A Global Optimization Approach to Fractional Optimal Control

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In this paper, we consider a fractional optimal control problem governed by system of linear differential equations, where its cost function is expressed as the ratio of a convex function and a concave function. This optimal control problem can, in principle, be solved by applying Dinkhelbach algorithm. However, it will lead to solving a sequence of hard DC programming problems. To overcome this difficulty, we introduce the reachable set for the linear system. In this way, the problem is reduced to a quasiconvex maximization problem in a finite demensional space. Based on a global optimality condition, we propose an effective algorithm for solving this fractional optimal control problem and we show that the algorithm generates a sequence of local optimal controls with improved cost values. The proposed algorithm is then applied to several test problems, where the global optimal cost value is obtained for each case.

Keywords: Fractional optimal control problem, global optimality conditions, reachable set, numerical algorithm

1 Introduction

Consider the fractional programming problem:

$$\max_{x \in D} \frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}, \qquad (1)$$

where $\boldsymbol{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, D is a convex compact subset in $\mathbb{R}^n$, $f(\boldsymbol{x})$ is convex on D and $g(\boldsymbol{x})$ is concave on D, while $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are positive definite for all $\boldsymbol{x} \in D$.

The above problem, which is referred to as Problem (P1), has many applications in engineering and economic. There are numerous methods in the literature for solving Problem (P1). They include variable transformation [1], direct nonlinear programming approach [2], and parametric approach [3]. Problem (P1) has been considered in [4-19], where f is concave and g is convex. Problem (P1) can, in principle, be solved by Dinkhelbach algorithm [12]. But in this way, the algorithm requires solving DC programming at each step which may be harder than solving the original Problem 1. In this paper, we consider a fractional optimal control problem governed by system of linear differential equations, where its cost function is expressed as the ratio of a convex function and a concave function. We introduce the reachable set for the linear system. In this way, the problem is reduced to a quasiconvex maximization problem in a finite demensional space. Based on a global optimality condition, we propose an effective algorithm for solving this fractional optimal control problem and we show that the algorithm generates a sequence of local optimal controls with decreasing cost values. The proposed algorithm is applied to several test problems, where the global optimal cost value is obtained for each case.

The rest of the paper is organized as follows. In Section 2, the formulation of the quadratic fractional optimal control problem is given. The algorithm and numerical results for several test problems are presented in Section 3. Some concluding remarks are stated in Section 4.

2 Fractional Optimal Control Problem

Consider the following system of linear differential equations over the time horizon $[t_0, t_f]$.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u}(t) + \boldsymbol{C}(t) \qquad (2a)$$
$$\boldsymbol{x}(t_0) = \boldsymbol{x}^0 \qquad (2b)$$

where t_0 and t_f are given with $-\infty < t_0 < t_f < +\infty$, $\boldsymbol{x}(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$, $\boldsymbol{u}(t) = [u_1(t), u_2(t), \cdots, u_r(t)]^T \in \mathbb{R}^r$ are, respectively, the state and control, and the elements of the matrix valued functions $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}(t) \in \mathbb{R}^{n\times r}$ and $\boldsymbol{C}(t) \in \mathbb{R}^{n\times 1}$ are piecewise continuous on $[t_0, t_f]$. Let $U \subset \mathrm{R}^r$ be a compact and convex subset. Then, the set of admissible controls is defined by

$$\boldsymbol{u} \in \mathrm{V} = \left\{ \boldsymbol{u} \in L_2^n([t_0, t_f]) \mid \boldsymbol{u}(t) \in U,\ t \in [t_0, t_f] \right\} \qquad (3)$$

where $L_2([t_0, t_f])$ denotes the set of all square integrable functions defined on $[t_0, t_f]$ with values in R^n.

The fractional optimal control problem, which is referred to as Problem (P2), may now be stated formally as follows.

Problem (P2): Given the dynamic system (2), find an admissible control $\boldsymbol{u} \in \mathrm{V}$ such that the cost function

$$\varphi\left(x(t_f)\right) = \frac{\langle \boldsymbol{D}_1 \boldsymbol{x}(t_f), \boldsymbol{x}(t_f) \rangle}{\langle \boldsymbol{D}_2 \boldsymbol{x}(t_f), \boldsymbol{x}(t_f) \rangle + d} \qquad (4)$$

is minimized over V, where $\langle \bullet, \bullet \rangle$ denotes the inner product, $\boldsymbol{D}_1$ is a symmetric positive definite matrix, $\boldsymbol{D}_2$ is a symmetric negative definite matrix, d is a positive constant such that $g_2\left(\boldsymbol{x}(t_f)\right) = \langle \boldsymbol{D}_2 \boldsymbol{x}(t_f), \boldsymbol{x}(t_f) \rangle + d > 0$ for all $\boldsymbol{u} \in \mathrm{V}$, and $g_1\left(\boldsymbol{x}(t_f)\right) = \langle \boldsymbol{D}_1 \boldsymbol{x}(t_f), \boldsymbol{x}(t_f) \rangle > 0$ for all $\boldsymbol{u} \in \mathrm{V}$.

It is well known [20] that the solution of system (2) can be written as:

$$\boldsymbol{x}(t|\boldsymbol{u}) = \boldsymbol{F}(t, t_0)\boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{F}(t, \tau)\left[\boldsymbol{B}(\tau)\boldsymbol{u}(\tau) + \boldsymbol{C}(\tau)\right] d\tau \qquad (5)$$

where $\boldsymbol{F}(t, \tau) \in \mathrm{R}^{n \times n}$ is the fundamental matrix solution of the matrix equation

$$\frac{\partial \boldsymbol{F}(t, \tau)}{\partial t} = \boldsymbol{A}(t)\boldsymbol{F}(t, \tau), \quad t \geq \tau \in [t_0, t_f] \qquad (6a)$$

$$\boldsymbol{F}(\tau, \tau) = \boldsymbol{I} \qquad (6b)$$

Here, $\mathbf{I}$ denotes the identity matix. Note that $\boldsymbol{x}(t|\boldsymbol{u})$ is an absolutely continuous vector-valued function of the time t. It satisfies (2a) almost everywhere on $(t_0, t_f]$ and the initial condition (2b). To continue, define

$$\mathrm{D} = \mathrm{D}(t_f) = \left\{ \boldsymbol{y} \in \mathrm{R}^n \mid \boldsymbol{y} = \boldsymbol{x}(t_f|\boldsymbol{u}),\ \boldsymbol{u} \in \mathrm{V} \right\} \quad (7)$$

D is called the rechable set of system (2) with respect to $\boldsymbol{u} \in \mathrm{V}$. Clearly, $\mathrm{D} \subset \mathrm{R}^n$ is a convex and compact set. Problem (P2) can be written as

$$\max_{x \in D} \varphi(\boldsymbol{x}) \qquad (8)$$

which is referred to as Problem (P3). Define

$$L(\varphi, c) = \{ \boldsymbol{x} \in D \mid \varphi(\boldsymbol{x}) \le c \} \qquad (9)$$

Clearly, $L(\varphi, c)$ is a convex set for each $c > 0$.

Definition 1. [1] A function $h : D \to R$ is said to be quasiconvex if the following inequality

$$h(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \max\{h(\boldsymbol{x}), h(\boldsymbol{y})\}$$

is satisfied for all $\boldsymbol{x}, \boldsymbol{y} \in D$ and $\alpha \in [0,1]$.

Lemma 1. [1] The function $h(\boldsymbol{x})$ is quasiconvex on D if and only if the set $L(h, c)$ is convex for each $c > 0$.

From Lemma 1, it is clear that the function $\varphi(\boldsymbol{x}(\bullet))$ is quasiconvex on D. Thus, Problem (P3) is a quasiconvex maximization problem. Now, we shall apply the global optimality conditions [1,3] to Problem (P3).

Theorem 1. [3] Let

$$E_c(\varphi) = \{ \boldsymbol{y} \in R^n \mid \varphi(\boldsymbol{y}) = c \} \quad (10)$$

Then,

$$\langle \varphi'(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \le 0 \qquad (11)$$

holds for all $\boldsymbol{y} \in E_{\varphi(\boldsymbol{z})}(\varphi)$ and $\boldsymbol{x} \in D$, where φ' denotes the gradient. In addition, suppose that $\varphi'(\boldsymbol{y}) \ne 0$ holds for all $\boldsymbol{y} \in E_{\varphi(\boldsymbol{z})}(\varphi)$. Then, condition (11) is a sufficient condition for $\boldsymbol{z} \in D$ to be a global solution to Problem (P3).

Clearly, condition (11) can be written equivalently as follows:

$$\sum_{i=1}^{n}\left\{\frac{\partial g_1(\boldsymbol{y})}{\partial x_i}g_2(\boldsymbol{y})-\frac{\partial g_2(\boldsymbol{y})}{\partial x_i}g_1(\boldsymbol{y})\right\}\left(\frac{x_i-y_i}{g_2^2(\boldsymbol{y})}\right)\leq 0 \quad (12)$$

for all $\boldsymbol{y}\in E_{\varphi(\boldsymbol{z})}(\varphi)$ and $\boldsymbol{x}\in D$.

Lemma 2. Suppose that for any feasible points $\boldsymbol{x},\boldsymbol{y}\in D$ such that the inequality

$$\langle\varphi'(\boldsymbol{y}),\boldsymbol{x}-\boldsymbol{y}\rangle>0$$

holds. Then, $\varphi(\boldsymbol{x})\geq\varphi(\boldsymbol{y})$.

Proof. On the contrary, assume that $\varphi(\boldsymbol{x})<\varphi(\boldsymbol{y})$. Since φ is quasiconvex, we have

$$\varphi\big(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y}\big)\leq\max\{\varphi(\boldsymbol{x}),\varphi(\boldsymbol{y})\}=\varphi(\boldsymbol{y})$$

By Taylor's formula, there is a neighborhood of the point $\boldsymbol{y}$ on which

$$\varphi\big(\boldsymbol{y}+\alpha(\boldsymbol{x}-\boldsymbol{y})\big)-\varphi(\boldsymbol{y})=\alpha\left(\langle\varphi'(\boldsymbol{y}),\boldsymbol{x}-\boldsymbol{y}\rangle+\frac{o(\alpha\|\boldsymbol{x}-\boldsymbol{y}\|)}{\alpha}\right)\leq 0,$$

for sufficiently small $\alpha>0$, where

$$\lim_{\alpha\to 0}\frac{o(\alpha\|\boldsymbol{x}-\boldsymbol{y}\|)}{\alpha}=0.$$

Therefore, $\langle\varphi'(\boldsymbol{y}),\boldsymbol{x}-\boldsymbol{y}\rangle\leq 0$ which contradicts $\langle\varphi'(\boldsymbol{y}),\boldsymbol{x}-\boldsymbol{y}\rangle>0$. This completes the proof.

Lemma 3. Let $\varphi(\boldsymbol{x})$ be a function defined by (4). Then, it holds that

$$\varphi'(\boldsymbol{x})=\frac{2(\boldsymbol{D}_1-\boldsymbol{D}_2\varphi(\boldsymbol{x}))\boldsymbol{x}}{\langle\boldsymbol{D}_2\boldsymbol{x},\boldsymbol{x}\rangle+d} \qquad (13)$$

Proof. The proof follows readily from the definition of the function $\varphi(\boldsymbol{x})$.

In the numerical computation, we need to find a point $\boldsymbol{y}\in E_{\varphi(\boldsymbol{z})}(\varphi)$ in order to check the validity of the optimality condition (11). To do this, we prove the following assertion.

Lemma 4. Let $\boldsymbol{z}\in D$ and $\boldsymbol{h}\in R^n$ such that $\langle\varphi'(\boldsymbol{z}),\boldsymbol{h}\rangle<0$. Then, there exists a positive number $\alpha>0$ such that

$$\boldsymbol{y} = \boldsymbol{z} + \alpha\boldsymbol{h} \in E_{\varphi(\boldsymbol{z})}(\varphi)$$

Proof. By the definition of $\varphi(\boldsymbol{x})$, we have $\varphi(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathsf{D}$. By Lemma 3, it follows that condition $\langle \varphi'(\boldsymbol{z}), \boldsymbol{h} \rangle < 0$ can be written as

$$\langle \left(\boldsymbol{D}_1 - \boldsymbol{D}_2\varphi(\boldsymbol{z})\right)\boldsymbol{z}, \boldsymbol{h} \rangle < 0 \quad (14)$$

In order to find an α satisfying $\boldsymbol{y} \in E_{\varphi(\boldsymbol{z})}(\varphi)$, we need to solve the equation

$$\varphi(\boldsymbol{z} + \alpha\boldsymbol{h}) = \varphi(\boldsymbol{z}) \quad (15)$$

where $\boldsymbol{z} \in \mathsf{D}$ and $\boldsymbol{h} \in \mathsf{R}^n$, while

$$\varphi(\boldsymbol{z}) = \frac{\langle \boldsymbol{D}_1\boldsymbol{z}, \boldsymbol{z} \rangle}{\langle \boldsymbol{D}_2\boldsymbol{z}, \boldsymbol{z} \rangle + d}. \quad (16)$$

Or equivalently,

$$\varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{z}, \boldsymbol{z} \rangle + \varphi(\boldsymbol{z})d = \langle \boldsymbol{D}_1\boldsymbol{z}, \boldsymbol{z} \rangle \quad (17)$$

Since $\boldsymbol{D}_1^T = \boldsymbol{D}_1$ and $\boldsymbol{D}_2^T = \boldsymbol{D}_2$, it follows that the expression

$$\frac{\langle \boldsymbol{D}_1(\boldsymbol{z} + \alpha\boldsymbol{h}), \boldsymbol{z} + \alpha\boldsymbol{h} \rangle}{\langle \boldsymbol{D}_2(\boldsymbol{z} + \alpha\boldsymbol{h}), (\boldsymbol{z} + \alpha\boldsymbol{h}) \rangle + d} = \varphi(\boldsymbol{z}) \, (18)$$

becomes

$$\begin{aligned} &\langle \boldsymbol{D}_1\boldsymbol{z}, \boldsymbol{z} \rangle + 2\alpha\langle \boldsymbol{D}_1\boldsymbol{z}, \boldsymbol{h} \rangle + \alpha^2\langle \boldsymbol{D}_1\boldsymbol{h}, \boldsymbol{h} \rangle \\ &= \varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{z}, \boldsymbol{z} \rangle + 2\alpha\varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{z}, \boldsymbol{h} \rangle + \alpha^2\varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{h}, \boldsymbol{h} \rangle + \varphi(\boldsymbol{z})d \end{aligned} \quad (19)$$

Substituting (17) into (19), we obtain

$$2\alpha\langle \boldsymbol{D}_1\boldsymbol{z}, \boldsymbol{h} \rangle + \alpha^2\langle \boldsymbol{D}_1\boldsymbol{h}, \boldsymbol{h} \rangle = 2\alpha\varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{z}, \boldsymbol{h} \rangle + \alpha^2\varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{h}, \boldsymbol{h} \rangle$$

Since $\alpha \neq 0$, we have

$$\alpha = \frac{2[\langle (\varphi(\boldsymbol{z})\boldsymbol{D}_2 - \boldsymbol{D}_1)\boldsymbol{z}, \boldsymbol{h} \rangle]}{\langle \boldsymbol{D}_1\boldsymbol{h}, \boldsymbol{h} \rangle - \varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{h}, \boldsymbol{h} \rangle} = -\frac{2\langle (\boldsymbol{D}_1 - \varphi(\boldsymbol{z})\boldsymbol{D}_2)\boldsymbol{z}, \boldsymbol{h} \rangle}{\langle \boldsymbol{D}_1\boldsymbol{h}, \boldsymbol{h} \rangle - \varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{h}, \boldsymbol{h} \rangle} \quad (20)$$

Furthermore, since $\varphi(\boldsymbol{z}) > 0$ and $\boldsymbol{D}_2 < 0$, it is clear that

$$\langle \boldsymbol{D}_1\boldsymbol{h}, \boldsymbol{h} \rangle - \varphi(\boldsymbol{z})\langle \boldsymbol{D}_2\boldsymbol{h}, \boldsymbol{h} \rangle > 0 \, (21)$$

This, in turn, implies that $\alpha > 0$. Thus, the proof is completed.

Remark 1. Note that the condition assumed in Lemma 4 (i.e., $\langle \varphi'(\boldsymbol{z}), \boldsymbol{h} \rangle < 0$) is fulfilled if we take $\boldsymbol{h} = \bar{\boldsymbol{x}} - \boldsymbol{z}$ where $\boldsymbol{z}$ is a local maximum point and $\bar{\boldsymbol{x}} \in \mathsf{D}$. Thus, it follows that $\langle \varphi'(\boldsymbol{z}), \boldsymbol{x}-\boldsymbol{z}\rangle \le 0$, $\forall \boldsymbol{x} \in \mathsf{D}$.

Let $\boldsymbol{u}^*$ be an admissible control which is a global optimal control to Problem (P3) and let $\boldsymbol{x}^*$ be the corresponding solution of system (2).

Introduce an auxiliary function $\Pi(\boldsymbol{y})$ defined by

$$\Pi(\boldsymbol{y}) = \max_{\boldsymbol{x}\in \mathsf{D}} \langle \varphi'(\boldsymbol{y}), \boldsymbol{x}-\boldsymbol{y}\rangle, \quad \boldsymbol{y} \in \mathsf{R}^n \qquad (22)$$

Then, based on Theorem 1, we can derive the global optimality conditions for Problem (P2) in the following theorem.

Theorem 2. A control $\boldsymbol{u}^* \in \mathsf{V}$ is a global optimal control to Problem (P2) if and only if

$$\max\left\{\Pi(\boldsymbol{y}) \mid \boldsymbol{y} \in E_{\varphi(\boldsymbol{x}^*)}(\varphi)\right\} \le 0 \, (23)$$

where $\boldsymbol{x}^* = \boldsymbol{x}^*(t_f, \boldsymbol{u}^*) \in \mathsf{D}(t_f)$.

Proof. The validity of Theorem 2 is equivalent to that of the optilimality condition (11).

From Theorem 2, we can conclude that if there exist a process $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ and $\tilde{\boldsymbol{y}} \in E_{\varphi(\bar{x})}(\varphi)$ such that

$$\langle \varphi'(\tilde{\boldsymbol{y}}), \tilde{\boldsymbol{x}}-\tilde{\boldsymbol{y}}\rangle > 0 \qquad (24)$$

then the control $\bar{\boldsymbol{u}}$ is not a global optimal control to Problem (P2), where $\tilde{\boldsymbol{x}} = \boldsymbol{x}(t_f \,|\tilde{\boldsymbol{u}})$, $\bar{\boldsymbol{x}} = \boldsymbol{x}(t_f \,|\bar{\boldsymbol{u}})$, and $\tilde{\boldsymbol{u}}, \bar{\boldsymbol{u}} \in \mathsf{V}$

Example 1. Consider the problem

$$\max\left\{\varphi\left(\boldsymbol{x}(1)\right) = \frac{x_1^2(1)+x_2^2(1)}{2x_1(1)+4x_2(1)}\right\} \qquad (25)$$

or the equivalent problem

$$\min\left\{-\varphi\left(\boldsymbol{x}(1)\right) = -\frac{x_1^2(1)+x_2^2(1)}{2x_1(1)+4x_2(1)}\right\} \qquad (26)$$

where

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \end{cases} \qquad (27a)$$

with initial condition

$$\begin{cases} x_1(0) = 0 \\ x_2(0) = 1 \end{cases} \text{(27b)}$$

The set of admissible control is defined by

$$\boldsymbol{u} \in \mathrm{V} = \left\{ \boldsymbol{u} \in L_2^2([0,1]) \mid 0 \le u_1(t) \le 7, 0 \le u_2(t) \le 4, t \in [0,1] \right\} \qquad (28)$$

We can easily check that this problem has three controls $\boldsymbol{u}^0 = [0,4]^T$, $\boldsymbol{u}^1 = [7,4]^T$ and $\boldsymbol{u}^2 = [7,0]^T$ which satisfy the maximum principle. Now let us check whether the control $\boldsymbol{u}^0 = [0,4]^T$ is a global optimal or not. To do this, we first solve system (27) for $\boldsymbol{u} = \boldsymbol{u}^0$, yielding

$$x_1(t) = 0, \quad x_2(t) = 1 + 4t, \quad t \in [0,1]$$

Thus, $x_1(1) = 0$, $x_2(1) = 5$ and $\varphi(\boldsymbol{x},1) = 5/4$. The level set $E_{\varphi(\boldsymbol{x})}(\varphi)$ at the point $\boldsymbol{x}(1,\boldsymbol{u}^0)$ is

$$E_{\varphi(\boldsymbol{x}(1))}(\varphi) = \left\{ \boldsymbol{y} \in \mathrm{R}^2 \mid \frac{y_1^2 + y_2^2}{2y_1 + 4y_2} = \frac{5}{4} \right\}.$$

It is easy to verifi that $\tilde{\boldsymbol{y}} = [5/2, 5]^T \in E_{\varphi(\boldsymbol{x}(1))}(\varphi)$.

Now, we find a point $\tilde{\boldsymbol{u}} = [6,4]^T \in \mathrm{V}$. Let $\tilde{\boldsymbol{x}}$ be the corresponding trajectory obtained from (27). That is,

$$\tilde{x}_1(t) = 6t, \ \tilde{x}_2(t) = 1 + 4t, \ t \in [0,1]$$

Then, $\tilde{x}_1(1) = 6$, $\tilde{x}_2(1) = 5$.

Now, taking the partial derivative of φ with respect to the components of $\boldsymbol{x}$, we obtain

$$\begin{cases} \dfrac{\partial \varphi\left(x(1)\right)}{\partial x_1} = \dfrac{2x_1^2 + 8x_1x_2 - 2x_2^2}{(2x_1 + 4x_2)^2} \\[2ex] \dfrac{\partial \varphi\left(x(1)\right)}{\partial x_2} = \dfrac{4x_2^2 + 4x_1x_2 - 4x_1^2}{(2x_1 + 4x_2)^2} \end{cases}$$

Computing $\langle \varphi'(\tilde{\boldsymbol{y}}), \tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}} \rangle$, we obtain

$$\langle \varphi'(\tilde{\boldsymbol{y}}), \tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}} \rangle = \frac{2\tilde{y}_1^2 + 8\tilde{y}_1\tilde{y}_2 - 2\tilde{y}_2^2}{(2\tilde{y}_1 + 4\tilde{y}_2)^2}\left(\tilde{x}_1 - \tilde{y}_1\right) + \frac{4\tilde{y}_2^2 + 4\tilde{y}_1\tilde{y}_2 - 4\tilde{y}_1^2}{(2\tilde{y}_1 + 4\tilde{y}_2)^2}\left(\tilde{x}_2 - \tilde{y}_2\right)$$

$$= \frac{875}{2500} > 0$$

This implies that the control $\boldsymbol{u}^0 = [0,4]^T$ is not a global control to our problem.

In fact, the control $\boldsymbol{u}^1 = [7,4]^T$ is a global control with the cost value of $\varphi\left(\boldsymbol{x}(1,\boldsymbol{u}^1)\right) = 74/37$.

Before we derive an algorithm to solving Problem (P2), we need to compute $\Pi(\boldsymbol{y})$ for any $\boldsymbol{y} \in \mathrm{R}^n$.

First, we consider the linear optimal control problem, which is referred to as Problem (P4).

$$\max_{\boldsymbol{x} \in \mathrm{D}} \langle \varphi'(\boldsymbol{y}), \boldsymbol{x} \rangle \qquad (29)$$

Consider the following system of differential equations

$$\begin{cases} \dot{\boldsymbol{\psi}} = -\boldsymbol{A}^T \boldsymbol{\psi} \\ \boldsymbol{\psi}(t_f) = -\boldsymbol{\varphi}'(\boldsymbol{y}) \end{cases} \qquad (30)$$

Corresponding to $\boldsymbol{y} \in \mathrm{R}^n$. This system, which is known as the co-state system, has a unique piecewise differentiable solution $\boldsymbol{\psi}(t) = \boldsymbol{y}(t|\boldsymbol{y})$, defined on $[t_0, t_f]$, where $\boldsymbol{\psi}(t,\boldsymbol{y}) = \left[\psi_1(t), \cdots, \psi_n(t)\right]^T$. $\boldsymbol{\psi}(t)$ is referred to as the co-state. Problem (P4) can be solved by using the results presented in the following theorem.

Theorem 3. [1] Let $\boldsymbol{\psi}(t) = \boldsymbol{\psi}(t|\boldsymbol{y})$, $t \in [t_0, t_f]$, be a solution of the co-state system (30) for $\boldsymbol{y} \in \mathrm{R}^n$. Let $\boldsymbol{z}(t) = \boldsymbol{z}(t|\boldsymbol{y})$ be an admissible control. $\boldsymbol{z}(t)$ is an optimal control to Problem (P4), then it is necessary and sufficient that

$$\langle \boldsymbol{\psi}(t|\boldsymbol{y}), \boldsymbol{B}(t)\boldsymbol{z}(t|\boldsymbol{y}) \rangle = \min_{\boldsymbol{u} \in V} \langle \boldsymbol{\psi}(t|\boldsymbol{y}), \boldsymbol{B}(t)\boldsymbol{u}(t) \rangle \quad (31)$$

for almost every $t \in [t_0, t_f]$.

On the basis of Theorem 3, the value $\Pi(\boldsymbol{y})$ can be computed by using the following algorithm.

Algorithm 1.

Step 1. Solve the co-state system (30) for a given $\boldsymbol{y} \in \mathsf{R}^n$. Let $\boldsymbol{\psi}(t) = \boldsymbol{\psi}(t \mid \boldsymbol{y})$ be the solution.

Step 2. Find the optimal control $\boldsymbol{z}(t) = \boldsymbol{z}(t \mid \boldsymbol{y})$ as the solution of the problem

$$\min_{\boldsymbol{u} \in \mathsf{U}} \langle \boldsymbol{\psi}(t), \boldsymbol{B}(t)\boldsymbol{u}(t) \rangle$$

at each $t \in [t_0, t_f]$.

Step 3. Find a solution $\boldsymbol{x}(t) = \boldsymbol{x}(t \mid \boldsymbol{z})$ of system (2) for $\boldsymbol{u}(t) = \boldsymbol{z}(t \mid \boldsymbol{y})$.

Step 4. Find $\boldsymbol{x}(t_f) = \boldsymbol{x}(t_f \mid \boldsymbol{z})$ by (5) with $t = t_f$.

Step 5. Compute $\Pi(\boldsymbol{y})$ by the formula $\Pi(\boldsymbol{y}) = \langle \varphi'(\boldsymbol{y}), \boldsymbol{x}(t_f) - \boldsymbol{y} \rangle$.

3 Solution Computation

Definition 2. For a given integer m, let $A_{\boldsymbol{z}}^m$ be the set defined by

$$A_{\boldsymbol{z}}^m = \left\{ \boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^m \mid \boldsymbol{y}^i \in E_{\varphi(\boldsymbol{z})}(\varphi) \cap \mathsf{D}, i = 1, 2, \ldots, m \right\}$$

Then, it is called an approximation set, where $\boldsymbol{z} = \boldsymbol{x}(t_f \mid \boldsymbol{u}),\ \boldsymbol{u} \in \mathsf{V}$.

Lemma 5. Suppose that there exist a feasible point $\boldsymbol{z} \in \mathsf{D}$ and a point $\boldsymbol{y}^i \in A_{\boldsymbol{z}}^m$ such that

$$\langle \varphi'(\boldsymbol{y}^j), \boldsymbol{u}^j - \boldsymbol{y}^j \rangle > 0$$

Then, $\varphi(\boldsymbol{u}^j) > \varphi(\boldsymbol{z})$, where $\langle \varphi'(\boldsymbol{y}^j), \boldsymbol{u}^j \rangle = \max_{\boldsymbol{x} \in \mathsf{D}} \langle \varphi'(\boldsymbol{y}^j), \boldsymbol{x} \rangle$.

Proof. The proof follows readily from Lemma 2.

The algorithm for solving Problem (P2) may now be stated as follows.

Algorithm 2

Step 1. Let $k := 0$ and let $\bar{\boldsymbol{u}}^k \in \mathsf{V}$ be an arbitary given control. Starting with the control $\bar{\boldsymbol{u}}^k$, we find a local optimal control $\boldsymbol{u}^k$ by using the optimal control software package, MISER3 [21, 22].

Step 2. Find $\boldsymbol{x}^k = \boldsymbol{x}(t_f \,|\, \boldsymbol{u}^k)$ by solving system (2) for $\boldsymbol{u} = \boldsymbol{u}^k$.

Step 3. Construct the approximation set $A^m_{\boldsymbol{x}^k}$ as follows:

$$A^m_{\boldsymbol{x}^k} == \left\{ \boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^m \mid \boldsymbol{y}^i \in E_{\varphi(\boldsymbol{x}^k)}(\varphi) \cap \mathsf{D}(t_f), i = 1, 2, \ldots, m \right\}.$$

Step 4. Solve the linear optimal control problems

$$\max_{\boldsymbol{x} \in \mathsf{D}(t_f)} \langle \varphi'(\boldsymbol{y}^i), \boldsymbol{x} \rangle, \quad i = 1, 2, \ldots, m.$$

Step 5. Compute $\Pi(\boldsymbol{y}^i)$, $i = 1, 2, \ldots, m$, by Algorithm 1.

Step 6. Compute η_k:

$$\eta_k = \Pi(\boldsymbol{y}^i) = \max_{1 \le i \le m} \Pi(\boldsymbol{y}^i).$$

Let $\boldsymbol{z}^j = \boldsymbol{z}^j(t \,|\, \boldsymbol{y}^j)$ be the solution of the problem:

$$\langle \boldsymbol{\psi}^j(t), \boldsymbol{B}(t) \boldsymbol{z}^j \rangle = \min_{\boldsymbol{u} \in \mathsf{U}} \langle \boldsymbol{\psi}^j(t), \boldsymbol{B}(t) \boldsymbol{u}(t) \rangle, \, t \in [t_0, t_f],$$

where

$$\begin{cases} \dot{\boldsymbol{\psi}}^j(t) = -\boldsymbol{A}^T(t) \boldsymbol{\psi}^j(t) \\ \boldsymbol{\psi}^j(t_f) = -\boldsymbol{\varphi}'(\boldsymbol{y}^j) \end{cases}$$

Step 7. If $\eta_k \le 0$, then terminate. $\boldsymbol{u}^k$ is a global approximate solution; otherwise, go to next step.

Step 8. Set $\bar{\boldsymbol{u}}^{k+1} := \boldsymbol{z}^j(t \,|\, \boldsymbol{y}^j)$ and $k := k + 1$. Then, go to Step 2.

Lemma 6. Suppose that there is a point $\boldsymbol{y}^j \in A^m_{\boldsymbol{x}^k}$ for $\boldsymbol{u}^k \in \mathsf{D}(t_f)$ such that

$\langle \varphi'(\boldsymbol{y}^j), \boldsymbol{x}(t_f, \boldsymbol{z}^j) - \boldsymbol{y}^j \rangle > 0$, where $\boldsymbol{z}^j$ satisfies $\langle \varphi'(\boldsymbol{y}^j), \boldsymbol{x}(t_f | \boldsymbol{z}^j) \rangle = \min_{\boldsymbol{x}(t_f) \in D} \langle \varphi'(\boldsymbol{y}^j), \boldsymbol{x} \rangle$.

Then, it holds that

$$\varphi\left(\boldsymbol{x}(t_f | \boldsymbol{z}^j) \right) > \varphi\left(\boldsymbol{x}^k(t_f | \boldsymbol{z}^k) \right)$$

Proof. From Lemma 2, we have

$$\langle \varphi'(\boldsymbol{y}^j), \boldsymbol{x}(t_f | \boldsymbol{z}^j) - \boldsymbol{y}^j \rangle > 0$$

Thus,

$$\varphi\left(\boldsymbol{x}(t_f | \boldsymbol{z}^j) \right) \geq \varphi\left(\boldsymbol{y}^j \right) = \varphi\left(\boldsymbol{x}^k(t_1 | \boldsymbol{u}^k) \right).$$

This completes the proof.

Theorem 4. If $\eta_k > 0$ for all $k = 1, 2, \ldots, s$, then the sequence $\left\{ \boldsymbol{J}(\boldsymbol{u}^k) \right\}$ constructed by Algorithm 2 is a monotonically increasing sequence, i.e.,

$$J(\boldsymbol{u}^{k+1}) \geq J(\boldsymbol{u}^k), k = 1, 2, \ldots, s$$

where $J(\boldsymbol{u}^k) = \varphi\left(\boldsymbol{x}(t_f | \boldsymbol{u}^k) \right)$.

Remark 2. If the functions $g_1\left(\boldsymbol{x}(t_f) \right) > 0$ and $g_2\left(\boldsymbol{x}(t_f) \right) > 0$ for all $\boldsymbol{x} \in \mathrm{R}^n$, then it follows from (20) that

$$A^m_{\boldsymbol{x}^k} = \left\{ \boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^m \mid \boldsymbol{y}^i \in E_{\varphi(\boldsymbol{x}^k)}(\varphi), i = 1, 2, \ldots, m \right\}$$

where $\boldsymbol{y}^i = \boldsymbol{x}^k + \alpha_i h^i$, $\alpha_i > 0$, $i = 1, 2, \ldots, m$, and

$$\alpha_i = \frac{2\left[\langle \left(\varphi\left(\boldsymbol{x}^k\left(t_f | \boldsymbol{u}^k \right) \right) \boldsymbol{D}_2 - \boldsymbol{D}_1 \right) \boldsymbol{x}^k\left(t_f | \boldsymbol{u}^k \right), \boldsymbol{h}^i \rangle \right]}{\langle \boldsymbol{D}_1 \boldsymbol{h}^i, \boldsymbol{h}^i \rangle - \varphi\left(\boldsymbol{x}^k\left(t_f | \boldsymbol{u}^k \right) \right) \langle \boldsymbol{D}_2 \boldsymbol{h}^i, \boldsymbol{h}^i \rangle}$$

Numerical Examples

In this section, Algorithm 2 is used to solve two fractional optimal control problems. The computer used in the numerical computation is DELL desktop computer with Intel Core I5 CPU (2.67GHZ) and 3GB RAM. All the calculations are done within the Matlab environment. The optimal control software package, MISER 3, is used for our local search in Step 1. The two fractional optimal control problems are shown as following.

Example 1. Consider an optimal control, where the cost function

$$\varphi\left(\boldsymbol{x}(2)\right) = \frac{\langle \boldsymbol{D}_1 \boldsymbol{x}(2), \boldsymbol{x}(2)\rangle}{\langle \boldsymbol{D}_2 \boldsymbol{x}(2), \boldsymbol{x}(2)\rangle + 15000},$$

where

$$\boldsymbol{D}_1 = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix},\ \boldsymbol{D}_2 = \begin{pmatrix} -2 & 1 \\ -1 & 4 \end{pmatrix}$$

is maximized over $\mathsf{V} = \left\{\boldsymbol{u} \in \mathsf{R}^2 \mid -4 \le u_i \le 7, i = 1, 2, t \in [0, 2]\right\}$ subject to the dynamic system

$$\begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = x_1 + u_2 \end{cases}$$

with initial condition $x_1(0) = x_2(0) = 0$.

Example 2. Consider an optimal control, where the cost function

$$\varphi\left(\boldsymbol{x}(1)\right) = \frac{\langle \boldsymbol{D}_1 \boldsymbol{x}(1), \boldsymbol{x}(1)\rangle}{\langle \boldsymbol{D}_2 \boldsymbol{x}(1), \boldsymbol{x}(1)\rangle + 10000}$$

is maximized over $\mathsf{V} = \left\{\boldsymbol{u} \in \mathsf{R}^2 \mid -1 \le u_i \le 1, i = 1, 2, 3u_1 + 4u_2 \le 6, t \in [0,1]\right\}$ subject to the dynamic system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 - u_1 + u_2 \\ \dot{x}_2 = -x_1 + 2x_2 + 2u_1 - 3u_2 \end{cases}$$

with initial condition $x_1(0) = 2,\ x_2(0) = 1$.

Based on the local optimal controls obtained by MISER 3, the two optimal control problems are solved using the optimization procedures listed in Algorithm 2. Global optimal solutions are found for both of the two problems, see Table 1.

Table 1

Example	Local value	Global value	Computing

	of ψ	of ψ	time (min:sec)
1	$1.4958e-004$	$3.3348e+000$	$00:11.1144$
2	$1.6639e-001$	$2.0531e-001$	$00:02.8093$

4 Conclusions

Fractional optimal control problem has been considered. The problem was reduced to a quasiconvex maximization problem subject to linear constraints. Based on the global optimality conditions and properties of the quasiconvex function, we derived an effective algorithm for solving the problem globally. The numerical results are given to illustrate the applicability of the algorithm proposed.

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References

[1] Enkhbat, R.: Quasiconvex progarmming, Lambert Publisher, 2009.

[2] Ibaraki, T.: Parametric approaches to fractional programs, Mathematical Programming, **26**, 345-362, 1983.

[3] Enkhbat, R., Ibaraki, T.: On the maximization and minimization of quasiconvex function, International Journal of Nonlinear and Convex Analysis, 43-76, 2003.

[4] Donald, W.K.: The Walrasion Vision of the Microeconomy, The university of Michigan Press, 1994.

[5] Bector, C.R.: Duality in nonlinear fractional programming, Zeitschrift fur Operations Research, **17**, 183-193, 1973.

[6] Nicolas, H., Sandar, K., and Siegfried, S.: Handbook of Generalized Convexity and Generalized Monotonicity, Springer, 2005.

[7] Almogy, Y. and Levin, O.: A class of fractional programming problems,

Operations Research, **19**, 57-67, 1971.

[8] Zhang, A. and Hayashi, S.: Celis-Dennis-Tapia based approach to quadratic fractional programming problems with two quadratic constraints, Journal of Industrial and Management Optimization, 1(1), 83 - 98, 2011.

[9] Benson, H.P.: Global optimization algorithm for the nonlinear sum of ratios problems, Journal of Optimization Theory and Applications, **112**, 1-2, 2002.

[10] Cambini, A., Castagnoli, E., Martein, L., Mazzoleni, P. and Schaible, S.: Generalized convexity and fractional programming with economic applications, Lecture Notes in Economics and Mathematical Systems, 345, Springer, Berlin, 1990.

[11] Kheirfam, B.: Multi-parametric sensitivity analysis of the constraint matrix in piecewise linear fractional programming, Journal of Industrial and Management Optimization, 6(2), 347 - 361, 2010.

[12] Crouzeix, J.P., Ferland, J.A. and Schaible, S.: Duality in generalized linear fractional programming, Mathematical Programming, **27**, 342-354, 1983.

[13] Craven, B.D.: fractional Programming, Sigma Series in Aplied Mathematics, 4, Heldermann Verlag, 1988.

[14] Dinkelbach, W.: On nonlinear fractional programming, Management Science, **13**, 492-498, 1967.

[15] Hadjisavvas, N., Martinez-Legaz, J.E. and Penot, J.P.: Generalized convexity and generalized monotonicity, Lecture Notes in Economics and Mathematical Systems, 502, Springer, Berlin, 2001.

[16] Long, X.J. and Quan, J.: Optimality conditions and duality for minimax fractional programming involving nonsmooth generalized univexity, Journal of Industrial and Management Optimization, 1(3), 361 - 370, 2011.

[17] Horst, R. and Tuy, H.: Global Optimization: Deterministic Approaches, Springer, Berlin, 1993.

[18] Konno, H. and Kuno, T.: Generalized linear multiplicative and fractional programming, Annals of Operations Research, **25**, 147-161, 1990.

[19] Schaible, S.: Fractional programming: applications and algorithms, European Journal of Operational Research, **7**, 111-120, 1981.

[20] Ahmed

[21] Jennings, L.S., Fisher, M.E., Teo, K.L. and Goh, C.J. (1990). MISER3 Optimal Control Software: Theory and User Manual, University of Western Australia, Perth, Australia.

[22] K. L. Teo, C. J. Goh and K. H. Wong, A Uni_ed Computational Approach to Optimal Control Problems, 1st Edition, Longman Scienti_c and Technical, New York, 1991.