Existence and uniqueness of solutions of piecewise non-linear systems*

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Abstract

In this paper, we consider the existence and uniqueness of solutions of piecewise non-linear systems. We will present some necessary and sufficient conditions for the existence and uniqueness of solutions of this class of systems.

Keywords: piecewise non-linear system, uniqueness, existence, lexicographic inequalities.

1 Introduction

Piecewise system is an important class of hybrid systems. The study of this class of systems dates back to the early works of Andronov on oscillation in non-linear systems, Kalman on saturated linear systems in 1950s. This field has received much attention over the past two decades [1-12]. However, most of these works are based on the assumption that this class of systems is well-posed.

In [11], the authors have studied the well posedness of piecewise linear systems in the sense of Carathéodory. They first used the lexicographic inequalities and the smooth continuation to derive necessary and sufficient conditions for the well posedness of a bimodal system with single criterion. They then extended those results to a multi-modal system with multiple criteria. In [11], an algorithm is proposed for solving these conditions. In [12], some sufficient conditions are obtained for the well posedness of the switch based control systems. In [13], necessary and sufficient conditions for the well posedness of piecewise linear systems with multiple modes and multiple criteria are derived. A computational procedure is then developed to

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solve these necessary and sufficient conditions by using the Fourier-Motzkin elimination rather than the linear programming method as in [11]. However, all the above results are for linear systems. It appears that few results are available for cases involving non-linear systems. In this paper, we study the existence and uniqueness of solutions of piecewise non-linear systems. Some necessary and sufficient conditions for this class of systems will be derived. Our results generalize those obtained in [11] and [13].

The organization of the paper is as follows. In Section 2, we formulate the problem. Section 3 is devoted to developing necessary and sufficient conditions. Some concluding remarks are given in Section 4.

In the following, we will use the lexicographic inequalities of $x \in \mathbb{R}^n$, i.e.,

$$x \succeq 0 \Leftrightarrow \text{for some } i,\ x_j = 0 \ \ (j = 1, 2, \cdots, i-1),\ x_i > 0 \text{ or } x = 0.$$

2 Piecewise non-linear system with Multi-modal and multi-criteria

First, let us carefully examine the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} f_1(x_1, x_2), & \text{if } x_2 - x_1^2 \geq 0, \\ f_2(x_1, x_2), & \text{if } x_2 - x_1^2 \leq 0,\ x_2 + x_1^2 \geq 0,\ x_1 \leq 0, \\ f_3(x_1, x_2), & \text{if } x_2 - x_1^2 \leq 0,\ x_2 + x_1^2 \geq 0,\ x_1 \geq 0, \\ f_4(x_1, x_2), & \text{if } x_2 + x_1^2 \leq 0. \end{cases} \tag{1}$$

Here, $\mathbb{R}^2$ is partitioned into four parts $R^i, i = 1, \cdots, 4$, which is shown in Figure 1. In each R^i, the system is evolved according to the model $\dot{x} = f_i(x)$.

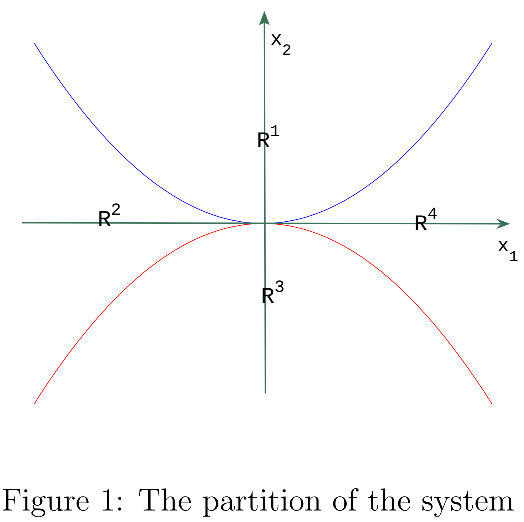

Figure 1: The partition of the system (1).

A question emerges naturally: when does the system (1) admit a unique solution in some sense for any initial condition $x_0 \in \mathbb{R}^2$.

To address this question, let us consider a general system given by

$$\Sigma : \dot{x}(t) = f_i(x), \text{ if } y = \left[h_i^1,\ h_i^2, \cdots, h_i^{p_i}(x)\right] \geq 0, \tag{2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \cdots, m$, and $h_i^j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, m$, $j = 1, \cdots, p_i$, are real functions defined on $\mathbb{R}^n$.

When $m = 2$, the system (2) is called bimodal. For a bimodal system, if $p_1 = p_2 = 1$, then the system is called a bimodal system with a single criterion.

Suppose that $x(t)$ satisfies $h_i^j(x) = 0$, $j = 1, \cdots, p_i$, and $h_k^l(x) = 0$, $l = 1, \cdots, p_k$, at some time $\hat{t}$. Then, which mode is to be applied is determined by the behavior of $x(t)$ in the time interval $\left[\hat{t}, \hat{t} + \varepsilon\right]$, where $\varepsilon > 0$ is a small constant.

Let us first recall some definitions given in [11].

Definition 2.1. For a given initial state $x(t_0)$, suppose that $x(t)$ satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau))\, d\tau \tag{3}$$

and is absolutely continuous on each compact subinterval of $[t_0,\ t_1)$, where $f(x)$ is the vector field given by the right hand of (2). If there exists no left-accumulation point [11] of event times on $[t_0,\ t_1)$, then $x(t)$ is said to be a solution of system (2) on $[t_0,\ t_1)$ in the sense of Carathéodory for the initial state $x(t_0)$.

Definition 2.2. Let $\mathcal{S}$ be a subset of $\mathbb{R}^n$. If for a given initial state x_0, there exists an $\varepsilon > 0$ such that for all $t \in [0, \varepsilon]$, $x(t) \in \mathcal{S}$, then we say that the system has the smooth continuation property at x_0 with respect to $\mathcal{S}$. Moreover, if from any $x_0 \in \mathcal{S}$, the smooth continuation is possible with respect to $\mathcal{S}$, then the system is said to have the smooth continuation property with respect to $\mathcal{S}$.

Throughout the paper, we assume that the following conditions are satisfied:

Assumption A. 1). f_i, h_i^j, $i = 1, \cdots, m$; $j = 1, \cdots, p_i$, are analytic functions.
2). For each $i = 1, \cdots, m$, and any $M > 0$, there exists a $K_{i,M}$ such that

$$\|f_i(x)\| \leq K_{i,M}(1 + \|x\|), \text{ for any } x \in \{x \in \mathbb{R}^n : \|x\| \leq M\}, \tag{4}$$

where $\|\cdot\|$ denotes the usual norm of $\mathbb{R}^n$.

3 Main Results

In this section, we will give some necessary and sufficient conditions for the existence and uniqueness of solutions of system (2).

Lemma 3.1 *If f_i, $i = 1, \cdots, m$, satisfy (4), then the following two statements are equivalent.*

i) The system (2) admits a unique solution on $[0,\infty)$ *for any initial state* x_0.
ii) For the system (2), the smooth continuation from every initial state $x_0 \in \mathbb{R}^n$ *is possible only in one of the* m *modes. In other words, the smooth continuation is possible only in one of the sets*

$$\left\{x \in \mathbb{R}^n : h_i^1(x) \geq 0,\ h_i^2(x) \geq 0,\ \cdots,\ h_i^{p_i}(x) \geq 0\right\},\ i = 1, \cdots, m. \tag{5}$$

The exception is for cases where solutions in any two of the sets above are the same in some time interval.

Proof. i)⇒ii). Since the system (2) admits a unique solution, it follows from Definition 2.1 that there exists no left-accumulation point of event times. Thus, the smooth continuation exists only in one of the m models.

ii)⇒i). From Definition 2.1 and Definition 2.2, we only need to prove that for any initial state $x(t_0)$, the system (2) has a unique absolutely continuous solution $x(t)$ satisfying (3) for any interval $[t_0, t_1] \subset [0,\infty)$. From ii) and Assumption A (ii), there exists a local unique solution from every initial state. Thus, we can construct a successively connected solution as follows.

$$x(t) = x_i(t),\ t \in [\tau_{i-1}, \tau_i],\ i = 1, 2, \cdots, N, \cdots$$

where, τ_i, for each $i = 1, 2, \cdots, N, \cdots$, is the switching time, and $x_i(t)$ is the part of the connected solution driven by the i-th m mode. If $\lim_{i\to\infty} \tau_i = \tau < t_1$, then it follows from (4) that $\{x(\tau_i), i = 1, 2, \cdots, N, \cdots\}$ is a bounded set and $x(t)$ is uniformly continuous on $[t_0, \tau)$. Thus, $\{x(\tau_i)\}$ has a well-defined limit at τ. From $x(\tau)$, there exists a unique solution again. Repeat this process, we obtain a unique state $x(t)$ satisfying (2) on $[t_0, t_1]$. The absolutely continuity of $x(t)$ is easily obtained from Assumption A and (3).

Finally, we shall verify that there exist no left-accumulation point of event times in $\{\tau_i\}$. On a contrary, we suppose that there exists a left-accumulation point τ. Then, $x(t)$ has a well-defined limit at that point. However, this contradicts to the fact that the smooth continuation is possible in only one of the m modes. Therefore, there exists a unique solution on $[t_0, t_1]$ for every initial state x_0. Thus, the system (2) exists a unique solution on $[0,\infty)$ for any initial state x_0. ■

Note that f and h are analytic functions. We recursively define the Lie derivative, $L_f^k h : \mathbb{R}^n \to \mathbb{R}$, of h along f as

$$L_f^k h(x) = \begin{cases} h(x), & \text{if } k = 0, \\ \left(\frac{\partial}{\partial x} L_f^{k-1} h(x)\right) f(x), & \text{if } k > 0, \end{cases} \tag{6}$$

where k is a non-negative integer. Define

$$T_i = \bigcap_{j=1}^{p_i} \bigcup_{k=1}^{\infty} T_{i,j,k} \tag{7}$$

$$T_{i,j,k} = \{x \in \mathbb{R}^n : H_{i,j,k} \succeq 0\}, \ i = 1, \cdots, m, \ j = 1, \cdots, p_i, \tag{8}$$

$$H_{i,j,k} = \left[h_i^j, L_{f_i} h_i^j, \cdots, L_{f_i}^{k-1} h_i^j\right]^\top, \ i = 1, \cdots, m, \ j = 1, \cdots, p_i. \tag{9}$$

and

$$K_{i,j} = \{x \in \mathbb{R}^n : f_i(x) = f_j(x)\}, \ i = 1, \cdots, m. \tag{10}$$

We have the following theorem.

Theorem 3.1 *Suppose that f_i, $i = 1, \cdots, m$, satisfy (4). Then, the system (2) admits a unique solution on $[0, \infty)$ for any initial state x_0 if and only if*

$$\bigcup_{i=1}^{m} T_i = \mathbb{R}^n, \quad T_i \bigcap T_j \subset K_{i,j}, \ \textit{for all } i \neq j. \tag{11}$$

Proof. We note that

$$x \in \left\{x \in \mathbb{R}^n : h_i^1(x) \geq 0, \ h_i^2(x) \geq 0, \ \cdots, \ h_i^{p_i}(x) \geq 0\right\}$$

if and only if

$$x \in T_i.$$

($\Rightarrow$) Since the system (2) admits a unique solution on $[0, \infty)$ for any initial state x_0, the smooth continuation is possible only in one of the m modes of (5) for each $x_0 \in \mathbb{R}^n$ by Lemma 3.1. For example, suppose the smooth continuation is only possible in the first mode. Then we can easily check that if $x_0 \in T_1$, then $\bigcup_{i=1}^{m} T_i = \mathbb{R}^n$. On the other hand, suppose $x_0 \in T_i \bigcap T_j$, that is to say the smooth continuation is possible in T_i and T_j, then, by the uniqueness of solutions of (2), the vector fields f_i and f_j must be the same. Thus, $T_i \bigcap T_j \subset K_{i,j}$, for all $i \neq j$.

($\Leftarrow$) Suppose $\bigcup_{i=1}^{m} T_i = \mathbb{R}^n$. For each $x_0 \in \mathbb{R}^n$, suppose that $x_0 \in T_i$. Then, the smooth continuation is possible in the i-th mode of (5). We note that $T_i \bigcap T_j \subset K_{i,j}$, that is to say if the smooth continuation is satisfied in two modes, then the two modes must be the same. Thus, the system (2) exists a unique solution on $[0, \infty)$ for any initial state x_0 by Lemma 3.1. ■

The conditions of Theorem 3.1 are diffcult to be verified for ensuring the existence and uniqueness of the solution of the system (2), because the definition of T_i is expressed in terms of the infinite number of intersections of $T_{i,j,k}$.

Suppose that the system (2) is with a single criterion, *i.e.*, $p_1 = p_2 = \cdots = p_m = 1$. Define

$$H_i = \left[h_i, L_{f_i} h_i, \cdots, L_{f_i}^{n-1} h_i\right]^\top, \ i = 1, \cdots, m. \tag{12}$$

Definition 3.1. For any i, $i = 1, \cdots, m$, we say that the Jacobian matrix $J_i(x)$ of (12) satisfies the ratio condition uniformly if there exists an $\varepsilon > 0$ such that the leading principal minors Δ_1^i, $\cdots$, Δ_n^i of $J_i(x)$ satisfy

$$\left|\Delta_1^i\right| \geq \varepsilon, \ \frac{|\Delta_2^i|}{|\Delta_1^i|} \geq \varepsilon, \ \cdots, \ \frac{|\Delta_n^i|}{|\Delta_{n-1}^i|} \geq \varepsilon. \tag{13}$$

We have the following theorem.

Theorem 3.2 *Consider the system (2) with only a single criterion and assume that Assumption A is satisfied. Suppose that for each* $i = 1, \cdots, m$*, the Jacobian matrix* $J_i(x)$ *of (12) satisfies the ratio condition (13) uniformly and the following condition is satisfied*

$$\{x : f_i(x) = 0\} = \{x : H_i(x) = 0\} = \{x_0\}, \ i = 1, \cdots, m, \tag{14}$$

Then, there exists a unique solution on $[0, \infty)$ *for any initial state* x_0 *if and only if*

$$\bigcup_{i=1}^{m} S_i = \mathbb{R}^n \ , \tag{15}$$

where

$$S_i = \{x \in \mathbb{R}^n : H_i \succeq 0\}, \ i = 1, \cdots, m.$$

Proof. If for each $i = 1, \cdots, m$, the Jacobian matrix $J_i(x)$ of (12) satisfies the ratio condition uniformly, it is clear that $H_i : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one map, $i = 1, \cdots, m$, (Theorem 1 [14]). To prove this theorem, we only need to show that

$$S_i = \bigcup_{k=1}^{\infty} T_{i,1,k}$$

by virtue of Theorem 3.1. Indeed, since

$$S_i \subset \bigcup_{k=1}^{\infty} T_{i,1,k}.$$

Thus, it remains to show that

$$S_i \supset \bigcup_{k=1}^{\infty} T_{i,1,k}.$$

However, this is assured by the condition (14) and $H_i : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one map. ■

To illustrate the applicability of Theorem 3.2, we consider the system

$$\dot{x} = \begin{cases} x, & \text{if } y = e^x - 1 + x \geq 0, \\ -x, & \text{if } y = e^x - 1 + x \leq 0. \end{cases} \tag{16}$$

Since $\frac{d}{dx}(e^x - 1 + x) = 1 + e^x$, we can choose $\varepsilon = 1$ in (13). Then, both of the sub-systems of (16) are observable (Theorem 1 [14]). We can easily verify that the system (16) satisfies (14) and (15), and thus the system (16) admits a unique solution on $[0, \infty)$ for any initial state x_0.

To illustrate it further, let us consider the following two dimensional case. The dynamics is given by

$$\begin{cases} \dot{x}_1 = \frac{1}{2}x_1^2 + e^{x_2} + x_2 \\ \dot{x}_2 = x_1^2 \end{cases} \quad \text{if } y = x_1 \geq 0, \qquad \begin{cases} \dot{x}_1 = 2x_2 + \cos x_2 + x_1 e^{x_1} \\ \dot{x}_2 = \sin x_1 \end{cases} \quad \text{if } y = x_1 \leq 0. \tag{17}$$

Then,

$$\begin{aligned} H_1 &= \begin{bmatrix} h_1 \\ L_{f_1} h_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{2}x_1^2 + e^{x_2} + x_2 \end{bmatrix}, \\ H_2 &= \begin{bmatrix} h_2 \\ L_{f_2} h_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -\left(2x_2 + \cos x_2 + x_1 e^{x_1}\right) \end{bmatrix}, \end{aligned}$$

The Jacobian matrix of H_1 and H_2 are

$$J_1 = \begin{bmatrix} 1 & 0 \\ x_1 & e^{x_2} + 1 \end{bmatrix}, J_2 = \begin{bmatrix} 1 & 0 \\ -\left(1 + x_1\right) e^{x_1} & -2 + \sin x_2 \end{bmatrix}.$$

$$\begin{aligned} \left|\Delta_1^1\right| &= 1, \ \frac{\left|\Delta_2^1\right|}{\left|\Delta_1^1\right|} = \left|e^{x_2} + 1\right| \geq 1, \\ \left|\Delta_1^2\right| &= 1, \ \frac{\left|\Delta_2^2\right|}{\left|\Delta_1^2\right|} = \left|2 - \sin x_2\right| \geq 1. \end{aligned}$$

Thus, (17) satisfies Assumption A and (13). Note that

$$S_1 = \{x : x_1 > 0\} \cup \{x : x_1 = 0 \text{ and } e^{x_2} + x_2 > 0\},$$

$$S_2 = \{x : x_1 < 0\} \cup \{x : x_1 = 0 \text{ and } 2x_2 + \cos x_2 < 0\},$$

and $S_1 \cup S_2 = \mathbb{R}^2$. Thus, (17) admits a unique solution on $[0, \infty)$ for any initial state $x_0 \in \mathbb{R}^2$.

In the remainder of this section, we consider a special system with a single criterion given by

$$\dot{x} = A_i x + b_i, \ \text{if } h_i\left(x\right) = p_{i,m_i}\left(x\right), \ i = 1, \cdots, m, \tag{18}$$

where $p_{i,m_i}\left(x\right)$, $i = 1, \cdots, m$, are polynomials of degree m_i. For consistency, let

$$\left\{x : L^0_{A_i x + b_i} p_{i,m_i}\left(x\right) = 0, \ L^1_{A_i x + b_i} p_{i,m_i}\left(x\right) > 0\right\} = \left\{x : p_{i,m_i}\left(x\right) > 0\right\}.$$

Define

$$H_{i,d_i} = \bigcup_{j=1}^{d_i-1} \left\{ x : L^k_{A_ix+b_i} p_{i,m_i}(x) = 0,\ L^j_{A_ix+b_i} p_{i,m_i}(x) > 0,\ k = 0, 1, \cdots, j-1 \right\}, \tag{19}$$

where

$$d_i = \sum_{j=0}^{m_i} (n+j-1)!/j!\,(n-1)!, \tag{20}$$

We have the following theorem.

Theorem 3.3 *The system (18) admits a unique solution on* $[0,\infty)$ *for any initial state* x_0 *if and only if*

$$\bigcup_{i=1}^{m} H_{i,d_i} = \mathbb{R}^n,\ H_{i,d_i} \bigcap H_{j,d_j} \subset K_{i,j},\ \text{for all } i \neq j, \tag{21}$$

where $K_{i,j}$ *is defined in (10) with* $f_i(x) = p_{i,m_i}(x)$, $f_j(x) = p_{j,m_j}(x)$.

Before the proof of this theorem, we need the following lemma.

Lemma 3.2 *Consider*

$$\dot{x} = Ax + b,\ y = p_m(x),$$

where A *is an* $n \times n$ *matrix,* b *is an* n*-vector and* $p_m(x)$ *is a polynomial of degree* m*. Then, there exists an output linear differential equation of order* d*, such that*

$$y^{(d)}(t) = \sum_{i=0}^{d-1} \alpha_i y^{(i)}(t),\ t \geq 0, \tag{22}$$

where $y^{(i)}(t) = L^i_{Ax+b} p_m(x)$, α_i *are real constants and the order* d *is such that*

$$d \leq d_{n,m} = \sum_{i=0}^{m} (n+i-1)!/i!\,(n-1)!. \tag{23}$$

Proof. Note that the set of polynomials of degree m defined on $\mathbb{R}^n$ can be considered as a subspace which dimension is less than $d_{n,m}$. Thus, $y^{(0)}$, $y^{(1)}$, $\cdots$, $y^{(d_{n,m})}$ are linear dependent. Hence, there exists some number $d \leq d_{n,m}$ such that (22) is satisfied. ■

Proof of Theorem 3.3. To prove Theorem 3.3, we only need to prove

$$H_{i,d_i} = \bigcup_{j=1}^{\infty} \left\{ x : L^k_{A_ix+b_i} p_{i,m_i}(x) = 0,\ L^j_{A_ix+b_i} p_{i,m_i}(x) > 0,\ k = 0, 1, \cdots, j-1 \right\} \tag{24}$$

in view of Theorem 3.1. Clearly,

$$H_{i,d_i} \subset \bigcup_{j=1}^{\infty} \left\{ x : L^k_{A_i x + b_i} p_{i,m_i}(x) = 0,\ L^j_{A_i x + b_i} p_{i,m_i}(x) > 0,\ k = 0, 1, \cdots, j-1 \right\}.$$

Thus, it remains to show that

$$H_{i,d_i} \supset \bigcup_{j=1}^{\infty} \left\{ x : L^k_{A_i x + b_i} p_{i,m_i}(x) = 0,\ L^j_{A_i x + b_i} p_{i,m_i}(x) > 0,\ k = 0, 1, \cdots, j-1 \right\}.$$

We prove it by contradiction. Suppose

$$x_0 \in \bigcup_{j=1}^{\infty} \left\{ x : L^k_{A_i x + b_i} p_{i,m_i}(x) = 0,\ L^j_{A_i x + b_i} p_{i,m_i}(x) > 0,\ k = 0, 1, \cdots, j-1 \right\}$$

but $x_0 \notin H_{i,d_i}$, i.e., there exists a $k > d_i$ such that

$$L^j_{A_i x + b_i} p_{i,m_i}(x) = 0, \text{ for } j = 1, \cdots k-1, \tag{25}$$

$$L^k_{A_i x + b_i} p_{i,m_i}(x) > 0. \tag{26}$$

By Lemma 3.2, there exists a $d \leq d_i$ such that

$$y^{i,(d)}(t) = \sum_{j=0}^{d-1} \alpha_j y^{i,(j)}(t), t \geq 0, \tag{27}$$

where $y^{i,(j)}(t) = L^j_{A_i x + b_i} p_{i,m_i}(x)$. We take the Lie differential at both sides of (27). We have

$$y^{i,(s)}(t) = \sum_{j=s-d}^{s-1} \alpha_j y^{i,(j)}(t), t \geq 0,\ s = d, d+1, \cdots, k. \tag{28}$$

On this basis, we have $L^k_{A_i x + b_i} p_{i,m_i}(x) = 0$. This contradicts (26). We complete the proof.■

To verify if the system (18) admits a unique solution on $[0, \infty)$ for any initial state x_0, we only need to verify the condition (21) in view of Theorem 3.3. To verify the condition (21), we only need to solve a sequence of polynomial equations and strict polynomial inequalities.

Let us consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \text{if } y = x_1 x_2 + x_2^2 \geq 0, \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \text{if } y = x_1 x_2 + x_2^2 \leq 0. \end{cases} \tag{29}$$

We have $y_1^{(1)} = L_{A_1x}y = x_2\left(2x_1 + 3x_2\right)$, $y_1^{(2)} = L_{A_1x}^2 y = 4x_2\left(x_1 + 2x_2\right) = 4y_1^{(1)} - 4y$,

$$\begin{aligned} H_1 &= \left\{x \in \mathbb{R}^2 : y > 0\right\} \bigcup \left\{x : y = 0 \text{ and } y_1^{(1)} > 0\right\} \\ &= \left\{x \in \mathbb{R}^2 :\ x_2 > 0 \text{ and } x_1 + x_2 > 0\right\} \bigcup \left\{x \in \mathbb{R}^2 :\ x_2 < 0 \text{ and } x_1 + x_2 < 0\right\} \\ &\quad \bigcup \left\{x \in \mathbb{R}^2 :\ x_2 = 0\ \right\}. \end{aligned}$$

We can compute

$$\begin{aligned} H_2 &= \left\{x \in \mathbb{R}^2 :\ x_2 < 0 \text{ and } x_1 + x_2 > 0\right\} \bigcup \left\{x \in \mathbb{R}^2 :\ x_2 > 0 \text{ and } x_1 + x_2 < 0\right\} \\ &\quad \bigcup \left\{x \in \mathbb{R}^2 :\ x_2 \neq 0 \text{ and } x_1 + x_2 = 0\right\} \bigcup \left\{(0,0)^\top\right\}. \end{aligned}$$

Since $[2,-2]^\top \notin H_1 \cup H_2$, we conclude that the solution of the system (29) does not exist at $[2,-2]^\top$. We note that $H_1 \cap H_2 = \left\{(0,0)^\top\right\}$. Thus, if the system (29) has a solution from $x_0 \in \mathbb{R}^2$, then this solution is unique.

Remark 3.1. We note that when $m_i = 1$, the system (18) reduces to the case considered in [13]. Thus, our results contain those obtained in [13].

4 Conclusion

In this paper, we have considered the existence and uniqueness of solutions of piecewise non-linear systems. We have developed some necessary and sufficient conditions for the existence and uniqueness of solutions of this class of systems.

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