SIAM J. CONTROL OPTIM.
Vol. 50, No. 1, pp. 334–356

STRUCTURAL INVARIANTS OF TWO-DIMENSIONAL SYSTEMS*

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Abstract. In this paper, some fundamental structural properties of two-dimensional (2-D) systems which remain invariant under feedback and output-injection transformation groups are identified and investigated for the first time. As is well known, structural invariants that follow from the definition of controlled and conditioned invariance, output-nulling, input-containing, self-bounded and self-hidden subspaces play pivotal roles in many theoretical studies of systems theory and in the solution of several control/estimation problems. These concepts are developed and studied within a 2-D context in this paper.

Key words. two-dimensional systems, geometric control, controlled and conditioned invariance, self-boundedness and self-hiddenness

AMS subject classifications. 93C05, 93B27, 93B52

DOI. 10.1137/100815153

1. Introduction. The fundamental notion upon which classic geometric control theory hinges is that of controlled invariance. For one-dimensional (1-D) systems governed by the standard linear time-invariant state difference equation $x_{k+1} = A\,x_k + B\,u_k$, controlled invariant subspaces—also known as (A, B)-invariant or $A(\mathrm{mod}\,B)$-invariant subspaces—were introduced in the pioneering paper [1] as the subspaces satisfying $A\,\mathcal{V} \subseteq \mathcal{V} + \mathrm{im}\;B$. These subspaces fully characterize the state trajectories generated by the system, in the sense that

(a) If the initial state x_0 lies on a controlled invariant subspace $\mathcal{V}$, a control u_k exists that maintains the entire state trajectory on the same subspace $\mathcal{V}$;

(b) Conversely, if for any initial condition on a subspace $\mathcal{L}$ the entire trajectory can be kept on $\mathcal{L}$ with a suitable control function, $\mathcal{L}$ is controlled invariant.

In the usual 1-D context, controlled invariance also enjoys a fundamental feedback property:

(c) The control input that maintains the state trajectory on a controlled invariant subspace can always be expressed in terms of a static state feedback input $u_k = F\,x_k$.

Conditioned invariance for 1-D systems—also referred to as (C, A)-invariance—was also introduced in [1] as the dual of controlled invariance. In the last 40 years, controlled and conditioned invariant subspaces have played a pivotal role in the solution of a vast number of control and estimation problems, including disturbance decoupling, unknown-input observation, model matching, noninteraction, and optimal control/filtering problems; see, e.g., [25, 3, 23] and the references cited therein. For this reason, several efforts have been devoted to extend the notion of controlled invariance to two-dimensional (2-D) systems. The first paper containing a definition of controlled invariance for 2-D Fornasini–Marchesini first-order models [9]

$$x_{i+1,j+1} = A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B_1\,u_{i+1,j} + B_2\,u_{i,j+1} \tag{1.1}$$

*Received by the editors November 16, 2010; accepted for publication (in revised form) October 13, 2011; published electronically January 26, 2012. This work was supported by the Australian Research Council (Discovery Grant DP0986577).
http://www.siam.org/journals/sicon/50-1/81515.html

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is [7], where a 2-D controlled invariant subspace is defined as a subspace $\mathcal{V}$ satisfying $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right] \mathcal{V} \subseteq (\mathcal{V} \oplus \mathcal{V}) + \operatorname{im} \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$. This subspace inclusion extends the 1-D counterpart in a natural way. While it is true that given boundary conditions on a subspace $\mathcal{V}$ satisfying this definition a control input can always be designed to maintain the entire solution of (1.1) on $\mathcal{V}$, and that such control can be expressed as a static feedback $u_{i,j} = F\,x_{i,j}$, the 2-D analogue of property (b) now does not hold, i.e., a control might exist maintaining the solution of (1.1) on a certain subspace $\mathcal{L}$ for any $\mathcal{L}$-valued boundary condition without $\mathcal{L}$ necessarily being controlled invariant for (1.1). Hence, this definition of controlled invariance enjoys good feedback properties as shown in [21], but it does not characterize the set of trajectories generated by (1.1) univocally. Therefore, when this definition is used in the solution of decoupling, control, and estimation problems, it can lead only to sufficient—and hence conservative—conditions; see Remark 3.2 in [7]. A second definition of controlled invariance for 2-D systems was given in [16] for the 2-D singular model

$$E\,x_{i+1,j+1} = A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,u_{i,j}. \tag{1.2}$$

This model was considered because the purpose was to ultimately characterize the solutions of a singular 2-D system in Roesser form over a bounded frame that could be brought back to the form (1.2). For the aims of this paper, we can consider E to be the identity matrix. According to the definition given in [16], a controlled invariant subspace for (1.2) with $E = I_n$ is a subspace $\mathcal{V}$ satisfying $A_1\,\mathcal{V} + A_2\,\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. Both properties (a) and (b) extend to this definition. But the drawback of this model is the lack of a static feedback characterization of controlled invariance since the structure of (1.2) is not closed under the feedback control $u_{i,j} = F\,x_{i,j}$, in the sense that the closed-loop model obtained applying such control has a different structure from the one in (1.2). Hence, property (c) has no meaning in this case. To combine the advantages of these two definitions without incurring in their drawbacks, here we define controlled invariance for the Fornasini–Marchesini original model

$$x_{i+1,j+1} = A_0\,x_{i,j} + A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,u_{i,j} \tag{1.3}$$

[8], which like the model (1.1) can realize any strictly causal bivariate rational function but where, different from (1.1), the input appears only once. This model is closed under static feedback controls $u_{i,j} = F\,x_{i,j}$; moreover, unlike (1.1), its dual is also well-defined, so that conditioned invariance can be introduced in a natural way. In this paper we show that the definition of controlled invariance given in [16] can be trivially extended to models in the form (1.3), thus retaining the fundamental properties (a) and (b) of the 1-D case. Even though for (1.3) a static feedback input is well-defined, such definition does not imply that the feedback property (c) automatically holds for 2-D controlled invariant subspaces. However, we will show that it is possible to introduce a well-characterized restriction of the set of 2-D controlled invariant subspaces of (1.3) which is locus of solutions of (1.3) generated by the input $u_{i,j} = F\,x_{i,j}$. In other words, we show that similar to what happens for systems over rings [12], there is no exact counterpart of the definition of controlled invariance for 2-D systems that retains all three properties (a), (b), (c), but we have to distinguish between the subspaces for which (a) and (b) are satisfied (thus leading to the notion of controlled invariance) and then restrict this set of subspaces to identify those where the solutions of (1.3) generated by a static feedback input lie (and this will lead to the notion of controlled invariance of feedback type). This definition, with respect to the one in [7], characterizes univocally and in finite terms the subspace

of trajectories of a 2-D system that are generated by static feedback controls. This enables, for example, a solution of the classic disturbance decoupling problem to be derived in terms of constructive conditions that eliminate the conservativeness of the solutions proposed so far in the literature; see [7] and [21]. Then, the problem of parameterizing the set of feedback inputs $u_{i,j} = F\,x_{i,j}$ that generate solutions of (1.3) on controlled invariant subspaces of feedback type is taken into account, along with the problem of internal and external stabilization of the corresponding solutions with respect to that subspace. The dual notion is called 2-D conditioned invariance of output-injection type and is shown to completely characterize the solution of local state reconstruction in presence of unknown inputs. The notion of self-boundedness [2] is also generalized for the first time to 2-D systems. It is also shown that the minimum of the set of self-bounded subspaces, which plays an important role in the solution of output-nulling and decoupling problems with maximum assignment of the closed-loop dynamics [18], is not given as in the 1-D case as the intersection of the largest output-nulling and the smallest input-containing subspace of a 2-D system, [19]. An algorithm is provided for the computation of such minimum. In this paper, several other structural properties of 2-D systems are introduced and investigated from a geometric perspective. Indeed, we show how reachability and observability can be characterized geometrically as invariants under the system dynamic maps contained in or containing suitable subspaces. A simple algorithm is presented for the computation of a basis of the reachability and nonobservability subspaces of a 2-D system, and it is shown how a 2-D counterpart of the Kalman canonical decomposition can be constructed. These considerations on reachability of 2-D systems are then exploited in conjunction with the definition of controlled invariance of feedback type to characterize—for the first time in a 2-D context—the notion of controllability subspaces.

2. Problem formulation.

Consider the Fornasini–Marchesini 2-D model [8]

$$x_{i+1,j+1} = A_0\,x_{i,j} + A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,u_{i,j}, \tag{2.1}$$

$$y_{i,j} = C\,x_{i,j}, \tag{2.2}$$

where for all $i,j \in \mathbb{Z}$, the vector $x_{i,j} \in \mathbb{R}^n$ is the *local state*, $u_{i,j} \in \mathbb{R}^m$ is the input, and $y_{i,j} \in \mathbb{R}^p$ is the output. Here, $A_0,\ A_1,\ A_2 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. For the sake of brevity, we identify the system (2.1)–(2.2) with the quintuple $\Sigma \stackrel{\text{def}}{=} (A_0, A_1, A_2; B; C)$. To define appropriate boundary conditions for Σ, we introduce the sets

$$\mathfrak{Q}_i \stackrel{\text{def}}{=} (\{i\} \times \{j \in \mathbb{Z}\,|\, j \geq i\}) \cup (\{j \in \mathbb{Z}\,|\, j \geq i\} \times \{i\}).$$

A suitable set of boundary conditions for (2.1)–(2.2) is given by assigning the local state $x_{i,j}$ for all $(i,j) \in \mathfrak{Q}_0$. An alternative set of boundary conditions for (2.1)–(2.2) is given by defining for each $k \in \mathbb{Z}$ the separation sets $\mathfrak{C}_k \stackrel{\text{def}}{=} \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \;\big|\; i+j=k\}$ (see also [9]) and defining boundary conditions for (2.1)–(2.2) as the assignment of the local state $x_{i,j}$ over two consecutive separation sets, e.g., for all $(i,j) \in \mathfrak{C}_{-1} \cup \mathfrak{C}_0$. The fact that for a valid definition of a boundary condition the local state must be assigned over two adjacent separation sets is due to the intrinsically second-order recursion in (2.1). To treat these boundary conditions in a unified framework, we introduce the symbol $\mathfrak{B}$ to identify any boundary condition for (2.1), including $\mathfrak{Q}_0$ or $\mathfrak{C}_{-1} \cup \mathfrak{C}_0$. We use the symbol $\mathfrak{B}_1$ to denote the first forward boundary, i.e., $\mathfrak{Q}_1$ or $\mathfrak{C}_0 \cup \mathfrak{C}_1$, respectively. Similarly, $\mathfrak{B}_i$ denotes the ith forward boundary. We also denote by $\mathfrak{B}^+$ the set of indexes defined by iterating the recursion (2.1) starting from indexes over $\mathfrak{B}$. Hence

- If $\mathfrak{B} = \mathfrak{Q}_0$, then $\mathfrak{B}^+ = \bigcup_{i=0}^{\infty} \mathfrak{Q}_i$;
- If $\mathfrak{B} = \mathfrak{C}_{-1} \cup \mathfrak{C}_0$, then $\mathfrak{B}^+ = \bigcup_{i=-1}^{\infty} \mathfrak{C}_i$.

Given a subspace $\mathcal{W}$ of $\mathbb{R}^n$, we say that (2.1) has a $\mathcal{W}$-valued boundary condition if $x_{i,j} \in \mathcal{W}$ for all $(i,j) \in \mathfrak{B}$ and that (2.1) has a $\mathcal{W}$-valued solution if $x_{i,j} \in \mathcal{W}$ for all $(i,j) \in \mathfrak{B}^+$.

3. 2-D controlled and conditioned invariance.

DEFINITION 3.1. *A subspace $\mathcal{V}$ is a 2-D controlled invariant subspace for Σ if it is simultaneously a 1-D controlled invariant subspace for the pairs (A_0, B), (A_1, B), and (A_2, B), i.e., if*

$$A_i \, \mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B, \quad i \in \{0,1,2\}. \tag{3.1}$$

The set of 2-D controlled invariants is always nonempty, as it contains at least $\{0\}$ and $\mathbb{R}^n$. Since the sum of 1-D controlled invariant subspaces is a controlled invariant subspace [3, Property 4.1.1], the same is trivially true for 2-D controlled invariant subspaces. In general, however, the intersection of two (1-D or 2-D) controlled invariant subspaces is not a (1-D or 2-D) controlled invariant subspace. The following theorem presents the fundamental system-theoretic interpretation of Definition 3.1 and is a simple extension of Theorem 2.2 in [16].

THEOREM 3.2. *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$. Equation* (2.1) *has a $\mathcal{V}$-valued solution for any $\mathcal{V}$-valued boundary condition if and only if $\mathcal{V}$ is a 2-D controlled invariant subspace.*

Proof. **(Necessity).** Suppose (3.1) does not hold. Then, there exist ξ', ξ'', ξ''' in $\mathcal{V}$ such that no $\omega \in \mathbb{R}^m$ exists for which $A_0 \,\xi' + A_1 \,\xi'' + A_2 \,\xi''' + B\,\omega \in \mathcal{V}$ holds. This means that for a boundary condition with $x_{0,0} = \xi'$, $x_{1,0} = \xi''$ and $x_{0,1} = \xi'''$, a control $u_{0,0}$ cannot be found such that $x_{1,1} \in \mathcal{V}$. Hence, (2.1) does not have a $\mathcal{V}$-valued solution with this $\mathcal{V}$-valued boundary condition.

(Sufficiency). The sufficiency is obvious. ∎

Remark 3.1. The definition of 2-D controlled invariance proposed in [7], while convenient for its feedback properties, does not guarantee that a subspace $\mathcal{V}$ for which the system has a $\mathcal{V}$-valued solution for any $\mathcal{V}$-valued boundary condition is a 2-D controlled invariant subspace. In other words, for the model (1.1) and the definitions currently available for 2-D controlled invariance, the *only if* part of Theorem 3.2 does not hold.

The dual concept of 2-D controlled invariance is called 2-D conditioned invariance.

DEFINITION 3.3. *A subspace $\mathcal{S}$ is a 2-D conditioned invariant subspace for Σ if it is simultaneously 1-D conditioned invariant for the pairs (A_0, C), (A_1, C), and (A_2, C), i.e., if*

$$A_i \,(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}, \qquad i \in \{0,1,2\}.$$

The duality between 2-D controlled and conditioned invariance can be stated in precise terms as follows. Let $\Sigma^\top$ identify the dual system of (2.1)–(2.2), i.e., $\Sigma^\top \stackrel{\text{def}}{=} (A_0^\top, A_1^\top, A_2^\top; C^\top; B^\top)$.

LEMMA 3.4. *The orthogonal complement of a 2-D controlled invariant subspace for Σ is a 2-D conditioned invariant subspace for $\Sigma^\top$, and* vice-versa.

Proof. Let $\mathcal{L}$ be a 2-D controlled invariant subspace for Σ. Let $i \in \{0,1,2\}$. From $A_i \,\mathcal{L} \subseteq \mathcal{L} + \operatorname{im} B$, we get $A_i^\top \,(\mathcal{L} + \operatorname{im} B)^\perp \subseteq \mathcal{L}^\perp$, which in turn yields $A_i^\top \,(\mathcal{L}^\perp \cap \ker B^\top) \subseteq \mathcal{L}^\perp$. Therefore, $\mathcal{L}$ is a 2-D conditioned invariant subspace for $\Sigma^\top$. The same steps can be easily reversed to show that the opposite implication holds as well. ∎

Since the intersection of 1-D conditioned invariant subspaces is a 1-D conditioned invariant subspace [3, Property 4.1.2], the same is trivially true for 2-D conditioned invariant subspaces. In general, however, this is not true for the sum of conditioned invariant subspaces.

4. Output-nulling and input-containing subspaces. In many control problems it is of interest to derive control laws that maintain certain outputs of a system at zero. The most famous example is the disturbance decoupling problem [3]. This requirement leads to the notion of output-nulling subspace. An output-nulling subspace for Σ is such that (2.1)–(2.2) have a $\mathcal{V}$-valued solution with an identically zero output for any $\mathcal{V}$-valued boundary condition. A solution of (2.1)–(2.2) yields zero output if and only if for all $(i,j) \in \mathfrak{B}^+$ the local state $x_{i,j}$ lies in $\ker C$. Hence, an output-nulling subspace is simply a 2-D controlled invariant subspace contained in $\ker C$. The set of output-nulling subspaces of (2.1)–(2.2) is denoted by $\mathcal{V}(\Sigma)$. As for the set of 2-D controlled invariant subspaces, this set is seen to be closed under subspace addition but not under subspace intersection. Therefore, $(\mathcal{V}(\Sigma), +; \subseteq)$ is a (nondistributive and modular) upper semilattice with respect to the binary operation $+$ and with respect to the partial ordering $\subseteq$. Thus, it admits a maximum $\mathcal{V}^\star$ given by the sum of all elements of $\mathcal{V}(\Sigma)$, i.e., $\mathcal{V}^\star \stackrel{\text{def}}{=} \max \mathcal{V}(\Sigma) = \sum_{\mathcal{V} \in \mathcal{V}(\Sigma)} \mathcal{V}$. The following lemma extends the famous algorithm for the computation of $\mathcal{V}^\star$ introduced in [1].

LEMMA 4.1. *$\mathcal{V}^\star$ is the last term of the monotonically nonincreasing sequence*

$$\{\mathcal{V}_i\}_i : \quad \begin{cases} \mathcal{V}_0 = \ker C, \\ \mathcal{V}_i = \displaystyle\bigcap_{j=0}^{2} A_j^{-1}(\mathcal{V}_{i-1} + \operatorname{im} B) \cap \ker C, \qquad i \in \{1, 2, \ldots, k_v\}, \end{cases}$$

where the integer $k_v \leq n-1$ is determined by the condition $\mathcal{V}_{k_v+1} = \mathcal{V}_{k_v}$, i.e., $\mathcal{V}^\star = \mathcal{V}_{k_v}$.

Proof. First, we show by induction that the sequence $\{\mathcal{V}_i\}_i$ is monotonically nonincreasing. Trivially $\mathcal{V}_0 \supseteq \mathcal{V}_1$. Let $\mathcal{V}_{h-1} \supseteq \mathcal{V}_h$. We show that $\mathcal{V}_h \supseteq \mathcal{V}_{h+1}$. From $\mathcal{V}_{h-1} \supseteq \mathcal{V}_h$ we get

$$\mathcal{V}_h = \bigcap_{j=0}^{2} A_j^{-1}(\mathcal{V}_{h-1} + \operatorname{im} B) \cap \ker C \supseteq \bigcap_{j=0}^{2} A_j^{-1}(\mathcal{V}_h + \operatorname{im} B) \cap \ker C = \mathcal{V}_{h+1}.$$

We have proved that the sequence $\{\mathcal{V}_i\}_i$ is monotonically nonincreasing. This also implies that if $\mathcal{V}_{k_v+1} = \mathcal{V}_{k_v}$, also $\mathcal{V}_j = \mathcal{V}_{k_v}$ for all $j \geq k_v$. As such, since two subsequent subspaces are equal if and only if they have equal dimensions and the dimension of $\mathcal{V}_0$ is at least one, the dimension of the subspace $\mathcal{V}_{k+1}$ of $\{\mathcal{V}_i\}_i$ must decrease by at least one with respect to the dimension of $\mathcal{V}_k$ before stationarity is reached. Therefore, stationary of the sequence $\{\mathcal{V}_i\}_i$ is reached in at most n steps. Let k_v denote the index at which the sequence $\{\mathcal{V}_i\}_i$ becomes stationary. We show that $\mathcal{V}_{k_v}$ is a 2-D controlled invariant subspace. First, notice that

$$\mathcal{V}_{k_v} = \bigcap_{j=0}^{2} A_j^{-1}(\mathcal{V}_{k_v} + \operatorname{im} B) \cap \ker C. \tag{4.1}$$

Let $h \in \{0,1,2\}$. By applying A_h to both sides of (4.1) we obtain

$$A_h\,\mathcal{V}_{k_v} = A_h\left(\bigcap_{j=0}^{2} A_j^{-1}(\mathcal{V}_{k_v} + \text{im } B) \cap \ker C\right) \subseteq \bigcap_{j=0}^{2} A_h\,A_j^{-1}(\mathcal{V}_{k_v} + \text{im } B) \cap (A_h \ker C)$$
$$\subseteq A_h\,A_h^{-1}(\mathcal{V}_{k_v} + \text{im } B) \subseteq (\mathcal{V}_{k_v} + \text{im } B) \cap \text{im}\,A_h \subseteq \mathcal{V}_{k_v} + \text{im } B.$$

Therefore, $\mathcal{V}_{k_v}$ is a 1-D controlled invariant subspace for (A_h, B) with $h \in \{0,1,2\}$. Hence, it is also a 2-D controlled invariant subspace. Given the way $\{\mathcal{V}_i\}_i$ has been constructed, $\mathcal{V}_i \subseteq \ker C$, so that $\mathcal{V}_{k_v}$ is also output-nulling. We show that $\mathcal{V}_{k_v}$ is the largest output-nulling subspace for Σ, so that it coincides with $\mathcal{V}^\star$. Let $\tilde{\mathcal{V}}$ be another output-nulling subspace, so that $\tilde{\mathcal{V}}$ is a 2-D controlled invariant subspace for (A_h, B) for $h \in \{0,1,2\}$ and $\tilde{\mathcal{V}}$ is contained in $\ker C$. Hence, $\tilde{\mathcal{V}} \subseteq A_h^{-1}(\tilde{\mathcal{V}} + \text{im } B)$. Since this is true for each $h \in \{0,1,2\}$, we find

$$\tilde{\mathcal{V}} \subseteq \bigcap_{j=0}^{2} A_j^{-1}(\tilde{\mathcal{V}} + \text{im } B) \cap \ker C. \tag{4.2}$$

Now we show that every term of $\{\mathcal{V}_i\}_i$ contains $\tilde{\mathcal{V}}$, so that, in particular, $\mathcal{V}_{k_v} \supseteq \tilde{\mathcal{V}}$. Clearly, $\mathcal{V}_0 \supseteq \tilde{\mathcal{V}}$. Suppose $\mathcal{V}_i \supseteq \tilde{\mathcal{V}}$. Thus

$$\mathcal{V}_{i+1} = \bigcap_{j=0}^{2} A_j^{-1}(\mathcal{V}_i + \text{im } B) \cap \ker C \supseteq \bigcap_{j=0}^{2} A_j^{-1}(\tilde{\mathcal{V}} + \text{im } B) \cap \ker C,$$

which in turn contains $\tilde{\mathcal{V}}$ in view of (4.2). Hence, $\mathcal{V}_{k_v} \supseteq \tilde{\mathcal{V}}$. This implies that $\mathcal{V}^\star = \mathcal{V}_{k_v}$. ∎

A further consequence of Remark 3.1 is that using the definition of 2-D controlled invariance given in [7] and the subsequent definition of $\mathcal{V}^\star$, the fact that a solution of (1.1) leads to an identically zero output does not necessarily imply that the local state lies on $\mathcal{V}^\star$. This is what causes the inherent conservativeness in the solutions of decoupling and observation problems that use such definition. On the contrary, the fact that the statement of Theorem 3.2 is both necessary and sufficient for the definition of 2-D controlled invariance used in this paper guarantees that $\mathcal{V}^\star$ univocally characterizes the solutions of Σ that lead to zero outputs, and this will lead to a solution of output-nulling and disturbance decoupling problems in terms of necessary and sufficient conditions. This will also lead to the new notion of self-boundedness (and, by duality, self-hiddenness) of 2-D systems.

The duals of 2-D output-nulling subspaces are the 2-D *input-containing* subspaces. A 2-D input-containing subspace $\mathcal{S}$ is a 2-D conditioned invariant subspace that contains $\text{im}\,B$.

The set of input-containing subspaces of Σ is denoted by $\mathcal{S}(\Sigma)$. This set is closed under subspace intersection but not under subspace addition. Therefore, $(\mathcal{S}(\Sigma), \cap; \subseteq)$ is a (nondistributive and modular) lower semilattice with respect to the binary operation $\cap$ and with respect to the partial ordering $\subseteq$. Thus, it admits a minimum given by $\mathcal{S}^\star = \min\,\mathcal{S}(\Sigma) = \bigcap_{\mathcal{S} \in \mathcal{S}(\Sigma)} \mathcal{S}$. By dualizing the algorithm for $\mathcal{V}^\star$, we have the following.

LEMMA 4.2. *$\mathcal{S}^\star$ is the last term of the monotonically nondecreasing sequence*

$$\{\mathcal{S}_i\}_i : \quad \begin{cases} \mathcal{S}_0 = \text{im}\, B, \\ \mathcal{S}_i = \displaystyle\sum_{j=0}^{2} A_j (\mathcal{S}_{i-1} \cap \ker C) + \text{im}\, B, \qquad i \in \{1, 2, \dots, k_s\}, \end{cases}$$

where the integer $k_s \le n-1$ is determined by the condition $\mathcal{S}_{k_s+1} = \mathcal{S}_{k_s}$, i.e., $\mathcal{S}_{k_s} = \mathcal{S}^\star$.

Using Lemma 3.4, it is straightforward to prove that the orthogonal complement of an output-nulling subspace for Σ is input-containing for $\Sigma^\top$ and vice versa. In particular, $\mathcal{V}^\star$ and $\mathcal{S}^\star$ are dual, i.e., $\max \mathcal{V}(\Sigma) = (\min \mathcal{S}(\Sigma^\top))^\perp$.

5. 2-D controlled invariant and output-nulling subspaces of feedback type. The Fornasini–Marchesini model (2.1)–(2.2), unlike the model used in [16], is closed under the feedback $u_{i,j} = F\,x_{i,j}$, which leads to the closed-loop equation

$$x_{i+1,j+1} = (A_0 + B\,F)\,x_{i,j} + A_1\,x_{i+1,j} + A_2\,x_{i,j+1}. \tag{5.1}$$

It is easy to see that the notion of 2-D controlled invariance alone is not sufficient to guarantee the existence of a feedback matrix F that maintains the local state $x_{i,j}$ on a 2-D controlled invariant subspace $\mathcal{V}$ for $\mathcal{V}$-valued boundary conditions. For this reason, we introduce the concept of 2-D *controlled invariance of feedback type*.

DEFINITION 5.1. *Subspace $\mathcal{W}$ is a 2-D controlled invariant of feedback type for Σ if*

- *$\mathcal{W}$ is a 1-D controlled invariant subspace for (A_0, B);*
- *$\mathcal{W}$ is both A_1- and A_2-invariant.*

The next theorem shows that this definition completely characterizes the subspaces of trajectories of a 2-D system generated by a feedback law in which the solutions of Σ lie.

THEOREM 5.2. *A subspace $\mathcal{W}$ is a 2-D controlled invariant subspace of feedback type for Σ if and only if there exists a static feedback control $u_{i,j} = F\,x_{i,j}$ such that for any $\mathcal{W}$-valued boundary condition of Σ, $x_{i,j} \in \mathcal{W}$ for all $(i,j) \in \mathfrak{B}^+$.*

Proof. **(Only if).** Since $\mathcal{W}$ is a 2-D controlled invariant subspace of feedback type for Σ, it is a 1-D controlled invariant subspace for (A_0, B), and this implies that two matrices X_0 and Ω exist such that $A_0\,W = W\,X_0 + B\,\Omega$, where W is a basis of $\mathcal{W}$ (i.e., im $W = \mathcal{W}$ and $\ker W = \{0\}$). Since $\mathcal{W}$ is A_1- and A_2-invariant, two matrices X_1 and X_2 exist such that $A_1\,W = W\,X_1$ and $A_2\,W = W\,X_2$. Since W is full column-rank, we can solve the linear equation $\Omega = -F\,W$ in F, and with its solution we get $(A_0 + B\,F)\,W = W\,X_0$. With this F in the closed-loop system (5.1), we see that if $x_{i,j}, x_{i+1,j}$ and $x_{i,j+1}$ are in $\mathcal{W}$, so is $x_{i+1,j+1}$, and then for any $\mathcal{W}$-valued boundary condition, $x_{i,j}$ remains in $\mathcal{W}$ for all $(i,j) \in \mathfrak{B}^+$.

(If). By virtue of (5.1), the inclusion $[\,A_0 + B\,F \quad A_1 \quad A_2\,](\mathcal{W} \oplus \mathcal{W} \oplus \mathcal{W}) \subseteq \mathcal{W}$ must hold; otherwise it would be possible to find $x_{0,0}$, $x_{1,0}$, $x_{0,1} \in \mathcal{W}$ such that $x_{1,1}$ does not lie on $\mathcal{W}$; this implies that $(A_0 + B\,F)\mathcal{W} \subseteq \mathcal{W}$—which means that $\mathcal{W}$ is a 1-D controlled invariant subspace for (A_0, B)—and $A_1\,\mathcal{W} \subseteq \mathcal{W}$ and $A_2\,\mathcal{W} \subseteq \mathcal{W}$. ∎

Given a 2-D controlled invariant subspace of feedback type for Σ, any feedback matrix $F \in \mathbb{R}^{m \times n}$ such that $u_{i,j} = F\,x_{i,j}$ generates a $\mathcal{W}$-valued solution for any $\mathcal{W}$-valued boundary condition is called a *friend* of $\mathcal{W}$. Notice that the **(If)** part of Theorem 5.2 does not hold for any other definition given so far for 2-D controlled invariant subspaces.

The notion of 2-D controlled invariant subspaces of feedback type can be extended to output-nulling subspaces. A subspace $\mathcal{W}$ is output-nulling of feedback type if it is a 2-D controlled invariant subspace of feedback type contained in the null-space of C. Also, $\mathcal{W}$ is output-nulling of feedback type if and only if F exists such that for any $\mathcal{W}$-valued boundary condition, the solution of (5.1) is $\mathcal{W}$-valued and $y_{i,j} = C\,x_{i,j}$ is zero for all $(i,j) \in \mathfrak{B}^+$. The set of output-nulling subspaces of feedback type, denoted by $\mathcal{W}(\Sigma)$, is closed under addition. Hence, the maximum output-nulling subspace $\mathcal{W}^\star$ of feedback type can still be defined as the sum of the elements of $\mathcal{W}(\Sigma)$. An algorithm for the computation of $\mathcal{W}^\star$ is given below.

LEMMA 5.3. *$\mathcal{W}^\star$ is the last term of the monotonically nonincreasing sequence*

$\{\mathcal{W}_i\}_i$:

$$\begin{cases} \mathcal{W}_0 = \ker C, \\ \mathcal{W}_i = \ker C \cap A_0^{-1}(\mathcal{W}_{i-1} + \operatorname{im} B) \cap A_1^{-1}\,\mathcal{W}_{i-1} \cap A_2^{-1}\,\mathcal{W}_{i-1}, \qquad i \in \{1,2,\ldots,k_w\}, \end{cases}$$

where the integer $k_w \le n-1$ is determined by the condition $\mathcal{W}_{k_w+1} = \mathcal{W}_{k_w}$, i.e., $\mathcal{W}^\star = \mathcal{W}_{k_w}$.

The proof follows from that of Lemma 4.1 with the obvious modifications.

Example 5.1. Consider the 2-D system with

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 3 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

In this example, the largest output-nulling subspace does not coincide with the largest output-nulling subspace of feedback type. In fact, by using Lemma 4.1 and 5.3, we find

$$\mathcal{V}^\star = \operatorname{im} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{W}^\star = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

A direct check shows that $F = \frac{1}{20}\left[\begin{smallmatrix} 0 & 0 & -38 \\ 0 & 1 & 0 \end{smallmatrix}\right]$ is a friend of $\mathcal{W}^\star$.

6. 2-D conditioned invariant subspaces of output-injection type. We consider the dual of 2-D controlled invariance of feedback type, which we name 2-D *conditioned invariance of output-injection type*. We relate this concept to the existence of certain types of observers that maintain the information of specific components of the local state vector in presence of unknown inputs; this approach closely follows the 1-D one developed in [24] and thoroughly discussed in Chapter 5 of [23].

DEFINITION 6.1. *Subspace $\mathcal{Z}$ is a 2-D conditioned invariant subspace of output-injection type for Σ if*

- *$\mathcal{Z}$ is a 1-D conditioned invariant subspace for (A_0, C);*
- *$\mathcal{Z}$ is both A_1- and A_2-invariant.*

Hence, $\mathcal{Z}$ is a 2-D conditioned invariant subspace of output-injection type for Σ if and only if an output-injection matrix $G \in \mathbb{R}^{n \times p}$ exists such that $(A_0 + G\,C)\,\mathcal{Z} \subseteq \mathcal{Z}$, $A_1\,\mathcal{Z} \subseteq \mathcal{Z}$, $A_2\,\mathcal{Z} \subseteq \mathcal{Z}$. Hence, a 2-D input-containing subspace of output-injection type is a 1-D conditioned invariant subspace for (A_0, C) which contains the image of B, and it is both A_1- and A_2-invariant.

The following lemma shows that 2-D conditioned invariant and input-containing subspaces of output-injection type and 2-D controlled invariant and output-nulling subspaces of feedback type are dual concepts.

LEMMA 6.2. *The orthogonal complement of a 2-D controlled invariant (resp., output-nulling) subspace of feedback type for Σ is a 2-D conditioned invariant (resp., input-containing) subspace of output-injection type for $\Sigma^\top$ and vice versa.*

Now we relate the concept of conditioned invariance of output-injection type with the existence of certain local state reconstructors that maintain information on the local state of Σ modulo a certain subspace. More precisely, given the subspace $\mathcal{Z}$ of $\mathbb{R}^n$ and a full row-rank matrix Q such that $\mathcal{Z} = \ker Q$, we define (with a slight abuse of nomenclature) a $\mathcal{Z}$-observer for (2.1)–(2.2) as a system ruled by the recursion

$$\omega_{i+1,j+1} = K_0\,\omega_{i,j} + K_1\,\omega_{i+1,j} + K_2\,\omega_{i,j+1} + L\,y_{i,j} \tag{6.1}$$

such that if $\omega_{i,j} = Q\,x_{i,j}$ for all $(i,j) \in \mathfrak{B}$, then $\omega_{i,j} = Q\,x_{i,j}$ for all $(i,j) \in \mathfrak{B}^+$. In other words, a $\mathcal{Z}$-observer maintains the knowledge of the components of the local state that are external to $\mathcal{Z}$, i.e., if $\omega_{i,j} = x_{i,j}/\mathcal{Z}$ on the boundary, then $\omega_{i,j} = x_{i,j}/\mathcal{Z}$ everywhere. Thus, if the initial conditions of the system and of the observer are equal modulo $\mathcal{Z}$, the state of the observer is always equal to the local state of the system modulo $\mathcal{Z}$. The 2-D system (6.1) is here referred to as an observer even if as a matter of fact it only maintains information on the local state of Σ, but it does not reconstruct such information in the case of mismatched boundary conditions of Σ and (6.1). In the case in which (6.1) is capable of recovering the local state of Σ modulo $\mathcal{Z}$ with greater accuracy as the index (i,j) evolves away from the boundary, system (6.1) will be referred to as an *asymptotic $\mathcal{Z}$-observer*; see section 6.2.

The following theorem provides a geometric characterization of conditioned invariance of output-injection type.

THEOREM 6.3. *A subspace $\mathcal{Z}$ is an input-containing subspace of output-injection type for Σ if and only if there exists a $\mathcal{Z}$-observer for Σ.*

Proof. **(Only if).** Let $\mathcal{Z}$ be an input-containing subspace for (A_0, C), and invariant with respect to A_1 and A_2. We can write $\mathcal{Z} = \ker Q$, where Q satisfies the linear equations $Q\,A_0 = \Gamma_0\,Q + \Lambda\,C$, $Q\,A_1 = \Gamma_1\,Q$, $Q\,A_2 = \Gamma_2\,Q$, $Q\,B = 0$. Consider (6.1) with $K_i = \Gamma_i$ for $i \in \{0,1,2\}$, and $L = \Lambda$. Define the error as $e_{i,j} \stackrel{\text{def}}{=} Q\,x_{i,j} - \omega_{i,j}$. Since it is assumed that $\omega_{i,j} = Q\,x_{i,j}$ over the boundary $\mathfrak{B}$, $e_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}$. Thus,

$$\begin{aligned} e_{i+1,j+1} &= Q\,x_{i+1,j+1} - \omega_{i+1,j+1} \\ &= Q\,A_0\,x_{i,j} + Q\,A_1\,x_{i+1,j} + Q\,A_2\,x_{i,j+1} \\ &\quad - \Gamma_0\,\omega_{i,j} - \Gamma_1\,\omega_{i+1,j} - \Gamma_2\,\omega_{i,j+1} - \Lambda\,C\,x_{i,j} \\ &= \Gamma_0\,e_{i,j} + \Gamma_1\,e_{i+1,j} + \Gamma_2\,e_{i,j+1}. \end{aligned} \tag{6.2}$$

Since these dynamics are autonomous, the error is zero everywhere if it is zero over $\mathfrak{B}$.

(If). There exists a $\mathcal{Z}$-observer for Σ. Therefore, given $\omega_{i,j} = Q\,x_{i,j}$ over the boundary $\mathfrak{B}$, we have $\omega_{i,j} = Q\,x_{i,j}$ over $\mathfrak{B}^+$. Let the boundary condition of (2.1) be such that $x_{0,0} \in \mathcal{Z} \cap \ker C$, $x_{1,0} \in \mathcal{Z}$ and $x_{0,1} \in \mathcal{Z}$. The boundary condition of the $\mathcal{Z}$-observer is such that $\omega_{0,0} = \omega_{1,0} = \omega_{0,1} = 0$. This is compatible with the fact that $\omega_{i,j} = Q\,x_{i,j}$ for $(i,j) \in \{(0,0),(1,0),(0,1)\}$, since for such pairs of indexes we have $x_{i,j} \in \mathcal{Z}$, and hence $Q\,x_{i,j} = 0$. Therefore, from (6.1) it is found that $\omega_{1,1} = K_0\,\omega_{0,0} + K_1\,\omega_{1,0} + K_2\,\omega_{0,1} + L\,C\,x_{0,0}$, which is zero since $x_{0,0} \in \ker C$. On the other hand, $x_{1,1} = A_0\,x_{0,0} + A_1\,x_{1,0} + A_2\,x_{0,1}$ leads to $Q\,x_{1,1} = Q\,A_0\,x_{0,0} + Q\,A_1\,x_{1,0} + Q\,A_2\,x_{0,1} + Q\,B\,u_{0,0} = \omega_{1,1}$, which is zero as shown above. For the arbitrariness of $x_{0,0}$, $x_{1,0}$, $x_{0,1}$ and $u_{0,0}$ we get $Q\,A_0(\mathcal{Z} \cap \ker C) + Q\,A_1\,\mathcal{Z} + Q\,A_2\,\mathcal{Z} = \{0\}$ and $Q\,B = 0$, which imply that $A_0(\mathcal{Z} \cap \ker C) + A_1\,\mathcal{Z} + A_2\,\mathcal{Z} \subseteq \mathcal{Z}$ and $\operatorname{im} B \subseteq \ker\, Q = \mathcal{Z}$. Hence, $\mathcal{Z}$ is an input-containing subspace of output-injection type. ∎

Conditioned invariant subspaces as defined in [6] guarantee the existence of a $\mathcal{Z}$-observer, but the converse is not necessarily true, and the condition in Theorem

6.1 for that definition is only sufficient. The set of 2-D input containing subspaces of output-injection type, denoted by $\mathcal{Z}(\Sigma)$, admits a minimum $\mathcal{Z}^\star$, which can be computed by duality as follows.

LEMMA 6.4. *$\mathcal{Z}^\star$ is the last term of the monotonically nondecreasing sequence*

$$\{\mathcal{Z}_i\}_i : \quad \begin{cases} \mathcal{Z}_0 = \operatorname{im} B, \\ \mathcal{Z}_i = \operatorname{im} B + A_0(\mathcal{Z}_{i-1} \cap \ker C) + A_1\, \mathcal{Z}_{i-1} + A_2\, \mathcal{Z}_{i-1}, & i \in \{1, 2, \ldots, k_z\}, \end{cases}$$

where the integer $k_z \leq n-1$ is determined by the condition $\mathcal{Z}_{k_z+1} = \mathcal{Z}_{k_z}$, i.e., $\mathcal{Z}^\star = \mathcal{Z}_{k_z}$.

6.1. Disturbance decoupling problem. The notion of output-nulling subspace of feedback type is useful in the solution of the disturbance decoupling problem. Consider the model

$$x_{i+1,j+1} = A_0\, x_{i,j} + A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B\, u_{i,j} + H\, w_{i,j}, \tag{6.3}$$
$$y_{i,j} = C\, x_{i,j}, \tag{6.4}$$

where $w_{i,j}$ is a disturbance to be rejected using a control $u_{i,j} = F\, x_{i,j}$, i.e., our aim is to find a feedback law $u_{i,j} = F\, x_{i,j}$ such that the output of the closed-loop system

$$x_{i+1,j+1} = (A_0 + B\, F)\, x_{i,j} + A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + H\, w_{i,j} \tag{6.5}$$

is not affected by the disturbance w, i.e., such that $y_{i,j} = C\, x_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}^+$ for zero boundary condition and for any disturbance $w_{i,j}$. This basic problem is solved without conservativeness for the first time in a 2-D framework in the following theorem.

THEOREM 6.5. *The disturbance decoupling problem admits solutions if and only if*

$$\operatorname{im} H \subseteq \mathcal{W}^\star. \tag{6.6}$$

Proof. **(If).** By taking F to be a friend of $\mathcal{W}^\star$, for each $\mathcal{W}^\star$-valued boundary condition, by (6.5) the local state lies on $\mathcal{W}^\star$ for all $(i,j) \in \mathfrak{B}^+$, and therefore it is contained in the null-space of C. Thus, the system is disturbance decoupled.

(Only if). Suppose the closed-loop system is disturbance decoupled, i.e., F exists such that $y_{i,j} = C\, x_{i,j} = 0$ for any disturbance $w_{i,j}$. When $w_{i,j}$ is zero, the closed-loop system is still disturbance decoupled, i.e., $x_{i,j}$ lies for all (i,j) on the largest subspace $\mathcal{T}$ satisfying $[\,(A_0 + B\, F) \;\; A_1 \;\; A_2\,](\mathcal{T} \oplus \mathcal{T} \oplus \mathcal{T}) \subseteq \mathcal{T} \subseteq \ker C$. This subspace is clearly $\mathcal{W}^\star$. Moreover, since the system must be decoupled for each value of $w_{i,j}$, H must satisfy $\operatorname{im} H \subseteq \mathcal{T} = \mathcal{W}^\star$. $\square$

Remark 6.1. The solution of the disturbance decoupling problem can easily be extended to the case where the disturbance to reject is measurable, so that the control can be expressed as $u_{i,j} = F\, x_{i,j} + S\, w_{i,j}$. The problem is to find matrices F and S such that the output of the closed-loop system

$$x_{i+1,j+1} = (A_0 + B\, F)\, x_{i,j} + A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B\, S\, w_{i,j} + H\, w_{i,j} \tag{6.7}$$

is zero for zero boundary conditions and for any disturbance $w_{i,j}$. Using the same arguments of Theorem 6.5, we can show that the problem is solvable if and only if im $H \subseteq \mathcal{W}^\star + \text{im}\, B$. In fact, if this condition holds, im H can be decomposed as im $H = \mathcal{H}_1 + \mathcal{H}_2$, where $\mathcal{H}_1 \subseteq \mathcal{W}^\star$ and $\mathcal{H}_2 \subseteq \text{im}\, B$. This means that the closed-loop system

$$x_{i+1,j+1} = (A_0 + B\,F)\, x_{i,j} + A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + (B\,S + H_2)\, w_{i,j} + H_1\, w_{i,j},$$

where H_1 and H_2 are basis matrices of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. By choosing S to satisfy $B\,S + H_2 = 0$—which is possible since $\text{im}\, H_2 = \mathcal{H}_2 \subseteq \text{im}\, B$—and F to be a friend of $\mathcal{W}^\star$, given a $\mathcal{W}^\star$-valued boundary condition, the local state remains on $\mathcal{W}^\star$, and $y_{i,j}$ is zero.

6.2. Friends and stabilization. In this section we show how to characterize the set of friends of a 2-D controlled invariant subspace of feedback type. A fundamental result for this purpose is the following theorem.

THEOREM 6.6. *The set of friends of the* 2*-D controlled invariant subspace of feedback type* $\mathcal{W}$ *with basis matrix* W *coincides with the set of matrices* F *such that* $\Omega = -F\,W$*, where* Ω *is a solution of* $A_0\,W = W\,X_0 + B\,\Omega$ *for some matrix* X_0*.*

Proof. Let F be such that $\Omega = -F\,W$, where Ω is a solution of $A_0\,W = W\,X_0 + B\,\Omega$ for a certain X_0. Therefore, $A_0\,W = W\,X_0 - B\,F\,W$. Moreover, $A_1\,W = W\,X_1$ and $A_2\,W = W\,X_2$. Therefore, F is a friend of $\mathcal{W}$. Conversely, let F be a friend of $\mathcal{W}$. Then, $(A_0 + B\,F)\mathcal{W} \subseteq \mathcal{W}$ can be written as $(A_0 + B\,F)W = W\,\Xi$ for a suitable Ξ. Hence, $A_0\,W = W\,X_0 + B\,\Omega$ holds with $X_0 = \Xi$ and $\Omega = -F\,W$. ∎

Theorem 6.6 basically says that the set of friends of a 2-D controlled invariant subspace of feedback type $\mathcal{W}$ coincides with the set of friends of $\mathcal{W}$ when the latter is viewed as a 1-D controlled invariant subspace of the pair (A_0, B). However, we will see in this section that the problem of the inner and outer stabilization of 2-D controlled invariant subspaces of feedback type does not follow as a simple consequence of the classical inner and outer stabilization of 1-D controlled invariant subspaces.

As in the 1-D case, two degrees of freedom can be identified in the computation of a friend F. The first follows from the computation of the solution X_0 and Ω of $A_0\,W = W\,X_0 + B\,\Omega$. In fact, the set of solutions X_0, Ω of $A_0\,W = W\,X_0 + B\,\Omega$ is[1]

$$\begin{bmatrix} X_0 \\ \Omega \end{bmatrix} = \begin{bmatrix} W & B \end{bmatrix}^\dagger A_0\,W + \begin{bmatrix} H_0 \\ H_1 \end{bmatrix} K_1, \tag{6.8}$$

where $\left[\begin{smallmatrix} H_0 \\ H_1 \end{smallmatrix}\right]$ is a basis matrix of the subspace $\ker[\,W \quad B\,]$ and K_1 is an arbitrary matrix of suitable size. Therefore, we can write $X_0 = X_0(K_1)$ and $\Omega = \Omega(K_1)$. The second degree of freedom comes from the solution of the linear equation $\Omega = -F\,W$, which can be written as $F = -\Omega(K_1)\,(W^\top W)^{-1}\,W^\top + K_2\,Z$, where $Z^\top$ is a basis of $\ker W^\top$ and K_2 is another arbitrary matrix of suitable size. Hence, we can write $F = F(K_1, K_2)$. Thus, as for the 1-D case, there are two degrees of freedom in the computation of a friend F, given by K_1 and K_2.

Consider the change of basis $T = [\,W \quad W_c\,]$, where W_c is such that T is nonsingular. Then

$$T^{-1}(A_0{+}B\,F)T = \begin{bmatrix} L_1(K_1, K_2) & L_2(K_1, K_2) \\ O & L_3(K_1, K_2) \end{bmatrix},$$

$$T^{-1}A_1\,T = \begin{bmatrix} M_1 & M_2 \\ O & M_3 \end{bmatrix}, \quad T^{-1}A_2\,T = \begin{bmatrix} N_1 & N_2 \\ O & N_3 \end{bmatrix}.$$

[1]Here the symbol † is used to denote the Moore–Penrose pseudo-inverse.

Since, as mentioned, the set of friends of $\mathcal{W}$ coincides with the set of friends of $\mathcal{W}$ when this is regarded as a 1-D controlled invariant subspace with respect to (A_0, B), it follows that—as in the 1-D case—matrix $L_1(K_1, K_2)$ does not depend on K_2, and matrix $L_3(K_1, K_2)$ does not depend on K_1, [3, 23, 25]. A more explicit proof of this fact can be obtained by adapting that of Lemma 3.3 in [21].

Hence, defining $\left[\begin{smallmatrix} x'_{i,j} \\ x''_{i,j} \end{smallmatrix}\right] \stackrel{\text{def}}{=} T^{-1} x_{i,j}$, the closed-loop update equation is in the new basis as

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} L_1(K_1) & L_2(K_1, K_2) \\ O & L_3(K_2) \end{bmatrix} \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} + \begin{bmatrix} M_1 & M_2 \\ O & M_3 \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix} + \begin{bmatrix} N_1 & N_2 \\ O & N_3 \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix},$$

where x' represent the components of the local state on the 2-D controlled invariant subspace of feedback type $\mathcal{W}$, while x'' represent the components of the local state that are external to $\mathcal{W}$. The zeros in the matrices above confirm that the local state of the closed-loop system lies on $\mathcal{W}$ for any $\mathcal{W}$-valued boundary condition. In fact, a $\mathcal{W}$-valued boundary condition in these coordinates is such that $x''_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}$. Hence, the components $x''_{i,j}$ are identically zero on $\mathfrak{B}^+$ since the update equation $x''_{i+1,j+1} = L_3(K_2)\, x''_{i,j} + M_3\, x''_{i+1,j} + N_3\, x''_{i,j+1}$ with the boundary condition $x''_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}$ generates $x''_{i,j} = 0$ for all $(i,j) \in \mathfrak{B}^+$.

A 2-D controlled invariant subspace of feedback type $\mathcal{W}$ is said to be inner stabilizable if a friend F of $\mathcal{W}$ can be found so that for any $\mathcal{W}$-valued boundary condition the local state $x_{i,j}$ lies on $\mathcal{W}$ for all $(i,j) \in \mathfrak{B}^+$ and converges to zero as the bi-index (i,j) evolves away from the boundary $\mathfrak{B}$. Similarly, $\mathcal{W}$ is said to be outer stabilizable if for any (not necessarily $\mathcal{W}$-valued) boundary condition a friend of $\mathcal{W}$ exists that makes the corresponding local state converge to $\mathcal{W}$ (without making it necessarily converge to zero) as (i,j) moves away from $\mathfrak{B}$. Using the change of basis described above, we see that

- $\mathcal{W}$ is inner stabilizable if and only if $x'_{i,j}$ converges to zero as (i,j) evolves away from $\mathfrak{B}$, i.e., if and only if a matrix K_1 exists such that $(L_1(K_1), M_1, N_1)$ is asymptotically stable;
- $\mathcal{W}$ is outer stabilizable if and only if $x''_{i,j}$ converges to zero as (i,j) evolves away from $\mathfrak{B}$, i.e., if and only if a matrix K_2 exists such that $(L_3(K_2), M_3, N_3)$ is asymptotically stable.

Hence, K_1 affects the inner stabilizability of $\mathcal{W}$ but not the outer stabilizability, while the K_2 affects the outer stabilizability of $\mathcal{W}$ but not the inner stabilizability. Let us now focus on the inner stabilization of $\mathcal{W} \in \mathcal{W}(\Sigma)$. If we construct the friend F with (6.8) and then solve $\Omega = -F\, W$, we find that the matrices X_0, $X_1 = W^\dagger A_1\, W$ and $X_2 = W^\dagger A_2\, W$ are the matrices of the Fornasini–Marchesini subsystem that represent the internal dynamics of (2.1) restricted to $\mathcal{W}$. In fact, let us consider a $\mathcal{W}$-valued boundary condition. Given $x_{i,j}, x_{i+1,j}, x_{i,j+1} \in \mathcal{W}$, there exist $z_{i,j}, z_{i+1,j}, z_{i,j+1}$ such that $z_{i,j} = W\, x_{i,j}$, $z_{i+1,j} = W\, x_{i+1,j}$, and $z_{i+1,j} = W\, x_{i+1,j}$. Therefore, we can write the closed-loop system as

$$\begin{aligned} x_{i+1,j+1} &= (A_0 + B\, F)\, W\, z_{i,j} + A_1\, W\, z_{i+1,j} + A_2\, W\, z_{i,j+1} \\ &= W\, (X_0\, z_{i,j} + X_1\, z_{i+1,j} + X_2\, z_{i,j+1}), \end{aligned}$$

which implies that $x_{i+1,j+1}$ lies on $\mathcal{W}$, and therefore by defining $z_{i+1,j+1} \stackrel{\text{def}}{=} X_0\, z_{i,j} + X_1\, z_{i+1,j} + X_2\, z_{i,j+1}$, given a solution $x_{i,j}$ on $\mathcal{W}$ we can construct the vector $z_{i,j}$ which represents the projection of the local state $x_{i,j}$ on $\mathcal{W}$. Hence, $\mathcal{W}$ is inner stabilizable if and only if $A_0\, W = W\, X_0 + B\, \Omega$ can be solved in X_0, and Ω in such a way that the triple $(X_0, W^\dagger A_1 W, W^\dagger A_2 W)$ is asymptotically stable in the usual 2-D sense, i.e., if and only if the determinant of $I - X_0 z_1 z_2 - W^\dagger A_1 W z_2 - W^\dagger A_2 W z_1$ differs from zero for all (z_1, z_2) in the unit bi-disc $\{(\zeta_1,\zeta_2) \in \mathbb{C}\times\mathbb{C} \mid |\zeta_1| \le 1 \text{ and } |\zeta_2| \le 1\}$; see [9, Proposition 3]. A computationally tractable method to compute an inner stabilizing friend is given in the following theorem.

THEOREM 6.7. *Let $\mathcal{W}$ be a 2-D controlled invariant subspace of feedback type of dimension r with basis $W \in \mathbb{R}^{n\times r}$. Let Q_0 denote the first r rows of $[\,W \quad B\,]^\dagger A_0 W$. Then, $\mathcal{W}$ is inner stabilizable if $\Phi = \Phi^\top > 0$, $\Psi = \Psi^\top > 0$, $\Theta = \Theta^\top > 0$ and Ξ exist such that the linear matrix inequality (LMI)*

$$\left[\begin{array}{ccc|c} \Psi & O & \star & \star \\ O & \Theta & \star & \star \\ O & O & \Phi - \Psi - \Theta & \star \\ \hline Q_0\,\Phi + H_0\,\Xi & W^\dagger A_1 W\,\Phi & W^\dagger A_2 W\,\Phi & \Phi \end{array}\right] > 0 \tag{6.9}$$

holds.[2] *Given a quadruple $(\Phi, \Psi, \Theta, \Xi)$ in the convex set defined by (6.9), a matrix K_1 such that the triple (X_0, X_1, X_2) is asymptotically stable is given by $K_1 = \Xi\,\Phi^{-1}$.*

Proof. From the properties of the Schur complements applied on the condition in [15], (X_0, X_1, X_2) is asymptotically stable if three symmetric positive semidefinite matrices P_0, P_1, and P_2 exist such that

$$\left[\begin{array}{ccc|c} P_0 & O & O & \begin{bmatrix} X_0^\top \\ X_1^\top \\ X_2^\top \end{bmatrix} P \\ O & P_1 & O & \\ O & O & P_2 & \\ \hline P\,[\,X_0 \quad X_1 \quad X_2\,] & & & P \end{array}\right] > 0, \tag{6.10}$$

where $P = P_0 + P_1 + P_2$. Since $\mathcal{W} \in \mathcal{W}(\Sigma)$, $X_1 = W^\dagger A_1 W$ and $X_2 = W^\dagger A_2 W$. Since Q_0 is equal to the first r rows of $[\,W \quad B\,]^\dagger A_0 W$, we can write $X_0 = Q_0 + H_0 K_1$. Pre- and postmultiplying the former by the block-diagonal matrix diag$(P^{-1}, P^{-1}, P^{-1}, P^{-1})$ and defining $\Phi = P^{-1}$, $\Psi = P^{-1} P_0 P^{-1}$, and $\Theta = P^{-1} P_1 P^{-1}$ along with $\Xi = K_1\,\Theta$ we get (6.9). ∎

By adapting the arguments used in [21, section 3.2], one can easily see that the problem of the outer stabilization of a 2-D controlled invariant subspace of feedback type is not convex, as this problem is equivalent to one of stabilization by static output feedback. However, various established numerical techniques can be used to find feasible points. For example, the so-called sequential linear programming matrix method developed in [17] along the same lines of [21].

Example 6.1. As already observed, the fact that a friend for a 2-D controlled invariant subspace can be computed as a friend of (A_0, B) seems to suggest that the inner and outer stabilization of 2-D controlled invariant subspaces of feedback type can be reduced to that of the 1-D system described by the pair (A_0, B). Unfortunately, this is not the case. Consider the 2-D system described by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C = [\,1 \quad 0 \quad 0\,].$$

[2]The symbol $\star$ is used to abbreviate off-diagonal blocks in symmetric matrices.

By using the sequence in Lemma 5.3, we find $\mathcal{W}^\star = \operatorname{im} W$, where $W = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. A direct inspection shows that $F_0 = \left[\begin{array}{ccc} 0 & -2 & \frac{1}{10} \end{array}\right]$ is a friend of the 1-D controlled invariant subspace $\mathcal{W}^\star$ with respect to the pair (A_0, B), since $(A_0 + B\,F_0)\,\mathcal{W}^\star = \operatorname{im}\left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]^\top \subseteq \mathcal{W}^\star$. Moreover, this friend is such that all the eigenvalues of the 1-D closed-loop system $(A_0 + B\,F_0)$ lie in the open unit disc, i.e., they are asymptotically stable in a 1-D sense. In fact, the eigenvalues of $(A_0 + B\,F_0)$ are equal to 0 (with double multiplicity) and 0.1. The inner eigenvalues of $\mathcal{W}^\star$ with respect to $(A_0 + B\,F_0)$ are 0.1 and 0. However, it can be seen that this friend is not inner stabilizing for the 2-D controlled invariant subspace of feedback type $\mathcal{W}^\star$ with respect to (2.1). To see this, we compute the triple (X_0, X_1, X_2) whose dynamics represent the dynamics of the 2-D system (2.1) restricted to $\mathcal{W}^\star$, i.e., such that $(A_0 + B\,F_0)\,W = W\,X_0$, $A_1\,W = W\,X_1$, $A_2\,W = W\,X_2$. Such a triple is given by

$$X_0 = W^\dagger\,(A_0 + B\,F_0)\,W = \begin{bmatrix} 0 & 0 \\ 2 & \frac{1}{10} \end{bmatrix},$$
$$X_1 = W^\dagger\,A_1\,W = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \quad X_2 = W^\dagger\,A_2\,W = 0.$$

This triple (X_0, X_1, X_2) is not asymptotically stable, i.e., there exist $\zeta_1 \in \mathbb{C}$ and $\zeta_2 \in \mathbb{C}$ in the unit bi-disc (i.e., such that $|\zeta_1| < 1$ and $|\zeta_2| < 1$) for which the determinant $\det(I - X_0\,\zeta_1\,\zeta_2 - X_1\,\zeta_2 - X_2\,\zeta_1) = \left[\begin{smallmatrix} 1 & 2\,\zeta_2 \\ -2\,\zeta_1\,\zeta_2 & 1-\frac{\zeta_1\,\zeta_2}{10} \end{smallmatrix}\right]$ is equal to zero. An example of such a pair is $(\zeta_1, \zeta_2) = \left(-\frac{2}{3}, -\frac{3}{5}\right)$. On the other hand, the problem of finding an inner stabilizing friend for $\mathcal{W}^\star$ is solvable with the results of Theorem 6.7. First, we compute

$$H = \left[\begin{array}{cc} W & B \end{array}\right]^\dagger = \frac{1}{2}\begin{bmatrix} 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \quad \text{and} \quad Q = \left[\begin{array}{cc} W & B \end{array}\right]^\dagger A_0\,W = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 2 & 0 \end{bmatrix}$$

so that $Q_0 = \left[\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right]$. The LMI (6.9) in Theorem 6.7 is satisfied with $\Phi = \operatorname{diag}(2.0649, 0.3607)$, $\Psi = \operatorname{diag}(2.0649, 0.3607)$, $\Theta = \operatorname{diag}(0.6883, 0.2734)$, $\Xi = \left[\begin{array}{cc} 5.8405 & 0 \end{array}\right]$. These values lead to $K_1 = \Xi\,\Phi^{-1} = \left[\begin{array}{cc} 2\sqrt{2} & 0 \end{array}\right]$, which gives

$$\begin{bmatrix} X_0 \\ \Omega \end{bmatrix} = \left[\begin{array}{cc} W & B \end{array}\right]^\dagger A_0\,W + H\,K_1 = Q + H\,K_1 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 4 & 0 \end{array}\right],$$

which in turn gives $F = -\Omega\,W^\dagger = \left[\begin{array}{ccc} 0 & -4 & 0 \end{array}\right]$. The new matrix F is indeed a friend of $\mathcal{W}^\star$ since $(A_0 + B\,F)\,\mathcal{W}^\star = \{0\}$ and is inner stabilizing for $\mathcal{W}^\star$. In fact, it is associated with the new matrix $X_0 = W^\dagger\,(A_0 + B\,F)\,W = 0$. The triple (X_0, X_1, X_2) is now asymptotically stable, as the determinant of the matrix $I - X_0\,\zeta_1\,\zeta_2 - X_1\,\zeta_2 - X_2\,\zeta_1 = \left[\begin{smallmatrix} 1 & 2\,\zeta_2 \\ 0 & 1 \end{smallmatrix}\right]$ never vanishes for any value $\zeta_2 \in \mathbb{C}$. Therefore, this new friend F is inner stabilizing for $\mathcal{W}^\star$.

A parallel (dual) theory for the outer stabilization of 2-D conditioned invariant subspaces of output-injection type can easily be established. First, we can see that the set of output-injection matrices of the 2-D conditioned invariant subspace of output-injection type $\mathcal{Z}$ such that $(A_0 + G\,C)\,\mathcal{Z} \subseteq \mathcal{Z}$, $A_1\,\mathcal{Z} \subseteq \mathcal{Z}$, and $A_2\,\mathcal{Z} \subseteq \mathcal{Z}$ coincides with the set of matrices G such that $\Lambda = -Q\,G$, where Λ is a solution of $Q\,A_0 = \Gamma_0\,Q + \Lambda\,C$ for a suitable Γ_0, where Q is a full row-rank matrix such that $\ker Q = \mathcal{Z}$. The set of solutions of $Q\,A_0 = \Gamma_0\,Q + \Lambda\,C$ is given by

$$\begin{bmatrix} \Gamma_0 & \Lambda \end{bmatrix} = Q\,A_0 \begin{bmatrix} Q \\ C \end{bmatrix}^{\dagger} + K_1 \begin{bmatrix} H_0 & H_1 \end{bmatrix}, \tag{6.11}$$

where $[\,H_0 \quad H_1\,]$ has linearly independent rows and its null-space spans the image of $\left[\begin{smallmatrix} Q \\ C \end{smallmatrix}\right]$. Matrix K_1 is arbitrary. The solution of $\Lambda = -Q\,G$ is given by $G = -Q^\top (Q\,Q^\top)^{-1}\,\Lambda + H\,K_2$, where $\operatorname{im} H = \ker Q$ and K_2 is arbitrary. Changing coordinates with $T = \left[\begin{smallmatrix} S_c \\ Q \end{smallmatrix}\right]$ where the rows of S_c are linearly independent of those of Q, we get

$$T(A_0 + B\,F)\,T^{-1} = \begin{bmatrix} L_1(K_1,K_2) & L_2(K_1,K_2) \\ O & L_3(K_1,K_2) \end{bmatrix},$$

$$T\,A_1\,T^{-1} = \begin{bmatrix} M_1 & M_2 \\ O & M_3 \end{bmatrix}, \quad T\,A_2\,T^{-1} = \begin{bmatrix} N_1 & N_2 \\ O & N_3 \end{bmatrix}.$$

It is easily proved that $L_1(K_1,K_2)$ does not depend on K_1 and $L_3(K_1,K_2)$ does not depend on K_2. Hence, we obtain the following.

- $\mathcal{Z}$ is inner stabilizable if and only if a matrix K_2 exists such that $(L_1(K_2), M_1, N_1)$ is asymptotically stable.
- $\mathcal{Z}$ is outer stabilizable if and only if a matrix K_1 exists such that $(L_3(K_1), M_3, N_3)$ is asymptotically stable.

Similar to the 1-D case [23], outer stabilizable 2-D conditioned invariant subspaces of output-injection type are referred to as 2-D *detectability subspaces*.

Again by using arguments based on duality, it can be seen that $\mathcal{Z}$ is outer stabilizable if and only if an output-injection matrix G exists such that $Q\,(A_0 + G\,C) = \Gamma_0\,Q$, $Q\,A_1 = \Gamma_1\,Q$, and $Q\,A_2 = \Gamma_2\,Q$, where the triple $(\Gamma_0, \Gamma_1, \Gamma_2)$ is asymptotically stable. Outer stabilizable input-containing subspaces of output-injection type are linked to the existence of asymptotic $\mathcal{Z}$-observers for Σ. The 2-D system ruled by (6.1) is an asymptotic $\mathcal{Z}$-observer if for any boundary condition of Σ and of (6.1), the local state of (6.1) asymptotically reconstructs the local state $x_{i,j}$ of Σ modulo the components of this vector on $\mathcal{Z}$. In other words, on the basis of the observations $y_{i,j}$, the vector $\omega_{i,j}$ asymptotically converges to $x_{i,j}/\mathcal{Z}$, as the indexes i and j evolve away from the boundary, regardless of the boundary conditions of Σ and (6.1). This is instrumental in the definitions of asymptotic $\mathcal{Z}$-observers for which, when there is a mismatch between the boundary conditions of Σ and (6.1), the estimation error converges to zero as the bi-index (i,j) evolves away from the boundary; see [22]. In fact, if $\mathcal{Z}$ is outer stabilizable, by considering an observer (6.1) with the asymptotically stable triple $(K_1, K_2, K_3) = (\Gamma_1, \Gamma_2, \Gamma_3)$ and $L = \Lambda$, by (6.2) we obtain that $e_{i,j}$ goes to zero as (i,j) evolves away from the boundary $\mathfrak{B}$. This leads to the following result.

THEOREM 6.8. *A subspace $\mathcal{Z}$ is a detectability subspace of Σ if and only if there exists an asymptotic $\mathcal{Z}$-observer for Σ.*

Again, the **(If)** part of Theorem 6.8 does not hold for the other definitions given so far in the literature of 2-D conditioned invariance. Hence, the definition given here appears to be the most suitable in the characterization of 2-D asymptotic observers with unknown inputs.

The following result presents a computationally tractable test for outer stabilizability that is the dual of the one in Theorem 6.7.

THEOREM 6.9. *Let $\mathcal{Z}$ be a 2-D conditioned invariant subspace of output-injection type of dimension r, and let Q be a full row-rank matrix such that* $\ker Q = \mathcal{Z}$. *Let*

Π_0 *denote the first* $n-r$ *columns of* $Q\,A_0 \left[\begin{smallmatrix} Q \\ C \end{smallmatrix}\right]^{\dagger}$. *Then,* $\mathcal{Z}$ *is outer stabilizable if* $\Phi=\Phi^\top>0$, $\Psi=\Psi^\top>0$, $\Theta=\Theta^\top>0$, *and* Ξ *exist such that the LMI*

$$
\left[\begin{array}{ccc|c}
\Psi & \star & \star & \star \\
O & \Theta & \star & \star \\
O & O & \Phi-\Psi-\Theta & \star \\
\hline
\Phi\,\Pi_0+\Xi\,H_0 & \Phi\,Q\,A_1\,Q^\dagger & \Phi\,Q\,A_2\,Q^\dagger & \Phi
\end{array}\right] > 0 \tag{6.12}
$$

holds. Given a quadruple (Φ,Ψ,Θ,Ξ) *in the convex set defined by* (6.12), *a matrix* K_1 *such that the triple* $(\Gamma_0,\Gamma_1,\Gamma_2)$ *is asymptotically stable is given by* $K_1=\Phi^{-1}\,\Xi$.

7. Self-boundedness and self-hiddenness. In this section, we introduce the notions of self-boundedness and self-hiddenness for 2-D systems. Self-bounded subspaces play a central role in disturbance decoupling problems since they allow such problems to be solved without necessarily making the closed-loop system maximally unobservable [7]. In fact, as shown in section 6.1, the common strategy to solve decoupling problems is to check if $\operatorname{im} H$ is contained in $\mathcal{W}^\star$ and, in this case, looking for a friend of $\mathcal{W}^\star$. However, a friend of *any* output-nulling subspace of feedback type that contains $\operatorname{im} H$ is a suitable solution of the problem. This observation is useful because sometimes in the solution of a disturbance decoupling problem it is convenient to look for output-nulling subspaces of smaller dimension than $\mathcal{W}^\star$, such as the self-bounded subspaces $\mathcal{W}^\star \cap \mathcal{Z}^\star$ or $\mathcal{R}_{\mathcal{W}^\star}$ that will be defined in the sequel. For example, in decoupling problems with dynamic feedback/feedforward control architectures, the size of the dynamic compensator is often equal to the dimension of the output-nulling subspace that contains $\operatorname{im} H$ (or that satisfies other structural conditions that depend on the knowledge on the disturbance to reject); see, e.g., the *full information* control [21].

As such, the use of self-bounded subspaces smaller than $\mathcal{W}^\star$ in the solution of decoupling problem often leads to feedforward compensators of smaller size. In the 1-D case, there are other advantages connected with the use of self-bounded subspaces in the solution of decoupling problems. The most important is the fact that these subspaces guarantee maximum freedom in the choice of the dynamics of the closed-loop system [18]. The extension of this property to the 2-D case can only be conjectured at this stage and is the object of intense ongoing investigation.

We begin by recalling some facts about reachability for 2-D Fornasini–Marchesini models. The notions of reachability and observability in the 2-D case are intrinsically richer than their 1-D counterpart, due to the difference that arises in such systems between the local state (which has been denoted in this paper by $x_{i,j}$ and represents the size of the update vector in (2.1)) and the global state (which represents the memory of the system and is an infinite-dimensional vector). Correspondingly, the reachability and observability of 2-D models can be introduced in a local form if they are referred to the local state or in a global form if it is referred to the global state; see, e.g., [4, 10, 13, 14]. In this section we focus our attention on 2-D reachability in a local sense.

As is well-known, the reachable subspace from the origin of a 1-D linear time-invariant (continuous or discrete-time) systems described by a pair (A,B) coincides with the smallest A-invariant subspace that contains the image of B. We extend this characterization to the 2-D case by showing that an analogous result holds. Consider the following 2-D sequence of subspaces

$$\begin{cases} \mathcal{R}_{i,j} = \{0\}, & (i,j) \in \mathfrak{B}, \\ \mathcal{R}_{i,j} = A_0\,\mathcal{R}_{i-1,j-1} + A_1\,\mathcal{R}_{i,j-1} + A_2\,\mathcal{R}_{i-1,j} + \mathrm{im}\ B, & (i,j) \in \mathfrak{B}^+ \setminus \mathfrak{B}. \end{cases}$$

The subspace $\mathcal{R}_{i,j}$ is the reachable subspace from zero boundary condition of the local state x at (i,j). The sequence gives rise to the subspace with greatest dimension at $i = j = n-1$. Therefore, we can define the reachability subspace from zero boundary condition as $\mathcal{R}^\star = \mathcal{R}_{n-1,n-1}$. When $\mathcal{R}^\star = \mathbb{R}^n$, the system is completely reachable from zero boundary conditions. Application of the 2-D Cayley–Hamilton theorem [5, 8, 11] leads to the following result, which provides a simpler and computationally attractive characterization of $\mathcal{R}^\star$.

THEOREM 7.1. *$\mathcal{R}^\star$ is the smallest subspace of $\mathbb{R}^n$ that is simultaneously invariant with respect to A_0, A_1, and A_2 and that contains the image of B. Moreover, $\mathcal{R}^\star$ is the last term of the monotonically nondecreasing sequence*

$$\{\mathcal{R}_i\}_i: \quad \begin{cases} \mathcal{R}_0 = \mathrm{im}\ B, \\ \mathcal{R}_i = \sum_{j=0}^{2} A_j\,\mathcal{R}_{i-1} + \mathrm{im}\ B, & i \in \{1,2,\ldots,k_r\}, \end{cases}$$

where the value of $k_r \le n-1$ is determined with the condition $\mathcal{R}_{k_r+1} = \mathcal{R}_{k_r}$. Hence, $\mathcal{R}^\star = \mathcal{R}_{k_r}$.

It is easy to see that the set of A_0-, A_1-, and A_2-invariant subspaces containing the image of B is closed under both subspace intersection and addition; therefore, its maximum is $\mathbb{R}^n$ and its minimum is given by the intersection of all its elements. The algorithm for $\mathcal{R}^\star$ then follows as a particular case of Lemma 4.2 with C equal to the null matrix.

COROLLARY 7.2. *Given zero initial conditions $x_{i,j} = 0$ for $(i,j) \in \mathfrak{B}$, for any $u_{i,j}$ with $(i,j) \in \mathfrak{B}^+$ we have $x_{i,j} \in \mathcal{R}^\star$ for all $(i,j) \in \mathfrak{B}^+$.*

If we change coordinates using a basis of $\mathbb{R}^n$ adapted to $\mathcal{R}^\star$, i.e., we consider a nonsingular matrix $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \in \mathbb{R}^n$ such that im $T_1 = \mathcal{R}^\star$, we find

$$T^{-1}\,A_i\,T = \begin{bmatrix} A_i^{11} & A_i^{12} \\ O & A_i^{22} \end{bmatrix}, \qquad T^{-1}\,B = \begin{bmatrix} B^1 \\ O \end{bmatrix},$$

where the submatrices are partitioned conformably with T_1 and T_2. In these new coordinates, the second block of components of the local state is structurally unaffected by the input, while the dynamics of the subsystem described by $(A_0^{11}, A_1^{11}, A_2^{11}; B^1)$ are completely reachable from zero boundary condition. Such subsystem can therefore be referred to as the reachable part of Σ (in a local sense). Dual arguments can be used to define the nonobservability subspace of Σ as the limiting subspace $\mathcal{Q}^\star$ obtained with the recursion

$$\{\mathcal{Q}_i\}_i: \quad \begin{cases} \mathcal{Q}_0 = \ker\ C, \\ \mathcal{Q}_i = \bigcap_{j=0}^{2} A_j^{-1}\,\mathcal{Q}_{i-1} \cap \ker\ C, & i \in \{1,2,\ldots,k\}, \end{cases}$$

so that an analogue of the Kalman canonical decomposition can be established.

Now, the concept of self-boundedness defined in [2] is extended to 2-D systems in Fornasini–Marchesini form. With no loss of generality, we can assume that B is of full column-rank.[3]

[3]If B has a nontrivial kernel, a subspace $\mathcal{U}_0$ of the input space exists that does not influence the local state state dynamics. By performing a suitable (orthogonal) change of basis in the input space, we may eliminate $\mathcal{U}_0$ and obtain an equivalent system for which this condition is satisfied.

DEFINITION 7.3. *The output-nulling subspace* $\mathcal{V}$ *is* self-bounded *if for any* $\mathcal{V}$*-valued boundary condition, any control yielding zero output is such that* $x_{i,j} \in \mathcal{V}$ *for all* $(i,j) \in \mathfrak{B}^+$. *Likewise, the output-nulling subspace of feedback type* $\mathcal{W}$ *is* self-bounded of feedback type *if for any* $\mathcal{W}$*-valued boundary condition, any control* $u_{i,j} = F\,x_{i,j}$ *yielding zero output is such that* $x_{i,j} \in \mathcal{W}$ *for all* $(i,j) \in \mathfrak{B}^+$.

In other words, given a boundary condition on a self-bounded subspace, there are no inputs that enable the local state to escape this subspace without leaving the kernel of C, i.e., without causing the output to become different from zero. Clearly, $\mathcal{V}^\star$ is self-bounded and $\mathcal{W}^\star$ is self-bounded of feedback type, since by definition these are the largest subspaces to which each boundary condition of Σ must belong so that the existence of an input (in the second case having the form $u_{i,j} = F\,x_{i,j}$) is guaranteed to maintain the output of Σ at zero. The following result provides a characterization of self-boundedness, similar to the one in [2].

THEOREM 7.4. *An output-nulling subspace* $\mathcal{V}$ *is self-bounded if and only if* $\mathcal{V}^\star \cap \operatorname{im} B \subseteq \mathcal{V}$.

Proof. **(If).** Suppose $\mathcal{V}^\star \cap \operatorname{im} B \subseteq \mathcal{V}$. Consider a $\mathcal{V}$-valued boundary condition for Σ. The update equation that delivers the local state on $\mathfrak{B}_1$ is (2.1), where $x_{i,j}, x_{i+1,j}, x_{i,j+1} \in \mathcal{V}$. Since $\mathcal{V}$ is a 2-D controlled invariant subspace, $A_0\,x_{i,j}$ can be decomposed into $\xi_{i,j} + \omega_{i,j}$, where $\xi_{i,j} \in \mathcal{V}$ and $\omega_{i,j} \in \operatorname{im} B$. The same is true for $A_1\,x_{i+1,j}$ and $A_2\,x_{i,j+1}$, so that $x_{i+1,j+1} = \phi_{i,j} + \psi_{i,j} + B\,u_{i,j}$, where $\phi_{i,j} \in \mathcal{V}$ and $\psi_{i,j} \in \operatorname{im} B$. In order for the local state in $\mathfrak{B}_1$ to be contained in $\ker C$, $\psi_{i,j} + B\,u_{i,j}$ must not cause $x_{i+1,j+1}$ to leave $\mathcal{V}^\star$, so we need $\psi_{i,j} + B\,u_{i,j} \in \mathcal{V}^\star \cap \operatorname{im} B$. However, if $\mathcal{V}^\star \cap \operatorname{im} B \subseteq \mathcal{V}$, the local state in $\mathfrak{B}_1$ is still in $\mathcal{V}$, as it is given by the sum of two vectors in $\mathcal{V}$. This argument can be repeated for all boundary regions $\mathfrak{B}_k$, and by induction the statement holds on $\mathfrak{B}^+$.

(Only if). Consider a vector $\eta \in \mathcal{V}^\star \cap \operatorname{im} B$ such that $\eta \notin \mathcal{V}$. Consider a $\mathcal{V}$-valued boundary condition for Σ, and let us use the local state update equation to compute the local state over $\mathfrak{B}_1$; using the same argument of the **(If)** part, we can write $x_{i+1,j+1} = \phi_{i,j} + \psi_{i,j} + B\,u_{i,j}$, where $\phi_{i,j} \in \mathcal{V}$ and $\psi_{i,j} \in \operatorname{im} B$. We can always choose $u_{i,j}$ to be such that $B\,u_{i,j} = \eta - \psi_{i,j}$, since $\eta \in \operatorname{im} B$ and $\psi_{i,j} \in \operatorname{im} B$, and with this choice $\psi_{i,j} + B\,u_{i,j}$ is equal to η, which is an element in $\mathcal{V}^\star$ but not of $\mathcal{V}$. Performing this operation recursively for all sets $\mathfrak{B}_k$, we construct a control input that generates a local state solution of (2.1) contained in $\mathcal{V}^\star$ but not in $\mathcal{V}$. Therefore, $\mathcal{V}$ is not self-bounded. ∎

Now we focus our attention on self-bounded subspaces of feedback type.

LEMMA 7.5. *Let* $\mathcal{W}$ *and* $\tilde{\mathcal{W}}$ *be two output-nulling subspaces of feedback type for* (2.1) *such that* $\mathcal{W} \supseteq \tilde{\mathcal{W}} \supseteq \mathcal{W}^\star \cap \operatorname{im} B$. *Then, any friend of* $\mathcal{W}$ *is also a friend of* $\tilde{\mathcal{W}}$.

Proof. Let F be a friend of $\mathcal{W}$. Since $\tilde{\mathcal{W}}$ is output-nulling, $A_0\,\tilde{\mathcal{W}} \subseteq \tilde{\mathcal{W}} + \operatorname{im} B$, which we add to the obvious inclusion $B\,F\,\tilde{\mathcal{W}} \subseteq \operatorname{im} B$ to get

$$(A_0 + B\,F)\,\tilde{\mathcal{W}} \subseteq \tilde{\mathcal{W}} + \operatorname{im} B. \tag{7.1}$$

Moreover, since $\mathcal{W}$ contains $\tilde{\mathcal{W}}$, we find

$$(A_0 + B\,F)\,\tilde{\mathcal{W}} \subseteq (A_0 + B\,F)\,\mathcal{W} \subseteq \mathcal{W}. \tag{7.2}$$

The intersection of (7.1) and (7.2) and application of the modular rule [23, p. 16] yield

$$(A_0 + B\,F)\,\tilde{\mathcal{W}} \subseteq \mathcal{W} \cap (\tilde{\mathcal{W}} + \operatorname{im} B) = \tilde{\mathcal{W}} + (\mathcal{W} \cap \operatorname{im} B). \tag{7.3}$$

Since $\tilde{\mathcal{W}} \supseteq \mathcal{W}^\star \cap \operatorname{im} B \supseteq \mathcal{W} \cap \operatorname{im} B$, from (7.3) we find $(A_0 + B\,F)\tilde{\mathcal{W}} \subseteq \tilde{\mathcal{W}}$. ∎

The following theorem provides a geometric characterization for self-boundedness, which is a direct extension to 2-D systems of the one given in [2].

THEOREM 7.6. *An output-nulling subspace of feedback type $\mathcal{W}$ is a self-bounded subspace of feeedback type if and only if $\mathcal{W}^\star \cap \operatorname{im} B \subseteq \mathcal{W}$.*

The proof of Theorem 7.6 follows as a simple adaptation of that of Theorem 1 in [20].

As a result of Lemma 7.5 and Theorem 7.6, we have the following corollary, whose proof is now obvious.

COROLLARY 7.7. *Let F be a friend of $\mathcal{W}^\star$. Then, F is a friend of any self-bounded subspace of feedback type.*

We denote by $\Phi(\Sigma)$ the set of self-bounded subspaces of feedback type of (2.1)–(2.2):

$$\Phi(\Sigma) \stackrel{\text{def}}{=} \{\mathcal{W} \in \mathcal{W}(\Sigma) \,|\, \mathcal{W} \supseteq \mathcal{W}^\star \cap \operatorname{im} B\}.$$

PROPOSITION 7.8. *The set $\Phi(\Sigma)$ is closed under sum and intersection.*

Proof. The fact that $\Phi(\Sigma)$ is closed under sum is obvious, since such is also $\mathcal{W}(\Sigma)$. Let us prove that it is closed under intersection. Let $\mathcal{W}_1, \mathcal{W}_2 \in \Phi(\Sigma)$, and let F be a friend of $\mathcal{W}^\star$. By Corollary 7.7, F is also a friend of $\mathcal{W}_1$ and $\mathcal{W}_2$. Therefore, $(A_0 + B\,F)(\mathcal{W}_1 \cap \mathcal{W}_2) \subseteq \mathcal{W}_1 \cap \mathcal{W}_2$. Therefore, $\mathcal{W}_1 \cap \mathcal{W}_2$ is a 2-D controlled invariant subspace of feedback type. It also contains $\mathcal{W}^\star \cap \operatorname{im} B$ and it is contained in $\ker C$ because both $\mathcal{W}_1$ and $\mathcal{W}_2$ are. □

Now, given $\mathcal{W}_1, \mathcal{W}_2 \in \Phi(\Sigma)$, it is easily seen that their sum $\mathcal{W}_1 + \mathcal{W}_2$ is the smallest element of $\Phi(\Sigma)$ containing both $\mathcal{W}_1$ and $\mathcal{W}_2$, and $\mathcal{W}_1 \cap \mathcal{W}_2$ is the largest element of $\Phi(\Sigma)$ contained in both $\mathcal{W}_1$ and $\mathcal{W}_2$. Hence, $(\Phi(\Sigma), +, \cap; \subseteq)$ is a nondistributive modular lattice. As such, it admits a maximum element, which is trivially $\mathcal{W}^\star$, and a minimum element, which we now want to characterize. In the 1-D case, such minimum coincides with the intersection of the largest output-nulling and the smallest input-containing subspaces of the system, [19]. We show that the corresponding element in the 2-D case, i.e., $\mathcal{W}^\star \cap \mathcal{Z}^\star$, is self-bounded of feedback type. We first show that it is output-nulling. Since $\mathcal{W}^\star \subseteq \ker C$, we find

$$A_0(\mathcal{W}^\star \cap \mathcal{Z}^\star) = A_0(\mathcal{W}^\star \cap \mathcal{Z}^\star \cap \ker C) \subseteq A_0\,\mathcal{W}^\star \cap A_0(\mathcal{Z}^\star \cap \ker C), \tag{7.4}$$

where the last inclusion holds in view of the modular rule, since $\mathcal{W}^\star \subseteq \ker C$. Since $\mathcal{Z}^\star$ is a 1-D conditioned invariant subspace with respect to (A_0, C) and $\mathcal{W}^\star$ is a 1-D controlled invariant subspace with respect to (A_0, B), relations $A_0(\mathcal{Z}^\star \cap \ker C) \subseteq \mathcal{Z}^\star$ and $A_0\,\mathcal{W}^\star \subseteq \mathcal{W}^\star + \operatorname{im} B$ can be used on (7.4) to get

$$A_0(\mathcal{W}^\star \cap \mathcal{Z}^\star) \subseteq (\mathcal{W}^\star + \operatorname{im} B) \cap \mathcal{Z}^\star \subseteq (\mathcal{W}^\star \cap \mathcal{Z}^\star) + (\operatorname{im} B \cap \mathcal{Z}^\star) = (\mathcal{W}^\star \cap \mathcal{Z}^\star) + \operatorname{im} B$$

since $\mathcal{Z}^\star \supseteq \operatorname{im} B$. Therefore $\mathcal{W}^\star \cap \mathcal{Z}^\star$ is a 1-D controlled invariant subspace for (A_0, B). Trivially, it is also A_1- and A_2-invariant. Moreover, $\mathcal{W}^\star \cap \mathcal{Z}^\star \subseteq \mathcal{W}^\star \subseteq \ker C$, which means that $\mathcal{W}^\star \cap \mathcal{Z}^\star$ is output-nulling. Finally, since $\mathcal{Z}^\star \supseteq \operatorname{im} B$, we also have that $\mathcal{W}^\star \cap \mathcal{Z}^\star \supseteq \mathcal{W}^\star \cap \operatorname{im} B$. Then, $\mathcal{W}^\star \cap \mathcal{Z}^\star \in \Phi(\Sigma)$.

Even if $\mathcal{W}^\star \cap \mathcal{Z}^\star$ seems to satisfy the analogous properties of its 1-D counterpart, in the 2-D case this subspace does not coincide with the minimum of $\Phi(\Sigma)$, as the following example shows. Consider Σ with the matrices A_0, A_1, A_2 given in Example 5.1 with

$$B = \begin{bmatrix} 2 & 0 \\ -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The sequence $\{\mathcal{W}_i\}_i$ becomes stationary at $k=0$, i.e., $\mathcal{W}_0 = \mathcal{W}^\star = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, whereas the sequence $\{\mathcal{Z}_i\}_i$ becomes stationary at $k=1$, and it yields $\mathcal{Z}^\star = \mathcal{Z}_1 = \mathbb{R}^3$. It follows that the intersection $\mathcal{W}^\star \cap \mathcal{Z}^\star$ is equal to $\mathcal{W}^\star$. Such intersection is therefore self-bounded, as it will certainly contain $\mathcal{W}^\star \cap \operatorname{im} B$. A friend of $\mathcal{W}^\star$ is given by $F = \frac{4}{5}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$. Hence, F is also a friend of $\mathcal{W}^\star \cap \mathcal{Z}^\star$, the latter being equal to $\mathcal{W}^\star$. However, $\mathcal{W}^\star$ is not the smallest self-bounded subspace. In fact, it is easy to see that the subspace $\mathcal{H} = \operatorname{im}\,[\,0\ \ 0\ \ 1\,]^\top$ is also an element of $\Phi(\Sigma)$. Indeed, it is output-nulling (as $(A_0 + B\,F)\mathcal{H} = \mathcal{H}$, $A_1\,\mathcal{H} = A_2\,\mathcal{H} = \{0\}$ and $\mathcal{H} \subseteq \ker C$), and it coincides with $\mathcal{W}^\star \cap \operatorname{im} B$, so that $\mathcal{H} \supseteq \mathcal{W}^\star \cap \operatorname{im} B$.

To find the right way to compute the minimum of $\Phi(\Sigma)$, we prove the following result.

LEMMA 7.9. *The set $\Phi(\Sigma)$ coincides with the set of $A_0 + B\,F$, A_1- and A_2-invariant subspaces containing $\mathcal{W}^\star \cap \operatorname{im} B$, where F is any friend of $\mathcal{W}^\star$.*

Proof. Given $\mathcal{W} \in \Phi(\Sigma)$ and a friend F of $\mathcal{W}^\star$, by Corollary 7.7 matrix F is also a friend of $\mathcal{W}$. Hence, $\mathcal{W}$ is an $A_0 + B\,F$, A_1- and A_2-invariant subspace containing $\mathcal{W}^\star \cap \operatorname{im} B$. Conversely, if $\mathcal{W}$ is $A_0 + B\,F$, A_1- and A_2-invariant subspace containing $\mathcal{W}^\star \cap \operatorname{im} B$, then it is output-nulling of feedback type for Σ and also self-bounded. ∎

As a consequence of Lemma 7.9, we can compute the minimum of $\Phi(\Sigma)$ as the minimum of the set of $A_0{+}B\,F$, A_1- and A_2-invariant subspaces containing $\mathcal{W}^\star{\cap}\operatorname{im} B$, where F is a friend of $\mathcal{W}^\star$.

LEMMA 7.10. *Let F be a friend of $\mathcal{W}^\star$. The monotonically nondecreasing sequence*

$$\{\mathcal{H}_i\}_i: \quad \begin{cases} \mathcal{H}_0 = \mathcal{W}^\star \cap \operatorname{im} B, \\ \mathcal{H}_i = (\mathcal{W}^\star \cap \operatorname{im} B) + (A_0 + B\,F)\mathcal{H}_{i-1} + A_1\,\mathcal{H}_{i-1} + A_2\,\mathcal{H}_{i-1}, \quad i \in \{1,\dots,k_h\}, \end{cases}$$

does not depend on the particular friend F of $\mathcal{W}^\star$. The smallest element $\mathcal{H}^\star$ of $\Phi(\Sigma)$ can be computed as the last term of the sequence $\{\mathcal{H}_i\}_i$, where the integer $k_h \le n-1$ is determined by the condition $\mathcal{H}_{k_h+1} = \mathcal{H}_{k_h}$, i.e., $\mathcal{H}^\star = \mathcal{H}_{k_h}$.

Proof. The algorithm follows from that of $\{\mathcal{R}_i\}_i$. We need to prove that this minimum is well-defined and does not depend on the particular choice of F (provided F is a friend of $\mathcal{W}^\star$). We prove by induction that the sequences $\{\mathcal{H}_i\}_i$ and $\{\tilde{\mathcal{H}}_i\}_i$ obtained as above but using two different friends F_1 and F_2 of $\mathcal{W}^\star$, respectively, coincide. This statement is true for $i=0$. Assume $\mathcal{H}_{i-1} = \tilde{\mathcal{H}}_{i-1}$, and let us prove that $\mathcal{H}_i = \tilde{\mathcal{H}}_i$. It is sufficient to show that $(\mathcal{W}^\star \cap \operatorname{im} B) + (A_0 + B\,F_1)\mathcal{H}_{i-1} = (\mathcal{W}^\star \cap \operatorname{im} B) + (A_0 + B\,F_2)\tilde{\mathcal{H}}_{i-1}$, and, since $\mathcal{H}_{i-1} = \tilde{\mathcal{H}}_{i-1}$, this is equivalent to proving that

$$[(A_0{+}B\,F_1) - (A_0{+}B\,F_2)]\ \mathcal{H}_{i-1} \subseteq \mathcal{W}^\star \cap \operatorname{im} B. \tag{7.5}$$

The two friends F_1 and F_2 can be written as $F_i = -\Omega_i\,W^\dagger$, where W is a basis for $\mathcal{W}^\star$ and Ω_i is a solution of $A_0\,W = W\,X_0^i + B\,\Omega_i$ for suitable matrices X_0^i, so that (7.5) is equivalent to $B\,(\Omega_1 - \Omega_2)\,W^\dagger\,\mathcal{H}_{i-1} \subseteq \mathcal{W}^\star \cap \operatorname{im} B$. From (6.8) it follows that $\left[\begin{smallmatrix} X_0^1 - X_0^i \\ \Omega_1 - \Omega_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} H_0 \\ H_1 \end{smallmatrix}\right](K_1^1 - K_1^2)$, where K_1^i are arbitrary and H_0 and H_1 are such that $W\,H_0 + B\,H_1$ is zero. Hence,

$$B\,H_1\,(K_1^1 - K_1^2)\,W^\dagger\,\mathcal{H}_{i-1} \subseteq \mathcal{W}^\star \cap \operatorname{im} B. \tag{7.6}$$

The left-hand side of the former is clearly a subspace of im B. It is easy to see that it is also a subspace of $\mathcal{W}^\star$. Since $W\,H_0 + B\,H_1 = 0$, (7.6) becomes $B\,H_1\,(K_1^1 - K_1^2)\,W^\dagger\,\mathcal{H}_{i-1} = W\,H_0\,(K_1^2 - K_1^1)\,W^\dagger\,\mathcal{H}_{i-1}$, and thus the left-hand side of (7.6) is also a subspace of $\mathcal{W}^\star$. □

By duality, we can define the concept of self-hidden subspaces. The set of self-hidden subspaces of Σ can be defined as $\Psi(\Sigma) \stackrel{\text{def}}{=} \{\mathcal{Z} \in \mathcal{Z}(\Sigma) \,|\, \mathcal{Z} \subseteq \mathcal{Z}^\star + \ker\, C\}$. This set is closed under addition, and its maximum is given by $\mathcal{Z}^\star + \mathcal{Q}^\star$, where $\mathcal{Q}^\star$ is the largest invariant with respect to A_0, A_1, and A_2 contained in $\ker C$.

8. Reachability subspaces on 2-D controlled invariant subspaces. A 2-D controlled invariant subspace of feedback type $\mathcal{W}$ is such that from any $\mathcal{W}$-valued boundary condition, at least one local state solution of (2.1) can be maintained on $\mathcal{W}$ by means of a suitable control action $u_{i,j} = F\,x_{i,j}$. As in the 1-D case, however, it is not possible to reach any point of $\mathcal{W}$ from any other point (in particular, from zero boundary condition) with a $\mathcal{W}$-valued solution of (2.1), but only a subspace $\mathcal{R}_\mathcal{W}$ of $\mathcal{W}$, which we name the *reachable subspace on* $\mathcal{W}$. The following theorem provides a method to characterize and compute this subspace.

THEOREM 8.1. *Let $\mathcal{W}$ be a 2-D controlled invariant subspace of feedback type. Let F be a friend of $\mathcal{W}$. The smallest element of the set of invariant subspaces with respect to $A_0 + B\,F$, A_1, and A_2 which contain $\mathcal{W} \cap \mathrm{im}\ B$ does not depend on F, and is $\mathcal{R}_\mathcal{W}$.*

Proof. Consider a $\mathcal{W}$-valued boundary condition. The input $u_{i,j}$ that generates a $\mathcal{W}$-valued solution of Σ can always be written as $u_{i,j} = F\,x_{i,j} + v_{i,j}$. In fact, we can always define the input $v_{i,j} \stackrel{\text{def}}{=} u_{i,j} - F\,x_{i,j}$. Then,

$$x_{i+1,j+1} = (A_0 + B\,F)\,x_{i,j} + A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,v_{i,j}. \tag{8.1}$$

If we want $x_{i+1,j+1} \in \mathcal{W}$, we need $B\,v_{i,j}$ to be an element of $\mathcal{W}$, and therefore also an element of $\mathcal{W} \cap \mathrm{im}\ B$. Hence, $v_{i,j} \in B^{-1}\,\mathcal{W}$ for all $(i,j) \in \mathfrak{B}^+$. The reachable subspace of (8.1) relative to the input $v_{i,j}$ is by definition the smallest invariant with respect to $A_0 + B\,F$, A_1, and A_2, containing the subspace of the local state space spanned by the input action on the local state, which in this case is $\mathrm{im}\ B \cap \mathcal{W}$. We call such minimum $\mathcal{R}_\mathcal{W}(F)$. Now we show that $\mathcal{R}_\mathcal{W}(F)$ does not depend on F. Let F^1 and F^2 be two friends of $\mathcal{W}$. Let $x_{i,j}, x_{i+1,j}, x_{i,j+1} \in \mathcal{W}$ and define $x^i_{i+1,j+1} = (A_0 + B\,F^i)\,x_{i,j} + A_1\,x_{i+1,j} + A_2\,x_{i,j+1} + B\,v^i_{i,j}$ for $i \in \{1,2\}$, where $v^i_{i,j} \in B^{-1}\,\mathcal{W}$ for all (i,j). We can always choose $v^2_{i,j}$ such that $x^1_{i+1,j+1} = x^2_{i+1,j+1}$. In fact, it is sufficient to choose $v^2_{i,j} = v^1_{i,j} + (F^1 - F^2)x_{i,j}$. Now, using the same argument of the proof of Lemma 7.10, we can show that $v^2_{i,j} \in B^{-1}\,\mathcal{W}$ for all (i,j). Thus, $\mathcal{R}_\mathcal{W}(F_1) = \mathcal{R}_\mathcal{W}(F_2)$. □

As a consequence of Theorem 8.1, we have that

- $\mathcal{R}_\mathcal{W}$ is a 2-D controlled invariant subspace of feedback type and has the same friends of $\mathcal{W}$;
- $\mathcal{R}_\mathcal{W} \subseteq \mathcal{R}^\star$; and
- $\mathcal{R}_{\mathcal{W}^\star} = \min \Phi(\Sigma)$.

While in the 1-D case the reachable subspace on a controlled invariant subspace $\mathcal{V}$ can be computed as the intersection of $\mathcal{V}$ with the smallest subspace $\mathcal{S}$ such that $A(\mathcal{S} \cap \mathcal{V}) \subseteq \mathcal{S}$ containing im B as shown in [19], this is not necessarily true for 2-D systems. Thus, in general, $\mathcal{R}_\mathcal{W}$ does not coincide with $\mathcal{W} \cap \mathcal{T}^\star$, where $\mathcal{T}^\star$ is the limiting subspace of

$$\{\mathcal{T}_i\}_i : \quad \begin{cases} \mathcal{T}_0 = \operatorname{im} B, \\ \mathcal{T}_i = \operatorname{im} B + A_0(\mathcal{T}_{i-1} \cap \mathcal{W}) + A_1\,\mathcal{T}_{i-1} + A_2\,\mathcal{T}_{i-1}, \qquad i \in \{1, 2, \dots, k_t\}. \end{cases}$$

In fact, even if $\mathcal{W} \cap \mathcal{T}^\star$ is invariant with respect to $A_0 + B\,F$, A_1, and A_2 and contains $\mathcal{W} \cap \operatorname{im} B$, it is not the smallest and therefore contains only $\mathcal{R}_\mathcal{W}$. In view of Theorem 8.1, the subspace $\mathcal{R}_\mathcal{W}$ can be computed by first computing *any* friend F of $\mathcal{W}$ and then applying the recursion given in Theorem 7.1 with $A_0 + B\,F$ in place of A_0 and a basis matrix for the subspace $\mathcal{W} \cap \operatorname{im} B$ in place of B.

Concluding remarks. In this paper, fundamental structural invariants of 2-D systems have been introduced and discussed. The most remarkable differences with respect to the 1-D case are (i) the need for a distinction between 2-D controlled invariance and controlled invariance of feedback type in order to fully characterize the trajectories of a 2-D system generated by arbitrary and feedback controls, and (ii) the fact that the minimum self-bounded subspace of a 2-D system—which is expected to play a pivotal role in the solution of disturbance decoupling problems with stability as the 1-D counterpart—does not coincide with the intersection of the largest output-nulling with the smallest input-containing subspaces of a 2-D system. An algorithm is also given for the computation of such a minimum.

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