Regularity for some backward heat equations*

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Abstract

We study boundary value problems for backward parabolic heat equations with Cauchy condition at the initial time, i.e., at the "wrong" end of the time interval. We found that existence, uniqueness, and regularity of solutions can be still achieved for the problem with boundary inputs.

Key words: parabolic equations, backward heat equation, inverse problems, regularity, frequency domain, Hardy spaces.

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Parabolic equations such as heat equations have fundamental significance for natural sciences, and various boundary value problems for them were widely studied including inverse and ill-posed problems (see, e.g., examples in Tikhonov and Arsenin (1977), Glasko (1984), Prilepko *et al* (1984), Beck (1985)). For ill-posed problems, solvability and regularity estimates for the solutions are not guaranteed for typical inputs. A classical example is the backward heat equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0, \quad t \in [0, T] \tag{0.1}$$

with the Cauchy condition at initial time $t = 0$. This problem is ill-posed, and the corresponding problem with the Cauchy condition at time $t = T$ is well-posed. For the

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parabolic equations, it is commonly recognized that the type of the boundary value conditions and the sign of the second order coefficient usually defines if a problem is well-posed or ill-posed.

Apparently there are boundary value problems that do not fit the framework given by the classical theory of parabolic equations and by the mentioned above examples. In Dokuchaev (2007), an example was given of a boundary value problem that appears to be ill-posed but allows solvability, uniqueness of the solutions, and regularity estimates, for an explicitly given set of inputs which is dense in a L_2-space of all possible inputs. This result was obtained in frequency domain for linear parabolic equations on $(0, +\infty)$ with the second order Cauchy condition at $x = 0$ and with zero initial condition at $t = 0$.

In the present paper, we give another example of a problem that looks similar to classical ill-posed problems but allows solvability and regularity estimates for the solutions. Using the approach similar to the one from Dokuchaev (2007), we investigate a boundary value problem in the interval $[0, +\infty)$ for linear parabolic equations similar to the backward heat equation (0.1), with Cauchy condition at initial time $t = 0$ and with mixed boundary condition at the boundary point $x = 0$. This parabolic equation is known to be solvable with Cauchy condition at terminal time but not at initial time. For these problems, we established existence, uniqueness, and regularity of solutions for the problem with boundary inputs. More precisely, we found that the boundary inputs from $W_2^1(\mathbf{R})$ vanishing for $t \le 0$ ensure solvability and regularity estimates for the solutions in a weighted Sobolev space; the weight is defined by the sign of the first order coefficient. As far as we know, this fact about backward heat equations was left unnoticed in the existing literature.

1 The problem setting

Let us consider the following boundary value problem:

$$
\begin{aligned}
& a\frac{\partial u}{\partial t}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) + b\frac{\partial u}{\partial x}(x,t) + cu(x,t) = 0, \quad x > 0,\ t > 0, \\
& u(x,0) \equiv 0, \quad x > 0, \\
& k_0 u(0,t) + k_1 \frac{\partial u}{\partial x}(0,t) \equiv g(t), \quad t > 0.
\end{aligned}
\tag{1.1}
$$

Here $a, b, c, k_0, k_1 \in \mathbf{R}$ are constants.

If $b=0$ then the parabolic equation may describe the heat propagation or diffusion along a semi-infinite rod with Robin type boundary condition at the end point, with the diffusion coefficient a^{-1}. The equation with $b\neq 0$ describes a diffusion process with drift.

We assume that

$$a>0, \qquad k_0^2+k_1^2>0. \tag{1.2}$$

The assumption that $a>0$ and the presence of the initial condition at $t=0$ makes problem (1.1) similar to the ill-posed backward heat equation.

Let Γ denote the set of all functions $g:\mathbf{R}\to\mathbf{R}$ such that $g(t)=0$ for $t\le 0$ and $\|g\|_{W_2^1(\mathbf{R})}<+\infty$, where

$$\|g\|_{W_2^1(\mathbf{R})}\triangleq\|g\|_{L_2(\mathbf{R})}+\left\|\frac{\partial g}{\partial t}\right\|_{L_2(\mathbf{R})}.$$

We mean that the corresponding derivative exists in $L_2(\mathbf{R})$.

Remark 1 *Note that the conditions on g are requires that $g(0+)=0$ and that g is is absolutely continuous on $\mathbf{R}$, since $dg(t)/dt\in L_2(\mathbf{R})$.*

For $q\in\mathbf{R}$, introduce a weight function $r(x)=r(x,q)\triangleq e^{-qx}$.

Let $\mathbf{R}^+\triangleq[0,+\infty)$, and let $L_2(\mathbf{R}^+,r)$ be the space of functions $v=v(x):\mathbf{R}^+\to\mathbf{R}$ with finite norm

$$\|v\|_{L_2(\mathbf{R}^+,r)}\triangleq\left(\int_0^\infty r(x)|v(x)|^2dx\right)^{1/2}.$$

Let $D\triangleq\mathbf{R}^+\times\mathbf{R}$, i.e., D is the domain of $\{(x,t)\}$ for equation (1.1). Let $L_2(D,r)$ be the space of functions $v=v(x,t):D\to\mathbf{R}$ such that $v(x,t)\equiv 0$ for $t<0$, with finite norm

$$\|v\|_{L_2(D,r)}\triangleq\left(\int_0^\infty r(x)\|v(x,\cdot)\|^2_{L_2(\mathbf{R})}dx\right)^{1/2}=\left(\int_0^\infty r(x)dx\int_{\mathbf{R}}|v(x,t)|^2dt\right)^{1/2}.$$

Let $\mathcal{W}=\mathcal{W}(q)$ be the space of the functions $v=v(x,t):D\to\mathbf{R}$ such that $v(x,t)\equiv 0$ for $t<0$ and with finite norm

$$\|v\|_{\mathcal{W}}\triangleq\|v\|_{L_2(D,r)}+\left\|\frac{\partial v}{\partial x}\right\|_{L_2(D,r)}+\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L_2(D,r)}+\left\|\frac{\partial v}{\partial t}\right\|_{L_2(D,r)}.$$

We mean that the corresponding derivatives exist in the class $L_2(D,r)$.

The equations presented in problem (1.1) for any $u \in \mathcal{W}$ are defined as equalities in some L_2 spaces. Let us show this. All the terms of the parabolic equation are elements of $L_2(D,r)$, so this equation is considered as an equality in $L_2(D,r)$. Since $u(\cdot,t)$ is continuous in $L_2(\mathbf{R}^+,r)$ as a function of $t \in \mathbf{R}$, the initial condition at time $t=0$ is well defined as an equality in $L_2(\mathbf{R}^+,r)$. Further, for any $y>0$, we have that $u|_{[0,y]\times\mathbf{R}^+} \in C([0,y],L_2(\mathbf{R}))$ and $\frac{\partial u}{\partial x}\Big|_{[0,y]\times\mathbf{R}^+} \in C([0,y],L_2(\mathbf{R}))$, i.e., $u(x,\cdot)$ and $\frac{\partial u}{\partial x}(x,\cdot)$ are continuous in $L_2(\mathbf{R})$ as functions of $x \in [0,y]$. Hence the functions $u(0,t)$ and $\frac{du}{dx}(x,t)|_{x=0}$ are well defined as elements of $L_2(\mathbf{R})$, and the boundary value condition at $x=0$ is well defined as an equality in $L_2(\mathbf{R})$. Therefore, all equations in (1.1) are defined for $u \in \mathcal{W}$ as equalities in the corresponding L_2-spaces. Technically, it is not a classical solution.

2 The main result

Theorem 1 *Let condition (1.2) holds, and let $c>0$. Let $q>-b$, for q from the definition of $\mathcal{W}=\mathcal{W}(q)$. Then there exists a unique solution $u(x,t)$ of problem (1.1) in the class $\mathcal{W}$ with this q for any $g \in \Gamma$. Moreover, there exists a constant $C=C(a,b,c,k_0,k_1,q)$ such that*

$$\|u\|_{\mathcal{W}} \le C \, \|g\|_{W_2^1(\mathbf{R})} . \tag{2.1}$$

Remark 2 *In fact, (2.1) is the estimate for the norms in $L_2(D)$ of the functions $e^{-qx/2}u(x,t)$, $e^{-qx/2}u'_t(x,t)$, $e^{-qx/2}u'_x(x,t)$, $e^{-qx/2}u''_{xx}(x,t)$. For $b>0$, negative $q>-b$ are allowed, and this estimate is stronger than the one for $q=0$, when $L_2(D,r)=L_2(D)$. For $b \le 0$, (2.1) holds for positive $q>-b\ge 0$ only. In that case, (2.1) provide estimates of $u(\cdot)$ in the corresponding weighted Sobolev space only.*

Remark 3 *It is essential for the proof given below that the only allowed inputs are boundary inputs.*

The following theorem shows that assumption that $c>0$ is not really restrictive if we are not interested in the properties of the solutions for $T \to +\infty$, as can happen if we deal with solutions on a finite time interval.

Theorem 2 *Let $c \le 0$, and let condition (1.2) holds. Let $M \in (-c, +\infty)$ be such that $g(t)e^{-Mt} \in \Gamma$. Then problem (1.1) has a unique solution u such that $u_M \in \mathcal{W}$, where $u_M(x,t) \triangleq e^{-Mt}u(x,t)$.*

In Theorem 2, we mean again that the solution u is such that the equations in (1.1) are satisfied as equalities in some L_2-spaces.

Proof of Theorem 1. (A) Let us assume first that

$$\mu \triangleq b^2/4 - c \le 0. \tag{2.2}$$

Let $\mathbf{C}^+ \triangleq \{z \in \mathbf{C}: \ \mathrm{Re}\, z > 0\}$. For $v \in L_2(\mathbf{R})$, we denote by $\mathcal{L}v$ the Laplace transform

$$V(p) = (\mathcal{L}v)(p) \triangleq \int_0^\infty e^{-pt}v(t)dt, \quad p \in \mathbf{C}^+. \tag{2.3}$$

Let H^r be the Hardy space of holomorphic functions $h(p)$ on $\mathbf{C}^+$ with finite norm $\|h\|_{H^r} = \sup_{\nu>0} \|h(\nu + i\omega)\|_{L^r(\mathbf{R})}$, $r \ge 1$ (see, e.g., Duren (1970)).

For functions $V: \mathbf{R}^+ \times \bar{\mathbf{C}}^+ \to \mathbf{C}$, where $\bar{\mathbf{C}}^+ = \{z: \ \mathrm{Re}\, z \ge 0\}$, we introduce norms

$$\|V\|_{L_{22}^H} \triangleq \left(\int_{\mathbf{R}^+} r(x)\|V(x,\cdot)\|_{H^2}^2 dx\right)^{1/2},$$
$$\|V\|_{\mathcal{H}} \triangleq \|V\|_{L_{22}^H} + \left\|\frac{\partial V}{\partial x}\right\|_{L_{22}^H} + \left\|\frac{\partial^2 V}{\partial x^2}\right\|_{L_{22}^H} + \left\|pV(\cdot,p)\right\|_{L_{22}^H}.$$

As usual, the statement $\|V\|_{\mathcal{H}} < +\infty$ means that the corresponding derivatives of V exist in the corresponding classes.

Let us assume first that there exists a solution $u \in \mathcal{W}$ of (1.1). Set $g_0(t) \triangleq u(0,t)$, $g_1(t) \triangleq \frac{du}{dx}(x,t)|_{x=0}$. As was discussed above, the functions g_k are well defined as elements of $L_2(\mathbf{R})$.

Let $G \triangleq \mathcal{L}g$, $G_k \triangleq \mathcal{L}g_k$ and $U \triangleq \mathcal{L}u$. Since $g \in \Gamma$, we have that $G \in H^2$. Since $u \in \mathcal{W}$, we have that the functions G_k are well defined and $G_k \in H^2$.

The assumption that u is a solution and $u \in \mathcal{W}$ implies that

$$apU(x,p) + \frac{\partial^2 U}{\partial x^2}(x,p) + b\frac{\partial U}{\partial x}(x,p) + cU(x,p) = 0, \quad x > 0,$$
$$k_0U(0,p) + k_1\frac{\partial U}{\partial x}(0,p) \equiv G(p), \quad p \in \mathbf{C}^+, \tag{2.4}$$

and

$$U(x,\cdot), \frac{\partial U}{\partial x}(x,\cdot), \frac{\partial^2 U}{\partial x^2}(x,\cdot) \in H^2 \quad \text{for a.e.} \quad x > 0, \qquad \|U\|_{\mathcal{H}} < +\infty. \tag{2.5}$$

Let $\lambda_k=\lambda_k(p)$ be the roots of the equation $\lambda^2+b\lambda+c+ap=0$ defined for $p\in\mathbf{C}^+$, i.e., $\lambda_1(p)\triangleq -b/2-\sqrt{\mu-ap}$ and $\lambda_2(p)\triangleq -b/2+\sqrt{\mu-ap}$, where $\mu=b^2/4-c<0$. Here we select the branch of the square root such that $\operatorname{Arg}\sqrt{\mu-ap}\in(-\pi/2,\pi/2]$, i.e., that $\operatorname{Re}\sqrt{\mu-ap}\ge 0$, where $\operatorname{Arg}z$ denote the principal value of the argument of z such that $\operatorname{Arg}z\in(-\pi,\pi]$. Under these assumptions, we have that $\operatorname{Re}(\mu-ap)<0$ for $p\in\mathbf{C}^+$, the function $\sqrt{\mu-ap}$ is holomorphic in $\mathbf{C}^+$, and

$$\operatorname{Re}\lambda_1(p)\le -\frac{b}{2},\quad \operatorname{Im}\lambda_1(p)\ne 0\quad\text{if}\quad p\in\mathbf{C}^+\quad\text{or}\quad p=i\omega,\ \omega\in\mathbf{R}, \tag{2.6}$$

$$\exists\delta>0,\ \nu>0,\ \omega_*>0:\ 2\operatorname{Re}\lambda_2(\nu+i\omega)-q>\delta\quad\text{if}\quad\omega\in\mathbf{R},\ |\omega|\ge\omega_*. \tag{2.7}$$

In addition, we have that the functions $\lambda_k(p)$ and $e^{x\lambda_k(p)}$ are holomorphic in $\mathbf{C}^+$ for any $x>0$, and

$$\begin{aligned}&(\lambda_1(p)-\lambda_2(p))^{-1}\in H^\infty,\quad \lambda_k(p)(\lambda_1(p)-\lambda_2(p))^{-1}\in H^\infty,\quad k=1,2,\\ &\qquad (k_0+k_1\lambda_1(p))^{-1}\in H^\infty,\quad \lambda_1(p)(k_0+k_1\lambda_1(p))^{-1}\in H^\infty.\end{aligned} \tag{2.8}$$

The last two statements in (2.8) follow from the fact that $k_0+k_1\lambda_1(p)\ne 0$ for $p\in\mathbf{C}^+$ or $p=i\omega$, $\omega\in\mathbf{R}$; this fact follows from the second inequality in (2.6).

Let

$$N\triangleq\left\|\frac{1}{\lambda_1-\lambda_2}\right\|_{H^\infty}+\sum_{k=1,2}\left\|\frac{\lambda_k}{\lambda_1-\lambda_2}\right\|_{H^\infty}+\left\|\frac{1}{k_0+k_1\lambda_1}\right\|_{H^\infty}+\left\|\frac{\lambda_1}{k_0+k_1\lambda_1}\right\|_{H^\infty}.$$

For $x>0$ and $p\in\mathbf{C}^+$, set

$$\begin{aligned}U(x,p)\triangleq\frac{1}{\lambda_1(p)-\lambda_2(p)}\Big((G_1(p)\ &-\ \lambda_2(p)G_0(p))e^{\lambda_1(p)x}\\ -(G_1(p)\ &-\ \lambda_1(p)G_0(p))e^{\lambda_2(p)x}\Big).\end{aligned} \tag{2.9}$$

It can be verified directly that, for every given $p\in\mathbf{C}^+$, this $U(\cdot,p)$ is the unique solution of the Cauchy problem

$$\begin{aligned}&apU(x,p)+\frac{\partial^2U}{\partial x^2}(x,p)+b\frac{\partial U}{\partial x}(x,p)+cU(x,p)=0,\quad x>0,\\ &U(0,p)=G_0(p),\quad \frac{\partial U}{\partial x}(0,p)=G_1(p).\end{aligned} \tag{2.10}$$

Equation (2.9) for U can be rewritten as $U(x,p)=U_1(x,p)+U_2(x,p)$, where

$$\begin{aligned}U_1(x,p)&=e^{\lambda_1(p)x}J_1(p),\qquad J_1(p)=\frac{1}{\lambda_1(p)-\lambda_2(p)}(G_1(p)-\lambda_2(p)G_0(p)),\\ U_2(x,p)&=e^{\lambda_2(p)x}J_2(p),\qquad J_2(p)=\frac{1}{\lambda_1(p)-\lambda_2(p)}(G_1(p)-\lambda_1(p)G_0(p)).\end{aligned}$$

Let us show that

$$\|U_1\|_{L_{22}^H} < +\infty. \tag{2.11}$$

By (2.7), $|e^{x\lambda_1(p))}| \le e^{-xb/2}$, $p \in \mathbf{C}^+$. It follows that

$$\|e^{x\lambda_1}J_1\|_{H^2} \le \sup_{p\in\mathbf{C}^+} |e^{x\lambda_1(p))}| \|J_1\|_{H^2} \le e^{-xb/2}\|J_1\|_{H^2} \le Ne^{-xb/2} \sum_{k=0,1} \|G_k\|_{H^2}.$$

Then (2.11) holds. The assumption that $u \in \mathcal{W}$ implies that (2.5) holds. This and (2.11) imply that

$$\|U_2\|_{L_{22}^H} < +\infty. \tag{2.12}$$

By (2.7) again,

$$\|U_2\|_{L_{22}^H}^2 \ge \int_{\mathbf{R}^+} dx\, r(x) \int_{\mathbf{R}} |U_2(x,\nu+i\omega)|^2 d\omega = \int_{\mathbf{R}^+} dx\, e^{-qx} \int_{\mathbf{R}} |e^{\lambda_2(\nu+i\omega)x} J_2(\nu+i\omega)|^2 d\omega$$
$$= \int_{\mathbf{R}^+} dx\, e^{-qx} \int_{\mathbf{R}} e^{2\mathrm{Re}\,\lambda_2(\nu+i\omega)x} |J_2(\nu+i\omega)|^2 d\omega \ge \int_{\mathbf{R}^+} dx\, e^{\delta x} \int_{\omega:\,|\omega|\ge\omega_*} |J_2(\nu+i\omega)|^2 d\omega.$$

By (2.12), it follows that $J_2(p)$ is vanishing on $\{p = \nu + i\omega :\ |\omega| \ge \omega_*\}$. Since $J_2 \in H^2$, it follows that $J_2(p) \equiv 0$. Therefore, we have obtained that, if $u \in \mathcal{W}$ is a solution of (1.1), then

$$G_1(p) = \lambda_1 G_0(p), \quad J_1(p) = G_0(p), \quad J_2(p) \equiv 0.$$

Remind that $k_0G_0(p) + k_1G_1(p) = G(p)$. It follows that $k_0G_0(p) + k_1\lambda_1G_0(p) = G(p)$. Therefore, we have proved that if $u \in \mathcal{W}$ is a solution of (1.1), then $U(x,p) = U_1(x,p)$, and

$$G_0(p) = (k_0 + k_1\lambda_1(p))^{-1}G(p),$$
$$U(x,p) = e^{\lambda_1(p)x}G_0(p). \tag{2.13}$$

In particular, we have proved the uniqueness of solution in the class $\mathcal{W}$.

Let us establish existence of a solution in the class $\mathcal{W}$.

Let the operator $\mathcal{F}^{-1} : H^2 \to L_2(\mathbf{R})$ be defined such that $v = \mathcal{F}^{-1}V$ is

$$v(t) = \lim_{M\to\infty} \frac{1}{2\pi} \int_{-M}^{M} e^{i\omega t} V(i\omega) d\omega,$$

where the limit is in $L_2(\mathbf{R})$. By Paley-Wiener Theorem, this operator $\mathcal{F}^{-1}$ is continuous (see, e.g., Yosida (1995), p.163). In addition, $v(t)$ vanishes for $t < 0$, and if $\overline{V}(i\omega) =$

$V(-i\omega)$ then v is real. In fact, this operator represents the inverse Fourier transform of the trace of V on $i\mathbf{R}$.

Let us show that $u(x,\cdot) \triangleq \mathcal{F}^{-1}U(x,\cdot)$ is a solution of (1.1) if U is defined by (2.13).

First, let us estimate $\|U\|_{\mathcal{H}}$.

Repeat that, by (2.7), $|e^{x\lambda_1(p))}| \le e^{-xb/2}$ for $p\in\mathbf{C}^+$. It follows that

$$\begin{aligned}&\|p^m e^{x\lambda_1(p)}G_0(p)\|_{H^2} \le e^{-xb/2}\|p^m G_0(p)\|_{H^2}\\ &\le e^{-xb/2}\|(k_0+k_1\lambda_1(p))^{-1}\|_{H^\infty}\|p^m G\|_{H^2} \le Ne^{-xb/2}\|p^m G\|_{H^2}, \quad m=0,1.\end{aligned}$$

It follows from the above estimate that

$$\|p^m U\|_{L^H_{22}} \le NC_1\,\|p^m G\|_{H^2}\,, \quad m=0,1. \tag{2.14}$$

Further, we have that

$$\frac{\partial U}{\partial x}(x,p) = G_0(p)\lambda_1 e^{\lambda_1 x} = \frac{\lambda_1}{k_0+k_1\lambda_1}G(p)\lambda_1 e^{\lambda_1 x}. \tag{2.15}$$

We obtain again that

$$\begin{aligned}\left\|\frac{\partial U}{\partial x}\right\|^2_{L^H_{22}} &= \int_{\mathbf{R}^+} r(x)\left\|\frac{\partial U}{\partial x}(x,p)\right\|^2_{H^2} dx \le NC_2\int_{\mathbf{R}^+} r(x)\left\|e^{\lambda_1(p)x}G(p)\right\|^2_{H^2} dx\\ &\le NC_2\int_{\mathbf{R}^+} r(x)e^{-bx}\,\|G(p)\|^2_{H^2}\,dx \le C_3\|G\|_{H^2}.\end{aligned} \tag{2.16}$$

By (2.4), $\partial^2 U/\partial x^2$ can be expressed as a linear combination of U, pU, and $\partial U/\partial x$. By (2.14)-(2.16),

$$\begin{aligned}&\int_{\mathbf{R}^+} r(x)\left\|\frac{\partial^2 U}{\partial x^2}(x,p)\right\|^2_{H^2} dx\\ &\le C_4\left(\int_{\mathbf{R}^+} r(x)\left\|\frac{\partial U}{\partial x}(x,p)\right\|^2_{H^2} dx + \sum_{m=0,1}\int_{\mathbf{R}^+} r(x)\,\|p^m U(x,p)\|^2_{H^2}\,dx\right).\end{aligned}$$

It follows that

$$\int_{\mathbf{R}^+} r(x)\left\|\frac{\partial^2 U}{\partial x^2}(x,p)\right\|^2_{H^2} dx \le C_5(\|G\|^2_{H^2} + \|pG(p)\|^2_{H^2}). \tag{2.17}$$

Here C_k are constants that depend on a,b,c,k_0,k_1,q. By (2.14)-(2.17), it follows that estimate (2.5) holds for U defined by (2.13).

By (2.14), it follows that $u(x,\cdot) = \mathcal{F}^{-1}U(x,\cdot)$ and $\frac{\partial u}{\partial t}(x,\cdot) = \mathcal{F}^{-1}(pU(x,\cdot)$ are well defined in $L_2(D,r)$ and are vanishing for $t < 0$. It follows that $u(\cdot)$ is continuous in $L_2(\mathbf{R}^+)$ as a function of t, and that $u(\cdot,0) = 0$ in $L_2(\mathbf{R}^+)$. In addition, we have that $\overline{U(x,i\omega)} = U(x,-i\omega)$ for $\omega \in \mathbf{R}$ (for instance, $\overline{G_k(i\omega)} = G_k(-i\omega)$, $\overline{e^{x\lambda_k(i\omega)}} = e^{x\lambda_k(-i\omega)}$, etc). It follows that $u(x,\cdot) = \mathcal{F}^{-1}U(x,\cdot)$ is real. By (2.5), estimate (2.1) holds. Therefore, u is the solution of (1.1) in $\mathcal{W}$. The uniqueness was already established above. This completes the proof of Theorem 1 for the case when (2.2) holds.

(B) Let us prove Theorem 1 for the case when (2.2) does not hold. Note that if $v \in \mathcal{W}$ and $u(x,t) \triangleq v(\varepsilon x,t)$, then $u \in \mathcal{W}$ for any $\varepsilon > 0$, and $\|u\|_{\mathcal{W}} \le C\|v\|_{\mathcal{W}}$, where $C = C(\varepsilon) > 0$ is a constant. For $\varepsilon \in (0, 2\sqrt{c}/b)$, consider the problem

$$\begin{aligned}
&a\frac{\partial v}{\partial t}(x,t) + \varepsilon^2\frac{\partial^2 v}{\partial x^2}(x,t) + \varepsilon b\frac{\partial v}{\partial x}(x,t) + cv(x,t) = 0, \quad x > 0,\ t > 0,\\
&v(x,0) \equiv 0, \quad x > 0,\\
&k_0 v(0,t) + k_1\frac{\partial v}{\partial x}(0,t) \equiv g(t), \quad t > 0.
\end{aligned} \tag{2.18}$$

Clearly, the analog of (2.2) is the condition $\varepsilon^2 b^2/4 < c$, and it is satisfied. By the part (A) of the proof, it follows that problem (2.18) has unique solution $v \in \mathcal{W}$, where $\mathcal{W}$ is defined for $q = q_\varepsilon > -\varepsilon b$, and an analog of estimate (2.1) holds. By (2.18), it follows that the function $u(x,t) \triangleq v(\varepsilon x,t)$ is the unique solution of (1.1), and (2.1) holds for the space $\mathcal{W}$ defined for $q = \varepsilon^{-1}q_\varepsilon$, i.e., for $q > -b$. It can be obtained by the change of variables such as

$$\begin{aligned}
\int_{\mathbf{R}^+} e^{-q_\varepsilon x}\|v(x,\cdot)\|^2_{L_2(\mathbf{R})}dx &= \int_{\mathbf{R}^+} e^{-q_\varepsilon x}\|u(\varepsilon x,\cdot)\|^2_{L_2(\mathbf{R})}dx\\
&= \varepsilon^{-1}\int_{\mathbf{R}^+} e^{-q_\varepsilon y/\varepsilon}\|u(y,\cdot)\|^2_{L_2(\mathbf{R})}dy < +\infty.
\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Rewrite the parabolic equation with c replaced by $c + M$ and $g(t)$ replaced by $g(t)e^{-Mt}$. By Theorem 1, solution $u_M \in \mathcal{M}$ of the new equation exists. Clearly, $u(x,t) = e^{Mt}u_M(x,t)$ is the solution of of the original problem. □

Remark 4 *A similar result can be obtained for the classical well-posed problem with $a < 0$. In that case, the conditions can be less restrictive, for example, with $b \le 0$ and $q = 0$. In contrast, we require that $q > -b$ for our case with $a > 0$.*

3 Some applications

Exact compensation of a boundary input by an initial value input

Let $T > 0$ be given. Let us consider the following well-posed boundary value problem on $\mathbf{R}^+ \times [0, T]$:

$$\begin{aligned} &a\frac{\partial v}{\partial t}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t) + b\frac{\partial v}{\partial x}(x,t) + cv(x,t), \qquad x > 0,\ t \in [0,T], \\ &v(x,0) \equiv v_*(x), \\ &k_0 v(0,t) + k_1 \frac{\partial v}{\partial x}(0,t) \equiv g(t), \quad t > 0. \end{aligned} \tag{3.1}$$

Here $a > 0, b > 0, c, k_0, k_1 \in \mathbf{R}$ are constants such that (1.2) holds.

Theorem 3 *For any $g \in \Gamma$, there exists $v_* \in L_2(\mathbf{R}^+)$ such that $v(x,T) \equiv 0$, where $v \in \mathcal{W}$ is the solution of problem (3.1).*

Proof. The time change $t \to T - t$ transforms problem (3.1) into the equation with initial time at time $t = 0$ in (1.1). It suffices to take the solution $u \in \mathcal{W}$ for $q = 0$ of problem (1.1) and take $v_*(x) = u(x,T)$, $v(x,t) = u(x, T-t)$. □

Theorem 3 gives a solution for the following inverse problem: find the distribution of the initial temperature along the rod such that the temperature at a given time T is a given constant even if there is a variable in time loss/gain of heat at the endpoint.

Restoration of the past distributions for diffusion processes

Assume that there is a probability space. Let $\mathbf{E}$ denote the expectation of a random variable.

Let $w(t)$ be a scalar Wiener process (i.e., it a pathwise continuous Gaussian process with independent increments such that $\mathbf{E}w(t) = 0$ and $\mathbf{E}w(t)^2 = t$). Consider the following stochastic process

$$y(t) = \xi + \beta t + \sigma w(t). \tag{3.2}$$

Here $\beta > 0$ and $\sigma > 0$ are constants, ξ is a random variable such that $\xi \geq 0$, $\mathbf{E}\xi^2 < +\infty$, and ξ is independent from $w(\cdot)$. We assume also that ξ has the probability density function $\rho \in L_2(\mathbf{R}^+)$. However, we assume that this ρ is unknown.

The process $y(t)$ describes the evoultion of a Brownian motion starting at ξ with drift β; it is the limit continuous time model of the random walk.

Set $\tau \triangleq \min\{t > 0 : y(t) = 0\}$.

Let function $p(x,t) : \mathbf{R}^+ \times [0,T] \to \mathbf{R}$ be such that $p(x,0) = \rho(x)$,

$$\int_B p(x,t)dx = \mathbf{E}e^{\lambda t}\mathbb{I}_{\{y(t)\in B\}}\mathbb{I}_{\{\tau\geq t\}}$$

for any interval $B \subset \mathbf{R}^+$. Here $\mathbb{I}$ denote the indicator function of an event.

In fact, $p(x,t)$ is the probability density function of the process $y(t)$ if this process is being killed (absorbed) at 0, and it is being killed inside $(0,+\infty)$, with the rate of killing $(-\lambda)$; the case of $\lambda > 0$ is not excluded. It is known that the evolution of p is described by the parabolic equation being ajoint to (1.1) with the boundary value conditions $p(x,0) \equiv \rho(x)$, $p(0,t) \equiv 0$, and with

$$a = 1/\sigma^2, \quad b = \beta/\sigma^2, \quad c = \lambda/\sigma^2, \quad k_0 = 1, \quad k_1 = 0. \tag{3.3}$$

It is also known that $p(\cdot,T) \in L_2(\mathbf{R}^+)$.

For $g \in \Gamma$, let $u_g = u_g(x,t)$ be the solution of the problem (1.1), (3.3).

Let $T > 0$ be given, and let $\Psi_g(x) \triangleq u_g(x,T)$. By Theorem 2 applied with $q = 0$, it follows that $\Psi_g \in L_2(\mathbf{R}^+)$.

Theorem 4 *For all functions $g \in \Gamma$ and all $\lambda \in \mathbf{R}$,*

$$\mathbf{E}e^{\lambda\tau}g(\tau)\mathbb{I}_{\{\tau<T\}} = -\mathbf{E}e^{\lambda T}\Psi_g(y(T))\mathbb{I}_{\{\tau\geq T\}} = -\int_{\mathbf{R}^+} p(x,T)\Psi_g(x)dx. \tag{3.4}$$

Theorem 4 allows to find the distribution of the first exit times for the Brownian motion when its initial position is random with an unknown probabilistic distribution and when the statistics of the observations is available at the terminal time T and only for particles that didn't achieve the boundary. More precisely, it allows to solve effectively the following inverse problem: find the distribution of $\tau\mathbb{I}_{\{\tau<T\}}$ using the terminal distribution $p(x,T)$ for the case when the distribution $\rho(x) = p(x,0)$ of $\xi = y(0)$ is unknown. Using this theorem, one can find the value of the expectation at the left hand side of (3.4) for any $g \in \Gamma$ via the following algorithm:

(a) Find u_g as the solution of (1.1), (3.3).

(b) Find $\Psi_g = u_g(\cdot, T)$.

(c) Using known $p(x,T)$, calculate the integral at the right hand side of (3.4).

This approach does not require regularization being used for ill-posed problem such as in Beck (1985) or Tikhonov and Arsenin (1977).

Proof of Theorem 4. By the definitions,

$$\mathbf{E}e^{\lambda(\tau\wedge T)}u_g(y(\tau\wedge T),\tau\wedge T) = \mathbf{E}e^{\lambda\tau}g(\tau)\mathbb{I}_{\{\tau<T\}} + \mathbf{E}e^{\lambda T}\Psi_g(y(T))\mathbb{I}_{\{\tau\geq T\}}.$$

By Itô formula,

$$\mathbf{E}e^{\lambda(\tau\wedge T)}u_g(y(\tau\wedge T),\tau\wedge T) = \mathbf{E}u_g(a,0) = 0,$$

since $u_g(x,0)\equiv 0$. In addition,

$$\mathbf{E}e^{\lambda T}\Psi_g(y(T))\mathbb{I}_{\{\tau\geq T\}} = \int_{\mathbf{R}^+} p(x,T)\Psi_g(x)dx.$$

Then the result follows. $\square$

References

Beck, J.V. (1985) *Inverse Heat Conduction.* John Wiley and Sons, Inc..

Dokuchaev, N. (2007) Parabolic equations with the second order Cauchy conditions on the boundary. *Journal of Physics A: Mathematical and Theoretical.* **40**, pp. 12409–12413.

Duren, P. (1970) *Theory of H^p-Spaces.* Academic Press, New York.

Glasko V.(1984). Inverse Problems of Mathematical physics. American Institute of Physics. New York.

Prilepko A.I., Orlovsky D.G., Vasin I.A. (1984). Methods for Solving Inverse Problems in Mathematical Physics. Dekker, New York.

Tikhonov, A. N. and Arsenin, V. Y. (1977) *Solutions of Ill-posed Problems.* W. H. Winston, Washington, D. C.

Yosida, K. (1995) *Functional Analysis.* Springer, Berlin Heilderberg New York.