrm{op}} = 1.1447$
		$t_2^* = 1.7000$	$t_2 = 1.7000$	$t_2^{\mathrm{op}} = 1.7000$
	Cost	0.6721	0.6702	0.6702
3	$\delta x^0 = [-0.2, 1, -10]^{\mathrm{T}}$	$t_1^* = 1.0972$	$t_1 = 1.0759$	$t_1^{\mathrm{op}} = 1.0506$
		$t_2^* = 1.7000$	$t_2 = 1.7000$	$t_2^{\mathrm{op}} = 1.7000$
	Cost	0.6796	0.6787	0.6783

In Table 3, the definitions of the nominal solution, the NE solution, and the optimal solution are the same as those in Table 2. We see that Assumption 2 is satisfied in all situations since each t_2^{op} in the optimal solutions is equal to 1.7000. The computed NE solutions are also satisfied as each t_2 computed is equal to 1.7000 and their corresponding costs are close to the optimal ones.

Until now, all the system's models in the simulations are accurate. In the following case, we will consider the situation where some of the system's parameters are perturbed.

4.3. **Case 3.** Revisit Example 1 again. Suppose that the model parameter γ in the system (45) is uncertain. Its nominal value is $\gamma^* = 0.00001$. Along the nominal trajectory for $N = 2, 3$, the NE solutions in the presence of parametric perturbation can be computed by our method in Theorem 3.1. Simulation results are presented in Table 4.

In the first two situations, there are only parametric perturbations. The NE solutions computed by the method in Theorem 3.1 approximate the optimal ones well in the presence of $\pm 20\%$ parametric perturbations. In the next two simulations, initial state perturbations are also presented. We can see that the NE solutions computed by our method have revenues close to those of the optimal solutions, while the revenues of the nominal solutions are much less in comparison.

5. **Conclusion.** In this paper, we have developed a NE solution for a class of switched impulsive systems with constraints on the switching times and parameters. The illustrated examples show that our NE solutions adapt to perturbations in the initial state, terminal condition and system's parameters. Further work would be of considerable importance if some of the assumptions can be relaxed.

Appendix A. The block terms of the symmetric matrix Γ^i and the matrix Λ^i are defined as:

$$\Gamma^i_{1,1} \triangleq \bar{\psi}^{i\mathrm{T}}_{x_i^-} S_{1,1}(t_i^{*+}) \bar{\psi}^i_{x_i^-} + \bar{\Phi}^i_{x_i^- x_i^-}, \ \Gamma^i_{1,2} \triangleq -\bar{\psi}^{i\mathrm{T}}_{x_i^-} S_{1,2}(t_i^{*+}), \ \Gamma^i_{1,3} \triangleq -\bar{\psi}^{i\mathrm{T}}_{x_i^-} S_{1,3}(t_i^{*+}),$$

$$\Gamma^i_{2,2} \triangleq S_{2,2}(t_i^{*+}), \quad \Gamma^i_{2,3} \triangleq S_{2,3}(t_i^{*+}), \quad \Gamma^i_{3,3} \triangleq S_{3,3}(t_i^{*+}),$$

TABLE 4. Simulation results of Case 3

No.	Perturbation	Nominal solution	NE solution	Optimal solution
1	$d\gamma = -2 \times 10^{-6}$	$s_1^* = 0.388$	$s_1 = 0.360$	$s_1^{op} = 0.356$
		$s_2^* = 0.454$	$s_2 = 0.421$	$s_2^{op} = 0.416$
		$t_1^* = 4.270$	$t_1 = 4.755$	$t_1^{op} = 4.846$
		$t_2^* = 7.810$	$t_2 = 8.205$	$t_2^{op} = 8.261$
	Revenue	3619	3634	3634
2	$d\gamma = 2 \times 10^{-6}$	$s_1^* = 0.289$	$s_1 = 0.311$	$s_1^{op} = 0.309$
		$s_2^* = 0.323$	$s_2 = 0.348$	$s_2^{op} = 0.345$
		$s_3^* = 0.374$	$s_3 = 0.405$	$s_3^{op} = 0.401$
		$t_1^* = 3.854$	$t_1 = 3.369$	$t_1^{op} = 3.434$
		$t_2^* = 6.120$	$t_2 = 5.667$	$t_2^{op} = 5.720$
		$t_3^* = 9.110$	$t_3 = 8.785$	$t_3^{op} = 8.818$
	Revenue	2831	2838	2838
3	$d\gamma = -2 \times 10^{-6}$	$s_1^* = 0.388$	$s_1 = 0.330$	$s_1^{op} = 0.322$
	$\delta x^0 = -[8000, 0.2]^{\mathrm{T}}$	$s_2^* = 0.454$	$s_2 = 0.386$	$s_2^{op} = 0.374$
		$t_1^* = 4.270$	$t_1 = 5.299$	$t_1^{op} = 5.547$
		$t_2^* = 7.810$	$t_2 = 8.641$	$t_2^{op} = 8.773$
	Revenue	3193	3248	3249
4	$d\gamma = 2 \times 10^{-6}$	$s_1^* = 0.289$	$s_1 = 0.334$	$s_1^{op} = 0.330$
	$\delta x^0 = [8000, 0.2]^{\mathrm{T}}$	$s_2^* = 0.323$	$s_2 = 0.374$	$s_2^{op} = 0.369$
		$s_3^* = 0.374$	$s_3 = 0.437$	$s_3^{op} = 0.429$
		$t_1^* = 3.854$	$t_1 = 2.822$	$t_1^{op} = 2.989$
		$t_2^* = 6.120$	$t_2 = 5.164$	$t_2^{op} = 5.290$
		$t_3^* = 9.110$	$t_3 = 8.429$	$t_3^{op} = 8.496$
	Revenue	3043	3077	3078

$$\Gamma_{1,3j+1}^i \triangleq \begin{cases} \bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,1}(t_i^{*+}) \bar{\psi}_{s_i}^i + \bar{\Phi}_{s_i x_i^-}^i, & \text{if } i = N-j+1, \\ -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,3j+1}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{1,3j+2}^i \triangleq \begin{cases} \tilde{\psi}_{x_i^-}^{i\mathrm{T}} - \left(\tilde{\psi}_{x_i^+}^i \bar{\psi}_{x_i^-}^i\right)^{\mathrm{T}}, & \text{if } i = N-j+1, \\ -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,3j+2}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{1,3j+3}^i \triangleq \begin{cases} \bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,1}(t_i^{*+}) \dot{\bar{\psi}}^i, & \text{if } i = N-j+1, \\ -\bar{\psi}_{x_i^-}^{i\mathrm{T}} S_{1,3j+3}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{2,3j+1}^i \triangleq \begin{cases} -S_{1,2}^{\mathrm{T}}(t_i^{*+}) \bar{\psi}_{s_i}^i, & \text{if } i = N-j+1, \\ S_{2,3j+1}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{2,3j+2}^i \triangleq \begin{cases} 0, & \text{if } i = N-j+1, \\ S_{2,3j+2}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{2,3j+3}^i \triangleq \begin{cases} -S_{1,2}^{\mathrm{T}}(t_i^{*+}) \dot{\bar{\psi}}^i, & \text{if } i = N-j+1, \\ S_{2,3j+3}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{3,3j+1}^i \triangleq \begin{cases} -S_{1,3}^{\mathrm{T}}(t_i^{*+}) \bar{\psi}_{s_i}^i, & \text{if } i = N-j+1, \\ S_{3,3j+1}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{3,3j+2}^i \triangleq \begin{cases} 0, & \text{if } i = N-j+1, \\ S_{3,3j+2}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma_{3,3j+3}^i \triangleq \begin{cases} -S_{1,3}^{\mathrm{T}}(t_i^{*+}) \dot{\bar{\psi}}^i, & \text{if } i = N-j+1, \\ S_{3,3j+3}(t_i^{*+}), & \text{if } i = N-j, \dots, 1, \end{cases}$$

$$\Gamma^i_{3j+1,3j+1} \triangleq \begin{cases} \bar{\psi}^{i\mathrm{T}}_{s_i} S_{1,1}(t_i^{*+})\bar{\psi}^i_{s_i} + \bar{\Phi}^i_{s_i s_i}, & \text{if } i = N-j+1, \\ S_{3j+1,3j+1}(t_i^{*+}), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j+1,3j+2} \triangleq \begin{cases} \tilde{\psi}^{i\mathrm{T}}_{s_i} - \left(\tilde{\psi}^i_{x_i^+}\bar{\psi}^i_{s_i}\right)^{\mathrm{T}}, & \text{if } i = N-j+1, \\ S_{3j+1,3j+2}(t_i^{*+}), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j+1,3j+3} \triangleq \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\Phi}^{i\mathrm{T}}_{s_i}\right) + \bar{\psi}^{i\mathrm{T}}_{s_i} S_{1,1}(t_i^{*+})\dot{\bar{\psi}}^i, & \text{if } i = N-j+1, \\ S_{3j+1,3j+3}(t_i^{*+}), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j+2,3j+2} \triangleq \begin{cases} 0, & \text{if } i = N-j+1, \\ S_{3j+2,3j+2}(t_i^{*+}), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j+2,3j+3} \triangleq \begin{cases} \dot{\tilde{\psi}}^i - \tilde{\psi}^i_{x_i^+}\dot{\bar{\psi}}^i, & \text{if } i = N-j+1, \\ S_{3j+2,3j+3}(t_i^{*+}), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3j+3,3j+3} \triangleq \begin{cases} \kappa_i + \dot{\bar{\psi}}^{i\mathrm{T}} S_{1,1}(t_i^{*+})\dot{\bar{\psi}}^i, & \text{if } i = N-j+1, \\ S_{3j+3,3j+3}(t_i^{*+}), & \text{if } i = N-j,\dots,1, \end{cases}$$

$$\Gamma^i_{3k+r,3l+1} \triangleq \begin{cases} -S^{\mathrm{T}}_{1,3k+r}(t_i^{*+})\bar{\psi}^i_{s_i}, & \text{if } i = N-l+1, \\ S_{3k+r,3l+1}(t_i^{*+}), & \text{if } i = N-l,\dots,1, \end{cases}$$

$$\Gamma^i_{3k+r,3l+2} \triangleq \begin{cases} 0, & \text{if } i = N-l+1, \\ S_{3k+r,3l+2}(t_i^{*+}), & \text{if } i = N-l,\dots,1, \end{cases}$$

$$\Gamma^i_{3k+r,3l+3} \triangleq \begin{cases} -S^{\mathrm{T}}_{1,3k+r}(t_i^{*+})\dot{\bar{\psi}}^i, & \text{if } i = N-l+1, \\ S_{3k+r,3l+3}(t_i^{*+}), & \text{if } i = N-l,\dots,1, \end{cases}$$

and

$$\Lambda^i_1 \triangleq -\bar{\psi}^{i\mathrm{T}}_{x_i^-} R_1(t_i^{*+}), \quad \Lambda^i_2 \triangleq R_2(t_i^{*+}), \quad \Lambda^i_3 \triangleq R_3(t_i^{*+}), \quad \Lambda^i_{3k+r} \triangleq R_{3k+r}(t_i^{*+}),$$

$$\Lambda^i_{3(N-i+1)+1} \triangleq -\bar{\psi}^{i\mathrm{T}}_{s_i} R_1(t_i^{*+}),\ \Lambda^i_{3(N-i+1)+2} \triangleq 0,\ \Lambda^i_{3(N-i+1)+3} \triangleq -\dot{\bar{\psi}}^{i\mathrm{T}} R_1(t_i^{*+}) + \epsilon_i,$$

where $j = 1,\dots,N-i+1$, $k = 1,\dots,N-i$, $l = k+1,\dots,N-i+1$, and $r = 1,2,3$. Note that block terms $\Gamma^i_{3k+r,3l+1}$, $\Gamma^i_{3k+r,3l+2}$ and $\Gamma^i_{3k+r,3l+3}$ exist only if $i \le N-1$.

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