Optimal replication of random vectors by ordinary integrals *

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Abstract

We consider a problem of replication of random vectors by ordinary integrals in the setting when an underlying random variable is generated by a Wiener process. The goal is to find an optimal adapted process such that its cumulative integral at a fixed terminal time matches this variable. The optimal process has to be minimal in an integral norm.

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1 Introduction

It is well known that random variables generated by a Wiener process can be represented via stochastic integrals, as is stated by the classical Martingale Representation Theorem. This important result led to the theory of backward stochastic differential equations and the martingale pricing method in Mathematical Finance.

We consider a problem of replication of random variables by ordinary integrals. The goal is to find an optimal adapted process such that its cumulative integral at a fixed terminal time matches this variable without error. The optimal process has to be minimal in an integral norm. An explicit solution of this problem is found.

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2 The problem setting and the main result

Consider a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and standard d-dimensional Wiener process $w(t)$ (with $w(0) = 0$) which generates the filtration $\mathcal{F}_t = \sigma\{w(r) : 0 \le r \le t\}$ augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$.

We denote by $|\cdot|$ the Euclidean norm for vectors and the Frobenius (i.e., Euclidean) norm for matrices.

For $p \ge 1$ and $q \ge 1$, we denote by $L^{n\times m}_{p,q}$ the class of random processes $v(t)$ adapted to $\mathcal{F}_t$ with values in $\mathbf{R}^{n\times m}$ such that $\mathbf{E}\left(\int_0^T |v(t)|^q dt\right)^{p/q} < +\infty$.

Let f be a $\mathcal{F}_T$-measurable random vector, $f \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n)$. By the Martingale Representation Theorem, there exists a unique $k_f \in L^{n\times d}_{2,2}$ such that

$$f = \mathbf{E}f + \int_0^T k_f(t)dw(t).$$

See, e.g., Theorem 4.2.4 in [11], p.67.

We assume that there exists $\theta \in (0, T)$ such that

$$\operatorname*{ess\,sup}_{t\in[\theta,T]} \mathbf{E}|k_f(t)|^2 < +\infty.$$

Let $g : [0, T) \to \mathbf{R}$ be a given measurable function such that there exist $c > 0$ and $\alpha \in (0.5, 1)$ such that

$$0 < g(t) \le c(T - t)^\alpha, \quad g(t)^{-1} \le c(1 + (T - t)^{-\alpha}), \quad t \in [0, T). \tag{1}$$

An example of such a function is $g(t) = 1$ for $t < T - \tau$, $g(t) = (T - t)^\alpha$ for $t \ge T - \tau$, where $\tau \in (0, T]$ can be any number.

Let U be the set of all processes from $L^{n\times 1}_{2,1}$ such that

$$\mathbf{E}\int_0^T g(t)|u(t)|^2 dt < +\infty. \tag{2}$$

By the definition of $L^{n\times 1}_{2,1}$, it follows that, for $u \in U$,

$$\mathbf{E}\left(\int_0^T |u(t)|dt\right)^2 < +\infty. \tag{3}$$

Let $\Gamma(t)$ be measurable matrix valued function in $\mathbf{R}^{n\times n}$, such that $\Gamma(t) = g(t)G(t)$, where $G(t) > 0$ is a symmetric positively defined matrix such that the matrices $G(t)$ and $G(t)^{-1}$ are both bounded. Clearly, $\mathbf{E}\int_0^T u(t)^\top \Gamma(t)u(t)dt < +\infty$ for $u \in U$.

Let $a \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$, and let $b \in \mathbf{R}^{n\times n}$ be a non-degenerate matrix.

Consider the problem

$$\text{Minimize} \quad \mathbf{E}\int_0^T u(t)^\top \Gamma(t) u(t)\, dt \quad \text{over} \quad u \in U \tag{4}$$

$$\text{subject to} \quad \begin{array}{l} \frac{dx}{dt}(t) = Ax(t) + bu(t), \quad t \in (0,T) \\ x(0) = a, \quad x(T) = f \quad \text{a.s.} \end{array} \tag{5}$$

Note that this problem is a modification of a stochastic control problem with terminal contingent claim. These problems were studied intensively in the setting that involve backward stochastic differential equations (BSDEs); a first problem of this type was introduced in [7]. In this setting, a non-zero diffusion coefficient is presented in the evolution equation for the plant process as an auxiliary control process. Our setting is different: a non-zero diffusion coefficient is not allowed. Problem (4)-(5) is a linear quadratic control problem. However, it has a potential to be extended on control problems of a general type, similarly to the theory of controlled backward stochastic differential equation.

Let

$$\widehat{k}_\mu(t) = R(t)^{-1} k_f(t), \qquad R(s) \triangleq \int_s^T Q(t) dt, \qquad Q(t) = e^{A(T-t)} b \Gamma(t)^{-1} b^\top e^{A^\top (T-t)}.$$

Lemma 1 $\widehat{k}_\mu(\cdot) \in L_{2,2}^{n\times d}$.

Theorem 1 *Problem (4)-(5) has a unique optimal solution in U. This solution is defined as $\widehat{u}(t) = \Gamma(t)^{-1} b^\top e^{A^\top (T-t)} \widehat{\mu}(t)$, where*

$$\widehat{\mu}(t) = R(0)^{-1}(\mathbf{E}f - e^{AT} a) + \int_0^t \widehat{k}_\mu(s) dw(s).$$

Remark 1 Restrictions (1) on the choice of $\Gamma(t) = g(t)G(t)$ mean that the penalty for the large size of $u(t)$ vanishes as $t \to T$. Thus, we do not exclude fast growing $u(t)$ as $t \to T$ such that $u(t)$ is not square integrable. This is why we select the class U of admissible controls to be wider than $L_{2,2}^{n\times 1}$. In [5], a related result was obtained for a simpler case when it was required to ensure that $x(T) = \mathbf{E}\{f|\mathcal{F}_\theta\}$ for some $\theta < T$. In this setting, the exact match could be achieved only for $\mathcal{F}_\theta$-measurable f; the optimal solution was found to be a square integrable process.

3 Proofs

Proof of Lemma 1. By the assumptions, we have that $Q(t) = g(t)^{-1}\mathcal{Q}(t)$, where

$$\mathcal{Q}(t) = e^{A(T-t)} b G(t)^{-1} b^\top e^{A^\top (T-t)}$$

is a bounded matrix, $g(t)^{-1} \geq c^{-1}(T-t)^{-\alpha}$ for $t \in [0,T)$. For a matrix $M = M^\top \geq 0$, set

$$\rho(M) = \inf_{x \in \mathbf{R}^n:\ |x|=1} x^\top M x.$$

Since b is a non-degenerate matrix, we have that $\zeta = \inf_{s\in[0,T]} \rho(\mathcal{Q}(s)) > 0$ and

$$\rho(R(t)) \geq \int_t^T \rho(Q(s))ds \geq c^{-1} \int_t^T (T-s)^{-\alpha} \rho(\mathcal{Q}(s))ds \geq -\frac{\zeta}{c} \frac{(T-s)^{1-\alpha}}{1-\alpha}\Big|_{s=t}^{s=T}, \tag{6}$$

where c is the constant from (1). Hence

$$\rho(R(t)) \geq \frac{\zeta(T-t)^{1-\alpha}}{c(1-\alpha)}. \tag{7}$$

It follows from (7) that

$$|R(t)^{-1}| \leq C \frac{(1-\alpha)}{(T-t)^{1-\alpha}},$$

for some constant $C > 0$ that is defined by ζ, c and n. Hence

$$\int_\theta^T \left|R(t)^{-1}\right|^2 dt \leq C^2(1-\alpha)^2 \int_\theta^T \frac{1}{(T-t)^{2-2\alpha}} dt = C^2(1-\alpha)^2 \frac{(T-\theta)^{2\alpha-1}}{2\alpha-1} < +\infty.$$

It follows that

$$\begin{aligned}
&\mathbf{E}\int_0^T |\widehat{k}_\mu(t)|^2 dt \leq \mathbf{E}\int_0^T \left|R(t)^{-1}\right|^2 |k_f(t)|^2 dt \\
&= \mathbf{E}\int_0^\theta \left|R(t)^{-1}\right|^2 |k_f(t)|^2 dt + \mathbf{E}\int_\theta^T \left|R(t)^{-1}\right|^2 |k_f(t)|^2 dt \\
&\leq \sup_{t\in[0,\theta]} \left|R(t)^{-1}\right|^2 \mathbf{E}\int_0^\theta |k_f(t)|^2 dt + \int_\theta^T \left|R(t)^{-1}\right|^2 \mathbf{E}|k_f(t)|^2 dt \\
&= \sup_{t\in[0,\theta]} \left|R(t)^{-1}\right|^2 \mathbf{E}\int_0^\theta |k_f(t)|^2 dt + \operatorname*{ess\,sup}_{t\in[\theta,T]} \mathbf{E}|k_f(t)|^2 \int_\theta^T \left|R(t)^{-1}\right|^2 dt
\end{aligned}$$

and

$$\mathbf{E}\int_0^T |\widehat{k}_\mu(t)|^2 dt < +\infty. \tag{8}$$

This completes the proof of Lemma 1. □

Remark 2 The assumption that b is non-degenerate was used to establish estimates (6)-(7). These estimates have some similarity with the classical criterion of controllability for the linear systems. However, these estimates are not covered immediately by the controllability approach since the matrix $\gamma(t)^{-1}G(t)^{-1}$ under the integral in (6) is time variable and has a singularity at $t = T$. It could be interesting to investigate if it is possible to replace the assumption that b is non-degenerate by a less restrictive assumption that the pair (A, b) is controllable. We leave this for future research.

Proof of Theorem 1. The solution of the linear equation in (5) is

$$x(t) = \int_0^t e^{A(t-s)} bu(s)ds + e^{At}a. \tag{9}$$

By the definition of U, it follows that (3) holds for any $u \in U$. Hence $x(T) \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n)$ for any $u \in U$.

Let the function $L(u, \mu) : U \times L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n) \to \mathbf{R}$ be defined as

$$L(u, \mu) \triangleq \frac{1}{2}\mathbf{E}\int_0^T u(t)^\top \Gamma(t) u(t)\, dt + \mathbf{E}\mu^\top (f - x(T)).$$

For a given μ, consider the following problem:

$$\text{Minimize } L(u, \mu) \text{ over } u \in U. \tag{10}$$

This problem does not have constraints on terminal value $x(T)$. Therefore, it can be solved by usual stochastic control methods for the forward plant equations. We solve problem (10) using the so-called stochastic maximum principle that gives a necessary condition of optimality; see, e.g., [1]-[4], [6]-[7], [9]-[10], [12]-[13]). For our problem (10), all versions of the stochastic maximum principle from the cited papers are equivalent and can be formulated as the following: if $u = u_\mu \in U$ is optimal then

$$\psi(t)^\top bu_\mu(t) - \frac{1}{2}u_\mu(t)^\top \Gamma(t) u_\mu(t) \ge \psi(t)^\top bv - \frac{1}{2}v^\top \Gamma(t) v \quad \text{for a.e. } t \text{ for all} \quad v \in \mathbf{R}^n \quad \text{a.s.}, \tag{11}$$

where $\psi(t)$ is a process from $L_{2,2}^{n\times 1}$ such that

$$d\psi(t) = -A^\top \psi(t)dt + \chi(t)dw(t),$$
$$\psi(T) = \mu,$$

for some process $\chi \in L_{2,2}^{n\times n}$. (See, e.g., Theorem 1.5 from [4], p.609). The only solution of the backward equation for ψ is

$$\psi(t) = e^{A^\top (T-t)}\mu(t), \quad \mu(t) = \mathbf{E}\{\mu | \mathcal{F}_t\}. \tag{12}$$

Necessary conditions of optimality (11) are satisfied for a unique up to equivalency process $u = u_\mu$ defined as

$$u_\mu(t) = \Gamma(t)^{-1} b^\top \psi(t). \tag{13}$$

Let us show that $u_\mu \in U$ for any μ. We have that

$$\mathbf{E}\left(\int_0^T |u_\mu(t)|dt\right)^2 \le C_1\mathbf{E}\left(\int_0^T |\Gamma(t)^{-1}||\mu(t)|dt\right)^2 \le C_2 \sup_{t\in[0,T]} \mathbf{E}|\mu(t)|^2 \int_0^T g(t)^{-1}dt < +\infty. \tag{14}$$

In addition,

$$\mathbf{E}\int_0^T g(t)|u_\mu(t)|^2dt \le C_3\mathbf{E}\int_0^T g(t)|\Gamma(t)^{-1}\mu(t)|^2dt \le C_4\mathbf{E}\int_0^T g(t)^{-1}|\mu(t)|^2dt$$
$$\le C_4 \sup_{t\in[0,T]} \mathbf{E}|\mu(t)|^2 \int_0^T g(t)^{-1}dt < +\infty. \tag{15}$$

Here $C_i > 0$ are constants defined by A, b, n, and T. Hence $u_\mu \in U$.

Clearly, the function $L(u,\mu)$ is strictly concave in u, and this minimization problem has a unique solution. Therefore, this $u = u_\mu$ is the unique solution of (10).

Further, we consider the following problem:

$$\text{Maximize } L(u_\mu, \mu) \text{ over } \mu \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n). \tag{16}$$

For $u = u_\mu$, equation (9) gives

$$x(T) = \int_0^T e^{A(T-t)}bu_\mu(t)dt + e^{AT}a.$$

Hence

$$L(u_\mu,\mu) = \frac{1}{2}\mathbf{E}\int_0^T u_\mu(t)^\top\Gamma(t)u_\mu(t)\,dt - \mathbf{E}\mu^\top\int_0^T e^{A(T-t)}bu_\mu(t)dt - \mathbf{E}\mu^\top e^{AT}a + \mathbf{E}\mu^\top f.$$

We have that

$$\mathbf{E}\mu^\top\int_0^T e^{A(T-t)}bu_\mu(t)dt = \mathbf{E}\mu^\top\int_0^T e^{A(T-t)}b\Gamma(t)^{-1}b^\top\psi(t)dt$$
$$= \mathbf{E}\mu^\top\int_0^T e^{A(T-t)}b\Gamma(t)^{-1}b^\top e^{A^\top(T-t)}\mu(t)dt = \mathbf{E}\mu^\top\int_0^T Q(t)\mu(t)dt$$
$$= \mathbf{E}\int_0^T \mu^\top Q(t)\mu(t)dt = \mathbf{E}\int_0^T \mathbf{E}\{\mu^\top Q(t)\mu(t)|\mathcal{F}_t\}dt = \mathbf{E}\int_0^T \mu(t)^\top Q(t)\mu(t)dt.$$

The fifth equality here holds by Fubini's Theorem. Further, we have that

$$\mathbf{E}\int_0^T u_\mu(t)^\top\Gamma(t)u_\mu(t)\,dt = \mathbf{E}\int_0^T (\Gamma(t)^{-1}b^\top\psi(t))^\top\Gamma(t)\Gamma(t)^{-1}b^\top\psi(t)\,dt$$
$$= \mathbf{E}\int_0^T \psi(t)^\top b\Gamma(t)^{-1}b^\top\psi(t)\,dt = \mathbf{E}\int_0^T (e^{A^\top(T-t)}\mu(t))^\top b\Gamma(t)^{-1}b^\top e^{A^\top(T-t)}\mu(t)\,dt$$
$$= \mathbf{E}\int_0^T \mu(t)^\top e^{A(T-t)}b\Gamma(t)^{-1}b^\top e^{A^\top(T-t)}\mu(t)\,dt = \mathbf{E}\int_0^T \mu(t)^\top Q(t)\mu(t)\,dt.$$

It follows that

$$L(u_\mu,\mu) = \mathbf{E}\mu^\top (f - e^{AT}a) - \frac{1}{2}\mathbf{E}\int_0^T \mu(t)^\top Q(t)\mu(t)\,dt.$$

By the Martingale Representation Theorem, there exists $k_\mu \in L^{n\times d}_{2,2}$ such that

$$\mu = \bar\mu + \int_0^T k_\mu(t)dw(t), \tag{17}$$

where $\bar\mu \triangleq \mathbf{E}\mu$. It follows that

$$\begin{aligned}\mathbf{E}\int_0^T \mu(t)^\top Q(t)\mu(t)\,dt &= \mathbf{E}\int_0^T \left(\bar\mu + \int_0^t k_\mu(s)dw(s)\right)^\top Q(t)\left(\bar\mu + \int_0^t k_\mu(s)dw(s)\right)dt\\ &= \int_0^T \bar\mu^\top Q(t)\bar\mu\,dt + \mathbf{E}\int_0^T \left(\int_0^t k_\mu(s)dw(s)\right)^\top Q(t)\left(\int_0^t k_\mu(s)dw(s)\right)dt\\ &= \bar\mu^\top\left(\int_0^T Q(t)\,dt\right)\bar\mu + \int_0^T \mathbf{E}\left(\int_0^t k_\mu(s)dw(s)\right)^\top Q(t)\left(\int_0^t k_\mu(s)dw(s)\right)dt\end{aligned}$$

$$= \bar\mu^\top R(0)\bar\mu + \int_0^T dt\,\mathbf{E}\int_0^t k_\mu(s)^\top Q(t)k_\mu(s)\,ds = \bar\mu^\top R(0)\bar\mu + \mathbf{E}\int_0^T ds\int_s^T k_\mu(s)^\top Q(t)k_\mu(s)\,dt$$

$$= \bar\mu^\top R(0)\bar\mu + \mathbf{E}\int_0^T k_\mu(s)^\top R(s)k_\mu(s)\,ds.$$

We have used Fubini's Theorem again to change the order of integration. Similarly,

$$\mathbf{E}\mu^\top f = \bar\mu^\top \bar f + \mathbf{E}\int_0^T k_\mu(t)^\top k_f(t)\,dt,$$

and $\mathbf{E}\mu^\top e^{AT}a = \bar\mu^\top e^{AT}a$. It follows that

$$\begin{aligned}&L(u_\mu,\mu)\\ &= \bar\mu^\top(\bar f - e^{AT}a) - \frac{1}{2}\bar\mu^\top R(0)\bar\mu - \frac{1}{2}\mathbf{E}\int_0^T k_\mu(\tau)^\top R(\tau)k_\mu(\tau)\,d\tau + \mathbf{E}\int_0^T k_\mu(t)^\top k_f(t)\,dt.\end{aligned}$$

Clearly, the maximum of this quadratic form is achieved for

$$\bar\mu = R(0)^{-1}(\bar f - e^{AT}a), \qquad \widehat{k}_\mu(t) = R(t)^{-1}k_f(t). \tag{18}$$

This means that the optimal solution $\widehat\mu$ of problem (16) is

$$\widehat\mu = R(0)^{-1}(\bar f - e^{AT}a) + \int_0^T \widehat{k}_\mu(t)dw(t).$$

Let $\widehat u(t)$ and $\widehat\mu(t)$ be defined by (12)-(13) for $\mu = \widehat\mu$, i.e., $\widehat u = u_{\widehat\mu}$. By Lemma 1 and (8), it follows that $\sup_{t\in[0,T]}\mathbf{E}|\widehat\mu(t)|^2 < +\infty$.

We found that $\sup_\mu \inf_u L(u,\mu)$ is achieved for $(\widehat{u},\widehat{\mu})$. We have that $L(u,\mu)$ is strictly convex in $u\in U$ and affine in $\mu\in L_2(\Omega,\mathcal{F},\mathbf{P},\mathbf{R}^n)$. In addition, $L(u,\mu)$ is continuous in $u\in L_{2,2}^{n\times 1}$ given $\mu\in L_2(\Omega,\mathcal{F},\mathbf{P},\mathbf{R}^n)$, and $L(u,\mu)$ is continuous in $\mu\in L_2(\Omega,\mathcal{F},\mathbf{P},\mathbf{R}^n)$ given $u\in U$. By Proposition 2.3 from [8], Chapter VI, p. 175, it follows that

$$\inf_{u\in U}\sup_{\mu} L(u,\mu)=\sup_{\mu}\inf_{u\in U} L(u,\mu). \tag{19}$$

Therefore, $(\widehat{u},\widehat{\mu})$ is the unique saddle point for (19).

Let U_f be the set of all $u(\cdot)\in U$ such that (5) holds. It is easy to see that

$$\inf_{u\in U_f}\frac{1}{2}\mathbf{E}\int_0^T u(t)^\top\Gamma(t)u(t)\,dt=\inf_{u\in U}\sup_{\mu} L(u,\mu),$$

and any solution (u,μ) of (19) is such that $u\in U_f$. It follows that $\widehat{u}\in U_f$ and it is the optimal solution for problem (4)-(5). Then the proof of Theorem 1 follows. □

Remark 3 If $\alpha=1$ then estimates (14)-(15) are not satisfied and $u_\mu\notin U$. This is why $\alpha=1$ is excluded.

Remark 4 The fact that the function $\widehat{u}$ ensures replication of f was obtained as a consequence of duality analysis for the Lagrangian. The replicating property can be also verified directly. Assume for simplicity that $A=0$ and $b=I$, where I is the unit matrix, then

$$\begin{aligned}
x(T)&=a+\int_0^T\widehat{u}(t)dt=a+\int_0^T\Gamma(t)^{-1}\widehat{\mu}(t)dt\\
&=a+\int_0^T\Gamma(t)^{-1}[R(0)^{-1}(\mathbf{E}f-a)+\int_0^t R(s)^{-1}k_f(s)dw(s)]dt\\
&=a+\Big(\int_0^T\Gamma(t)^{-1}dt\Big)R(0)^{-1}(\mathbf{E}f-a)+\int_0^T\left(\int_s^T\Gamma(t)^{-1}dt\right)R(s)^{-1}k_f(s)dw(s)\\
&=a+R(0)R(0)^{-1}(\mathbf{E}f-a)+\int_0^T R(s)R(s)^{-1}k_f(s)dw(s)=\mathbf{E}f+\int_0^T k_f(s)dw(s)=f.
\end{aligned}$$

4 Example of calculation of $\widehat{u}$

Consider a model where $f=F(\eta(T))$, where $\eta(t)$ satisfies the Ito equation

$$d\eta(t)=h(\eta(t),t)dt+\beta(\eta(t),t)dw(t).$$

Here $h(y,t):\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}^n$, $\beta(y,t):\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}^{n\times n}$ are measurable bounded functions such that the derivative $\partial\beta(y,t)/\partial y$ is bounded, $B(y,t)=\frac{1}{2}\beta(y,t)\,\beta(y,t)^\top\ge\delta I>0$ for all y,t, where $\delta>0$, I is the unit matrix.

Theorem 1 can be applied as the following.

Assume first that $h(y,t) \equiv 0$. In this case, Theorem 1 ensures that $x(T) = f$ and that $\mathbf{E}\int_0^T u(t)^\top \Gamma(t) u(t) dt$ is minimal for $\widehat{u}(t) = \Gamma(t)^{-1} b^\top e^{A^\top (T-t)} \widehat{\mu}(t)$, where

$$\widehat{\mu}(t) = R(0)^{-1}(H(\eta(0),0) - e^{AT}x(0)) + \int_0^t R(s)^{-1} \frac{\partial H}{\partial y}(\eta(s),s) d\eta(s). \tag{20}$$

Here H is the solution of the Cauchy problem for the parabolic equation

$$\begin{aligned} &\frac{\partial H}{\partial t}(y,t) + \sum_{i,j=1}^n B_{ij}(y,t) \frac{\partial^2 H}{\partial y_i \partial y_j}(y,t) = 0, \quad t < T, \\ &H(y,T) = F(y). \end{aligned} \tag{21}$$

In (21), B_{ij}, y_i are the components of the matrix B and the vector y. We assume that $F(x)$ is a regular enough function to ensure that (21) has a regular enough solution such that $\widehat{u} \in U$; for this, it suffices to ensure that $\partial H/\partial y$ is bounded. It can be noted that $H(y,t) = \mathbf{E}\{F(\eta(T))|\eta(t) = y\}$.

Assume now that $h(\cdot) \neq 0$. We still have that $x(T) = f$ for $u(t)$ defined by (20)-(21); in this case, $H(y,t) = \mathbf{E}_Q\{F(\eta(T))|\eta(t) = y\}$, where $\mathbf{E}_Q$ is the expectation under a probability measure Q such that the process $\eta(t)$ is a martingale under Q. By the Girsanov Theorem, this measure exists, it is equivalent to the original measure $\mathbf{P}$ and it is unique under our assumptions on h and β (see, e.g., Theorem 4.2.2 in [11], p. 66). In this case, the value $\mathbf{E}\int_0^T \widehat{u}(t)^\top \Gamma(t) \widehat{u}(t) dt$ is not minimal over $u \in U_f$ anymore. Instead, $\mathbf{E}_Q \int_0^T \widehat{u}(t)^\top \Gamma(t) \widehat{u}(t) dt$ is minimal over $u \in U_f$. This still means that the deviations of u are minimal but in a different sense. It can be also noted that the definition of the class U for the original measure has to be adjusted for the new measure Q, with the expectations $\mathbf{E}$ replaced by $\mathbf{E}_Q$.

This model could have applications in goal achieving problems, where the goal is to match a controlled differentiable process at time T with a random vector $f = F(\eta(T))$ generated by an uncontrolled observable stochastic process $\eta(t)$. For instance, $x(t)$ may represent a path of a missile controlled by an anti-aircraft command, and the process $\eta(t)$ may represent an observed uncontrolled parameter process describing the movement of an airborne target such that $f = F(\eta(T))$ represents the target coordinates at time T.

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