On the Cauchy problem for a generalized Boussinesq equation

Qun Lin[*], Yong Hong Wu and Ryan Loxton

Department of Mathematics and Statistics,

Curtin University of Technology, Perth, Australia

Abstract

In this paper, we consider the Cauchy problem for a generalized Boussinesq equation. We show that, under suitable conditions, a global solution for the initial value problem exists. In addition, we derive the sufficient conditions for the blow-up of the solution to the problem.

Keywords: Generalized Boussinesq equation, Cauchy problem, Ground state, Global existence, Solution blow-up.

1. Introduction

Over the last couple of decades, a great deal of work has been carried out worldwide to study the properties and solutions of Boussinesq type equations

[*]Corresponding Author: q.lin@curtin.edu.au

(see [3,6,7,8,9,10,11]). In this paper, we study the following Cauchy problem:

$$u_{tt} - \omega u_{xx} + u_{xxxx} + [f(u)]_{xx} = 0, \tag{1.1}$$

and

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.2}$$

where $u := u(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\omega > 0$ is a constant, f, u_0, $u_1 : \mathbb{R} \to \mathbb{R}$ are given functions and the subscripts denote partial differentiation. Note that the partial differential equation (1.1) is a well-known generalized Boussinesq equation that arises in the study of water waves (see [12, 17]), dense lattices (see [13]) and anharmonic lattice waves (see [15]).

Problem (1.1)-(1.2) with $\omega = 1$ has been previously considered in [3, 11]. Specially, the authors in [3] used Kato's theory developed in [4, 5] to show that the Cauchy problem (1.1)-(1.2) is locally well-posed. The solitary wave solutions of equation (1.1) were also investigated and it was found that within a certain range of phase speeds, those solutions are non-linearly stable. In [11], based on the ground state of a corresponding non-linear Euclidean scalar field equation (see Section 2 for a definition), sufficient conditions for solution blow-up were established. In addition, when $f(s) = |s|^{p-1}s$ for some $p > 1$ in (1.1), conditions guaranteeing the existence of a global solution for problem (1.1)-(1.2) were derived.

One of the aims of this paper is to construct sufficient conditions for the existence of a global solution for problem (1.1)-(1.2) when f is in a more general form and ω is an arbitrary constant. To do this, we first generalize Theorem 2.6 of [11]. As the method of proof employed in [11] is not suitable for the generalized problem considered here, we use a different approach to establish this result. Based on the new result, sufficient conditions for the existence of a global solution are established. The other aim of the work is to derive conditions for the blow-up of the solution to problem (1.1)-(1.2) for some more general cases of f. For this purpose, we propose a different approach to derive a necessary inequality and consequently establish the blow-up results. It should be addressed here that our blow-up results extend those reported in [11] which is for the case $f(s) = |s|^{p-1}s$ $(p > 1)$.

2. Preliminary results

Before proving our main results relating to problem (1.1)-(1.2), we will first need to establish some preliminary lemmas involving a corresponding non-linear Euclidean scalar field equation. Although the space domain of (1.1) is $\mathbb{R}$, we will study this corresponding equation in the more general setting $\mathbb{R}^N$.

The non-linear Euclidean scalar field equation that we will consider is

$$-\Delta\phi + \omega\phi = f(\phi), \tag{2.1}$$

where $\phi \in H^1(\mathbb{R}^N)\backslash\{0\}$, $\omega > 0$ is a constant and f is a given function. The function f is required to satisfy some conditions. More specifically, we consider the following two cases:

Case 1. $f(s) = |s|^{p-1}s - |s|^{q-1}s$ for some real numbers p and q satisfying

$1 < q < p < \kappa$, where

$$\kappa = \begin{cases} \frac{N+2}{N-2}, & N \geq 3, \\ +\infty, & N = 1, 2. \end{cases}$$

Case 2. f satisfies the following hypotheses:

(H$_1$). $f \in C^1(\mathbb{R})$; f is odd; $f'(0) = 0$ and $f(s) \geq 0$ for all $s \geq 0$.

(H$_2$). If $N \geq 3$, then $\lim\limits_{s \to +\infty} \dfrac{f(s)}{s^\ell} = 0$ and $\limsup\limits_{s \to +\infty} \dfrac{f'(s)}{s^{\ell-1}} < +\infty$, where $\ell = \frac{N+2}{N-2}$; otherwise, there exists an $\ell \in (1, \infty)$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{s^\ell} = 0 \text{ and } \limsup_{s \to +\infty} \frac{f'(s)}{s^{\ell-1}} < +\infty.$$

(H$_3$). There exists a real number $\theta \in \left(0, \frac{1}{2}\right)$ such that

$$F(s) := \int_0^s f(\tau) d\tau \leq \theta s f(s)$$

for all $s \geq 0$.

(H$_4$). The function $\dfrac{f(s)}{s}$ is strictly increasing on $(0, +\infty)$.

In this paper, $|\cdot|_l$ will denote the norm of $L^l(\mathbb{R}^N)$, while $\|\cdot\|_{H^1(\mathbb{R}^N)}$ will denote the norm of $H^1(\mathbb{R}^N)$. According to [1], if f is a continuously differentiable function satisfying (H$_2$) and $f(0) = f'(0) = 0$, then the functionals

$$S(\psi; f, \omega) := \int_{\mathbb{R}^N} \left[\frac{1}{2}\big|\nabla\psi(\mathbf{x})\big|^2 + \frac{\omega}{2}\big|\psi(\mathbf{x})\big|^2 - F\big(\psi(\mathbf{x})\big)\right] d\mathbf{x}$$

and

$$R(\psi; f, \omega) := \int_{\mathbb{R}^N} \Big[\big|\nabla\psi(\mathbf{x})\big|^2 + \omega\big|\psi(\mathbf{x})\big|^2 - \psi(\mathbf{x}) f\big(\psi(\mathbf{x})\big) \Big] d\mathbf{x}$$

are well-defined on $H^1(\mathbb{R}^N)$. Normally, we will omit f and ω when referring to those functions if the dependence is obvious.

Recall that a function $\varphi \in H^1(\mathbb{R}^N)\backslash\{0\}$ is called a *ground state* of equation (2.1) if

(i) φ is a solution of (2.1); and

(ii) $S(\varphi; f, \omega) \leq S(\psi; f, \omega)$ whenever ψ is a solution of (2.1).

In other words, φ minimizes S over the class of solutions of (2.1). For Case 2, it has been shown in reference [2] that such a ground state exists. This result is extended further in the following two lemmas.

Lemma 1. *Suppose that f satisfies the conditions listed in either Case 1 or Case 2, and that $\omega > 0$ and $\psi \in H^1(\mathbb{R}^N)\backslash\{0\}$. Then, there exists a unique $\lambda^* \in (0, +\infty)$ such that*

$$R(\lambda\psi; f, \omega) \begin{cases} > 0, & \text{if } 0 < \lambda < \lambda^*, \\ = 0, & \text{if } \lambda = \lambda^*, \\ < 0, & \text{if } \lambda > \lambda^*. \end{cases}$$

In addition, $S(\lambda^*\psi; f, \omega) > S(\lambda\psi; f, \omega)$ whenever $\lambda \neq \lambda^*$.

Lemma 2. *Let* $M := \{\psi \in H^1(\mathbb{R}^N)\backslash\{0\} : R(\psi; f, \omega) = 0\}$*,* $\omega > 0$ *and suppose that* f *satisfies the conditions listed in either Case 1 or Case 2. Then, there exists a solution* φ *to the following problem:*

$$\min_{\psi \in M} S(\psi; f, \omega). \tag{2.2}$$

Moreover, the set of solutions of problem (2.2) coincides with the set of ground states of equation (2.1).

Note that the results in the above two lemmas have been proved in [2] for Case 2 and in [14] for Case 1 with $\omega = 1$ and $N \geq 2$. The proofs for the remaining cases are given in the appendix.

In view of Lemma 2, we see that equation (2.1) has a ground state if $\omega > 0$ and f satisfies the conditions listed in either Case 1 or Case 2. Accordingly, set

$$d := \min_{\psi \in M} S(\psi). \tag{2.3}$$

Next we will prove a preliminary result that will be used in derivation of the conditions for the blow-up of the solution to problem (1.1)-(1.2). To do this, the following additional condition is required for Case 2:

- (H_4') There exists a real number $\beta > 1$ such that the function $\frac{f(s)}{s^\beta}$ is increasing on $(0, \infty)$.

Note that the condition (H_4') is stronger than the condition (H_4). If f satisfies the hypotheses (H_1), (H_2), (H_3) and (H_4'), we refer to it as Case 2^+. Hence, Case 2^+ is included in Case 2. It is also noted that if $f(s) = |s|^{p-1}s$ for some real number $p > 1$, then f satisfies all the conditions listed in Case 2^+.

Lemma 3. *Suppose that $\omega > 0$ and f satisfies the conditions listed in either Case 1 or Case 2^+. If $\psi \in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfying $R(\psi) < 0$, then, $R(\psi) < (\rho+1)\big[S(\psi) - d\big]$, where $\rho = q$ for Case 1 and $\rho = \beta$ for Case 2^+.*

Proof. From Lemma 1, it follows that there exists a unique number $\lambda^* \in (0,1)$ such that $R(\lambda^*\psi) = 0$. Let

$$G(\lambda) := (\rho+1)S(\lambda\psi) - R(\lambda\psi).$$

Now, we are in the position to prove that $G(\lambda)$ is strictly increasing on $(0,\infty)$. Noting that the function f is odd, we have

$$\begin{aligned} G(\lambda) =& \frac{\rho-1}{2}\lambda^2\big[\omega|\psi(\mathbf{x})|_2^2 + |\nabla\psi(\mathbf{x})|_2^2\big] \\ &+ \int_{\mathbb{R}^N}\Big[\lambda|\psi(\mathbf{x})|f\big(\lambda|\psi(\mathbf{x})|\big) - (\rho+1)F\big(\lambda|\psi(\mathbf{x})|\big)\Big]d\mathbf{x} \end{aligned}$$

and

$$\begin{aligned} G'(\lambda) =& \lambda(\rho-1)\big[\omega|\psi(\mathbf{x})|_2^2 + |\nabla\psi(\mathbf{x})|_2^2\big] \\ &+ \lambda\int_{\mathbb{R}^N}|\psi(\mathbf{x})|^2\left[f'\big(\lambda|\psi(\mathbf{x})|\big) - \rho\frac{f\big(\lambda|\psi(\mathbf{x})|\big)}{\lambda|\psi(\mathbf{x})|}\right]d\mathbf{x}. \end{aligned}$$

Note that, for both Case 1 and Case 2^+, the function $\frac{f(s)}{s^\rho}$ is increasing on $(0,\infty)$. Thus, $f'(s) - \rho\frac{f(s)}{s} \geq 0$ for each $s > 0$. Hence, $G'(\lambda) > 0$ for each $\lambda > 0$. Consequently, we have that $G(1) > G(\lambda^*)$. That is,

$$(\rho+1)S(\psi) - R(\psi) > (\rho+1)S(\lambda^*\psi) - R(\lambda^*\psi).$$

Using the fact that $R(\lambda^*\psi) = 0$ and $S(\lambda^*\psi) > d$, we can obtain that

$$(\rho+1)\big[S(\psi) - d\big] > R(\psi).$$

$\square$

3. Main results

In this section, we will introduce an equivalent form for problem (1.1)-(1.2). Then, on the basis of an existing local existence theorem, we construct conditions for the existence of global solution for problem (1.1)-(1.2) under Case 1 and Case 2, and then establish the sufficient conditions for the blow-up of the solution to problem (1.1)-(1.2) under Case 1 and Case 2^+.

Now, we consider the following problem which is equivalent to problem (1.1)-(1.2):

$$\left.\begin{aligned} &u_t = v_x, \\ &v_t = \omega u_x - u_{xxx} - \left[f(u)\right]_x, \end{aligned}\right\} \tag{3.1}$$

subject to the initial conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \tag{3.2}$$

Note that $u_1(x)$ in problem (1.1)-(1.2) and $v_0(x)$ in problem (3.1)-(3.2) satisfy $u_1(x) = v_0'(x)$.

Set

$$
\begin{aligned}
E(u,v) &:= \int_{-\infty}^{+\infty} \left[\frac{\omega}{2}u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}v^2 - F(u)\right] dx, \\
V(u,v) &:= \int_{-\infty}^{+\infty} uvdx, \\
I_1(u,v) &:= \int_{-\infty}^{+\infty} udx, \\
I_2(u,v) &:= \int_{-\infty}^{+\infty} vdx.
\end{aligned}
$$

According to [10, 11], it can be easily established that problem (3.1)-(3.2) is always locally well-posed, and the above four functionals are invariant.

Theorem 1. ***(Local existence)*** [10, 11] *If $f \in C^1(\mathbb{R})$ such that $f(0) = 0$ and $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, then problem (3.1)-(3.2) possesses a unique weak solution (u, v) in $C\big([0,T); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$ such that $E(u, v) = E(u_0, v_0)$, $V(u, v) = V(u_0, v_0)$, $I_1(u, v) = I_1(u_0, v_0)$ and $I_2(u, v) = I_2(u_0, v_0)$. Moreover, the interval of existence $[0, T)$ can be extended to a maximal interval $[0, T_{\max})$ such that either*

(i) $T_{\max} = +\infty$; *or*

(ii) $T_{\max} < +\infty$, $\lim\limits_{t\to T_{\max}^-} \|(u, v)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} = +\infty$,

where $\|(u, v)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} = \|u\|_{H^1(\mathbb{R})} + |v|_2$ denotes the norm of $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

Remark 1. Note that Theorem 1 is slightly different from the ones reported in [10,11] where $\omega = 1$. Let $g(s) := f(s) - \omega s + s$ for each $s \in \mathbb{R}$. If $f \in C^1(\mathbb{R})$ such that $f(0) = 0$, then $g \in C^1(\mathbb{R})$ and $g(0) = 0$.

Now, we define two subsets of $H^1(\mathbb{R})$ which will be proved to be invariant under the flow generated by problem (3.1)-(3.2) for Cases 1 and 2. Let

$$K_1 := \{\psi \in H^1(\mathbb{R}) : \quad S(\psi) < d, R(\psi) > 0\}$$

and

$$K_2 := \{\psi \in H^1(\mathbb{R}) : \quad S(\psi) < d, R(\psi) < 0\},$$

where d is defined as (2.3). Suppose that $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ are such that $E(u_0, v_0) < d$. We will show that if $\omega > 0$, f satisfies the conditions listed in either Case 1 or Case 2 and $u_0 \in K_1$, then the corresponding solution exists globally. Furthermore, if, in addition to satisfying the conditions listed in either Case 1 or Case 2^+, $\omega > 0$ and $u_0 \in K_2$, then the corresponding solution blows up in finite time. All these results are furnished precisely in the following theorems.

To simplify the presentation, for the remainder of this section we will use the following notation:

$$
\begin{aligned}
u(t) &:= u(t,x),\\
u_x(t) &:= u_x(t,x),\\
v(t) &:= v(t,x).
\end{aligned}
$$

Theorem 2. ***(Invariant sets)*** *Suppose that $\omega > 0$ and f satisfies the conditions listed in either Case 1 or Case 2, and that $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfying $E(u_0, v_0) < d$. Let $(u, v) \in C\big([0, T_{\max}); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$ be the weak solution of problem (3.1)-(3.2). If, for each $i \in \{1, 2\}$, $u_0 \in K_i$, then $u(t) \in K_i$ for $0 \le t < T_{\max}$.*

Proof. By virtue of Theorem 1, we have that $E\big(u(t), v(t)\big) = E(u_0, v_0) < d$ for each $t \in [0, T_{\max})$, which implies that $S\big(u(t)\big) < d$ for each $t \in [0, T_{\max})$. Hence, to prove $u(t) \in K_1$, it suffices to show $R\big(u(t)\big) > 0$. Arguing by contradiction, one can prove that $R\big(u(t)\big) > 0$ for each $t \in [0, T_{\max})$. Assume that there is a $\bar{t} \in [0, T_{\max})$ such that $R(\bar{t}) \le 0$. Noting that $u_0 \in K_1$, we see $R(u_0) > 0$. According to the continuity of $R\big(u(t)\big)$ with respect to t, there is a $t^* \in (0, \bar{t}]$ such that $R\big(u(t^*)\big) = 0$. It follows from Lemma 2 and the definition of d that $S\big(u(t^*)\big) \ge d$, which contradicts $S\big(u(t^*)\big) < d$.

Similarly, we can derive that if $u_0 \in K_2$, then $u(t) \in K_2$, $t \in [0, T_{\max})$. $\square$

Theorem 3. ***(Global existence in*** K_1***)*** *Suppose that* $\omega > 0$ *and* f *satisfies the conditions listed in either Case 1 or Case 2. Then, if* $u_0 \in K_1$ *and* $v_0 \in L^2(\mathbb{R})$ *such that* $E(u_0, v_0) < d$*, problem (3.1)-(3.2) possesses a unique weak solution* $(u, v) \in C\big([0, +\infty); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$.

Proof. As stated by Theorem 1, it suffices to prove that $\|u(t)\|_{H^1(\mathbb{R})} + |v(t)|_2$ is bounded for $0 \le t < T_{\max}$. Since f satisfies (H_3) (note that if $f(s) = |s|^{p-1}s - |s|^{q-1}s$, then f satisfies (H_3) by choosing $\theta = \frac{1}{q+1}$), we have

$$\begin{aligned} S\big(u(t)\big) \geq& \frac{1}{2}\int_{-\infty}^{+\infty} \left[|u_x(t,x)|^2 + \omega\,|u(t,x)|^2\right] dx - \theta \int_{-\infty}^{+\infty} u(t,x) f\big(u(t,x)\big) dx \\ =& \left(\frac{1}{2} - \theta\right) \int_{-\infty}^{+\infty} \left[|u_x(t,x)|^2 + \omega\,|u(t,x)|^2\right] dx + \theta R\big(u(t)\big) \\ \geq& \left(\frac{1}{2} - \theta\right) \min\{1, \omega\} \|u(t)\|^2_{H^1(\mathbb{R})} + \theta R\big(u(t)\big). \end{aligned}$$

Applying Theorem 2 yields $u(t) \in K_1$, i.e. $S\big(u(t)\big) < d$ and $R\big(u(t)\big) > 0$ for $0 \le t < T_{\max}$. Thus, $\|u(t)\|_{H^1(\mathbb{R})}$ is bounded on $[0, T_{\max})$ and $S\big(u(t)\big) > 0$. On the other hand, combining $E\big(u(t), v(t)\big) < d$ and $S\big(u(t)\big) > 0$, it is easily verified that $|v(t)|_2^2 < 2d$ for $0 \le t < T_{\max}$. □

Theorem 4. ***(Solution blow-up in*** K_2***)*** *Let* $\omega > 0$ *and* f *satisfy the conditions listed in either Case 1 or Case* 2^+*. Suppose that* $u_0 \in K_2$ *and* $v_0 \in L^2(\mathbb{R})$ *such that* $E(u_0, v_0) < d$ *and* $\xi^{-1}\hat{u}_0 \in L^2(\mathbb{R})$*, where* $\hat{u}_0$ *denotes the Fourier transform of* u_0*. Let* $(u, v) \in C\big([0, T_{\max}); H^1(\mathbb{R}) \times L^2(\mathbb{R})\big)$ *be the* *weak solution of problem (3.1)-(3.2). Then* $T_{\max} < +\infty$ *and*

$$\lim_{t \to T_{\max}^-} \left(\|u(t)\|_{H^1(\mathbb{R})} + |v(t)|_2 \right) = +\infty.$$

Proof. Here we use proof by contradiction. Suppose that $T_{\max} = +\infty$. According to [11], it follows from $\xi^{-1}\hat{u}_0 \in L^2(\mathbb{R})$ that

$$\xi^{-1}\hat{u} \in C^1\left([0,\infty); L^2(\mathbb{R})\right).$$

Let

$$I(t) := |\xi^{-1}\hat{u}(t,\xi)|_2^2, \ t \in [0,\infty).$$

Then,

$$I'(t) = 2\big(\xi^{-1}\hat{u}(t,\xi), \xi^{-1}\hat{u}_t(t,\xi)\big) \tag{3.3}$$

and

$$I''(t) = 2|v(t)|_2^2 - 2R(u(t)), \tag{3.4}$$

where $\big(\xi^{-1}\hat{u}(t,\xi), \xi^{-1}\hat{u}_t(t,\xi)\big) = \int_{-\infty}^{+\infty} \xi^{-1}\hat{u}(t,\xi)\overline{\xi^{-1}\hat{u}_t(t,\xi)}d\xi$. Using the Cauchy-Schwarz inequality, it follows from (3.3) that $[I'(t)]^2 \le 4I(t)|v(t)|_2^2$ for $t \in [0,\infty)$. Let $\rho = q$ for Case 1 and $\rho = \beta$ for Case 2^+. We have for each

$t \in [0, \infty)$ that

$$
\begin{aligned}
&I''(t)I(t) - \frac{\rho+3}{4}[I'(t)]^2 \\
&\qquad \geq -I(t)\Big[(\rho+1)|v(t)|_2^2 + 2R\big(u(t)\big)\Big] \\
&\qquad = -I(t)\Big\{2(\rho+1)\Big[E(u_0, v_0) - S\big(u(t)\big)\Big] + 2R\big(u(t)\big)\Big\}.
\end{aligned}
$$

Noting that $E(u_0, v_0) < d$, we have from above that

$$
\begin{aligned}
&I''(t)I(t) - \frac{\rho+3}{4}[I'(t)]^2 \\
&\qquad \geq -I(t)\Big\{2(\rho+1)\Big[d - S\big(u(t)\big)\Big] + 2R\big(u(t)\big)\Big\}.
\end{aligned}
$$

It follows from Theorem 2 that $R\big(u(t)\big) < 0$. Thus, using Lemma 3, we can obtain that $I''(t)I(t) - \frac{\rho+3}{4}[I'(t)]^2 > 0$. Define $J(t) := [I(t)]^{-\frac{\rho-1}{4}}$, then $J''(t) < 0$ for each $t \geq 0$.

Now, we will prove that there exists a $t^* > 0$ such that $I'(t^*) > 0$. If not, then, for all $t \geq 0$, $I'(t) \leq 0$. From (3.4) and $R\big(u(t)\big) < 0$, it follows that $I''(t) > 0$ for all $t \geq 0$. Note that

$$\lim_{t \to \infty} I'(t) = I'(0) + \int_0^\infty I''(s)ds$$

exists. Hence, there is a sequence $\{t_n\}$ such that

$$\lim_{n \to \infty} I''(t_n) = 0.$$

Combining (3.4) and $R\big(u(t)\big) < 0$, we get

$$\lim_{n\to\infty} R\big(u(t_n)\big) = 0. \tag{3.5}$$

Using Lemma 3 again yields that

$$(\rho+1)\big[E(u_0,v_0)-d\big] \geq (\rho+1)\big[S\big(u(t_n)\big)-d\big] > R\big(u(t_n)\big).$$

By virtue of (3.5), we have $E(u_0,v_0) \geq d$, which leads to a contradiction.

For such a t^*, $J(t^*) > 0$ and $J'(t^*) < 0$. Noting that $J''(t) < 0$ for $t \geq 0$, there exists a $\hat{t} \in \Big(0, -\frac{J(t^*)}{J'(t^*)}\Big]$ such that $J(\hat{t}) = 0$. Hence,

$$\lim_{t\to\hat{t}^-} I(t) = +\infty. \tag{3.6}$$

Combining (3.3) and the Cauchy-Schwarz inequality, we see that, for each $t \in [0,\hat{t})$,

$$\frac{d\,[I(t)]^{\frac{1}{2}}}{dt} = \frac{1}{2}\,[I(t)]^{-\frac{1}{2}}\,I'(t) \leq \frac{1}{2}\,[I(t)]^{-\frac{1}{2}}\,2\,[I(t)]^{\frac{1}{2}}\,|v(t)|_2 = |v(t)|_2,$$

from which we obtain that, for each $t \in [0,\hat{t})$,

$$[I(t)]^{\frac{1}{2}} < [I(0)]^{\frac{1}{2}} + \int_0^t |v(\tau)|_2 d\tau.$$

Thus, in view of (3.6), we obtain

$$\int_0^{\hat{t}} |v(\tau)|_2 d\tau = +\infty,$$

which implies that there exists a sequence $\{\tau_n\}$ such that $0 < \tau_n < \hat{t}$, $\lim_{n\to\infty} \tau_n = \hat{t}$ and

$$\lim_{n\to+\infty} |v(\tau_n)|_2 = +\infty.$$

This contradicts $T_{\max} = +\infty$. Therefore, $T_{\max} < +\infty$ and

$$\lim_{t\to T_{\max}^-} \left(\|u(t)\|_{H^1(\mathbb{R})} + |v(t)|_2\right) = +\infty.$$

□

4. Conclusion

In this paper, we have studied the solution to the Cauchy problem for a generalized Boussinesq equation. Based on the ground state of a corresponding non-linear Euclidean scalar field equation, we constructed two invariant sets. We have then established the sufficient conditions under which a unique solution exists globally if the initial function u_0 belongs to the first invariant set, while the solution blows up if u_0 belongs to the second invariant set.

5. Acknowledgement

The authors wish to acknowledge the reviewers for their comments and suggestions which have led to significant improvement of the paper.

6. Appendix

Proof of Lemma 1. We prove the lemma for Case 1. Firstly, it follows from the definitions of S and R that, for each $\lambda \in [0, \infty)$,

$$S(\lambda\psi) = \int_{\mathbb{R}^N} \Big(\frac{1}{2}\lambda^2|\nabla\psi(\mathbf{x})|^2 + \frac{\omega}{2}\lambda^2|\psi(\mathbf{x})|^2 - \frac{1}{p+1}\lambda^{p+1}|\psi(\mathbf{x})|^{p+1} + \frac{1}{q+1}\lambda^{q+1}|\psi(\mathbf{x})|^{q+1}\Big) d\mathbf{x}$$

and

$$R(\lambda\psi) = \int_{\mathbb{R}^N} \big(\lambda^2|\nabla\psi(\mathbf{x})|^2 + \omega\lambda^2|\psi(\mathbf{x})|^2 - \lambda^{p+1}|\psi(\mathbf{x})|^{p+1} + \lambda^{q+1}|\psi(\mathbf{x})|^{q+1}\big)\, d\mathbf{x}.$$

A straightforward calculation shows that

$$\frac{dS(\lambda\psi)}{d\lambda} = \frac{R(\lambda\psi)}{\lambda}. \tag{5.1}$$

Now, define

$$g(\lambda) := \lambda^{p-1} - a\lambda^{q-1} - b,$$

where $a = \dfrac{|\psi|_{q+1}^{q+1}}{|\psi|_{p+1}^{p+1}}$ and $b = \dfrac{\omega|\psi|_2^2 + |\nabla\psi|_2^2}{|\psi|_{p+1}^{p+1}}$. Then,

$$g'(\lambda) = (p-1)\lambda^{p-2} - a(q-1)\lambda^{q-2} = (p-1)\lambda^{q-2}\left[\lambda^{p-q} - \frac{a(q-1)}{p-1}\right]. \tag{5.2}$$

Set $\lambda_0 := \left[\frac{(q-1)|\psi|_{q+1}^{q+1}}{(p-1)|\psi|_{p+1}^{p+1}}\right]^{\frac{1}{p-q}} > 0$. It is clear from (5.2) that

$$g'(\lambda) \begin{cases} < 0, & \text{if } \lambda \in (0, \lambda_0), \\ = 0, & \text{if } \lambda = \lambda_0, \\ > 0, & \text{if } \lambda \in (\lambda_0, +\infty). \end{cases}$$

Consequently, $g(\lambda)$ is strictly decreasing on $[0, \lambda_0]$ and strictly increasing on $(\lambda_0, +\infty)$. Since $g(0) < 0$ and $\lim_{\lambda \to +\infty} g(\lambda) = +\infty$, there exists a unique $\lambda^* \in (\lambda_0, +\infty)$ such that

$$g(\lambda) \begin{cases} < 0, & \text{if } \lambda \in (0, \lambda^*), \\ = 0, & \text{if } \lambda = \lambda^*, \\ > 0, & \text{if } \lambda \in (\lambda^*, +\infty). \end{cases}$$

As $R(\lambda\psi) = -\lambda^2 |\psi|_{p+1}^{p+1} g(\lambda)$, we derive that $R(\lambda^*\psi) = 0$, $R(\lambda\psi) > 0$ for $\lambda \in (0, \lambda^*)$, and $R(\lambda\psi) < 0$ for $\lambda > \lambda^*$. In addition, from (5.1), we have

$$\frac{dS(\lambda\psi)}{d\lambda} \begin{cases} > 0, & \text{if } \lambda \in (0, \lambda^*), \\ = 0, & \text{if } \lambda = \lambda^*, \\ < 0, & \text{if } \lambda \in (\lambda^*, +\infty). \end{cases}$$

Hence, it follows that $S(\lambda^*\psi) > S(\lambda\psi)$ whenever $\lambda \neq \lambda^*$. $\square$

Proof of Lemma 2. Similar to Lemma 1, we prove this lemma for Case 1. Multiplying both sides of (2.1) by ϕ, integrating over $\mathbb{R}^N$ and using Green formula, we see that any solution of (2.1) belongs to M. If $\psi \in M$, then we have that, for $1 < q < p$,

$$
\begin{aligned}
S(\psi) &= \frac{1}{2}|\nabla\psi|_2^2 + \frac{\omega}{2}|\psi|_2^2 - \frac{1}{p+1}|\psi|_{p+1}^{p+1} + \frac{1}{q+1}|\psi|_{q+1}^{q+1} \\
&> \frac{1}{2}|\nabla\psi|_2^2 + \frac{\omega}{2}|\psi|_2^2 - \frac{1}{p+1}\left(|\psi|_{p+1}^{p+1} - |\psi|_{q+1}^{q+1}\right) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(|\nabla\psi|_2^2 + \omega|\psi|_2^2\right) \qquad (5.3) \\
&> 0.
\end{aligned}
$$

Hence, S is bounded below on M. Accordingly, let $\{v_n\} \subset M$ be a minimizing sequence such that $\lim_{n\to+\infty} S(v_n) = \inf_{\psi\in M} S(\psi)$.

Let ψ^* denote the Schwarz spherical rearrangement of a function $|\psi|$. From [2], ψ^* is the spherically symmetric non-increasing (with respect to $|\mathbf{x}|$) function having the same distribution function as $|\psi|$ such that

$$\int_{\mathbb{R}^N} |\nabla\psi^*(\mathbf{x})|^2 d\mathbf{x} \leq \int_{\mathbb{R}^N} |\nabla\psi(\mathbf{x})|^2 d\mathbf{x}$$

and

$$\int_{\mathbb{R}^N} |\psi^*(\mathbf{x})|^l d\mathbf{x} = \int_{\mathbb{R}^N} |\psi(\mathbf{x})|^l d\mathbf{x}$$

for each $l \in (1, \infty)$. Therefore,

$$S(\psi^*) \leq S(\psi) \qquad (5.4)$$

for each $\psi \in H^1(\mathbb{R}^N)$. In addition, it is easy to check that, for each real number $\gamma > 0$, $(\gamma\psi)^* = \gamma\psi^*$.

For a given n, it follows from Lemma 1 that there exists a unique real number $\nu_n > 0$ such that $R(\nu_n(v_n^*)) = 0$. Let $u_n = \nu_n(v_n)^* = (\nu_n(v_n))^*$. Then, according to (5.4) and Lemma 1, we get

$$S(u_n) = S\Big(\big(\nu_n(v_n)\big)^*\Big) \le S\big(\nu_n(v_n)\big) \le S\big(v_n\big).$$

Therefore, the spherically symmetric non-increasing sequence $\{u_n\}$ is a minimizing sequence in M as well.

By virtue of (5.3), we have $S(u_n) > \left(\frac{1}{2} - \frac{1}{p+1}\right)(|\nabla u_n|_2^2 + \omega|u_n|_2^2)$. Hence, the boundness of sequence $\{S(u_n)\}$ implies that sequence $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Applying the compactness lemma of W. Strauss [16] (see also [1]), there exists a subsequence of $\{u_n\}$, relabeled by $\{u_n\}$ for notational convenience, such that, for $1 < l < \ell$,

$$\begin{aligned}
&u_n \rightharpoonup u_\infty \text{ weakly in } H^1(\mathbb{R}^N),\\
&u_n \to u_\infty \text{ a.e. in } \mathbb{R}^N,\\
&u_n \to u_\infty \text{ strongly in } L^{l+1}(\mathbb{R}^N),
\end{aligned} \tag{5.5}$$

where ℓ is as defined in assumption (H_2). Arguing by contradiction, we can conclude that $u_\infty \neq 0$. Suppose that $u_\infty = 0$. Noting that u_n converges almost everywhere to 0 as $n \to \infty$, it is clear from $R(u_n) = 0$ that $\lim\limits_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^N)} = 0$. Thus, u_n strongly converges to 0 in $H^1(\mathbb{R}^N)$ as $n \to +\infty$. On the other hand, $R(u_n) = 0$ implies that

$$\min\{1,\omega\}\|u_n\|_{H^1(\mathbb{R}^N)}^2 \le |\nabla u_n|_2^2 + \omega|u_n|_2^2 + |u_n|_{q+1}^{q+1} = |u_n|_{p+1}^{p+1}.$$

Applying Sobolev's inequality, it follows

$$|u_n|_{p+1} \leq c\|u_n\|_{H^1(\mathbb{R}^N)},$$

here and thereafter, c denotes various positive constants. Noting that $p > 1$, we can obtain that

$$c \leq \|u_n\|_{H^1(\mathbb{R}^N)},$$

which leads to a contradiction.

According to Lemma 1, there is a unique real number $\mu > 0$ such that $R(\mu u_\infty) = 0$. Let $\phi := \mu u_\infty$. In view of (5.5), we have

$$\begin{aligned} &\mu u_n \rightharpoonup \phi \text{ weakly in } H^1(\mathbb{R}^N),\\ &\mu u_n \to \phi \text{ a.e. in } \mathbb{R}^N,\\ &\mu u_n \to \phi \text{ strongly in } L^{l+1}(\mathbb{R}^N). \end{aligned} \tag{5.6}$$

Noticing that $R(u_n) = 0$, Lemma 1 gives that $S(\mu u_n) \leq S(u_n)$. As S is weakly sequential lower semi-continuous on $H^1(\mathbb{R}^N)$, we have

$$S(\phi) \leq \liminf_{n\to+\infty} S(\mu u_n) \leq \lim_{n\to+\infty} S(u_n) = \inf_{\psi\in M} S(\psi).$$

Note that $\phi \in M$. Hence ϕ is a solution of problem (2.2).

Now, we will prove that ϕ satisfies (2.1). Since ϕ solves problem (2.2), there exists a Lagrange multiplier Λ such that

$$S'(\phi) = \Lambda R'(\phi). \tag{5.7}$$

We claim that $\Lambda = 0$, which implies that ϕ is a solution of (2.1). Indeed, it follows from [1] that S and R are continuously Frechet-differentiable and

$$
\begin{aligned}
< S'(\phi), \phi > &= \int_{\mathbb{R}^N} \Big[|\nabla\phi(\mathbf{x})|^2 + \omega|\phi(\mathbf{x})|^2 - |\phi(\mathbf{x})|^{p+1} + |\phi(\mathbf{x})|^{q+1} \Big] d\mathbf{x} = R(\phi) = 0, \\
< R'(\phi), \phi > &= 2|\nabla\phi|_2^2 + 2\omega|\phi|_2^2 - (p+1)|\phi|_{p+1}^{p+1} + (q+1)|\phi|_{q+1}^{q+1} \\
&< 2|\nabla\phi|_2^2 + 2\omega|\phi|_2^2 - (p+1)|\phi|_{p+1}^{p+1} + (p+1)|\phi|_{q+1}^{q+1} \\
&= (1-p)(|\nabla\phi|_2^2 + \omega|\phi|_2^2) \\
&< 0,
\end{aligned}
$$

where $< \cdot, \cdot > = < \cdot, \cdot >_{(H^{-1}(\mathbb{R}^N), H^1(\mathbb{R}^N))}$. Therefore, the solutions of problem of (2.2) are also ground states of (2.1). Recalling that each solution of (2.1) belongs to M, we can conclude that the set of ground states of (2.1) coincides with the set of solutions of problem (2.2) . $\square$

References

[1] H. Berestycki and P. L. Lions, Nonlinear scalar field equations I and II, existence of a ground state, Archive for Rational Mechanics and Analysis, 82, (1983) 313-375.

[2] H. Berestycki and T. Cazenave, Instability of stationary state in nonlinear Schrödinger and Klein-Gordon equations, CNRS, Université Paris, France, (1983), 1-52.

[3] J. L. Bona and R. L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Comm.

Math. Phys., 118 (1988) 15-29.

[4] T. Kato, Quasilinear equations of evolution, with applications to partial differential equations. Lecture notes in Mathematics, 448 (1974) 25-70.

[5] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation. Studies in Applied Mathematics. Ad. Math. Suppl. Stud. 8 (1983) 93-128.

[6] S. Lai and Y. H. Wu, The asymptotic solution of the Cauchy problem for a generalized Boussinesq equation, Discrete and Continuous Dynamical Systems - Series B, 3 (2003) 401-408.

[7] S. Lai, Y. H. Wu and X. Yang, The global solution of an initial boundary value problem for the damped Boussinesq equation, Communications on Pure and Applied Analysis, 3 (2004) 319-328.

[8] S. Lai, Y. H. Wu and Y. Zhou, Some physical structures for the (2 + 1)-dimensional Boussinesq water equation with positive and negative exponents, Computers and Mathematics with Applications, 56 (2008) 339-345.

[9] Q. Lin, Y. H. Wu, R. Loxton and S. Lai, Linear B-spline finite element method for the improved Boussinesq equation, Journal of Computational and Applied Mathematics (2008), doi:10.1016/j.cam.2008.05.049

[10] Y. Liu, Instability of solitary waves for generalized Boussinesq euqations, J. Dynamics Differential Equations, (1993), 537-558.

[11] Y. Liu, Instability and blow-up of solutions to a generalized Boussinesq equation, SIAM J. Math. Anal., 26(1995), 1527-1546.

[12] V. G. Makhankov, Dynamics of classical solitons (in non-integable systems), Physics Reports, 35 (1987) 1-128.

[13] P. Rosenau, Dynamics of dense lattices, Phyiscal Review B, 36 (1987) 5868-5876.

[14] J. Shu and J. Zhang, Instability of standing waves for a class of nonlinear Schrödinger equations, J. Math. Anal. Appl. 327 (2007), 878-890.

[15] S. J. Smith and R. Chatterjee,The generalized Korteweg-de Vries equation for interacting phonons in a coherent state basis, Physics Letters A 125 (1987) 129-133.

[16] W. Strauss, Existence of solitary waves in higher dimensions. Comm. Math. Phys., 55 (1977), 149-162.

[17] G. B. Whitham, Linear and nonlinear waves, New York: Wiley, 1974.